SOME INVARIANCE PRINCIPLES FOR MIXED RANK STATISTICS AND INDUCED ORDER STATISTICS*

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SUMMARY

Along with the affinity of mixed rank statistics and linear combinations of induced order statistics, some weak as well as strong invariance principles for these statistics are established. A variety of models (depending on the nature of dependence of the two variates) is considered and the regularity conditions are tailored for these diverse situations. Some applications to some problems in statistical inference are also stressed.

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1. INTRODUCTION

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independent and identically distributed random vectors (i.i.d.r.v.) with a (bivariate) distribution function (d.f.) F, defined on the Euclidean plan E^2. Let $G(x) = F(x, \infty)$ and $H(y) = F(\infty, y)$ be the two marginal d.f.'s and we assume that both G and H are nondegenerate and, in addition, H is continuous almost everywhere (a.e.). In the context of nonparametric tests for regression with stochastic predictors, Ghosh and Sen (1971) have considered some *mixed rank statistics* which may be defined as follows. For a sample of size $n(\geq 1)$, let $R_{ni} (= \sum_{j=1}^{n} u(Y_i - Y_j)$ where $u(t) = 1$ or 0 according as t is $\geq$ or $< 0)$ be the rank of Y_i among $Y_i, \ldots, Y_n$ $(1 \leq i \leq n)$; ties among the Y_i are neglected, in probability, because of the assumed continuity of H. Then, a mixed rank statistic M_n is defined by

$$M_n = \sum_{i=1}^{n} b(X_i) a_n(R_{ni}), \tag{1.1}$$

where $\{a_n(1), \ldots, a_n(n);\ n \geq 1\}$ is a triangular array of *scores* (to be defined formally later on) and $b(\cdot)$ is some given function. Asymptotic normality of such a mixed rank statistic [under H_0: $F = GH$ as well as local (contiguous) alternatives] has been studied by Ghosh and Sen (1971, Sections 3 and 4) under appropriate regularity conditons. There is a renewed interest in M_n because of its affinity to *induced* (or *concomitants of*) *order statistics* [c.f., Bhattacharya (1974), David and Galambos (1974)], though this affinity has apparently not been noticed. Let $Y_{n,1} < \cdots < Y_{n,n}$ be the ordered r.v.'s corresponding to $Y_1, \ldots, Y_n$ and let $X_{n:i} = X_k$ if $Y_{n,i} = Y_k$ for $i = 1, \ldots, n$, $k = 1, \ldots, n$. Then, the $X_{n:i}$ are termed the induced order statistics. Consider now a *linear combination of a function of induced order statistics* defined by

$$S_n = \sum_{i=1}^{n} c_{ni} b(X_{n:i}) \tag{1.2}$$

where the c_{ni} are (non-stochastic) constants. Note that, by definition (prior to (1.1)), $Y_i = Y_{n,R_{ni}}$, $1 \le i \le n$, so that $X_{n:R_{ni}} = X_i$, for $i = 1,\dots,n$. Thus, we may rewrite (1.2) as

$$S_n = \sum_{i=1}^{n} c_{nR_{ni}} b(X_i), \tag{1.3}$$

and hence, its affinity to M_n is readily identified by letting $c_{ni} = a_n(i)$, for $i = 1,\dots,n$.

Let $m(y) = E\{b(X_i) | Y_i = y\}$, $y \in E$ and let

$$S_n^* = \sum_{i=1}^{n} c_{ni}\{b(X_{n:i}) - m(Y_{n,k})\} \tag{1.4}$$

Bhattacharya (1974, 1976) and Sen (1976) have considered some invariance principles for $\{S_n^*\}$ under suitable regularity conditions. Their results are based on the basic fact that given the order statistics, the induced order statistics are conditionally independent (but, not necessarily identically distributed). If, we let

$$L_n = \sum_{i=1}^{n} c_{ni} m(Y_{n,i}) \tag{1.5}$$

then, L_n is a *linear combination of a function of order statistics*, and by (1.2), (1.4) and (1.5),

$$S_n = L_n + S_n^* . \tag{1.6}$$

Thus, intuitively, the existing invariance principles for $\{S_n^*\}$ and $\{L_n\}$ [viz., Sen (1978)] suggest that parallel results should hold for $\{S_n\}$ (or $\{M_n\}$). Note that where as S_n^* and L_n are uncorrelated, they need not be independent and this may cause certain difficulties in adapting the above approach. A representation of M_n (or S_n) as a functional of the empirical distributions along with existing invariance principles for these empirical processes gives us a convenient way of deriving the desired results.

Along with the preliminary notions, the basic theorems are presented in Section 2, while their proofs are deferred to Sections 3 and 4. The concluding section deals with some general remarks and some applications of the main theorems in certain problems of statistical inference.

2. INVARIANCE PRINCIPLES FOR $\{S_n\}$

We shall consider here both weak and strong invariance principles and in this context, the regularity conditions vary with the depth of the results as well as with the extent of stochastic dependence of X and Y.

In the conventional way, we define the scores by letting

$$a_n(i) = E\phi(U_{ni}), \quad 1 \le i \le n \quad (n \ge 1) \tag{2.1}$$

where $U_{n1} < \cdots < U_{nn}$ are the ordered r.v.s. of a sample of size n from the uniform (0, 1) distribution and

$$\phi(u) = \phi_1(u) - \phi_2(u), \quad 0 < u < 1, \tag{2.2}$$

where both ϕ_1 and ϕ_2 are nondecreasing and square integrable inside (0, 1). Let then

$$\bar{a}_n = n^{-1}\sum_{i=1}^n a_n(i), \quad A_n^2 = (n-1)^{-1}\sum_{i=1}^n [a_n(i) - \bar{a}_n]^2, \tag{2.3}$$

$$\bar{\phi} = \int_0^1 \phi(u)du \quad \text{and} \quad A^2 = \int_0^1 \phi^2(u)du - \bar{\phi}^2 . \tag{2.4}$$

Note that, by definition,

$$\bar{a}_n = \bar{\phi}_n , \quad A_n^2 \le [n/(n-1)]A^2, \quad \forall\ n, \quad \text{and} \quad A_n^2 \to A^2 \text{ as } n \to \infty \tag{2.5}$$

Also, let

$$\xi = Eb(X) = \int_E b(x)dG(u) \quad \text{and} \quad \zeta^2 = \int_E b^2(x)dG(x) - \xi^2. \tag{2.6}$$

We assume that $0 < \zeta < \infty$.

