RANK PROCEDURES FOR SOME TWO-POPULATIONS MULTIVARIATE EXTENDED CLASSIFICATION PROBLEMS

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Multivariate Extended Classification

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Abstract

Given independent samples from three multivariate populations with cumulative distribution functions $F^{(1)}(\underset{\sim}{x})$, $F^{(2)}(\underset{\sim}{x})$, and $F^{(0)}(\underset{\sim}{x}) = \theta F^{(1)}(\underset{\sim}{x}) + (1-\theta)F^{(2)}(\underset{\sim}{x})$, where $0 \leq \theta \leq 1$ is unknown, the three-action problem involving decision as to whether the value of θ is high, low, or intermediate, is considered. A class of consistent procedures based on the relative spacing of three sample averages of linearly compounded rank scores is formulated. The asymptotic operating characteristics of the procedures when $F^{(1)}$ and $F^{(2)}$ come close together are studied and the best choice of the compounding coefficients in terms of these considered. The consequence of using estimates of the best coefficients on the asymptotic operating characteristics is also examined.

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1. INTRODUCTION

Suppose there are three p-variate populations with continuous cumulative distribution functions (cdf) $F^{(0)}(\underset{\sim}{x})$, $F^{(1)}(\underset{\sim}{x})$, and $F^{(2)}(\underset{\sim}{x})$, $\underset{\sim}{x} = (x_1,\ldots,x_p)'$. Of these $F^{(1)}$ and $F^{(2)}$ are known to be distinct and $F^{(0)}$ is known to be an unknown mixture of $F^{(1)}$ and $F^{(2)}$. That is

$$F^{(0)}(\underset{\sim}{x}) = \theta\, F^{(1)}(\underset{\sim}{x}) + (1-\theta)F^{(2)}(\underset{\sim}{x}) \tag{1.1}$$

for some unknown θ, $(0\leq\theta\leq1)$. There are three possible decisions or actions d_0, d_1, d_2: d_1 is preferred when the value of θ is high, d_2 is preferred when the value of θ is low, and d_0 is preferred when the value of θ is intermediate. Independent random samples of sizes n_0, n_1, n_2 are available from $F^{(0)}$, $F^{(1)}$, and $F^{(2)}$ respectively. Let these be

$$\underset{\sim}{X}_{\alpha}^{(k)} = (X_{1\alpha}^{(k)},\ldots,X_{p\alpha}^{(k)}),\quad \alpha=1,\ldots,n_k,\quad k=0,1,2. \tag{1.2}$$

Let the $N = n_0+n_1+n_2$ observations corresponding to the i-th variable be ranked together and let the rank obtained by $X_{i\alpha}^{(k)}$ be $R_{i\alpha}^{(k)}$. Thus the samples (1.2) give rise to the rank vectors

$$(R_{1\alpha}^{(k)},\ldots,R_{p\alpha}^{(k)})',\quad \alpha=1,\ldots,n_k,\quad k=0,1,2. \tag{1.3}$$

In this paper we shall formulate and study certain decision rules for choosing one of d_0, d_1, and d_2 on the basis of the rank vectors (1.3).

The above decision problem can obviously be considered as an extended version of the nonparametric classification problem. (For the standard classification problem θ can have only 1 or 0 as its possible values.) In many practical situations it would be realistic to assume that the 0-th sample contains observations from the two basic populations in unknown numbers and our interest would be to determine, which, if any, of the two populations is preponderantly represented. There, a formulation such as above would be natural.

Further, we shall make the preference pattern more specific as follows. We shall suppose that there are two pairs of numbers (L_1, U_1), (L_2, U_2), $0 \leq L_2 < U_2 < L_1 < U_1 \leq 1$, such that (i) d_1 is preferred to both d_0, d_2 for $\theta \geq U_1$ and only to d_2 for $L_1 < \theta < U_1$, (ii) d_2 is preferred to both d_0, d_1 for $\theta \leq L_2$ and only to d_1 for $L_2 < \theta < U_2$, (iii) d_0 is preferred to both d_1, d_2 for $U_2 \leq \theta \leq L_1$, only to d_1 for $\theta < U_2$, and only to d_2 for $\theta > L_1$. There is no preference between d_1 and d_0 for $L_1 < \theta < U_1$, and similarly no preference between d_2 and d_0 for $L_2 < \theta < U_2$. With such specification the above problem can be considered as a monotone three-decision problem (see, for instance Ferguson [3], chapter 6).

The results of the present paper are closely related to those of Chatterjee [1], where rank methods for estimation of θ under the above set up are considered, and we would have many occasions to refer to [1].

2. A CLASS OF DECISION RULES

We start with a compounding vector $\underset{\sim}{\ell}(p\times 1)\neq\underset{\sim}{o}$, two numbers θ_1, θ_2 $(0<\theta_2<\theta_1<1)$, and a score matrix

$$\underset{\sim}{A}(p\times N) = (a_i(\alpha))_{\substack{i=1,\ldots,p \\ \alpha=1,\ldots,N}}. \tag{2.1}$$

With the help of the score matrix we derive from the rank vectors the (random) rank score vectors

$$\begin{aligned} \underset{\sim}{a}^{(k)} &= (a_1(R_{1\alpha}^{(k)}),\ldots,a_p(R_{p\alpha}^{(k)}))' \\ &= (a_{1\alpha}^{(k)},\ldots,a_{p\alpha}^{(k)})' \text{ (say)}, \quad \alpha=1,\ldots,n_k;\ k=0,1,2 \end{aligned} \tag{2.2}$$

and hence their means for the three samples

$$\bar{\underset{\sim}{a}}^{(k)} = (\bar{a}_1^{(k)},\ldots,\bar{a}_p^{(k)}) = \frac{1}{n_k}\sum_{\alpha=1}^{n_k} \underset{\sim}{a}_\alpha^{(k)}, \quad k=0,1,2. \tag{2.3}$$

Whenever, $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)} \neq \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$, we write

$$(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)} - \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}) \Big/ (\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)} - \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}) = T(\underset{\sim}{\ell}) \tag{2.4}$$

The decision rule is formulated as follows.

(a) When $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)} \neq \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$, (i) choose d_1 if $T(\underset{\sim}{\ell}) \geq \theta_1$

(ii) choose d_2 if $T(\underset{\sim}{\ell}) \leq \theta_2$

(iii) choose d_0 if $\theta_2 < T(\underset{\sim}{\ell}) < \theta_1$

(b) When $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)} = \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$ choose one of d_1, d_2, d_0 at random. (2.5)

An examination of (2.5) shows that what is really suggested here is: choose d_1 when $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}$ is close to $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}$ and away from $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$; choose d_2 when $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}$ is close to $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$ and away from $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}$; choose d_0 otherwise. In fact, for any real number q if we write

$$\begin{aligned} [q]_{0,1} &= q \quad \text{when} \quad 0 \leq q \leq 1 \\ &= 0 \quad \text{when} \quad q < 0 \\ &= 1 \quad \text{when} \quad q > 1, \end{aligned} \tag{2.6}$$

then $\tilde{\theta}(\underset{\sim}{\ell}) = [T(\underset{\sim}{\ell})]_{0,1}$ can be considered as a reasonable estimate of θ (see [1]). Since $0 < \theta_2 < \theta_1 < 1$, the above decision rule can be described equivalently by replacing $T(\underset{\sim}{\ell})$ by $\tilde{\theta}(\underset{\sim}{\ell})$ in (2.5) and this explains the motivation directly.

For future use we note that the above decision rule can be reformulated as follows.

(a) When $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)} \neq \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$, (i) choose d_1 if $(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)})(\theta_1\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}+(1-\theta_1)\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}\bar{\underset{\sim}{a}}^{(0)}) \geq 0$

(ii) choose d_2 if $(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)})(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}-\theta_2\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}-(1-\theta_2)\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}) \geq 0$

(iii) choose d_0 if $(\theta_1\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}+(1-\theta_1)\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}) \times (\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}-\theta_2\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}-(1-\theta_2)\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}) > 0$

(b) When $\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)} = \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}$, choose one of d_1, d_2, d_0 at random. (2.7)

In case (a), writing

$$T(\underset{\sim}{\ell})= (\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)})(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}) \Big/ (\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)} - \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)})^2$$

and noting that

$$(\theta_1\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}+(1-\theta_1)\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)})+(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(0)}-\theta_2\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)}-(1-\theta_2)\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}) = -(\theta_1-\theta_2)(\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}^{(1)})$$

it may be verified that the rules (2.5) and (2.7) are equivalent.

In practice, before using a procedure such as above, one will have to specify, apart from the score matrix $\underset{\sim}{A}$, the numbers θ_1, θ_2 and the vector $\underset{\sim}{\ell}$. As regards θ_1, θ_2, it is obvious that by increasing the value of θ_1 and decreasing the value of θ_2, one can increase the probability of choosing d_0 at the cost respectively of the probabilities of choosing d_1 and d_2. In the next section we shall see how θ_1, θ_2 should be chosen in relation to the preference scale so as to achieve consistency. The best choice of $\underset{\sim}{\ell}$ for the above rule is not immediately apparent. In [1], for the problem of estimation of θ under the above set-up, the best choice of $\underset{\sim}{\ell}$ in the estimate $\tilde{\theta}(\underset{\sim}{\ell})$ defined above was considered from the point of view of minimisation of asymptotic variance when $0<\theta<1$. This led to the simultaneous determination of $\underset{\sim}{\ell}$ and the estimate of θ by solving $(p+1)$ (nonlinear) equations.

In the present context, we shall take up this problem after we formulate and study the local operating characteristics for procedures of the above form.

Before concluding this section we note here an interpretation of the above decision rule. Under (1.1) the process of taking a sample of size n_0 from $F^{(0)}$ could be interpreted as that of performing n_0 Bernoulli trials with success probability θ and taking r and n_0-r observations from $F^{(1)}$ and $F^{(2)}$ respectively if the trials show just r successes. If r were observable it would be a sufficient statistic for θ and it is known that here the class of monotone procedures based on r is essentially complete (see e.g., [3], Chapter 6). The above procedure can be looked upon as a (behavioral) randomized procedure based on r, which we have to adopt due to the unobservability of r.

3. CONSISTENCY OF THE DECISION RULES

For the problem of section 1, consider a sequence of triplets of sample sizes $(n_{0\nu},n_{1\nu},n_{2\nu})$, $\nu=1,2\ldots$, $n_{0\nu}+n_{1\nu}+n_{2\nu} = N_\nu$, $N_\nu\to\infty$ as $\nu\to\infty$, and a corresponding sequence of decision rules of the form (2.4)-(2.5), based on fixed θ_1, θ_2, and $\underset{\sim}{\ell}$, and a sequence of score matiices $\underset{\sim}{A}_\nu(p\times N_\nu)$. As in [1], we make the following assumptions.

ASSUMPTION 3.1 There is a number $\lambda^*(0<\lambda^*<\frac{1}{3})$ such that

$$\lambda^* < \frac{n_{k\nu}}{N_\nu} < 1-\lambda^*, \quad k=0,1,2, \text{ for all } \nu.$$

ASSUMPTION 3.2 $\underset{\sim}{A}_\nu = (a_{\nu i}(\alpha))_{\substack{i=1,\ldots,p \\ \alpha=1,\ldots,N_\nu}}$ (3.1)

is given by

$$a_{\nu i}(\alpha) = \text{either } \varphi_i\left(\frac{\alpha}{N_\nu+1}\right) \text{or } E\,\varphi_i(U_{N_\nu}^{(\alpha)}),\ \alpha=1,\ldots,N_\nu,\ i=1,\ldots,p,$$

where $U_{N_\nu}^{(1)} < \ldots < U_{N_\nu}^{(N_\nu)}$ are the order statistics of a sample of size N_ν from the uniform distribution over (0,1), and, for each i, inside (0,1) $\varphi_i(u)$ is expressible as the difference of two square integrable absolutely continuous functions.

Let

$$H_\nu(\underset{\sim}{x}) = \frac{1}{N_\nu} \sum_{k=0}^{2} n_{k\nu} F^{(k)}(\underset{\sim}{x}), \tag{3.2}$$

and further, let the i-th-coordinate marginal cdf's of $F^{(k)}(\underset{\sim}{x})$, $H_\nu(\underset{\sim}{x})$ be $F_{[i]}(x)$, $H_{\nu[i]}(x)$. Write

$$\mu_{\nu i}^{(k)} = \int_{-\infty}^{\infty} \varphi_i(H_{\nu[i]}(x))dF_{[i]}^{(k)}(x),\ i = 1, \ldots, p$$

$$\underset{\sim}{\mu}_\nu^{(k)} = (\mu_{\nu 1}^{(k)}, \ldots, \mu_{\nu p}^{(k)})' \tag{3.3}$$

Suppose $\underset{\sim}{\ell}$ is so chosen that the following holds

ASSUMPTION 3.3 $\liminf_{\nu\to\infty} \underset{\sim}{\ell}'(\underset{\sim}{\mu}_\nu^{(2)} - \underset{\sim}{\mu}_\nu^{(1)}) > 0$

If we have some knowledge about how $F^{(1)}$ and $F^{(2)}$ differ, choosing the score functions $\varphi_i(u)$, i=1,...,p, and $\underset{\sim}{\ell}$, it would be possible to ensure that Assumption 3.3 holds (see [1] section 3).