For every $n(\ge 1)$, we define a stochastic process

$W_n^{(1)} = \{W_n^{(1)}(t),\ t \in [0, 1]\}$ by letting

$$W_n^{(1)}(t) = (\sqrt{n}A\zeta)^{-1}\{M_{[nt]} - [nt]\xi\phi\},\ t \in I = [0, 1], \tag{2.7}$$

where $[s]$ denotes the largest integer $\leq s$. Also, let $W = \{W(t),\ t \in I\}$ be a standard Wiener process on I. Then, we have the following

Theorem 2.1. Under H_0: $F \equiv GH$ *and the regularity conditions stated above,* $W_n^{(1)} \xrightarrow{\mathcal{D}} W$, *in the Skorokhod* J_1*-topology on* $D[0, 1]$. (2.8)

Let us now express F as

$$F(x, y) = G(x)H(y)[1 + \Omega(G(x), H(y))],\quad (x, y) \in E^2, \tag{2.9}$$

where [c.f., Sibuya (1959)] Ω may be regarded as a dependence function. $\Omega \equiv 0$ a.e. if H_0 holds. Consider now a sequence $\{K_n\}$ of alternative hypotheses, where

$$K_n:\ (2.9) \text{ holds for } \Omega = \Omega_{(n)}, \tag{2.10}$$

and $\{\Omega_{(n)}\}$ is a sequence of functions on E^2, such that $\lim_{n\to\infty} \Omega_{(n)} \equiv 0$. Let P_n and Q_n be the joint distributions of $\{(X_i, Y_i),\ 1 \leq i \leq n\}$ under H_0 and K_n, respectively. We assume for the next theorem that

$$\{Q_n\} \text{ is contiguous to } \{P_n\}, \tag{2.11}$$

where for an elaborate discussion of *contiguity*, we refer to Hájek and Šidák (1967, Chapter VI).

Theorem 2.2. Under (2.10), (2.11) *and the same regularity conditions on* $b(\cdot)$ *and* $\phi(\cdot)$ *as in Theorem 2.1, there exists a sequence* $\{\omega_n\}$ *(where* $\omega_n = \{\omega_n(t),\ t \in I\}$*) of* $D[0, 1]$ *valued functions (non-stochastic), such that*

$$W_n^{(1)} - \omega_n \xrightarrow{\mathcal{D}} W, \text{ in the } J_1\text{-topology on } D[0, 1]. \tag{2.12}$$

For the weak invariance principle considered above, it appears that for the null hypothesis as well as local (contiguous) alternatives, the regularity conditions on $b(\cdot)$ and $\phi(\cdot)$ are minimal. In this context, a martingale property of $\{M_n\}$ plays an important role and this will be considered in Section 3. This martingale property along with the strong invariance principle for martingales [c.f., Strassen (1967)] enables us to formulate the following

Theorem 2.3. *If, in addition to* (2.1), (2.2) *and* (2.6), *for some* $r > 2$, $\int |\phi|^r < \infty$ *or* $\int |b|^r dG < \infty$, *then under* H_0: $F \equiv GH$, *(in the Skorokhod-Strassen Sense)*,

$$\frac{1}{A\zeta}(M_n - n\bar{\phi}\xi) = W(n) + 0(n^{\frac{1}{2}-\eta}) \text{ a.s., as } n \to \infty , \tag{2.13}$$

where $\eta > 0$ *and* $\{W(t),\ t > 0\}$ *is a standard Wiener process on* $[0, \infty)$. *Hence, with probability* 1,

$$\lim_{n\to\infty} \sup(M_n - n\varepsilon\bar{\phi})/(2A^2\zeta^2 \log\log n)^{\frac{1}{2}} = +1, \tag{2.14}$$

$$\lim_{n\to\infty} \inf(M_n - n\xi\bar{\phi})/(2A^2\zeta^2 \log\log n)^{\frac{1}{2}} = -1 . \tag{2.15}$$

The proofs of these theorems are deferred to Section 3.

Next, we consider the case where (2.10) - (2.11) may not hold. Some extra regularity conditions are needed in this context . Let us define

$$\mu = \mu(b, \phi) = \int_{E^2}\int b(x)\phi(H(y))dF(x, y) . \tag{2.16}$$

(Under H_0: $F \equiv GH$, $\mu = \bar{\phi}\xi$). Also, let

$$V_i = b(X_i)\phi(H(Y_i)) + \int_{E^2}\int b(x)\phi^{(1)}(H(y))[u(y - Y_i) - H(y)]dF(x, y), \quad i \geq 1, \tag{2.17}$$

where we assume that $\phi^{(1)}$ exists (a.e.) and both $\phi^{(1)}$ and F are continuous. Let then

$$\sigma^2 = \text{Var}(V_1), \quad \phi^{(r)}(u) = \frac{d^r}{du^r}\phi(u), \quad r = 0, 1, 2 . \tag{2.18}$$

We need either of the two assumptions (depending on the nature of the invariance principle).

(A) $b(x)$ is continuous in x a.e. and for some $s > 2$,

$$\nu_s = \int_E |b(x)|^s dG(x) = E|b(X)|^s < \infty . \tag{2.19}$$

Also, there exist positive constants K and $\delta(>s^{-1})$, such that

$$|\phi^{(r)}(u)| \le K[u(1-u)]^{-\frac{1}{2}+\delta-r}, \quad \text{for } r = 0, 1, \quad u \in (0, 1). \tag{2.20}$$

(B) The d.f.F admits of a continuous and bounded (a.e.) density function f, (2.20) holds for $r = 0, 1, 2$ and

$$\left|\frac{d^r b(G^{-1}(t))}{dt^r}\right| \le K[t(1-t)]^{\delta-r}, \quad \text{for } r = 0, 1; \quad t \in (0, 1), \tag{2.21}$$

where $\delta(>s^{-1})$ and K are defined in (2.20).