Let $\bar{a}_\nu^{(k)}$ and $T_\nu(\underset{\sim}{\ell})$ be defined as in (2.3) and (2.4) corresponding to $(n_{0\nu}, n_{1\nu}, n_{2\nu})$ and $\underset{\sim}{A}_\nu$. From theorem 2.1 of [1] under Assumptions 3.1 and 3.2, as $\nu\to\infty$

$$\underset{\sim}{\ell}'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}) - \underset{\sim}{\ell}'(\underset{\sim}{\mu}_\nu^{(2)} - \underset{\sim}{\mu}_\nu^{(1)}) \overset{P}{\to} 0. \tag{3.4}$$

Hence, by Assumption 3.3,

$$\text{Prob}\{\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(2)} > \underset{\sim}{\ell}\bar{\underset{\sim}{a}}_\nu^{(1)}\} \to 1 \tag{3.5}$$

From (2.5) and (3.5), it follows that, to study the limiting probabilities of choosing d_1, d_2, and d_0, for the sequence of decision rules, we have only to study the limits of the probabilities of the three events (i) $T_\nu(\underset{\sim}{\ell}) \geq \theta_1$, (ii) $T_\nu(\underset{\sim}{\ell}) \leq \theta_2$, (iii) $\theta_2 < T_\nu(\underset{\sim}{\ell}) < \theta_1$ respectively. As shown in [1], under the assumptions made $T_\nu(\underset{\sim}{\ell}) \overset{P}{\to} \theta$. Hence, we conclude that the probabilities of choosing d_1, d_2, and d_0 tend to 1 for $\theta_1 < \theta \leq 1$, $0 \leq \theta < \theta_2$, and $\theta_2 < \theta < \theta_1$ respectively. Further, at $\theta=\theta_1$, the probability of choosing either d_1 or d_0, and at $\theta=\theta_2$, the probability of choosing either d_2 or d_0 tends to 1. Thus if

$$L_1 < \theta_1 < U_1, \quad \text{and} \quad L_2 < \theta_2 < U_2, \tag{3.6}$$

where $0 \leq L_2 < U_2 < L_1 < U_1 \leq 1$ are as defined in section 1, then the sequence of decision rules is consistent in the sense that for each θ the probability of choosing a preferred decision tends to 1.

The conclusions above remain true if $\underset{\sim}{\ell}$ is replaced by $\underset{\sim}{\ell}_\nu$, where $\underset{\sim}{\ell}_\nu$ is a sequence of non-null vectors varying with ν, provided (i) the elements of $\underset{\sim}{\ell}_\nu$ are uniformly bounded, and (ii) Assumption 3.3 holds when $\underset{\sim}{\ell}_\nu$ replaces $\underset{\sim}{\ell}$. (See remark at the end of section 3 in [1].) Further, even when θ_1, θ_2 are replaced by $\theta_{1\nu}$, $\theta_{2\nu}$, the sequence of decision rules is consistent in the above sense provided

$$L_1 < \liminf \theta_{1\nu} \leq \limsup \theta_{1\nu} < U_1, \quad L_2 < \liminf \theta_{2\nu} \leq \limsup \theta_{2\nu} < U_2 \tag{3.7}$$

whatever ν.

4. ASYMPTOTIC LOCAL OPERATING CHARACTERISTICS

We first consider some general results about sequences of random vectors. Let $\underset{\sim}{\mu}_\nu$ be a sequence of p-vectors, $\underset{\sim}{\Sigma}_\nu(p\times p)$ be a sequence of positive definite matrices, and $\underset{\sim}{X}_\nu$ be a sequence of random p-vectors asymptotically distributed as $\underset{\sim}{N}(\underset{\sim}{\mu}_\nu, \underset{\sim}{\Sigma}_\nu)$ in the sense that, for every p-vector $\underset{\sim}{\ell}\neq\underset{\sim}{o}$, $\underset{\sim}{\ell}'\underset{\sim}{X}_\nu$ is asymptotically distributed as $N(\underset{\sim}{\ell}'\underset{\sim}{\mu}_\nu, \underset{\sim}{\ell}'\underset{\sim}{\Sigma}_\nu\underset{\sim}{\ell})$. We denote the minimum and maximum characteristic roots of any positive definite matrix $\underset{\sim}{B}$ by $m(\underset{\sim}{B})$ and $M(B)$ respectively. The following lemma is implied by lemma 1.2 in [1].

LEMMA 4.1 *If*

$$\liminf_{\nu\to\infty} \frac{m(\underset{\sim}{\Sigma}_\nu)}{M(\underset{\sim}{\Sigma}_\nu)} > 0 \tag{4.1}$$

then for any sequence of non-null vectors $\underset{\sim}{\ell}_\nu$, $\underset{\sim}{\ell}'_\nu \underset{\sim}{X}_\nu$ *is asymptotically distributed as* $N(\underset{\sim}{\ell}'_\nu \underset{\sim}{\mu}_\nu, \underset{\sim}{\ell}'_\nu \underset{\sim}{\Sigma}_\nu \underset{\sim}{\ell}_\nu)$.

LEMMA 4.2 *If* $p = p_1+p_2$,

$$\underset{\sim}{X}_\nu = \begin{bmatrix} \underset{\sim}{X}_{\nu 1}(p_1\times 1) \\ \underset{\sim}{X}_{\nu 2}(p_2\times 1) \end{bmatrix}, \quad \underset{\sim}{\mu}_\nu = \begin{pmatrix} \underset{\sim}{\mu}_{\nu 1} \\ \underset{\sim}{\mu}_{\nu 2} \end{pmatrix}, \quad \underset{\sim}{\Sigma}_\nu = \begin{pmatrix} \underset{\sim}{\Sigma}_{\nu 11} \; (p_1\times p_1) & \underset{\sim}{\Sigma}_{\nu 12} \; (p_1\times p_2) \\ \underset{\sim}{\Sigma}_{\nu 21} \; (p_2\times p_2) & \underset{\sim}{\Sigma}_{\nu 22} \; p_2\times p_2) \end{pmatrix},$$

then for any two sequences of non-null vectors $\underset{\sim}{\ell}_{\nu 1}(p_1\times 1)$, $\underset{\sim}{\ell}_{\nu 2}(p_2\times 1)$, *under condition* (4.1), $(\underset{\sim}{\ell}'_{\nu 1}\underset{\sim}{X}_{\nu 1}, \underset{\sim}{\ell}'_{\nu 2}\underset{\sim}{X}_{\nu 2})$ *has asymptotically a bivariate normal distribution with means* $\underset{\sim}{\ell}'_{\nu 1}\underset{\sim}{\mu}_{\nu 1}$, $\underset{\sim}{\ell}'_{\nu 2}\underset{\sim}{\mu}_{\nu 2}$ *and dispersion*

$$\begin{pmatrix} \underset{\sim}{\ell}'_{\nu 1} \underset{\sim}{\Sigma}_{\nu 11} \underset{\sim}{\ell}_{\nu 1} & \underset{\sim}{\ell}'_{\nu 1} \underset{\sim}{\Sigma}_{\nu 12} \underset{\sim}{\ell}_{\nu 2} \\ \underset{\sim}{\ell}'_{\nu 2} \underset{\sim}{\Sigma}_{\nu 21} \underset{\sim}{\ell}_{\nu 1} & \underset{\sim}{\ell}'_{\nu 2} \underset{\sim}{\Sigma}_{\nu 22} \underset{\sim}{\ell}_{\nu 2} \end{pmatrix}$$

Proof: Follows from Lemma 4.1.

Let

$$\psi(\underset{\sim}{x}) = \sum_{r_1+\ldots+r_p\leq g} c_{r_1 r_2 \ldots r_p} \, x_1^{r_1} \, x_2^{r_2} \ldots x_p^{r_p}$$

be a g-th degree polynomial in the elements of $\underset{\sim}{x}=(x_1,\ldots,x_p)'$ and let Ψ_g denote the class of all such g-th degree polynomials. Let $\underset{\sim}{X}_\nu$ be as above asymptotically distributed as $N(\underset{\sim}{\mu}_\nu, \underset{\sim}{\Sigma}_\nu)$. Further, for each ν, let $\underset{\sim}{\xi}_\nu$ be exactly distributed as $N(\underset{\sim}{\mu}_\nu, \underset{\sim}{\Sigma}_\nu)$.

LEMMA 4.3 *Under condition* (4.1), *as* $\nu\to\infty$,

$$\sup_{\psi \in \Psi_g} |P\{\psi(\underset{\sim}{X}_\nu) \geq 0\} - P\{\psi(\underset{\sim}{\xi}_\nu) \geq 0\}| \to 0$$

Proof: If $\underset{\sim}{C}_\nu(p\times p)$ is a matrix such that $\underset{\sim}{C}_\nu \underset{\sim}{\Sigma}_\nu C_\nu' = I$, then $\underset{\sim}{C}_\nu(\underset{\sim}{\xi}_\nu - \underset{\sim}{\mu}_\nu)$ is distributed as $N(\underset{\sim}{o}, \underset{\sim}{I})$. Writing $\underset{\sim}{\eta}(p\times 1)$ for a random vector following the distribution $N(\underset{\sim}{o}, \underset{\sim}{I})$ and denoting $\underset{\sim}{C}_\nu(\underset{\sim}{X}_\nu - \underset{\sim}{\mu}_\nu) = \underset{\sim}{Y}_\nu$, we have to show

$$\sup_{\psi \in \Psi_g} |P\{\psi(C_\nu^{-1} Y_\nu + \mu_\nu) \geq 0\} - P\{\psi(\underset{\sim}{C}_\nu^{-1} \eta + \underset{\sim}{\mu}_\nu) \geq 0\}$$

$$= \sup_{\psi \in \Psi_g} |P\{\psi(\underset{\sim}{Y}_\nu) \geq 0\} - P\{\psi(\underset{\sim}{\eta}) \geq 0\}| \to 0 \tag{4.2}$$

as $\nu\to\infty$. Now, as (4.1) holds, by Lemma 4.1, Y_ν is asymptotically distributed as $N(\underset{\sim}{o}, \underset{\sim}{I})$. From a result of Ranga Rao ([11] Theorem 4.1) we get that if the random vector $(Z_{\nu 1}, \ldots, Z_{\nu m})$ converges in law to a random vector $(\xi_1, \ldots, \xi_m)$ and if the cdf of every linear combination of $\xi_1, \ldots, \xi_m$ is continuous, then for any numbers $\ell_0, \ell_1, \ldots, \ell_m$

$$\sup_{\ell_0, \ell_1, \ldots, \ell_m} |P\{\ell_0 + \ell_1 Z_{\nu 1} + \ldots + \ell_m Z_{\nu m} \geq 0\} - P\{\ell_0 + \ell_1 \xi_1 + \ldots + \ell_m \xi_m \geq 0\}| \to 0.$$

Here $\psi(\underset{\sim}{Y}_\nu)$ can be considered as a linear combination of the power products of the elements of $\underset{\sim}{Y}_\nu$. Since $\underset{\sim}{Y}_\nu$ converges in law to $\underset{\sim}{\eta}$, the set of power products of the elements of $\underset{\sim}{Y}_\nu$ converges in law to the corresponding set of power products of the elements of $\underset{\sim}{\eta}$. Since $\underset{\sim}{\eta}$ is distributed as $N(\underset{\sim}{o}, \underset{\sim}{I})$, the required continuity condition obviously holds. Hence, (4.2) follows. Q.E.D.

Now consider, as in section 3, a sequence of triplets of sample sizes $(n_{0\nu}, n_{1\nu}, n_{2\nu})$, $n_{0\nu} + n_{1\nu} + n_{2\nu} = N_\nu \to \infty$ and a sequence of score matrices $\underset{\sim}{A}_\nu(p\times N_\nu)$ subject to assumptions 3.1 and 3.2. Here, we suppose there is a corresponding sequence of triplets of p-variate cdf's $(F_\nu^{(0)}, F_\nu^{(1)}, F_\nu^{(2)})$ such that

$$F_\nu^{(0)}(\underset{\sim}{x}) = \theta\, F_\nu^{(1)}(\underset{\sim}{x}) + (1-\theta) F_\nu^{(2)}(\underset{\sim}{x}) \tag{4.3}$$

for some fixed θ, for all ν. Let independent samples

$$\underset{\sim}{X}_{\nu\alpha}^{(k)},\ \alpha=1,\ldots,n_{k\nu},\ k=0,1,2$$

be taken from $F_\nu^{(k)}$, k=0,1,2 and on the basis of these and $\underset{\sim}{A}_\nu$, $\underset{\sim}{a}_{\nu\alpha}^{(k)}$ and $\bar{\underset{\sim}{a}}_\nu^{(k)}$ be defined as in (2.2) and (2.3). We consider the decision rule (2.5) (where $\underset{\sim}{\ell}$, θ_1, θ_2 are given) based on $\bar{\underset{\sim}{a}}_\nu^{(k)}$, k=0,1,2 and denote the corresponding probability of choosing d_k given θ by $L_{k\nu}(\theta)$, k=0,1,2, $\sum_{k=0}^{2} L_{k\nu}(\theta) = 1$. As seen in section 3, if $F_\nu^{(0)}$, $F_\nu^{(1)}$, $F_\nu^{(2)}$ remain fixed, then as $\nu\to\infty$, $L_{0\nu}(\theta)$, $L_{1\nu}(\theta)$, and $L_{2\nu}(\theta)$ tend to 1 for $\theta_2 < \theta < \theta_1$, $\theta_1 < \theta$, and $\theta < \theta_2$ respectively. Therefore to keep $L_{k\nu}(\theta)$, k=0,1,2 informative, as $\nu\to\infty$ we impose the condition

$$\operatorname*{Sup}_{\underset{\sim}{x}} |F_\nu^{(1)}(\underset{\sim}{x}) - F_\nu^{(2)}(\underset{\sim}{x})| \to 0. \tag{4.4}$$

We shall show that under some further assumptions, it will be possible to find certain mathematically tractable, meaningful functions $L_{k\nu}^*(\theta)$, such that for every θ, as $\nu\to\infty$, $|L_{k\nu}(\theta)-L_{k\nu}^*(\theta)|\to 0$, k=0,1,2. We shall call $L_{k\nu}^*(\theta)$, k=0,1,2 the asymptotic local operating characteristic (ALOC) functions for the sequence of decision rules.

Let us write

$$\underset{\sim}{Z}_{\nu 1} = \sqrt{N_\nu}\ (\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(1)})$$

$$\underset{\sim}{Z}_{\nu 2} = \sqrt{N_\nu}\ \{\theta_1\bar{\underset{\sim}{a}}_\nu^{(1)}+(1-\theta_1)\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(0)}\}$$

$$\underset{\sim}{Z}_{\nu 3} = \sqrt{N_\nu}\ \{\bar{\underset{\sim}{a}}_\nu^{(0)}-\theta_2\bar{\underset{\sim}{a}}_\nu^{(1)}-(1-\theta_2)\bar{\underset{\sim}{a}}_\nu^{(2)}\} \tag{4.5}$$

Clearly, $\underset{\sim}{Z}_{\nu 1}$, $\underset{\sim}{Z}_{\nu 2}$, $\underset{\sim}{Z}_{\nu 3}$ are related by

$$\underset{\sim}{Z}_{\nu 2} + \underset{\sim}{Z}_{\nu 3} = -(\theta_1-\theta_2)\underset{\sim}{Z}_{\nu 1} \tag{4.6}$$

From (2.7) and (4.5), our sequence of rules is given by

(a) When $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1} \neq 0$, (i) choose d_1 if $(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1})(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 2}) \geq 0$

(ii) choose d_2 if $(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1})(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 3}) \geq 0$

(iii) choose d_0 if $(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 2})(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 3}) > 0$

(b) When $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1} = 0$, choose one of d_1, d_2, d_0 at random (4.7)

Thus the asymptotic behaviour of $L_{k\nu}(\theta)$, k=0,1,2, will be known if we can find the asymptotic form of the three pairwise marginal distributions of $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1}$, $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 2}$, $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 3}$. For this, we shall make use of the following result which is a straightforward multivariate extension of Hajek's [4] theorem 2.4. (The result is essentially contained in Puri and Sen [10]. In Proving it one should first consider the case of score functions with bounded second derivatives and then use the polynomial approximation technique of Hajek [4].)

For each $\nu=1,2,\ldots$, let $\underset{\sim}{Y}_{\nu\alpha} = (Y_{\nu 1\alpha},\ldots,Y_{\nu p\alpha})'$ $\alpha=1,\ldots,N_\nu$, be independently distributed random p-vectors, $\underset{\sim}{Y}_{\nu\alpha}$ having a continuous cdf $F_{\nu\alpha}(\underset{\sim}{y})$. Let the rank vector obtained from $\underset{\sim}{Y}_{\nu\alpha}$ by replacing the variate values by their coordinate-wise ranks be

$$\underset{\sim}{I}_{\nu\alpha} = (I_{\nu 1\alpha},\ldots,I_{\nu p\alpha})', \quad \alpha=1,\ldots,N_\nu.$$

For each ν, let $\underset{\sim}{A}_\nu = (a_{\nu i}(\alpha))$ be a score matrix subject to Assumption 3.2, and let

$$\underset{\sim}{a}_{\nu\alpha} = (a_{\nu 1}(I_{\nu 1\alpha}),\ldots,a_{\nu p}(I_{\nu p\alpha})), \quad \alpha=1,\ldots,N_\nu.$$

Define

$$\underset{\sim}{S}_\nu^{(r)} = \sum_{\alpha=1}^{N_\nu} c_{\nu\alpha}^{(r)}\, \underset{\sim}{a}_{\nu\alpha}, \quad r=1,\ldots,h,$$

$$\underset{\sim}{S}_\nu' = (\underset{\sim}{S}_\nu^{(1)'},\ldots,\underset{\sim}{S}_\nu^{(h)'})$$

where for each ν,

$$\begin{pmatrix} c_{\nu 1}^{(1)} & \cdots & c_{\nu N_\nu}^{(1)} \\ c_{\nu 1}^{(h)} & \cdots & c_{\nu N_\nu}^{(h)} \end{pmatrix}$$

is a matrix such that in at least one row all the elements are not equal.

Write

$$\bar{c}_\nu^{(r)} = \frac{1}{N_\nu} \sum_{\alpha=1}^{N_\nu} c_{\nu\alpha}^{(r)},$$

$$b_{\nu.rs} = \sum_{\alpha=1}^{N_\nu} (c_{\nu\alpha}^{(r)} - \bar{c}_\nu^{(r)})(c_{\nu\alpha}^{(s)} - \bar{c}_\nu^{(s)})$$

$$\underset{\sim}{B}_\nu = (b_{\nu.rs})_{r,s=1,\ldots,h}.$$

Suppose there is a sequence of p-variate cdf's $F_\nu(\underset{\sim}{y})$ such that

$$\max_{1 \leq \alpha \leq N_\nu} \sup_{\underset{\sim}{y}} |F_{\nu\alpha}(\underset{\sim}{y}) - F_\nu(\underset{\sim}{y})| \to 0. \tag{4.8}$$

Write

$$\bar{\varphi}_i = \int_0^1 \varphi_i(u)du, \quad \lambda_{\nu.ij} = \int_{-\infty}^{\infty} \varphi_i(F_{\nu[i]}(y_1))\, \varphi_j(F_{\nu[j]}(y_2))dF_{\nu[i,j]}(y_1,y_2) - \bar{\varphi}_i\, \bar{\varphi}_j$$

$$\underset{\sim}{\Lambda}_\nu = (\lambda_{\nu.ij})_{i,j=1,\ldots,p} \tag{4.9}$$

where $F_{\nu[i]}$, and $F_{\nu[i,j]}$ denote the marginal cdf's of F_ν corresponding to the i-th coordinate and i-th and j-th coordinates.

THEOREM 4.1 <u>If</u> (i) $b_{\nu.rr}$, $r=1,\ldots,h$ <u>are uniformly bounded for all</u> ν

(ii) $\liminf_{\nu\to\infty} m(\underset{\sim}{B}_\nu) > 0$

(iii) $\liminf_{\nu\to\infty} \dfrac{m(\underset{\sim}{B}_\nu)}{N_\nu \max_{r,\alpha} (c_{\nu\alpha}^{(r)} - \bar{c}_\nu^{(r)})^2} > 0$

(iv) $\liminf_{\nu\to\infty} m(\underset{\sim}{\Lambda}_\nu) > 0.$ (4.10) then $\underset{\sim}{S}_\nu - E\,\underset{\sim}{S}_\nu$ is asymptotically distributed as $N(\underset{\sim}{0}, \underset{\sim}{B}_\nu \otimes \underset{\sim}{\Lambda}_\nu)$ ($\otimes$ stands for Kronecker product).

Note that condition (i) can always be realized by suitably normalizing the elements of $\underset{\sim}{S}_\nu$. Because of (i), in the statement of conditions (ii) and (iii) we can equivalently replace $m(\underset{\sim}{B}_\nu)$ by $|\underset{\sim}{B}_\nu|$. Similarly, as by Assumption 3.2, the elements of $\underset{\sim}{\Lambda}_\nu$ are all bounded, in (iv) we can replace $m(\underset{\sim}{\Lambda}_\nu)$ by $|\underset{\sim}{\Lambda}_\nu|$.

Now, going back to the problem of finding the asymptotic behaviour of the pairwise marginals of $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1}$, $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 2}$ and $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 3}$, let us set up

$$F_\nu(\underset{\sim}{x}) = \tfrac{1}{2}\{F_\nu^{(1)}(\underset{\sim}{x}) + F_\nu^{(2)}(\underset{\sim}{x})\}. \tag{4.11}$$

From (4.3) and (4.4) we then have, as $\nu\to\infty$

$$\max_{k=0,1,2}\ \sup_{\underset{\sim}{x}} |F_\nu^{(k)}(\underset{\sim}{x}) - F_\nu(\underset{\sim}{x})| \to 0 \tag{4.12}$$

Let $\underset{\sim}{\Lambda}_\nu$ be defined with respect to (4.11) as in (4.9). We impose the following.

ASSUMPTION 4.1 $\liminf_{\nu\to\infty} m(\underset{\sim}{\Lambda}_\nu) > 0$.

Now if we write $\underset{\sim}{Z}_{\nu r}$ given by (4.5) in the form

$$\underset{\sim}{Z}_{\nu r} = \sum_{k=0}^{2} \sum_{\alpha=1}^{n_k} c_{\nu k\alpha}^{(r)}\, \underset{\sim}{a}_{\nu\alpha}^{(k)}, \quad r=1,2,3,$$

it is easily checked that

$$\sum_{k=0}^{2} \sum_{\alpha=1}^{n_k} c_{\nu k\alpha}^{(r)} = 0, \quad r=1,2,3$$

and that

$$b_{\nu.rs} = \sum_{k=0}^{2} \sum_{\alpha=1}^{n_k} c_{\nu k\alpha}^{(r)}\, c_{\nu k\alpha}^{(s)}, \quad r,s=1,2,3$$

are given by

$$b_{\nu.11} = N_\nu\left(\frac{1}{n_{1\nu}} + \frac{1}{n_{2\nu}}\right),$$

$$b_{\nu.12} = -N_\nu\left\{\frac{\theta_1}{n_{1\nu}} - \frac{(1-\theta_1)}{n_{2\nu}}\right\},\quad b_{\nu.13} = -N_\nu\left\{\frac{(1-\theta_2)}{n_{2\nu}} - \frac{\theta_2}{n_{1\nu}}\right\}$$

$$b_{\nu.22} = N_\nu\left\{\frac{1}{n_{0\nu}} + \frac{\theta_1^2}{n_{1\nu}} + \frac{(1-\theta_1)^2}{n_{2\nu}}\right\},\quad b_{\nu.23} = -N_\nu\left\{\frac{1}{n_{0\nu}} + \frac{\theta_1\theta_2}{n_{1\nu}} + \frac{(1-\theta_1)(1-\theta_2)}{n_{2\nu}}\right\}$$

$$b_{\nu.33} = N_\nu\left\{\frac{1}{n_{0\nu}} + \frac{\theta_2^2}{n_{1\nu}} + \frac{(1-\theta_2)^2}{n_{2\nu}}\right\} \tag{4.13}$$

Because of (4.6), the 3×3 matrix $(b_{\nu.rs})$ is singular. However, it is easily checked that the 3 principal submatrices

$$(b_{\nu.rs})_{1,2},\ (b_{\nu.rs})_{1,3},\ (b_{\nu.rs})_{2,3} \tag{4.14}$$

are positive definite. Further, it may be seen that by virtue of Assumption 3.1 and the fact $\theta_1 > \theta_2$ each of the three matrices (4.14) satisfies conditions (i)-(iii) of Theorem 4.1. By our Assumption 4.1, we can hence conclude that each of the three 2p-vectors

$$\begin{pmatrix} \underset{\sim}{Z}_{\nu1} - E\ \underset{\sim}{Z}_{\nu1} \\ \underset{\sim}{Z}_{\nu2} - E\ \underset{\sim}{Z}_{\nu2} \end{pmatrix},\quad \begin{pmatrix} \underset{\sim}{Z}_{\nu1} - E\ \underset{\sim}{Z}_{\nu1} \\ \underset{\sim}{Z}_{\nu3} - E\ \underset{\sim}{Z}_{\nu3} \end{pmatrix},\quad \begin{pmatrix} \underset{\sim}{Z}_{\nu2} - E\ \underset{\sim}{Z}_{\nu2} \\ \underset{\sim}{Z}_{\nu3} - E\ \underset{\sim}{Z}_{\nu3} \end{pmatrix}$$

is asymptotically normal.

Now let us write

$$E\ \underset{\sim}{Z}_{\nu1} = \sqrt{N_\nu}\ E(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}) = \underset{\sim}{\delta}_\nu \tag{4.15}$$

From Lemma 2.1 in [1] it follows that under (4.3) and Assumption 3.2, as $\nu\to\infty$,

$$\sqrt{N_\nu}\ E\{\bar{\underset{\sim}{a}}_\nu^{(0)} - \theta\ \bar{\underset{\sim}{a}}_\nu^{(1)} - (1-\theta)\bar{\underset{\sim}{a}}_\nu^{(2)}\} \to o.$$

(In [1], the result is actually proved for fixed $F^{(1)}$, $F^{(2)}$, and $F^{(0)} = \theta\ F^{(1)} + (1-\theta)F^{(2)}$. But a close examination of the proof shows that the result remains true even when $F^{(k)}$ is replaced by $F_\nu^{(k)}$ k=0,1,2). From (4.15)-(4.16)

$$E\, \underset{\sim}{Z}_{\nu 2}-(\theta-\theta_1)\underset{\sim}{\delta}_\nu \to \underset{\sim}{o}, \quad E\, \underset{\sim}{Z}_{\nu 3}-(\theta_2-\theta)\underset{\sim}{\delta}_\nu \to \underset{\sim}{o} \tag{4.17}$$

Hence by Theorem 4.1 each of the 2p-vectors

$$\begin{pmatrix} \underset{\sim}{Z}_{\nu 1} - \underset{\sim}{\delta}_\nu \\ \underset{\sim}{Z}_{\nu 2} - (\theta-\theta_1)\underset{\sim}{\delta}_\nu \end{pmatrix}, \begin{pmatrix} \underset{\sim}{Z}_{\nu 1} - \underset{\sim}{\delta}_\nu \\ \underset{\sim}{Z}_{\nu 3} - (\theta_2-\theta)\underset{\sim}{\delta}_\nu \end{pmatrix}, \begin{pmatrix} \underset{\sim}{Z}_{\nu 2} - (\theta-\theta_1)\underset{\sim}{\delta}_\nu \\ \underset{\sim}{Z}_{\nu 3} - (\theta_2-\theta)\underset{\sim}{\delta}_\nu \end{pmatrix} \tag{4.18}$$

is asymptotically normal with null mean vectors and dispersion matrices

$$(b_{\nu.rs})_{1,2} \otimes \underset{\sim}{\Lambda}_\nu,\ (b_{\nu.rs})_{1,3} \otimes \underset{\sim}{\Lambda}_\nu,\ (b_{\nu.rs})_{2,3} \otimes \underset{\sim}{\Lambda}_\nu \tag{4.19}$$

respectively. So for any $\underset{\sim}{\ell}\neq\underset{\sim}{o}$, each of the three pairs $(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1}-\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 2}-(\theta-\theta_1)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)$, $(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1}-\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 3}-(\theta_2-\theta)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)$, $(\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 2}-(\theta-\theta_1)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 3}-(\theta_2-\theta)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)$ is asymptotically bivariate normal with means 0 and dispersion matrices

$$(b_{\nu.rs})_{1,2}\, \underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell},\ (b_{\nu.rs})_{1,3}\, \underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell},\ (b_{\nu.rs})_{2,3}\, \underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell} \tag{4.20}$$

respectively.