Let us now introduce a sequence $\{W_n^{(2)}\}$ of stochastic processes $W_n^{(2)} = \{W_n^{(2)}(t),\ t \in I\}$ by letting

$$W_n^{(2)}(t) = \{M_{[nt]} - [nt]\mu\}/(\sigma\sqrt{n}), \quad t \in I. \tag{2.22}$$

Then, we have the following

Theorem 2.4. Under assumption (A) *and for* $\sigma > 0$,

$$W_n^{(2)} \xrightarrow{\mathcal{D}} W, \quad \textit{in the } J_1\textit{-topology on } D[0, 1]. \tag{2.23}$$

Theorem 2.5. Under the hypothesis of Theorem 2.4, (2.14) *and* (2.15) *hold with* $\overline{\phi}\xi$ *being replaced by* μ.

In the final theorem, we consider an almost sure (a.s.) representation for $\{M_n\}$ which extends Theorem 2.3 to the general case.

Theorem 2.6. Under (B), *there exists an* $\eta > 0$, *such that*

$$M_n = \sum_{i=1}^n V_i + \xi_n^*; \quad n^{-\frac{1}{2}}|\xi_n^*| = O(n^{-\eta}) \quad \textit{a.s., as } n \to \infty; \tag{2.24}$$

$$\sigma^{-1}\left(\sum_{i=1}^n V_i - n\mu\right) = W(n) + o(\sqrt{n}) \quad \textit{a.s., as } n \to \infty, \tag{2.25}$$

where $W = \{W(t),\ t \in [0, \infty]\}$ *is a standard Wiener process on* $E^+ = [0, \infty)$.

Proofs of the last three theorems are presented in Section 4.

So long, we have assumed that the scores are defined by (2.1). It is possible to replace the scores by some arbitrary scores $\{a_n^*(i)\}$ provided the $a_n(i)$ are $a_n^*(i)$ are sufficiently close to each other. Suppose that

$$\sum_{i=1}^n [a_n(i) - a_n^*(i)]^2 \text{ is (a) } o(1) \text{ or (b) } O(n^{-\eta'}), \text{ as } n \to \infty, \tag{2.26}$$

where $\eta' > 0$. Also, let $M_n^* = \sum_{i=1}^n b(X_i) a_n^*(R_{ni})$. Then

$$n^{-1}(M_n - M_n^*)^2 \le (n^{-1}\sum_{i=1}^n b^2(X_i))(\sum_{i=1}^n [a_n(i) - a_n^*(i)]^2) \tag{2.27}$$

where by the Kintchine law of large numbers, as $n \to \infty$

$$n^{-1}\sum_{i=1}^n b^2(X_i) \xrightarrow{a.s.} Eb^2(X_1) < \infty, \text{ by (2.19)}. \tag{2.28}$$

Moreover, by (2.19),

$$\max_{1 \le k \le n} |\sum_{i=1}^k b^2(X_i)|/n = O_p(1), \text{ as } n \to \infty. \tag{2.29}$$

Hence, by (2.26) - (2.29), under (a) in (2.26),

$$\max_{1 \le k \le n} |M_k - M_k^*|/\sqrt{n} \xrightarrow{p} 0, \text{ as } n \to \infty, \tag{2.30}$$

$$\sup_{k \ge n} |M_k - M_k^*|/\sqrt{n} \to 0 \text{ a.s., as } n \to \infty, \tag{2.31}$$

while, under (b) in (2.26),

$$|M_n - M_n^*| = O(n^{\frac{1}{2}-\eta'}) \text{ a.s., as } n \to \infty. \tag{2.32}$$

Hence, Theorems 2.1 through 2.6 also hold for $\{M_n\}$ being replaced by $\{M_n^*\}$. Now, (2.26) holds under fairly general conditions [see, for example, Appendix of Puri and Sen (1971)]. In particular, if $a_n(i) = E\phi(V_{ni})$ and $a_n^*(i) = \phi(i/(n+1)$, then, (2.26) holds under (2.20). Hence, in the sequel, we may use $a_n^*(i) = \phi(i/(n+1))$ instead of $a_n(i)$, whenever this leads to some simplification in the proofs to follow.

3. PROOFS OF THEOREMS 2.1, 2.2 AND 2.3

Let $F_n^{(1)}$ be the σ-field generated by $(X_1, \ldots, X_n)$, $F_n^{(2)}$ be the σ-field generated by $(R_{n1}, \ldots, R_{nn})$ and F_n^o be the σ-field generated by $(X_1,\ldots,X_n;\ R_{n1},\ldots,R_{nn})$ when H_0: $F\equiv GH$ holds. Then, F_n^o is nondecreasing in n.

Lemma 3.1. Under H_0: $F\equiv GH$ *and* (2.1), $\{M_n - n\bar{\phi}\xi,\ F_n^o;\ n\geq 1\}$ *is a martingale whenever* $b(\ )\in L_1$ *and* $\phi\subset L_1$.

Proof. Note that by (1.1), for every $n\geq 1$,

$$M_{n+1} = M_n + \sum_{i=1}^n b(X_i)[a_{n+1}(R_{n+1i}) - a_n(i)] + b(X_{n+1})a_{n+1}(R_{n+1n+1}). \tag{3.1}$$

Now, under H_0, given F_n^o, X_{n+1} and R_{n+1n+1} are (conditionally) independent where R_{n+1n+1} can assume the values $1,\ldots,n+1$ with the common probability $(n+1)^{-1}$ and $E\{b(X_{n+1})|F_n^o\} = E\{b(X_{n+1})|F_n^{(1)}\} = Eb(X_{n+1}) = \xi$. Thus,

$$\begin{aligned} E\{b(X_{n+1})a_{n+1}(R_{n+1n+1})|F_n^o\} &= E\{b(X_{n+1})|F_n^{(1)}\}E\{a_{n+1}(R_{n+1n+1})|F_n^{(2)}\} \\ &= \xi\cdot(n+1)^{-1}\sum_{i=1}^{n+1}a_{n+1}(i) = \xi\bar{a}_{n+1} \\ &= \xi\bar{\phi}\ ,\ \text{by (2.5)}. \end{aligned} \tag{3.2}$$