Since $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1}$ is asymptotically normal, for finding the ALOC functions of the sequence of rules (4.7), we can neglect the possibility $\underset{\sim}{\ell}'\underset{\sim}{Z}_{\nu 1}=0$. Let us denote by $\underset{\sim}{\zeta}_\nu$:

$$\underset{\sim}{\zeta}'_\nu = (\underset{\sim}{\zeta}'_{\nu 1}(1\times p),\ \underset{\sim}{\zeta}'_{\nu 2}(1\times p),\ \underset{\sim}{\zeta}'_{\nu 3}(1\times p)) \tag{4.21}$$

a random 3p-vector such that for each ν, $\underset{\sim}{\zeta}_\nu$ follows a normal distribution with mean vector

$$(\delta'_\nu,\ (\theta-\theta_1)\underset{\sim}{\delta}'_\nu\ ,\ (\theta_2-\theta)\underset{\sim}{\delta}'_\nu)' \tag{4.22}$$

and dispersion matrix

$$(b_{\nu.rs})_{1,2,3} \otimes \underset{\sim}{\Lambda}_\nu \tag{4.23}$$

As $(b_{\nu.rs})_{1,2,3}$ is of rank 2, this is a singular normal distribution of rank 2p. In fact, it may be checked that just as in (4.6) we have

$$\underset{\sim}{\zeta}_{\nu 2} + \underset{\sim}{\zeta}_{\nu 3} = -(\theta_1-\theta_2)\underset{\sim}{\zeta}_{\nu 1} \quad \text{a.s.} \tag{4.24}$$

For any $\underset{\sim}{\ell}\neq\underset{\sim}{0}$, the pairs $(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 1}-\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 2}-(\theta-\theta_1)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)$, $(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 1}-\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu,$ $\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 3}-(\theta_2-\theta)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)$, $(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 2}-(\theta-\theta_1)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 3}-(\theta_2-\theta)\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)$ follow bivariate normal distributions with zero means and dispersions given by the matrices (4.20) respectively. From (4.13) and Assumption 3.1 it follows that each of the three matrices (4.20) satisfies condition (4.1). Therefore, using Lemma 4.3, we get that for the rules (4.7) we have

$$L_{k\nu}(\theta) \simeq L^*_{k\nu}(\theta),\ k=0,1,2$$

($\simeq$ means the two sides have a vanishing difference), where

$$L^*_{1\nu}(\theta) = P\{(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 1})(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 2}) \geq 0\}$$

$$L^*_{2\nu}(\theta) = P\{(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 1})(\underset{\sim}{\ell}'\underset{\sim}{\zeta}_{\nu 3}) \geq 0\}$$

$$L^*_{0\nu}(\theta) = P\{(\ell'\zeta_{\nu 2})(\ell'\zeta_{\nu 3}) \geq 0\} \tag{4.25}$$

Using (4.24) it may be checked that

$$L^*_{0\nu}(\theta) + L^*_{1\nu}(\theta) + L^*_{2\nu}(\theta) = 1$$

For any two numbers h, k and $|\rho|<1$, let us write

$$H(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{h}\int_{-\infty}^{k} e^{-[x^2-2\rho xy+y^2]/2(1-\rho^2)}dx\,dy \tag{4.27}$$

$$J(h,k;\rho) = H(h,k;\rho) + H(-h,-k;\rho) = J(-h,-k;\rho) \tag{4.28}$$

If X,Y have a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation coefficient $\rho(-1 < \rho < 1)$, we have,

$$P\{XY \geq 0\} = J(\frac{\mu_1}{\sigma_1}, \frac{\mu_2}{\sigma_2}; \rho) \tag{4.29}$$

So if we write

$$b_{\nu.rr} = b^2_{\nu.r}$$

$$b_{\nu.rs}/b_{\nu.r}.b_{\nu.s} = \rho_{\nu.rs}, \; r\neq s=1,2,3 \tag{4.30}$$

$$\frac{\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu}{(\underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell})^{\frac{1}{2}}} = \gamma_\nu(\underset{\sim}{\ell}) \tag{4.31}$$

from (4.25) and (4.29) we get the expressions

$$L^*_{1\nu}(\theta) = J(\frac{1}{b_{\nu.1}}\,|\gamma_\nu(\underset{\sim}{\ell})|,\, \frac{(\theta-\theta_1)}{b_{\nu.2}}\,|\gamma_\nu(\underset{\sim}{\ell})|;\, \rho_{\nu.12})$$

$$L^*_{2\nu}(\theta) = J(\frac{1}{b_{\nu.1}}\,|\gamma_\nu(\underset{\sim}{\ell})|,\, \frac{(\theta_2-\theta)}{b_{\nu.3}}\,|\gamma_\nu(\underset{\sim}{\ell})|;\, \rho_{\nu.13})$$

$$L^*_{0\nu}(\theta) = J(\frac{(\theta-\theta_1)}{b_{\nu.2}}|\gamma_\nu(\underset{\sim}{\ell})|,\, \frac{(\theta_2-\theta)}{b_{\nu.3}}\,|\gamma_\nu(\underset{\sim}{\ell})|;\, \rho_{\nu.23}). \tag{4.32}$$

In order that $L^*_{k\nu}(\theta)$, k=0,1,2 may remain meaningful as $\nu\to\infty$, we require that $|\gamma_\nu(\underset{\sim}{\ell})|/b_{\nu.r}$, r=1,2,3 remain bounded away from zero and ∞. From (4.13) and (4.30) it is seen that by Assumption 3.1, $b_{\nu.r}>1$, r=1,2,3 remain bounded as $\nu\to\infty$. By Assumptions 3.2 and 4.1, $\underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell}$ is bounded away from both 0 and ∞. So the following assupption would ensure the meaningfulness of $L^*_{k\nu}$.

ASSUMPTION 4.2 (i) $0 < \liminf_{\nu\to\infty} |\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu|$, (ii) $\limsup_{\nu\to\infty} |\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu| < \infty$

For future reference we note here the modified assumptions

ASSUMPTION 4.2A (i) $0 < \liminf \underset{\sim}{\delta}'_\nu\underset{\sim}{\delta}'_\nu$, (ii) $\limsup \underset{\sim}{\delta}'_\nu\underset{\sim}{\delta}_\nu < \infty$

For Assumption 4.2 (i) to hold 4.2A(i) is necessary, and unless some of the elements of $\underset{\sim}{\ell}$ are zero, for 4.2(ii) to hold, 4.2A(ii) is necessary. If $\varphi_i(u)$, i=1,2,...,p and $F^{(1)}_\nu(\underset{\sim}{x})$, $F^{(2)}_\nu(\underset{\sim}{x})$ are so chosen that, while (4.4) holds, Assumption 4.2A is realised, then we can always find $\underset{\sim}{\ell}$ appropriately to satisfy

Assumption 4.2. $\underset{\sim}{\delta}_\nu$ given by (4.15) can be simply expressed in terms of φ_i and $F_\nu^{(k)}$ if Assumption 3.2 is slightly strengthened (see Hoeffding [5]).

So far we have assumed that, for the sequence of decision rules, $\underset{\sim}{\ell}$ remains fixed. If instead in the rule corresponding to sample sizes $(n_{0\nu}, n_{1\nu}, n_{2\nu})$ i.e., in (4.7) $\underset{\sim}{\ell}$ is replaced by $\underset{\sim}{\ell}_\nu$, where $\underset{\sim}{\ell}_\nu$ is a sequence of non-null vectors, the ALOC functions would still be given by (4.25) (or equivalently (4.31)-(4.32)) with $\underset{\sim}{\ell}$ replaced by $\underset{\sim}{\ell}_\nu$. To see this note that by Assumptions 3.1, 3.2 and 4.1 and (4.13), each of the three matrices (4.19) satisfies condition (4.1). (The latent roots of a Kronecker product are obtained by multiplying the roots of one factor matrix by those of the other). Hence, applying Lemma 4.2 we get that $(\underset{\sim}{\ell}_\nu' \underset{\sim}{Z}_{\nu 1} - \underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}_{\nu 1}' \underset{\sim}{Z}_{\nu 2} - (\theta-\theta_1) \underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu)$, $(\underset{\sim}{\ell}_\nu' \underset{\sim}{Z}_{\nu 1} - \underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}_\nu' \underset{\sim}{Z}_{\nu 3} - (\theta_2-\theta) \underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu)$, $(\underset{\sim}{\ell}_\nu' \underset{\sim}{Z}_{\nu 2} - (\theta-\theta_1) \underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu,\ \underset{\sim}{\ell}_\nu' \underset{\sim}{Z}_{\nu 3} - (\theta_2-\theta) \underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu)$ each have bivariate normal distributions with means 0 and dispersion matrices obtained by replacing $\underset{\sim}{\ell}$ in (4.20) by $\underset{\sim}{\ell}_\nu$. The rest of the argument is as before. Of course, here, in order that the ALOC functions may be meaningful, $\underset{\sim}{\ell}_\nu$ should be such that $|\gamma_\nu(\underset{\sim}{\ell}_\nu)|$ remains bounded away from both 0 and ∞. In view of Assumptions 3.2 and 4.1 this means we require the following.

ASSUMPTION 4.2B (i) $0 < \liminf_{\nu\to\infty} \dfrac{|\underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu|}{(\underset{\sim}{\ell}_\nu' \underset{\sim}{\ell}_\nu)^{\frac{1}{2}}}$ (ii) $\limsup_{\nu\to\infty} \dfrac{|\underset{\sim}{\ell}_\nu' \underset{\sim}{\delta}_\nu|}{(\underset{\sim}{\ell}_\nu' \underset{\sim}{\ell}_\nu)^{\frac{1}{2}}} < \infty$

Before concluding this section we prove a lemma which throws light on the forms of the ALOC functions.

LEMMA 4.4. For fixed h and ρ, $J(h,k;\rho)$ is increasing in k for $\rho k<h$ and decreasing in k for $\rho k>h$.

Proof: From (4.27), (4.28)

$$J(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \left[\int_{-\infty}^{k} e^{-\frac{1}{2}y^2} \left\{ \int_{-\infty}^{h} e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}} dx \right\} dy + \int_{k}^{\infty} e^{-\frac{1}{2}y^2} \left\{ \int_{h}^{\infty} e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}} dx \right\} dy \right]$$

Hence,

$$\frac{\partial J}{\partial k}(h,k;\rho) = \frac{e^{-k^2/2}}{2\pi\sqrt{1-\rho^2}} \left[\int_{-\infty}^{h} e^{-\frac{(x-\rho k)^2}{2(1-\rho^2)}} dx - \int_{h}^{\infty} e^{-\frac{(x-\rho k)^2}{2(1-\rho^2)}} dx \right]$$

Hence $\frac{\partial J}{\partial k} >$, = or < 0 according as $\rho k<h$, $\rho k = h$ or $\rho k>h$. Q.E.D.

From (4.32) and Lemma 4.3, it follows that for any ν, $L^*_{1\nu}(\theta)$ would be increasing in θ provided $b_{\nu.2} > b_{\nu.1} \cdot \rho_{\nu.12}(\theta-\theta_1)$, or by (4.30), provided

$$b_{\nu.22} > b_{\nu.12}(\theta-\theta_1) \tag{4.33}$$

From (4.13) it is seen that (4.33) holds for all θ. Similarly for any ν, $L^*_{2\nu}(\theta)$ would be decreasing in θ provided $b_{\nu 3} > b_{\nu 1}\ \rho_{\nu.13}(\theta_2-\theta)$ which always holds. The behaviour of $L^*_{0\nu}(\theta)$ follows from those of $L^*_{1\nu}(\theta)$ and $L^*_{2\nu}(\theta)$ and relation (4.26).

5. THE BEST CHOICE OF COEFFICIENTS

In this section, under the set up of Section 4 we consider the best choice of the compounding vector $\underset{\sim}{\ell}$ from the point of view of ALOC functions.

We first prove a lemma involving the function $J(h,k;\rho)$ defined by (4.27)-(4.28).

LEMMA 5.1 <u>For fixed</u> $h\geq 0$, $k\geq 0$, $(h,k) \neq (0,0)$, <u>and</u> ρ, $J(uh, uk;\rho)$ <u>is monotonically increasing in</u> $u>0$ <u>provided</u>

$$\rho \leq \min(\frac{h}{k}, \frac{k}{h}). \tag{5.1}$$

Proof: Let us define for any two numbers b,a

$$T(b,a) = \frac{1}{2\pi} \int_0^a \frac{e^{-\frac{1}{2}b^2(1+x^2)}}{1+x^2}\, dx \tag{5.2}$$

Clearly then

$$T(b,a) = T(-b,a), \quad T(b,-a) = -T(b,a) \tag{5.3}$$

Let $\Phi(x)$ stand for the standard normal cdf. Owen [8] has shown that whatever the numbers h, k, $H(h,k;\rho)$ given by (4.27) can be expressed as

$$H(h,k;\rho) = \tfrac{1}{2}[\Phi(h)+\Phi(k)] - T(h, (\tfrac{k}{h} - \rho)(1-\rho^2)^{-\frac{1}{2}})$$

$$-T(k, (\tfrac{h}{k} - \rho)(1-\rho^2)^{-\frac{1}{2}}) - R(h,k), \tag{5.4}$$

where

$$R(h,k) = \tfrac{1}{2}, \quad \text{if} \quad hk<0, \text{ or } hk=0 \text{ but } h+k<0$$

$$= 0, \quad \text{otherwise.}$$

From (4.28), (5.3) and (5.4) we get that for $h \geq 0$, $k \geq 0$, $(h,k) \neq (0,0)$
$J(h,k;\rho) = 1-2T(h, (\frac{k}{h} - \rho)(1-\rho^2)^{-\frac{1}{2}}) - 2T(k, (\frac{h}{k} - \rho)(1-\rho^2)^{-\frac{1}{2}}) - R^*(h,k)$ where

$$R^*(h,k) = 0 \quad \text{if} \quad h>0, \ k>0$$

$$= \tfrac{1}{2} \quad \text{if} \quad hk=0 \tag{5.5}$$

Hence for $h \geq 0$, $k \geq 0$, $(h,k) \neq (0,0)$, $u>0$

$$J(uh,uk;\rho) = 1-2T(uh, (\tfrac{k}{h} - \rho)(1-\rho^2)^{-\frac{1}{2}}) - 2T(uk, (\tfrac{h}{k} - \rho)(1-\rho^2)^{-\frac{1}{2}}) - R^*(h,k) \tag{5.6}$$

where $R^*(h,k)$ is as in (5.5).

From (5.2) we see that for $a>0$, $b>0$, $T(ub,a)$ decreases monotonically with u. Since under (5.1), $(\frac{k}{h} - \rho) \geq 0$, $\frac{h}{k} - \rho \geq 0$ with at least one inequality strict, from (5.6) the lemma follows. Q.E.D.

Note Given $h \geq 0$, $k \geq 0$, $(h,k) \neq (0,0)$, whatever ρ, $I(uh,uk;\rho)$ tends to the limit 1 (when both h, k > 0) or ½ (when h or k=0) as $u \to \infty$. As proved above for $\rho \leq \min(\frac{h}{k}, \frac{k}{h})$ this approach to limit is monotonic. However for $\rho > \min(\frac{h}{k}, \frac{k}{h})$ the approach may not be monotonic. To see this suppose $0<k<h$ and $\rho > \frac{k}{h}$. Then from (5.5), (5.6) and (5.2), we can write

$$J(uh,uk;\rho) = 1 + \frac{1}{\pi} \int_0^{(1-\rho^2)^{-\frac{1}{2}}(\rho - \frac{k}{h})} \frac{e^{-\frac{1}{2}u^2h^2(1+x^2)}}{1+x^2} dx - \frac{1}{\pi} \int_0^{(1-\rho^2)^{-\frac{1}{2}}(\frac{h}{k} - \rho)} \frac{e^{-\frac{1}{2}u^2k^2(1+x^2)}}{1+x^2} dx \tag{5.7}$$

Differentiating (5.7) with respect to u, we get

$$\pi u \frac{d}{du} J(uh,uk;\rho) = uke^{-\frac{1}{2}u^2k^2} \int_0^{u(h-\rho k)(1-\rho^2)^{-\frac{1}{2}}} e^{-x^2/2} dx - uhe^{-\frac{1}{2}u^2h^2} \int_0^{u(h\rho-k)(1-\rho^2)^{-\frac{1}{2}}} e^{-x^2/2} dx \tag{5.8}$$

Now we can choose $u>0$ small enough to make $uke^{-\frac{1}{2}u^2k^2} < uhe^{-\frac{1}{2}u^2h^2}$. Then we can choose $\rho > \frac{k}{h}$ close to 1 so as to make the two integrals in (5.8) sufficiently close making (5.8) negative.

We now make the following assumption.

ASSUMPTION 5.1 For sufficiently large ν,

$$\frac{\theta_2}{1-\theta_2} \leq \frac{n_{1\nu}}{n_{2\nu}} \leq \frac{\theta_1}{1-\theta_1} .$$

As θ_1 and θ_2 are close to 1 and 0 respectively, in view of Assumption 3.1, this is not very restrictive. By (4.13) and (4.30), in (4.32) we have $\rho_{\nu.23}<0$ and Assumption 5.1 ensures that for large ν, $\rho_{\nu.12} \leq 0$, $\rho_{\nu.13} \leq 0$. Thus $L^*_{1\nu}(\theta)$ (for $\theta \geq \theta_1$), $L^*_{2\nu}(\theta)$ (for $\theta \leq \theta_2$), and $L^*_{0\nu}(\theta)$ (for $\theta_2<\theta<\theta_1$) all satisfy the conditions of Lemma 5.1. (It is easily checked that without Assumption 5.1 this is not true) Hence, it follows that, for each ν, if we choose $\underset{\sim}{\ell}$ so that

$$[\gamma_\nu(\underset{\sim}{\ell})]^2 = (\underset{\sim}{\ell}'\delta_\nu)^2/(\underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell})$$

is maximized, $L^*_{1\nu}(\theta)$, $L^*_{2\nu}(\theta)$, and $L^*_{0\nu}(\theta)$ given (4.32) are respectively maximized in the domains $\theta_1 \le \theta \le 1$, $0 \le \theta \le \theta_2$, and $\theta_2 < \theta < \theta_1$. It is known that

$$\max_{\underset{\sim}{\ell} \ne \underset{\sim}{o}} (\underset{\sim}{\ell}'\underset{\sim}{\delta}_\nu)^2/(\underset{\sim}{\ell}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{\ell}) = \underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\delta}_\nu = \Delta_\nu^2 \quad \text{(say)} \tag{5.9}$$

and this is attained for

$$\underset{\sim}{\ell} = \underset{\sim}{\ell}_\nu = (g_\nu\underset{\sim}{\Lambda}_\nu)^{-1}\underset{\sim}{\delta}_\nu \tag{5.10}$$

where $g_\nu \ne 0$ are arbitrary numbers.

The ALOC functions of the sequence of decision rules corresponding to this $\underset{\sim}{\ell}_\nu$ are obtained from (4.25) and (4.32) as

$$\begin{aligned}
L^*_{1\nu}(\theta) &= P\{(\underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\zeta}_{\nu1})(\underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\zeta}_{\nu2}) \ge 0\} = J(\frac{\Delta_\nu}{b_{\nu1}},\ (\theta-\theta_1)\frac{\Delta_\nu}{b_{\nu2}}\ ;\ \rho_{\nu.12}) \\
L^*_{2\nu}(\theta) &= P\{(\underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\zeta}_{\nu1})(\underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\zeta}_{\nu3}) \ge 0\} = J(\frac{\Delta_\nu}{b_{\nu1}},\ (\theta_2-\theta)\frac{\Delta_\nu}{b_{\nu.3}};\ \rho_{\nu.13}) \\
L^*_{0\nu}(\theta) &= P\{(\underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\zeta}_{\nu2})(\underset{\sim}{\delta}_\nu'\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{\zeta}_{\nu3}) \ge 0\} = J((\theta-\theta_1)\frac{\Delta_\nu}{b_{\nu.2}},\ (\theta_2-\theta)\frac{\Delta_\nu}{b_{\nu.3}};\ \rho_{\nu.23})
\end{aligned} \tag{5.11}$$

where Δ_ν is given by (5.9) and $b_{\nu.r}$, $\rho_{\nu.rs}$, r,s=1,2,3 are given by (4.13) and (4.30). In order that these may remain meaningful as $\nu \to \infty$ we require Assumption 4.2A which together with Assumptions 3.2 and 4.1 ensures that Δ_ν remains bounded away from both 0 and ∞. Assumption 3.1 ensures that $b_{\nu.r} > 1$ remains bounded.

We call (5.11) as the ideal ALOC functions. As (5.11) is obtained from (4.32) by replacing $|\gamma_\nu(\underset{\sim}{\ell})|$ by Δ_ν, from the remarks at the end of section 4, $L^*_{1\nu}(\theta)$ is monotonic increasing and $L^*_{2\nu}(\theta)$ is monotinic decreasing in θ. Further, from Lemma 5.1 it follows that under Assumption 5.1 we can increase the value of $L^*_{1\nu}(\theta)$ for $\theta_1 \le \theta \le 1$, the value of $L^*_{2\nu}(\theta)$ for $0 \le \theta \le \theta_2$ and the value of $L^*_{0\nu}(\theta)$ for $\theta_2 < \theta < \theta_1$ by making Δ_ν larger. In practice by enlarging the set of observed variables, generally, the value of Δ_ν would increase (see the remarks in Section 5 of [1]), and hence, the performance of the procedure would improve.

Clearly for $\theta_1 \leq \theta \leq 1$, $L^*_{1\nu}(\theta) \geq L^*_{1\nu}(\theta_1) = J(\frac{\Delta_\nu}{b_{\nu 1}}, 0; \rho_{\nu.12})$ and as Δ_ν becomes large this lower bound approaches ½. (For $\rho_{\nu.12}=0$, the lower bound is exactly ½ whatever Δ_ν). For any $\theta > \theta_1$, however $L^*_{1\nu}(\theta)$ would approach 1 as Δ_ν becomes large. Similar observations apply to $L^*_{2\nu}(\theta)$ over $0 \leq \theta \leq \theta_2$. As regards $L^*_{0\nu}(\theta)$ we observe that as Δ_ν increases its value at both θ_2 and θ_1 approaches ½ and at any intermediate point approaches 1.

6. DECISION RULE USING ESTIMATED COEFFICIENTS

In this section we consider decision rules that would be obtained if in (2.4)-(2.5) we substitute for the coefficient vector $\underset{\sim}{\ell}$ some 'sample estimate' of the best choice $\underset{\sim}{\ell}_\nu$ given by (5.10).

Given a sequence of samples

$$\underset{\sim}{X}^{(k)}_{\nu\alpha}, \quad \alpha=1,\ldots,n_{k\nu}, \quad k=0,1,2, \tag{6.1}$$

Let

$$\underset{\sim}{a}^{(k)}_{\nu\alpha} = (a^{(k)}_{\nu.1\alpha},\ldots,a^{(k)}_{\nu.p\alpha})'$$
$$\bar{\underset{\sim}{a}}^{(k)}_{\nu} = (\bar{a}^{(k)}_{\nu.1},\ldots,\bar{a}^{(k)}_{\nu.p})'$$

be defined as in (2.2) and (2.3). Further, let

$$\hat{\sigma}^{(k)}_{\nu.ij} = \frac{1}{n_{k\nu}} \sum_{\alpha=1}^{n_{k\nu}} a^{(k)}_{\nu.i\alpha}\, a^{(k)}_{\nu.j\alpha} - \bar{a}^{(k)}_{\nu.i}\, \bar{a}^{(k)}_{\nu.j}$$

$$\hat{\underset{\sim}{\Sigma}}^{(k)}_{\nu} = (\hat{\sigma}^{(k)}_{\nu.ij})_{i,j=1,\ldots,p}, \quad k=0,1,2 \tag{6.2}$$

Let $q^{(k)}_\nu$, k=0,1,2 stand, in general, for some nonnegative valued random variables (possibly depending on the samples) such that $\sum_{k=0}^{2} q^{(k)}_\nu$ is uniformly bounded away from 0 and ∞, say

$$0 < Q_1 < \sum_{k=0}^{2} q_\nu^{(k)} < Q_2 < \infty \tag{6.3}$$

Define the sequence of vectors $\hat{\ell}_\nu$ by

$$\hat{\underset{\sim}{\ell}}_\nu = (\sum_{k=0}^{2} q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}), \text{ when } \sum_k q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)} \text{ is p.d.} \tag{6.4}$$

$$= \text{some arbitrary non-null vector, when } \sum_k q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)} \text{ is singular.}$$

Set up

$$\hat{T}_\nu = \hat{\underset{\sim}{\ell}}_\nu'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(0)}) / \hat{\underset{\sim}{\ell}}_\nu'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}) \tag{6.5}$$

whenever the denominator is nonzero. Consider the following sequence of decision rules obtained by replacing $\underset{\sim}{\ell}$ in (2.5) by $\hat{\underset{\sim}{\ell}}_\nu$:

(a) When $\hat{\underset{\sim}{\ell}}_\nu'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}) \neq 0$, (i) choose d_1, if $\hat{T}_\nu \geq \theta_1$

(ii) choose d_2, if $\hat{T}_\nu \leq \theta_2$

(iii) choose d_0, if $\theta_2 < \hat{T}_\nu < \theta_1$.