On the other hand, given $F_n^{(1)}$, the $b(X_i)$, $i\leq n$ are fixed, while as in Sen and Ghosh (1974), for every $1\leq i\leq n$,

$$E\{a_{n+1}(R_{n+1i})|F_n^{(2)}\} = \frac{R_{ni}}{n+1}a_{n+1}(R_{ni}+1) + \frac{n+1+R_{ni}}{n+1}a_{n+1}(R_{ni}) = a_n(R_{ni}), \tag{3.3}$$

where the last step follows from the recursive relation of expected order statistics, granted by the definition (2.1). Hence,

$$E\{\sum_{i=1}^n b(X_i)[a_{n+1}(R_{n+1i}) - a_n(R_{ni})]|F_n^o\} = 0\ . \tag{3.4}$$

From, (3.1), (3.2) and (3.4), we have

$$E\{M_{n+1}|F_n^o\} = M_n + \bar{\phi}\xi,\ \forall\ n\geq 1, \tag{3.5}$$

and (3.5) insures the lemma.

Let then

$$M_n^o = \sum_{i=1}^n b(X_i)\phi(H(Y_i)), \quad n \geq 1. \tag{3.6}$$

Then, we note that $E\{M_n^o | F_n\} = M_n$, $n \geq 1$ and

$$n^{-1}E_0(M_n - M_n^o)^2 = n^{-1}\{V(M_n^o) - V(M_n)\} = A^2 - A_n^2 \to 0 \quad \text{as} \quad n \to \infty \tag{3.7}$$

where E_0 stands for the expectation under H_0: $F \equiv GH$. (3.7) insures that for every (fixed) $m(\geq 1)$ and $t_1, \ldots, t_m$ (εI), as $n \to \infty$, (under H_0)

$$\max_{1 \leq j \leq m} n^{-\frac{1}{2}} |M_{[nt_j]} - M^o_{[nt_j]}| \xrightarrow{p} 0. \tag{3.8}$$

On the other hand, $\{M_k^o, k \geq 1\}$ involves a sum of i.i.d.r.v. with mean $\overline{\phi}\xi$ and variance $\zeta^2 A^2$ and the Donsker theorem applies to $\{M_k^o - k\overline{\phi}\xi\}$. Hence, (3.8) insures the convergence of *finite-dimensional distributions* (f.d.d.) of $\{W_n^{(1)}\}$ to those of W when H_0 holds. By our Lemma 3.1 and Lemma 4 of Brown (1971), the *tightness* of $\{W_n^{(1)}\}$ is insured by the convergence of f.d.d.'s, and hence, the proof of Theorem 2.1 is complete.

Note that the contiguity in (2.11) along with (3.8) implies that

$$\max_{1 \leq j \leq m} n^{-\frac{1}{2}} |M_{[nt_j]} - M^o_{[nt_j]}| \xrightarrow{p} 0, \text{ under } \{K_n\} \text{ as well,} \tag{3.9}$$

while by an appeal to the central limit theorem for a triangular array of independent r.v.s., the asymptotic multinormality of $\{n^{-\frac{1}{2}}(M^o_{[nt_j]} - \overline{\phi}\xi[nt_j]), 1 \leq j \leq m\}$ follows directly (where $\omega_n(t) = n^{-\frac{1}{2}}\{EM^o_{[nt]} - [nt]\overline{\phi}\xi\}$, $t \in I$) and this insures the convergence of f.d.d.'s of $\{W_n^{(1)} - \omega_n\}$ to those of W. Further, as in the proof of Theorem 3.1 [viz., (3.8) thorugh (3.12)] of Sen (1977), tightness under H_0 and contiguity insure the tightness under $\{K_n\}$ as well. Hence, the proof of Theorem 2.2 is complete.

Let $Z_{n+1} = M_{n+1} - M_n - \bar{\phi}\xi$. Then, by Lemma 3.1, $E(Z_{n+1} | F_n^o) = 0$, $\forall\ n \geq 0$, and by (3.11), we have

$$E_0\{Z_{n+1}^2 | F_n^o\} = E_0\{[b(X_{n+1})a_{n+1}(R_{n+1n+1}) - \bar{\phi}\xi]^2 | F_n^o\} +$$
$$E_0\{(\textstyle\sum_{i=1}^n b(X_i)[a_{n+1}(R_{n+1i}) - a_n(R_{ni})])^2 | F_n^o\} + 0$$
$$= \zeta^2 A_{n+1}^2 + Q_n^*, \quad \text{say}, \tag{3.10}$$

where proceeding as in Lemma 3.2 of Sen and Ghosh (1974), it follows that $Q_n^* \to 0$ a.s. as $n \to \infty$. Hence, by (2.5) and (3.10)

$$(\textstyle\sum_{k=0}^n E_0\{Z_{k+1}^2 | F_k^o\}) / (nA^2\zeta^2) \to 1 \text{ a.s., as } n \to \infty, \tag{3.11}$$

so that

$$\textstyle\sum_{k=0}^n E_0\{Z_{k+1}^2 | F_n^o\} \to \infty \text{ a.s., as } n \to \infty\ . \tag{3.12}$$

Note that under (2.1), (2.2) and for $\int|\phi|^r < \infty$, $r \geq 2$,

$$\max_{1 \leq i \leq n} |a_n(i)| = o(n^{1/r}), \tag{3.13}$$

and, similarly, under $\int|b|^r dG < \infty$ for some $r \geq 2$,

$$\max_{1 \leq i \leq n} |b(X_i)| = o(n^{1/r}) \text{ a.s., as } n \to \infty\ . \tag{3.14}$$