(b) When $\hat{\underset{\sim}{\ell}}_\nu'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}) = 0$, choose one of d_1, d_2, d_0 at random. (6.6)

We shall see below that when the samples (6.1) are taken from sequences of populations subject to (4.3) and (4.4) and $\underset{\sim}{\Lambda}_\nu$ is defined as in (4.9), $\hat{\underset{\sim}{\Sigma}}_\nu^{(k)} - \underset{\sim}{\Lambda}_\nu \overset{P}{\to} \underset{\sim}{0}$, so that (6.4) seems a natural 'estimate' of $\underset{\sim}{\ell}_\nu$ given by (5.10). Further, the choice (6.4) includes the following special case. Consider the set of (p+1) simultaneous equations in the elements s and $\underset{\sim}{\ell} = (\ell_1, \ldots, \ell_p)'$ given by

$$\underset{\sim}{\ell}'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(0)}) - s\, \underset{\sim}{\ell}'(\bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}) = 0$$

$$[s(\frac{N_\nu}{n_{0\nu}} + s\frac{N_\nu}{n_{1\nu}})\hat{\underset{\sim}{\Sigma}}_\nu^{(1)} + (1-s)(\frac{N_\nu}{n_{0\nu}} + (1-s)\frac{N_\nu}{n_{2\nu}})\hat{\underset{\sim}{\Sigma}}_\nu^{(2)}]\underset{\sim}{\ell} = \bar{\underset{\sim}{a}}_\nu^{(2)} - \bar{\underset{\sim}{a}}_\nu^{(1)}. \tag{6.7}$$

Let us consider an interval $(-\varepsilon, 1+\varepsilon)$ where $\varepsilon > 0$ is a number to be chosen suitably small. Let $\hat{\theta}_\nu$ stand for a solution of s of (6.7) in this interval, whenever such a solution exists, and otherwise, be arbitrarily defined in (0,1).

Then taking

$$q_\nu^{(0)} = 0, \qquad q_\nu^{(1)} = \hat{\theta}_\nu(\frac{N_\nu}{n_{0\nu}} + \hat{\theta}_\nu \frac{N_\nu}{n_{1\nu}}),$$

$$q_\nu^{(2)} = (1-\hat{\theta}_\nu)(\frac{N_\nu}{n_{0\nu}} + (1-\hat{\theta}_\nu) \frac{N_\nu}{n_{2\nu}}), \tag{6.8}$$

by Assumption 3.1, for a suitably small ε, $q_\nu^{(k)}$ would be nonnegative and (6.3) would hold. Further, $\hat{\underset{\sim}{\ell}}_\nu$ given by (6.4) would be equal to the $\underset{\sim}{\ell}$-solution of (6.7) whenever $\Sigma q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)}$ is positive definite. Hence, in this case $\hat{T}_\nu$ given by (6.5) would equal $\hat{\theta}_\nu$. From the results of [1] it follows that under mild assumptions, for a suitably small ε, with probability approaching 1 the equations (6.7) have a solution in $(-\varepsilon, 1+\varepsilon)$ and $\Sigma q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)}$ is positive definite, whatever be $0\leq\theta\leq 1$. Also, as shown in [1] within a particular class of rank based estimates of θ, $\hat{\theta}_\nu$ has the minimum asymptotic variance. Thus, for $q_\nu^{(k)}$ given by (6.8), (6.6) can be interpreted as the rule based on a 'best estimate.' (In [1], it was implicitly assumed that $0<\theta<1$ and accordingly s was allowed to lie in [0,1]. Here, as $0\leq\theta\leq 1$, we allow s to lie in $(-\varepsilon, 1+\varepsilon)$, and for a suitably small ε the results carry over).

<u>Consistency</u> First suppose that the samples (6.1) are taken from fixed populations $F^{(0)}$, $F^{(1)}$, $F^{(2)}$ subject to (1.1). Set up

$$\sigma_{\nu.ij}^{(k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_i(H_{\nu[i]}(x))\varphi_j(H_{\nu[j]}(y))dF_{[i,j]}^{(k)}(x,y)-\mu_{\nu i}^{(k)}\mu_{\nu j}^{(k)}$$

$$\underset{\sim}{\Sigma}_\nu^{(k)} = (\sigma_{\nu.ij}^{(k)})_{i,j=1,\ldots,p} \tag{6.9}$$

where $H_\nu(\underset{\sim}{x})$, $\mu_{\nu i}^{(k)}$ are as in (3.2) and (3.3). From Theorem 2.1 and 2.2 in [1] we have, as $\nu\to\infty$

$$\bar{\underset{\sim}{a}}_\nu^{(k)}-\underset{\sim}{\mu}_\nu^{(k)} \overset{P}{\to} \underset{\sim}{o}, \quad \hat{\underset{\sim}{\Sigma}}_\nu^{(k)}-\underset{\sim}{\Sigma}_\nu^{(k)} \overset{P}{\to} \underset{\sim}{0}, \; k=0,1,2 \tag{6.10}$$

Now, if the characteristic roots of $\underset{\sim}{\Sigma}_\nu^{(k)}$, k=0,1,2 are bounded away from 0, and lim inf $|\mu_{\nu i}^{(2)}-\mu_{\nu i}^{(1)}|$ is positive for at least one i (see Assumption III in [1]), from (6.3) and (6.10) it follows that $\sum_k q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)}$ is p.d. in probability and $\hat{\underset{\sim}{\ell}}'(\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(1)})$ is bounded away from 0 in probability. Further $\hat{\underset{\sim}{\ell}}_\nu'\{\theta\bar{\underset{\sim}{a}}_\nu^{(1)}+(1-\theta)\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(0)}\}$ converges to 0 in probability (see the argument in Section 4 of [1]). As, when $\sum_k q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)}$ is p.d. and $\hat{\underset{\sim}{\ell}}_\nu'(\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(1)}) \neq 0$

$$\hat{T}_\nu-\theta = \hat{\underset{\sim}{\ell}}_\nu'\{\theta\bar{\underset{\sim}{a}}_\nu^{(1)}+(1-\theta)\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(0)}\}/\hat{\underset{\sim}{\ell}}_\nu'(\bar{\underset{\sim}{a}}_\nu^{(2)}-\bar{\underset{\sim}{a}}_\nu^{(1)}),$$

it follows that $\hat{T}_\nu \xrightarrow{P} \theta$. Hence the rules (6.6) are consistent in the sense of Section 3, provided (3.6) holds.

<u>ALOC functions</u> Now suppose as in Section 4 that the samples (6.1) are taken from sequences of populations $F_\nu^{(k)}$, k=0,1,2 subject to (4.3) and (4.4). Then proceeding as in the proof of Theorem 2.2 of [1] and making use of (4.12) it may be shown that for $\underset{\sim}{\Lambda}_\nu$ given by (4.9) and (4.11),

$$\hat{\underset{\sim}{\Sigma}}_\nu^{(k)} - \underset{\sim}{\Lambda}_\nu \xrightarrow{P} \underset{\sim}{0}, \quad k=0,1,2, \tag{6.11}$$

and hence by (6.3),

$$\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)} - (\sum_k q_\nu^{(k)})\underset{\sim}{\Lambda}_\nu \xrightarrow{P} \underset{\sim}{0}$$

By Assumption 4.1, this implies that $\Sigma q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)}$ is p.d. in probability. Hence, by Assumption 3.2 and Lemma 1.1 of [1] we get

$$(\sum_k q_\nu^{(k)} \hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1} - (\sum_k q_\nu^{(k)})^{-1} \underset{\sim}{\Lambda}_\nu^{-1} \xrightarrow{P} \underset{\sim}{0} \tag{6.12}$$

From the above it follows that, for finding the ALOC functions of the rule (6.6), we may proceed as if $\Sigma q_\nu^{(k)} \hat{\Sigma}_\nu^{(k)}$ is p.d. Also, as proved in Section 4, $\sqrt{N_\nu}\ (\bar{a}_\nu^{(2)}-\bar{a}_\nu^{(1)})-\delta_\nu$ is asymptotically normal so that for asymptotic purposes we may neglect the possibility (b) in (6.6). Thus, defining $\underset{\sim}{Z}_{\nu r}$, r=1,2,3, as in (4.5), as regards ALOC functions, the rule (6.6) is equivalent to:

$$\text{choose } d_1, \text{ if } \underset{\sim}{Z}'_{\nu 1}(\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}\underset{\sim}{Z}_{\nu 2} \geq 0.$$

$$\text{choose } d_2, \text{ if } \underset{\sim}{Z}'_{\nu 1}(\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}\underset{\sim}{Z}_{\nu 3} \geq 0.$$

$$\text{choose } d_0, \text{ if } \underset{\sim}{Z}'_{\nu 1}(\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}\underset{\sim}{Z}_{\nu 2} < 0,\ \underset{\sim}{Z}'_{\nu 1}(\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}\underset{\sim}{Z}_{\nu 3} < 0 \qquad (6.13)$$

From the results of Section 4, we get that $\underset{\sim}{Z}_{\nu r}$, r=1,2,3 are bounded in probability. Hence, by (6.3) and (6.12),

$$(\sum_k q_\nu^{(k)})\underset{\sim}{Z}'_{\nu 1}(\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}\underset{\sim}{Z}_{\nu r}-\underset{\sim}{Z}'_{\nu 1}\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{Z}_{\nu r} \overset{P}{\to} 0,\ r=2,3 \qquad (6.14)$$

From this we can deduce that

$$P\{\underset{\sim}{Z}'_{\nu 1}(\sum_k q_\nu^{(k)}\hat{\underset{\sim}{\Sigma}}_\nu^{(k)})^{-1}\underset{\sim}{Z}_{\nu r}\geq 0\}-P\{\underset{\sim}{Z}'_{\nu 1}\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{Z}_{\nu r}\geq 0\} \to 0,\ r=2,3 \qquad (6.15)$$

(This follows by noting that $(\underset{\sim}{Z}'_{\nu 1},\underset{\sim}{Z}'_{\nu r})$ has asymptotically a 2p-variate normal distribution with uniformly bounded means and variances [Assumptions 3.1, 3.2, 4.1 and 4.2A]. Hence given any subsequence of $(\underset{\sim}{Z}'_{\nu 1},\underset{\sim}{Z}'_{\nu r})$ we can pick a sub-subsequence which converges in law to a <u>fixed</u> 2p-dimensional normal distribution, and hence, for which $\underset{\sim}{Z}'_{\nu 1}\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{Z}_{\nu r}$ has a limiting cdf continuous at 0. In view of this and (6.14) there can not be a subsequence for which the absolute value of the difference in (6.15) is bounded away from 0).

Now let $\underset{\sim}{\zeta}_{\nu 1}$, $\underset{\sim}{\zeta}_{\nu 2}$, $\underset{\sim}{\zeta}_{\nu 3}$ be defined as in (4.21)-(4.23). The dispersion matrices (4.19) of the 2p-vectors satisfy the condition (4.1). Hence from (6.15) and Lemma 4.3 we get that the ALOC functions of the rules (6.6) are given by

$$L^*_{1\nu}(\theta) = P\{\underset{\sim}{\zeta}'_{\nu 1}\, \underset{\sim}{\Lambda}^{-1}_{\nu}\, \underset{\sim}{\zeta}_{\nu 2} \geq 0\}$$

$$L^*_{2\nu}(\theta) = P\{\underset{\sim}{\zeta}'_{\nu 1}\, \underset{\sim}{\Lambda}^{-1}_{\nu}\, \underset{\sim}{\zeta}_{\nu 3} \geq 0\}$$

$$L^*_{0\nu}(\theta) = 1 - \hat{L}^*_{1\nu}(\theta) - \hat{L}^*_{2\nu}(\theta) \tag{6.16}$$

For convenience, hereafter we keep the subscript ν understood. Let C(p×p) be such that

$$\underset{\sim}{C}\, \Lambda\, \underset{\sim}{C}' = I \tag{6.17}$$

For $\underset{\sim}{\delta}$, and Δ^2 defined as in (4.15) and (5.9), $\Delta^{-1}\, \underset{\sim}{C}\, \underset{\sim}{\delta}$ is then a vector of unit length. Let P(p×p) be an orthogonal matrix whose first row is $\Delta^{-1}(\underset{\sim}{C}\underset{\sim}{\delta})'$. Denote

$$\underset{\sim}{\eta}_r = \underset{\sim}{P}\, \underset{\sim}{C}\, \underset{\sim}{\zeta}_r, \quad r=1,2,3$$

$$= (\eta_{r1},\ldots,\eta_{rp})' \text{ (say)} \tag{6.18}$$

From (4.21)-(4.23), it may be checked that $(\eta_{1i},\eta_{2i},\eta_{3i})$, i=1,2,...,p are independently normally distributed with means given by

$$E\,\eta_{11} = \Delta,\ E\,\eta_{21} = (\theta-\theta_1)\Delta,\ E\,\eta_{31} = (\theta_2-\theta)\Delta$$

$$E\,\eta_{1i} = E\,\eta_{2i} = E\,\eta_{3i} = 0, \quad i=2,3,\ldots,p \tag{6.19}$$

and with $(b_{rs})_{1,2,3}$ defined by (4.13) as the common dispersion matrix. From (6.16) and (6.18) we have

$$\hat{L}^*_1(\theta) = P\{\underset{\sim}{\eta}'_1\underset{\sim}{\eta}_2 \geq 0\}$$

$$\hat{L}^*_2(\theta) = P\{\underset{\sim}{\eta}'_1\underset{\sim}{\eta}_3 \geq 0\} \tag{6.20}$$

Or using notations as in (4.30) and writing

$$\underset{\sim}{\xi}_r = (\xi_{r1},\ldots,\xi_{rp})' = \frac{1}{b_r}\underset{\sim}{\eta}_r \quad r=1,2,3 \tag{6.21}$$

$$\hat{L}^*_1(\theta) = P\{\underset{\sim}{\xi}'_1\underset{\sim}{\xi}_2 \geq 0\} = P\{(\underset{\sim}{\xi}_1+\underset{\sim}{\xi}_2)'(\underset{\sim}{\xi}_1+\underset{\sim}{\xi}_2)-(\underset{\sim}{\xi}_1-\underset{\sim}{\xi}_2)'(\underset{\sim}{\xi}_1-\underset{\sim}{\xi}_2) \geq 0\}$$

$$\hat{L}^*_2(\theta) = P\{\underset{\sim}{\xi}'_1\underset{\sim}{\xi}_3 \geq 0\} = P\{(\underset{\sim}{\xi}_1+\underset{\sim}{\xi}_3)'(\underset{\sim}{\xi}_1+\underset{\sim}{\xi}_3)-(\underset{\sim}{\xi}_1-\underset{\sim}{\xi}_3)'(\underset{\sim}{\xi}_1-\underset{\sim}{\xi}_3) \geq 0\} \tag{6.22}$$

where $(\xi_{1i}, \xi_{2i}, \xi_{3i})$, i=1,2,...,p are independently normally distributed with means given by

$$E\ \xi_{11} = \frac{\Delta}{b_1}, \quad E\ \xi_{21} = (\theta-\theta_1)\frac{\Delta}{b_2}, \quad E\ \xi_{31} = (\theta_2-\theta)\frac{\Delta}{b_3},$$

$$E\ \xi_{1i} = E\ \xi_{2i} = E\ \xi_{3i} = 0, \quad i=2,3,\ldots,p \tag{6.23}$$

and with common dispersion $(\rho_{rs})_{1,2,3}$. Hence, it is easy to see that, for r=2,3,

$$U_{r-1} = \frac{(\underset{\sim}{\xi}_1+\underset{\sim}{\xi}_r)'(\underset{\sim}{\xi}_1+\underset{\sim}{\xi}_r)}{2(1+\rho_{1r})}, \quad V_{r-1} = \frac{(\underset{\sim}{\xi}_1-\underset{\sim}{\xi}_r)'(\underset{\sim}{\xi}_1-\underset{\sim}{\xi}_r)}{2(1-\rho_{1r})}$$

are <u>independently</u> distributed as noncentral χ^2's with same d.f. p. The non-centrality parameters, for r=2, are respectively

$$\lambda_1 = \frac{\Delta^2}{2(1+\rho_{12})}\{\frac{1}{b_1} + \frac{(\theta-\theta_1)}{b_2}\}^2, \quad \pi_1^* = \frac{\Delta^2}{2(1-\rho_{12})}\{\frac{1}{b_1} - \frac{(\theta-\theta_1)}{b_2}\}^2 \tag{6.24}$$

and for r=3, respectively

$$\pi_2 = \frac{\Delta^2}{2(1+\rho_{13})}\{\frac{1}{b_1} + \frac{(\theta_2-\theta)}{b_3}\}^2, \quad \pi_2^* = \frac{\Delta^2}{2(1-\rho_{13})}\{\frac{1}{b_1} - \frac{(\theta_2-\theta)}{b_3}\}^2 \tag{6.25}$$

So, from (6.22), we get

$$\begin{aligned} \hat{L}_1^*(\theta) &= P\{(1+\rho_{12})U_1 - (1-\rho_{12})V_1 \geq 0\} \\ \hat{L}_2^*(\theta) &= P\{(1+\rho_{13})U_2 - (1-\rho_{13})V_2 \geq 0\} \end{aligned} \tag{6.26}$$

Or writing

$$\frac{V_r}{U_r+V_r} = W_r, \quad r=1,2 \tag{6.27}$$

$$\hat{L}_1^*(\theta) = P\{W_1 \leq \frac{1+\rho_{12}}{2}\}, \quad \hat{L}_2^*(\theta) = P\{W_2 \leq \frac{1+\rho_{13}}{2}\} \tag{6.28}$$

From the above, it follows that W_r (r=1,2) has the doubly noncentral Beta distribution (see, for instance [7] pp. 197-198) with shape parameters $\frac{p}{2}$, $\frac{p}{2}$, and noncentrality parameters π_r^*, π_r i.e., with denisty function

$$e^{-\frac{1}{2}(\pi_r^*+\pi_r)} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!} \left(\frac{\pi_r^*}{2}\right)^s \frac{1}{t!} \left(\frac{\pi_r}{2}\right)^t \tag{6.29}$$

$$\frac{1}{B\left(\frac{p+2s}{2}, \frac{p+2t}{2}\right)} w^{\frac{p+2s}{2}-1} (1-w)^{\frac{p+2t}{2}-1} \qquad 0\leq w\leq 1 \quad r=1,2$$

Hence, using the incomplete Beta-function ratio notation

$$I_x(p,q) = \frac{1}{B(p,q)} \int_0^x w^{p-1}(1-w)^{q-1}dw, \quad 0\leq x\leq 1 \tag{6.30}$$

we have, from (6.28)

$$\hat{L}_1^*(\theta) = e^{-\frac{1}{2}(\pi_1^*+\pi_1)} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!}\left(\frac{\pi_1^*}{2}\right)^s \frac{1}{t!}\left(\frac{\pi_1}{2}\right)^t I_{\frac{1}{2}(1+\rho_{12})}\left(\frac{p}{2} + s, \frac{p}{2} + t\right) \tag{6.31}$$

$$\hat{L}_2^*(\theta) = e^{-\frac{1}{2}(\pi_2^*+\pi_2)} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!}\left(\frac{\pi_2^*}{2}\right)^s \frac{1}{t!}\left(\frac{\pi_2}{2}\right)^t I_{\frac{1}{2}(1+\rho_{13})}\left(\frac{p}{2} + s, \frac{p}{2} + t\right) \tag{6.32}$$

where π_1^*, π_1 and π_2^*, π_2 are given by (6.24) and (6.25).

It is interesting to note that for the classical problem of classification between two multinormal populations with unknown means but a known common dispersion matrix, John [6] obtained similar series expressions for the unconditional probabilities of misclassification.

To study the shapes of these ALOC functions, differentiating (6.31) with respect to θ we get

$$\frac{d\hat{L}_1^*(\theta)}{d\theta} = \tfrac{1}{2}\, e^{-\frac{1}{2}(\pi_1^*+\pi_1)} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!}\left(\frac{\pi_1^*}{2}\right)^s \frac{1}{t!}\left(\frac{\pi_1}{2}\right)^t \Big[\frac{d\pi_1^*}{d\theta}\{I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2} + s + 1, \tfrac{p}{2} + t)$$

$$- I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2} + s, \tfrac{p}{2} + t)\} + \frac{d\pi_1}{d\theta}\{I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2}+s, \tfrac{p}{2}+t+1) -$$

$$- I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2} + s, \tfrac{p}{2} + t)\}\Big] \qquad (6.33)$$

Or using the identity

$$I_x(p,q) = \frac{p}{p+q} I_x(p+1,q) + \frac{q}{p+q} I_x(p,q+1),$$

$$\frac{d\hat{L}_1^*(\theta)}{d\theta} = \tfrac{1}{2}\, e^{-\frac{1}{2}(\pi_1^*+\pi_1)} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!}\left(\frac{\pi_1^*}{2}\right)^s \frac{1}{t!}\left(\frac{\pi_1}{2}\right)^t \frac{1}{p+s+t}\{(\tfrac{p}{2} + s)\frac{d\pi_1}{d\theta} - (\tfrac{p}{2} + t)\frac{d\pi_1^*}{d\theta}\}$$

$$\{I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2}+s, \tfrac{p}{2} + t + 1) - I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2} + s + 1, \tfrac{p}{2} + t)\} \qquad (6.34)$$

As for all s,t

$$I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2} + s, \tfrac{p}{2} + t + 1) - I_{\frac{1}{2}(1+\rho_{12})}(\tfrac{p}{2} + s + 1, \tfrac{p}{2} + t) > 0,$$

a sufficient condition for (6.34) to be non-negative is

$$(\tfrac{p}{2} + s)\, \frac{d\pi_1}{d\theta} - (\tfrac{p}{2} + t)\, \frac{d\pi_1^*}{d\theta} \geq 0 \quad \text{for all } s,t$$

or equivalently

$$\frac{d\pi_1}{d\theta} \geq 0, \quad \frac{d\pi_1^*}{d\theta} \leq 0 \qquad (6.35)$$

Using (6.24), conditions (6.35) reduce to

$$-\frac{b_2}{b_1} \leq \theta-\theta_1 \leq \frac{b_2}{b_1}$$

By (4.13) and (4.30) this can be written as

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right)(\theta-\theta_1)^2 \le \frac{1}{n_0} + \frac{\theta_1^2}{n_1} + \frac{(1-\theta_1)^2}{n_2} \tag{6.36}$$

As $\theta_1 > \frac{1}{2}$, (6.36) is always satisfied for $\theta_1 \le \theta \le 1$. Generally, there is a number θ_1^* such that

$$\theta_1^* < \theta_1,\ (\theta_1^*-\theta_1)^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right) = \frac{1}{n_0} + \frac{\theta_1^2}{n_1} + \frac{(1-\theta_1)^2}{n_2} \tag{6.37}$$

for which (6.36) holds for all $\theta_1^* \le \theta \le 1$. From (6.37) we get

$$\theta_1^* = \theta_1-\left[\theta_1^2 + \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}\left(\frac{1}{n_0} + \frac{1-2\theta_1}{n_2}\right)\right]^{\frac{1}{2}} \tag{6.38}$$

If

$$\frac{1}{n_0} + \frac{1-2\theta_1}{n_2} \ge 0,$$

or equivalently,

$$\frac{n_2}{n_0} \ge 2\theta_1-1, \tag{6.39}$$

we have $\theta_1^* \le 0$. So we have proved that if (6.39) holds, then $\frac{d\hat{L}_1^*(\theta)}{d\theta} \ge 0$ for all $\theta\varepsilon[0,1]$. In general, we have established that $\frac{d\hat{L}_1^*(\theta)}{d\theta} \ge 0$ for all $\theta\varepsilon[\theta_1^*,1]$. It may be noted that (6.35) is only a sufficient condition for (6.34) to be nonnegative and it is possible that when $\theta_1^* > 0$, $\frac{d\hat{L}_1^*(\theta)}{d\theta}$ may remain nonnegative to the left of θ_1^* as well. The author has been unable to get a full answer to the question whether $\hat{L}_1^*(\theta)$ is nondecreasing in the entire interval [0,1] in general. In any case, the above shows that, quite generally, to the value of $\hat{L}_1^*(\theta)$, $\hat{L}_1^*(\theta_1)$ gives a lower bound in $[\theta_1,1]$, and if $L_1<\theta_1<U_1$, $\hat{L}_1^*(U_1)$ gives a lower bound in $[U_1,1]$.

As regards $\hat{L}_2^*(\theta)$, we can similarly show that a sufficient condition for $\frac{d}{d\theta}\hat{L}_2^*(\theta) \le 0$ is

$$(\frac{1}{n_1}+\frac{1}{n_2})(\theta_2-\theta)^2 \le \frac{1}{n_0}+\frac{\theta_2^2}{n_1}+\frac{(1-\theta_2)^2}{n_2}$$

and as $\theta_2 < \frac{1}{2}$, this is always satisfied for $0 \le \theta \le \theta_2$. Generally, writing

$$\theta_2^* = \theta_2 + [(1-\theta_2)^2 + (\frac{1}{n_1}+\frac{1}{n_2})^{-1}(\frac{1}{n_0}+\frac{2\theta_2-1}{n_1})]^{\frac{1}{2}}$$

we can show that $\frac{d\hat{L}_2^*(\theta)}{d\theta} \le 0$ in $[0,\theta_2^*]$. If

$$\frac{n_1}{n_0} \ge 1-2\theta_2 \tag{6.42}$$

we have $\theta_2^* \ge 1$, so that $\hat{L}_2^*(\theta)$ is non-increasing in [0,1]. Here also it is possible $\hat{L}_2^*(\theta)$ is nonincreasing in [0,1] more generally, but this could not be proved. Quite generally, to the value of $\hat{L}_2^*(\theta)$, $\hat{L}_2^*(\theta_2)$ gives a lower bound in $[0,\theta_2]$ and, if $L_2<\theta_2<U_2$, $\hat{L}_2^*(L_2)$ gives a lower bound in $[0,L_2]$.

To see how the ALOC functions of the procedure of this section are related to the ideal ALOC functions, we note that from (5.11), (6.18) and (6.21), we can write the latter as

$$L_1^*(\theta) = P\{\xi_{11}\xi_{21} \ge 0\}. \quad L_2^*(\theta) = P\{\xi_{11}\xi_{31} \ge 0\}. \tag{6.43}$$

whereas from (6.20)

$$\hat{L}_1^*(\theta) = P\{\xi_{11}\xi_{21} + \sum_2^p \xi_{1i}\xi_{2i} \ge 0\}$$
$$\hat{L}_2^*(\theta) = P\{\xi_{11}\xi_{31} + \sum_2^p \xi_{1i}\xi_{3i} \ge 0\} \tag{6.44}$$

where $(\xi_{1i}, \xi_{2i}, \xi_{3i})$, i=1,2,...,p are independently normally distributed with means given by (6.23) and with common dispersion (ρ_{rs}). Exact analytical comparison of (6.43) and (6.44) seems difficult.