Note that by (3.1), for every $n \geq 1$,

$$|Z_{n+1}| \leq |b(X_{n+1}) - \xi|\,|a_{n+1}(R_{n+1n+1})| + |\xi|\ |a_{n+1}(R_{n+1n+1}) - \bar{\phi}|$$
$$+ \max_{1 \leq i \leq n} |b(X_i)| \textstyle\sum_{i=1}^n |a_{n+1}(R_{n+1i}) - a_n(R_{ni})|, \tag{3.15}$$

where by (2.1), (2.2) and (3.13), we have

$$\textstyle\sum_{i=1}^n |a_{n+1}(R_{n+1i}) - a_n(R_{ni})| = 0(n^{1/r}), \quad \text{as } n \to \infty \tag{3.16}$$

[Actually, take first $\phi \uparrow$ and note that $a_{n+1}(R_{ni}) \leq a_{n+1}(R_{n+1i}) \leq a_{n+1}(R_{ni} + 1)$, $\forall\ 1 \leq i \leq n$, and then use (3.13). For (2.2), use the same technique for each component of ϕ.] Thus, by (3.13) through (3.16), we conclude that

$$|Z_{n+1}| = o(n^{2/r}) \text{ a.s., as } n \to \infty\ , \tag{3.17}$$

when either $\int|\phi|^r < \infty$ or $\int|b|^r dG < \infty$ for some $r > 2$. Thus, if $\psi(n)$ is any $\uparrow$ in n, for which

$$\psi(n)/n^{2/r} \not\to \text{ as } n \to \infty, \tag{3.18}$$

then by virtue of (3.12), (3.17) and (3.18), on denoting by $s_n^2 = \sum_{i=0}^{n-1} E_0\{Z_{i+1}^2 | F_i^o\}$, we have

$$\sum_{n\geq 1} \frac{1}{\psi(s_n^2)} E_0\{Z_{n+1}^2 I(Z_{n+1}^2 > \psi(s_n)) | F_n^o\} < \infty \text{ a.s.} \tag{3.19}$$

By virtue of (3.12), (3.18), (3.19) and Lemma 3.1, (2.13) follows from Theorem 4.4 of Strassen (1967). (2.14) and (2.15) follows directly by using the law of iterated logarithm for the Wiener process W. This completes the proof of Theorem 2.3.

4. PROOF OF THEOREMS 2.4, 2.5 AND 2.6

Let $G_n(x) = n^{-1}\sum_{i=1}^n u(x - X_i)$, $x \in E$, $H_n(y) = n^{-1}\sum_{i=1}^n u(y - Y_i)$, $y \in E$ and $F_n(x, y) = n^{-1}\sum_{i=1}^n u(x - X_i)u(y - Y_i)$, $(x, y) \in E^2$, be respectively, the empirical d.f.'s corresponding to G, H and F. Also, since (2.25) holds under (2.20), in the case of the scores defined by $\phi(i/(n+1))$, $1 \leq i \leq n$, we shall work with these in the sequel. Then, we may write

$$n^{-1}M_n = \int_{E^2}\int b(x)\phi(\frac{n}{n+1} H_n(y))dF_n(x, y) . \tag{4.1}$$

By expanding $\phi(\frac{n}{n+1} H_n)$ around $\phi(H)$ and F_n around F, we obtain from (4.1) that

$$\begin{aligned} n^{-1}M_n &= \int_{E^2} \{b(x)\phi(H(y))dF_n(x, y) + \\ &\qquad b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y))dF(x, y)\} + n^{-1}\xi_n^* \text{ (say)} \\ &= n^{-1}\sum_{i=1}^n V_i + n^{-1}\xi_n^* \end{aligned} \tag{4.2}$$

where the V_i are defined by (2.17) and ξ_n^* consists of the remainder terms.

Now, under (2.19) and (2.20), $EV_i = \mu$ and $\mathrm{Var}(V_i) = \sigma^2 < \infty$. Thus, the Donsker theorem applies to the sequence $\{V_i\}$ of i.i.d.r.v., and hence, to prove Theorem 2.4, all we need to show is that

$$n^{-\frac{1}{2}} \max_{1 \le k \le n} |\xi_k^*| \xrightarrow{p} 0, \tag{4.3}$$

while to prove Theorem 2.5, by virtue of the law of iterated logarithm for the $\sum_{i=1}^n (V_i - \mu)$, it suffices to show that

$$|\xi_n^*| (2n \log \log n)^{-\frac{1}{2}} \to 0, \text{ a.s., as } n \to \infty . \tag{4.4}$$

Finally, to prove Theorem 2.6, we need to verify (2.24). For this reason, in the remaining of this section, we consider explicitly the remainder terms $\{\xi_n^*\}$ and verify (4.3), (4.4) and (2.24) under appropriate regularity conditions.

Note that by (4.2),

$$\begin{aligned} n^{-1}\xi_n^* &= \int_{E^2}\!\!\int b(x)[\phi(\tfrac{n}{n+1} H_n(y)) - \phi(H(y))]dF_n(x, y) \\ &\quad - \int_{E^2}\!\!\int b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y))dF(x, y) \\ &= \int_{E^2}\!\!\int b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y))d[F_n(x, y) - F(x, y)] \\ &\quad + \int_{E^2}\!\!\int b(x)[\phi(\tfrac{n}{n+1} H_n(y)) - \phi(H(y)) - \{H_n(y) - H(y)\}\phi^{(1)}(H(y))]dF_n(x, y) \\ &= C_{n1} + C_{n2}, \quad \text{say.} \end{aligned} \tag{4.5}$$

In this context, the following result due to Csaki (1977) will be repeatedly used. Let $\{U_i, i \ge 1\}$ be a sequence of i.i.d.r.v. having the uniform (0, 1) d.f. and let $I_n(t) = n^{-1}\sum_{i=1}^n u(t - U_i)$, $t \in I$, $n \ge 1$. Then, for every $\varepsilon > 0$, $n \to \infty$,

$$n^{\frac{1}{2}}(\log \log n)^{-\frac{1}{2}} \sup_{t \in I}\{t(1-t)\}^{-\frac{1}{2}+\varepsilon} |I_n(t) - t| = O(1) \text{ a.s.}, \tag{4.6}$$

Further, Csörgö and Rèvèsz (1975) have the following result

$$P\{\overline{\lim_{n\to\infty}} \sup_{\varepsilon_n < t < 1-\varepsilon_n} \frac{\sqrt{n}|I_n(t) - t|}{\sqrt{2t(1-t)\log \log n}} = \sqrt{2}\} = 1 , \tag{4.7}$$

where

$$\varepsilon_n = (\log n)^4/n . \tag{4.8}$$

Also, by virtue of Theorem 3.1 of Braun (1976), for every $\varepsilon > 0$,

$$\max_{1\le k\le n} \sup_{t\in I} \frac{k|I_k(t) - t|}{\sqrt{n}\{t(1-t)\}^{\frac{1}{2}-\varepsilon}} = 0_p(1) . \tag{4.9}$$

Now, by (2.20) and (4.6), for every $\varepsilon > 0$, as $n \to \infty$,

$$\sup_{y\in E} |[H_n(y) - H(y)]\phi^{(1)}(H(y))\{H(y)[1-H(y))]\}^{1-\delta+\varepsilon}| = 0(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}) \text{ a.s.}, \tag{4.10}$$

while by (2.20) and (4.7), as $n \to \infty$,

$$\sup_{H^{-1}(\varepsilon_n)\le t\le H^{-1}(1-\varepsilon_n)} |[H_n(y) - H(y)]\phi^{(1)}(H(y))\{H(y)[1-H(y)]\}^{1-\delta}|$$
$$= 0(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}) \text{ a.s.} \tag{4.11}$$

and by (2.20) and (4.8),

$$\max_{k\le n} \sup_{y\in E} \frac{k}{\sqrt{n}} |\{H_k(y) - H(y)\}\phi^{(1)}(H(y))\{H(y)[1-H(y)]\}^{1-\delta+\varepsilon}| = 0_p(1), \tag{4.12}$$

where in (4.10), (4.11) and (4.12), $\delta > s^{-1}$ (by (2.19) - (2.20)), so that on setting $\varepsilon = \frac{1}{2}(\delta - s^{-1})$, we have

$$(1-\delta+\varepsilon)s/(s-1) = 1-\gamma, \quad \text{where } \gamma > 0. \tag{4.13}$$

By (2.19), for every $\eta > 0$, there exists a $K_\eta(<\infty)$, such that

$$\int_{|x|>K_\eta} |b(x)|^s dG(x) < \eta \tag{4.14}$$

Let

$$E_\eta^{(1)} = \{x: |x| \le K_\eta\}, \quad E_\eta^{(2)} = \{y: \eta \le H(y) \le 1-\eta\}, \; E_\eta = E_\eta^{(1)} \times E_\eta^{(2)}, \; E_\eta^c = E^2 \backslash E_\eta. \tag{4.15}$$

Then, by (4.5) and (4.15), we have

$$|C_{n1}| \le C_{n11} + C_{n12}^{(1)} + C_{n12}^{(2)} \tag{4.16}$$

where

$$C_{n11} = \left|\int_{E_\eta}\int b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y))d[F_n(x, y) - F(x, y)]\right| \quad (4.17)$$

$$C_{n12}^{(1)} = \left|\int_{E_\eta^c}\int b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y))dF_n(x, y)\right| \quad (4.18)$$

$$C_{n12}^{(2)} = \left|\int_{E_\eta^c}\int b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y))dF(x, y)\right| . \quad (4.19)$$

Now, by the Hölder inequality

$$C_{n12}^{(1)} \le \{\int_E |b(x)|^s dG_n(x)\}^{1/s} \{\int_{E\setminus E_\eta^{(2)}} |[H_n(y) - H(y)]\phi^{(1)}(H(y))|^{\frac{s}{s-1}} dH_n(y)\}^{\frac{s-1}{s}}$$

$$+ \{\int_{|x|\ge K_\eta} |b(x)|^s dG_n(x)\}^{1/s} \{\int_E |[H_n(y) - H(y)]\phi^{(1)}(H(y))|^{\frac{s}{s-1}} dH_n(y)\}^{\frac{s-1}{s}} . \quad (4.20)$$

Now, $\int_E |b(x)|^s dG_n(x)$ (being a reverse martingale) converges a.s. to $\int_E |b(x)|^s dG(x) [<\infty$, by (2.19)] and, similarly, $\int_{|x|\ge K_\eta} |b(x)|^s dG_n(x) \xrightarrow{a.s.} \int_{|x|>K_\eta} |b(x)|^s dG_n(x) < \eta$. Also, by (4.10),

$$n^{\frac{1}{2}}(\log\log n)^{-\frac{1}{2}} \int_E \{|H_n - H||\phi^{(1)}(H)|\}^{\frac{s}{s-1}} dH_n = O(1)\int_E \{H(y)[1 - H(y)]\}^{-1+\gamma} dH_n(y),$$

where $\gamma > 0$ and the last integral is also a reverse martingale, and hence, a.s. converges to $\int_0^1 [u(1-u)]^{-1+\gamma} du < \infty$. Similarly, $n^{\frac{1}{2}}(\log\log n)^{-\frac{1}{2}} \int_{E\setminus E_\eta^{(2)}} |\{H_n - H\}\phi^{(1)}(H)\}|^{\frac{s}{s-1}} dH_n \to 0$ a.s., as $n\to\infty$. Hence, $C_{n/2}^{(1)} = o((n^{-1}\log\log n)^{\frac{1}{2}})$ a.s., as $n\to\infty$. A similar treatment holds for $C_{n12}^{(2)}$. Thus, as $n\to\infty$,

$$C_{n12}^{(1)} + C_{n12}^{(2)} = o((n^{-1}\log\log n)^{\frac{1}{2}}) . \quad (4.21)$$

Moreover, by using (4.12) instead of (4.10) in the above manipulations, we have on parallel lines

$$\max_{k\le n}\{n^{-\frac{1}{2}}k[C_{k12}^{(1)} + C_{k12}^{(2)}]\} = o_p(1), \quad \text{as } n\to\infty . \quad (4.22)$$

Let $A_r = \{x, y): a_{r1} \le x \le a_{r2},\ c_{r1} \le y \le c_{r2}\}$ be a block (εE^2).