If we write $B_x(p: \pi^*,\pi)$ for the tail probability to the left of x of the noncentral Beta distribution with shape parameters $\frac{p}{2}$, $\frac{p}{2}$ and noncentrality parameters π^*, π, i.e., if

$$B_x(p;\pi^*,\pi) = e^{-\frac{1}{2}(\pi^*+\pi)} \sum_{x=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!}(\frac{\pi^*}{2})^s \frac{1}{t!}(\frac{\pi}{2})^t I_x(\frac{p}{2} + s, \frac{p}{2} + t)$$

Then, from (6.31) and (6.32), we have

$$\hat{L}_1^*(\theta) = B_{\frac{1}{2}(1+\rho_{12})}(p;\pi_1^*,\pi_1), \hat{L}_2^*(\theta) = B_{\frac{1}{2}(1+\rho_{13})}(p;\pi_2^*,\pi_2). \tag{6.45}$$

Proceeding just as in (6.22)-(6.26), from (6.43) we can similarly show

$$L_1^*(\theta) = B_{\frac{1}{2}(1+\rho_{12})}(1;\pi_1^*,\pi_1), L_2^*(\theta) = B_{\frac{1}{2}(1+\rho_{13})}(1,\pi_2^*,\pi_2). \tag{6.46}$$

where π_1^*, π_1, π_2^*, π_2 are as before given by (6.24) and (6.25). Table 1 gives the values of the function $B_x(p;\pi^*,\pi)$ for a few selected values of x, π^*, π and p=1,2,3. As by our Assumption 5.1, $\frac{1}{2}(1+\rho_{12})$ and $\frac{1}{2}(1+\rho_{13})$ are both $\leq \frac{1}{2}$, we have taken only values $x \leq 0.5$. Also, from (6.24)-(6.25), we have for $\theta \geq \theta_1$, $(1+\rho_{12})\pi_1 \geq (1-\rho_{12})\pi_1^*$, and for $\theta \leq \theta_2$, $(1+\rho_{13})\pi_2 \geq (1-\rho_{13})\pi_2^*$. Hence, we have given the values of $B_x(p;\pi^*,\pi)$ only for $x\pi \geq (1-x)\pi^*$. The values of $B_x(1;\pi^*,\pi)$ were computed by using the relation

$$B_x(1;\pi^*,\pi) = J(\sqrt{x\pi} + \sqrt{(1-x)\pi^*}, \sqrt{x\pi} - \sqrt{(1-x)\pi^*}; 2x-1)$$

where J is given by (4.27) and (4.28) and then using Table 8.5 in [9]. The values of $B_x(p;\pi^*,\pi)$ $p \geq 2$ were computed by using the fact that if X is a random variable following the ordinary Beta distribution with parameters $\frac{1}{2}f_1$, $\frac{1}{2}f_2$, where $f_1 = (p+\pi^*)^2(p+2\pi^*)^{-1}$, $f_2 = (p+\pi)^2(p+2\pi)^{-1}$, then the transform $Xg/[Xg+1-X]$, where $g = (p+2\pi^*)(p+\pi)/(p+\pi^*)(p+2\pi)$, is approximately distributed as a non-central Beta variable with parameters $\frac{1}{2}p$, $\frac{1}{2}p$, π^*, π. Dasgupta [2] has found this approximation adequate and in our case it was found to give surprisingly close values for p=1.

From an examination of the values in the table, it seems legitimate to conclude that in $[\theta_1,1]$ generally a high value for L_1^* is attended by a high value of $\hat{L}_1^*$, the value of $\hat{L}_1^*$ is less than that of L_1^*, and the difference increases with p. Similar observations hold for $[0,\theta_2]$, L_2^* and $\hat{L}_2^*$. The falling away of the values of $\hat{L}_1^*$ and $\hat{L}_2^*$ with increasing p is understandable, since the number of estimated parameters increases with p.

7. A SPECIAL CASE

In this section, we consider the situation where the preference scale of Section 1 holds with $L_2=0$, $U_1=1$. Such a situation would arise in practice when the values $\theta=0$ and $\theta=1$ are of special interest and d_1 and d_2 are respectively the most preferred decisions only in these cases. All that has been said in the earlier sections still hold. But here we can also approach the problem of best determination of $\underset{\sim}{\ell}$ from another angle.

As the sample sizes increase, here it would be realistic to take θ_1 and θ_2 closer and closer to 1 and 0 respectively so as to make the procedure more and more discriminating as regards the choice of d_1 and d_2. So, to judge the performance of a particular compounding vector $\underset{\sim}{\ell}$, and hence, to find the best choice of $\underset{\sim}{\ell}$, we can alternatively proceed as follows. Given <u>fixed</u> $F^{(1)}$, $F^{(2)}$, and $F^{(0)}$ subject to (1.1) and sequences of sample sizes $n_{1\nu}$, $n_{2\nu}$, $n_{0\nu}$ as in Section 3, we can consider the sequence of decision rules based on $\underset{\sim}{\ell}$, and $\theta_{1\nu}(\to 1)$, $\theta_{2\nu}(\to 0)$. The limiting behaviour of the corresponding probabilities of choosing d_1, when $\theta=1$, and d_2, when $\theta=0$, can then be studied. Of course $\theta_{1\nu}$, $\theta_{2\nu}$ are to be taken so that these probabilities remain bounded away from 1 in large samples.

We write $P_{1\nu}$ and $P_{2\nu}$ respectively for the probabilities of choosing d_1 when $\theta=1$ and d_2 when $\theta=0$ for the decision rule (2.7) based on $n_{0\nu}$, $n_{1\nu}$, $n_{2\nu}$, $\underset{\sim}{\ell}$, $\theta_{1\nu}$ and $\theta_{2\nu}$. Using notations as in Section 3 under Assumption 3.3, we have from (2.7) and (3.5)

$$\begin{aligned} P_{1\nu} &\simeq P\{\theta_{1\nu}\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(1)} + (1-\theta_{1\nu})\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(2)} - \underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(0)} \geq 0|\theta=1\} \\ P_{2\nu} &\simeq P\{\ell'\bar{\underset{\sim}{a}}_\nu^{(0)} - \theta_{2\nu}\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(1)} - (1-\theta_{2\nu})\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(2)} \geq 0|\theta=0\} \end{aligned} \tag{7.1}$$

Now, by Theorem 2.3 of [1],

$$P\{\theta\ell'\bar{\underset{\sim}{a}}_\nu^{(1)}+(1-\theta)\ell'\bar{\underset{\sim}{a}}_\nu^{(2)}-\underset{\sim}{\ell}'\bar{\underset{\sim}{a}}_\nu^{(0)}\leq x[\underset{\sim}{\ell}'(\frac{\theta^2}{n_{1\nu}}\underset{\sim}{\Sigma}_\nu^{(1)} + \frac{(1-\theta)^2}{n_{2\nu}}\underset{\sim}{\Sigma}_\nu^{(2)} + \frac{1}{n_{0\nu}}\underset{\sim}{\Sigma}_\nu^{(0)})\underset{\sim}{\ell}]^{\frac{1}{2}}|\theta\}$$

where $\underset{\sim}{\Sigma}_\nu^{(k)}$, k=0,1,2 are defined by (6.9), converges to the standard normal cdf $\Phi(x)$, and as is well known, this convergence is uniform with respect to x. So, in (7.1), taking $\theta_{1\nu} = 1 - \frac{\kappa_1}{\sqrt{N_\nu}}$, $\theta_{2\nu} = \frac{\kappa_2}{\sqrt{N_\nu}}$, $\kappa_1, \kappa_2 > 0$, from (3.4) we get that

$$P_{1\nu} \simeq \Phi(\kappa_1[\underset{\sim}{\ell}'(\underset{\sim}{\mu}_\nu^{(2)}-\underset{\sim}{\mu}_\nu^{(1)})][\underset{\sim}{\ell}'(\frac{N_\nu}{n_{1\nu}}\underset{\sim}{\Sigma}_\nu^{(1)} + \frac{N_\nu}{n_{0\nu}}\underset{\sim}{\Sigma}_\nu^{(0)})\underset{\sim}{\ell}]^{-\frac{1}{2}}) \tag{7.2}$$

$$P_{2\nu} \simeq \Phi(\kappa_2[\underset{\sim}{\ell}'(\underset{\sim}{\mu}_\nu^{(2)}-\underset{\sim}{\mu}_\nu^{(1)})][\underset{\sim}{\ell}'(\frac{N_\nu}{n_{2\nu}}\underset{\sim}{\Sigma}_\nu^{(2)} + \frac{N_\nu}{n_{0\nu}}\underset{\sim}{\Sigma}_\nu^{(0)})\underset{\sim}{\ell}]^{-\frac{1}{2}}) \tag{7.3}$$

As in Section 3, the expressions (7.2)-(7.3) remain valid even when we replace $\underset{\sim}{\ell}$ by $\underset{\sim}{\ell}_\nu$, where $\underset{\sim}{\ell}_\nu$ is a sequence of uniformly bounded vectors subject to Assumption 3.3.

From the above expressions it is clear that when we are interested only in the values θ=1,0, for large sample sizes the best choice of $\underset{\sim}{\ell}$ in terms of $P_{1\nu}$, $P_{2\nu}$ would be that for which, subject to Assumption 3.3

$$[\underset{\sim}{\ell}'(\underset{\sim}{\mu}^{(2)}-\underset{\sim}{\mu}^{(1)})]^2[\underset{\sim}{\ell}'(\frac{N_\nu}{n_{0\nu}}\underset{\sim}{\Sigma}_\nu^{(0)} + \theta^2\frac{N_\nu}{n_{1\nu}}\underset{\sim}{\Sigma}_\nu^{(1)} + (1-\theta)^2\frac{N_\nu}{n_{2\nu}}\underset{\sim}{\Sigma}_\nu^{(2)})\underset{\sim}{\ell}]^{-1}$$

is maximized, θ being taken 1 or 0 according to the situation. The problem then becomes the same as that considered in [1] in connection with the estimation of θ with the restriction that $\theta=0$ or 1. In practice, we may estimate this best $\underset{\sim}{\ell}$ by solving the equations (6.7) simultaneously. The corresponding rule (see the discussion after (6.7)) is the rule based on the 'best' estimate of θ proposed in [1]. Just as in the proof of Theorem 4.2 [1], it can be shown that for this rule $P_{1\nu}$ when $\theta=1$ and $P_{2\nu}$ when $\theta=0$ are maximized in large samples.

8. CONCLUDING REMARKS

In the foregoing sections we have considered solutions to certain classification-type problems specific to the mixture set-up (1.1). In practice, these would be appropriate where we have apriori knowledge of the validity of (1.1). The following questions naturally arise: (i) How is one to test the validity of (1.1)? (ii) What happens if one tries to apply the proposed procedures even though (1.1) does not hold? So far as the author is aware, for ungrouped data, no satisfactory answer to the first question has yet been found even in the univariate case. If the three samples are grouped to give contingency tables based on the same system of cells, (1.1) implies a composite hypothesis involving the cell probabilities, and this can be tested by standard methods. To consider the second question, for simplicity, suppose, as $\nu\to\infty$, subject to Assumption 3.1 $n_{k\nu}/N_\nu$, k=0,1,2, converge to some limits. Then write $H(\underset{\sim}{x})$ for the limit of (3.2) and define $\underset{\sim}{\mu}^{(k)}$ with respect to it as in (3.3). For a given $\underset{\sim}{\ell}$, if $F^{(k)}$, k=0,1,2 are such that $\underset{\sim}{\ell}'\underset{\sim}{\mu}^{(1)} < \underset{\sim}{\ell}'\underset{\sim}{\mu}^{(0)} < \underset{\sim}{\ell}'\underset{\sim}{\mu}^{(2)}$, and if we are prepared to treat $|\underset{\sim}{\ell}'\underset{\sim}{\mu}^{(k)}-\underset{\sim}{\ell}'\underset{\sim}{\mu}^{(0)}|/(\underset{\sim}{\ell}'\underset{\sim}{\mu}^{(2)}-\underset{\sim}{\ell}'\underset{\sim}{\mu}^{(1)})$ as a measure of the distance between $F^{(k)}$ and $F^{(0)}$, k=1,2, then clearly the rule (2.5) has some meaning. The asymptotic theory of Sections 4 and 5, however, no longer applies. If we still try to apply the rule (6.6) with $\underset{\sim}{\ell}$ determined from (6.7), trouble may arise in that (6.7) may not have real solutions. A real solution would exist with probability approaching 1, if $\underset{\sim}{\mu}^{(0)}$ is a convex linear combination of $\underset{\sim}{\mu}^{(1)}$ and $\underset{\sim}{\mu}^{(2)}$.

We have seen that performance as judged by the ideal ALOC functions (5.11) has an all-round improvement as Δ_ν increases. The author has not so far been able to prove the corresponding result for the actual ALOC functions (6.16) of the procedure of Section 6. The difficulty in the proof arises from the fact that no compact expressions for probabilities of the type (6.44) (such as that used for (6.43) in the proof of Lemma 5.1) could be found and the series expressions (6.31)-(6.32) are not of much help. However, as the figures in Table 1 suggest, an increase in the value of the ideal ALOC function is attended by an increase in the value of the corresponding actual ALOC function and thus it seems that we can improve the performance of the procedure of Section 6 by making Δ larger.

In this paper we have considered rules based on the variate wise ranks. These may be useful in situations where ranks only are available or dependable. Of course, the model (1.1) is invariant under a transformation of the set of p variates into another set containing any number of variates. So, when the original variate values are at hand we can first apply a suitable transformation and then use the procedures of this paper on the transformed variates. This opens up a wide range of possibilities. Further, it may be noted that, although we have based our rules on rank scores, when the original values are available similar rules based on the values and their transforms can be formulated. In that case, to develop the asymptotic theory we would require conditions (like existence of moments up to a certain order) which would guarantee asymptotic normality, convergence in probability etc. Investigations into some of these various possibilities have been undertaken in the context of the estimation problem and the results are hoped to be published later.

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Table 1

Values of $B_x(p;\pi^*,\pi)$[a]

x \ p	$\pi^*=0.10,\ \pi=0.20$			$\pi^*=10.00,\ \pi=20.00$			$\pi^*=0.60,\ \pi=0.80$			$\pi^*=6.00,\ \pi=8.00$			$\pi^*=0.10,\ \pi=8.00$		
	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
.10	-	-	-	-	-	-	-	-	-	-	-	-	.608	.383	.238
.40	.451	.411	.383	.641	.634	.601	-	-	-	-	-	-	.919	.877	.834
.45	.477	.462	.447	.744	.736	.713	.492	.470	.454	.532	.521	.509	.936	.911	.880
.50	.515	.512	.510	.823	.821	.807	.526	.521	.518	.606	.602	.599	.950	.936	.916

[a] For the empty cells $x\pi < (1-x)\pi^*$. See Section 6.