Then

$$\sup_{A_r} |b(x)[H_n(y) - H(y)]\phi^{(1)}(H(y)) - b(a_{r1})[H_n(c_{r1}) - H(c_{r1})]\phi^{(1)}(H(c_{r1}))||$$

$$\le \sup_{a_{r1}\le x\le a_{r2}} |b(x) - b(a_{r1})| \sup_{c_{r1}\le y\le c_{r2}} |[H_n(y) - H(y)]\phi^{(1)}(H(y))|$$

$$+ |b(a_{r1})| \sup_{c_{r1}\le y\le c_{r2}} |[H_n(y) - H(y)][\phi^{(1)}(H(y)) - \phi^{(1)}(H(c_{r1}))]|$$

$$+ |b(a_{r1})\phi^{(1)}(H(c_{r1}))| \sup_{c_{r1}\le y\le c_{r2}} |[H_n(y) - H(y)] - [H_n(c_{r1}) - H(c_{r1})]| .$$

Also, F is assumed to be continuous everywhere, and hence,

$$\int_{A_r}\int d[F_n(x, y) - F(x, y)] \to 0 \text{ a.s., as } n\to\infty . \tag{4.24}$$

Further using Theorem F of Csörgö and Rèvèsz (1975) along with the compactness of the Kiefer process, we have for small $H(c_{r2}) - H(c_{r1})$,

$$n^{\frac{1}{2}} \sup_{c_{r1}\le y\le c_{r2}} |[H_n(y) - H(y)] - [H_n(c_{r1}) - H(c_{r1})]|$$
$$= o((\log\log n)^{\frac{1}{2}}) \text{ a.s., as } n\to\infty . \tag{4.25}$$

Now E_η is a compact subspace of E^2, $b(x)$ is continuous inside $E_\eta^{(1)}$ $\phi^{(1)}(H(y))$ is bounded and continuous inside $E_\eta^{(2)}$ and (4.10) holds. Thus, for every $\varepsilon' > 0$, one can choose a set $\{A_r,\ r = 1,\dots,m\}$ (where $m = m_{\varepsilon'}$) of non-overlapping and exhaustive blocks, such that $E_\eta = \bigcup_r A_r$, $\sup_{a_{r1}\le x\le a_{r2}} |b(x) - b(a_{r1})| < \varepsilon'/3$ and $H(c_{r2}) - H(c_{r1}) < \varepsilon'/3$.

Then, by using (4.23), (4.24) and (4.25), we obtain from (4.17) that as $n\to\infty$,

$$C_{n11} = o(n^{-1}\log\log n)^{\frac{1}{2}}) \text{ a.s., as } n\to\infty . \tag{4.26}$$

Also, we note that by the tightness part of Theorem 3.1 of Braun (1976), for small $H(c_{r2}) - H(c_{r1})$,

$$\max_{k\le n}\ \sup_{c_{r1}\le y\le c_{r2}} n^{-\frac{1}{2}}k\,|[H_k(y)-H(y)]-[H_k(c_{r1})-H(c_{r1})]| = o_p(1), \tag{4.27}$$

and hence by (4.9), (4.12), (4.17), (4.23), (4.24) and (4.27),

$$\max_{k\le n} n^{-\frac{1}{2}}\{kC_{k11}\} \xrightarrow{p} 0, \quad \text{as} \quad n\to\infty \quad . \tag{4.28}$$

Thus, by (4.16), (4.21), (4.22), (4.26) and (4.28), as $n\to\infty$,

$$|C_{n1}| = o((n^{-1}\log\log n)^{\frac{1}{2}}) \text{ a.s.}, \quad \max_{k\le n} n^{-\frac{1}{2}}k|C_{k1}| = o_p(1). \tag{4.29}$$

For the treatment of C_{n2} in (4.5), we have by the Hölder inequality,

$$|C_{n2}| \le \{\int_E |b(x)|^s dG_n(x)\}^{1/s}\{\int_E |\phi(\tfrac{n}{n+1}H_n)-\phi(H)-[H_n-H]\phi^{(1)}(H)|^{\frac{s}{s-1}}dH_n\}^{\frac{s-1}{s}}, \tag{4.30}$$

where, as in after (4.20), $\int_E |b(x)|^s dG_n(x) \xrightarrow{a.s.} E|b(X)|^s < \infty$, by (2.19). Hence, it suffices to show that as $n\to\infty$,

$$\int_E |\phi(\tfrac{n}{n+1}H_n)-\phi(H)-[H_n-H]\phi^{(1)}(H)|^{\frac{s}{s-1}}dH_n = o((n^{-1}\log\log n)^{\frac{s}{2(s-1)}}) \text{ a.s.}, \tag{4.31}$$

where $s>2$ and (2.19) - (2.20) hold. Here also, we define $E_\eta^{(2)}$ as in (4.15). Then, by the first order Taylor's expansion of $\phi(\frac{n}{n+1}H_n)$ along with the continuity of $\phi^{(1)}(H)$ (inside $E_\eta^{(2)}$), by using (4.11), we obtain that

$$\int_{E_\eta^{(2)}} |\phi(\tfrac{n}{n+1}H_n)-\phi(H)-[H_n-H]\phi^{(1)}(H)|^{\frac{s}{s-1}}dH_n$$
$$= o((n^{-1}\log\log n)^{\frac{1}{2}}) \text{ a.s., as } n\to\infty. \tag{4.32}$$

For the complementary part, we have

$$\int_{E\setminus E_\eta^{(2)}} |\phi(\tfrac{n}{n+1}H_n)-\phi(H)-[H_n-H]\phi^{(1)}(H)|^{s/(s-1)}dH_n$$
$$\le C^*\{\int_{E\setminus E_\eta^{(2)}} |\phi(\tfrac{n}{n+1}H_n)-\phi(H)|^{s/(s-1)}dH_n + \int_{E\setminus E_\eta^{(2)}} |(H_n-H)\phi^{(1)}(H)|^{\frac{s}{s-1}}dH_n\}$$
$$= C_{n21}^{(2)} + C_{n22}^{(2)}, \text{ say, where } C^*(<\infty) \text{ depends only on } s. \tag{4.33}$$

By using (4.10) and proceeding as in after (4.20), it readily follows that $C_{n22}^{(2)} = o((n^{-1}\log\log n)^{s/2(s-1)})$ a.s., as $n \to \infty$. Similarly, by using (4.12) we have $\max_{k\le n} n^{-\frac{1}{2}} k |C_{k22}^{(2)}| = o_p(1)$. The treatment of $C_{n21}^{(2)}$ deserves extra care. Note that by (2.20) (4 = 0),

$$\max_{1\le i\le x} |\phi(\tfrac{n}{n+1})| = 0(n^{\frac{1}{2}-\delta}) \ . \tag{4.34}$$

Also the $H(Y_i)$ are i.i.d.r.v. having the uniform (0, 1) d.f. Hence, by some standard arguments, we have under (2.20) $(r = 0)$,

$$\max_{1\le i\le n} |\phi(H(Y_i))| = 0(n^{\frac{1}{2}-\delta}(\log n)^{\frac{1}{2}}) \text{ a.s., as } n \to \infty \ . \tag{4.35}$$

Let now $\{a_n\}$ be defined by $H(a_n) = n^{-1+\varepsilon}$, where $\varepsilon = \frac{1}{2}(\delta - s^{-1}) > 0$, by (2.19). Then, using the fact that by (4.7), $H_n(a_n) = n^{-1+\varepsilon}\{1 + 0(n^{-\varepsilon/2}\sqrt{\log\log n})\}$ a.s., as $n \to \infty$, we obtain that, with probability 1,

$$\begin{aligned}
&\int_{-\infty}^{a_n} |\phi(\tfrac{n}{n+1} H_n) - \phi(H)|^{s/(s-1)} dH_n \\
&\le [0(n^{\frac{1}{2}-\delta}\sqrt{\log n})]^{s/(s-1)} \cdot H_n(a_n) \\
&\overset{a.s.}{=} [0(n^{\frac{1}{2}-\delta}\sqrt{\log n})]^{s/(s-1)} \cdot [0(n^{-1+\varepsilon})] \\
&= [0(n^{-s/2(s-1)})][0(n^{-\varepsilon(s+1)/(s-1)}(\log n)^{s/2(s-1)})] \\
&= o(n^{-s/2(s-1)}), \quad \text{as } \varepsilon > 0.
\end{aligned} \tag{4.36}$$

On the other hand, by (4.7) $\sup_{a_n \le y \le H^{-1}(\eta)} |H_n(y)/H(y) - 1| \to 0$ a.s., as $n \to \infty$, and hence, on writing $\phi(\frac{n}{n+1} H_n) - \phi(H) = (\frac{n}{n+1} H_n - H)\phi^{(1)}(H_n^*)$ where $H_n^* = \theta(\frac{}{n+1} H_n + (1-\theta)H$ (for $0 < \theta < 1$) and bounding $|H_n^*/H - 1|$ by an arbitrary number less than 1, we have for large n,

$$\int_{a_n}^{H^{-1}(\eta)} |\phi(\frac{n}{n+1} H_n) - \phi(H)|^{s/(s-1)} dH_n$$

$$\leq K^* \int_{a_n}^{H^{-1}(\eta)} |\frac{n}{n+1} H_n - H|^{s/(s-1)} \{[H(1-H)]^{-3/2+\delta}\}^{s/(s-1)} dH_n \quad \text{(by (2.20)}$$

$$\stackrel{a.s.}{=} [0((n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})^{s/(s-1)}] \int_{a_n}^{H^{-1}(\eta)} [H(1-H)]^{-1+\gamma} dH_n \quad \text{(by (4.7))}$$

$$\leq [0((n^{-1}\log\log n)^{s/2(s-1)})] \int_{-\infty}^{H^{-1}(\eta)} [H(1-H)]^{-1+\gamma} dH_n$$

$$= o(n^{-1}\log\log n)^{s/2(s-1)}) \quad \text{a.s.,} \qquad (4.37)$$

as $\eta(>0)$ is arbitrary and by arguments following (4.20) $\int_{-\infty}^{H^{-1}(\eta)} \{H(1-H)\}^{-1+\gamma} dH_n \xrightarrow{a.s.} \int_0^\eta [u(1-u)]^{-1+\gamma} du$, which can be made arbitrarily small. A similar treatment applies to the upper tail $\{H^{-1}(1-\eta) \leq y < \infty\}$. Instead of using (4.4), (4.7), (4.10) and (4.11), if we use (4.9) and (4.12), then by very similar arguments, we obtain that $\max_{k\leq n} n^{-\frac{1}{2}}|k|C_{k2}| \xrightarrow{P} 0$, as $n \to \infty$. Thus, we conclude that (4.3) and (4.4) hold under assumption (A) in (2.19)-(2.20) and the proofs of Theorems 2.4 and 2.5 are complete.

For the proof of (2.24) [under the stronger assumption (B)], for C_{n2}, we use (4.30). Here, we expand $\phi(\frac{n}{n+1} H_n)$ around $\phi(H)$ by a second order Taylor's expansion, permissible under (2.20) (for $r=2$). This will make (4.32) as well as (4.37) $0(n^{-\frac{1}{2}-\eta})$ a.s. for some $\eta > 0$, while (4.36) does not need any adjustment. For the treatment of C_{n1} in (4.5), we make use of the Bahadur (1966) result (extended to the bivariate case) that for every $(x_0, y_0) \varepsilon E^2$,

$$\sup\{n^{\frac{1}{2}}|[F_n(x,y) - F(x,y)] - [F_n(x_0,y_0) - F(x_0,y_0)]| :$$
$$|[F(x,\infty) - F(x_0,\infty)]| + |[F(\infty,y) - F(\infty,y_0)]| \leq n^{-\frac{1}{2}}\log n\}$$
$$= 0(n^{-\frac{1}{4}}(\log n)^2) \text{ a.s., as } n \to \infty . \qquad (4.38)$$

This is a stronger result than (4.24)-(4.25). On the other hand, for