Transactions of the **14th International Conference on Structural Mechanics in Reactor Technology** (SMiRT 14), Lyon, France, August 17-22, 1997

B03/5

Shape sensitivity analysis and energy release rate in three dimensional cracked bodies

Saliba R.(1), Venere M.(1), Feijoo R.(2), Taroco E.(2)
(1) Instituto Balseiro and Centro Atomico Bariloche, Argentina
(2) Laboratório Nacional de Computaçao Cientifica, Brazil

ABSTRACT - This paper describes how Shape Sensitivity Analysis could be used in Fracture Mechanics to simulate crack advance and to obtain the expression of the potential energy release rate or, more precisely, the decrease of the potential energy due to the crack advance. The Finite Element technique is used to model three-dimensional cracked bodies and to evaluate the state of strain, stress and the shape change velocities that represent the crack advances. Finally, to show the potentiality of the present approach, numerical results of two well known examples, a cracked plate and a penny-shaped cracked bar, are presented and compared with those obtained by others authors.

1. INTRODUCTION

An important concept in the analysis of elastic cracked bodies is the energy release rate, introduced by Griffiths (1921) in his renowned work in which he applies the principle of energy conservation in the study of crack growth. In other words, this parameter, denoted by G, is the rate of change with respect to *crack advance* of the energy available for fracture.

Given the difficulty in obtaining expressions of potential energy as explicit functions of crack length, which enable us to obtain derivatives in a direct form, several procedures both numerically and experimentally have been developed in fracture mechanics.

The aim of this paper is to apply shape sensitivity analysis as a systematic methodology to obtain the expression of potential energy release rate in three dimensional cracked bodies. First, we study the variation of the potential energy in the initial configuration of a body, assuming arbitrary change of the initial shape. Second, introducing a properly velocity field, we simulate *crack advance* as the modification of the body's shape. In this way, we obtain the general expression of the energy release rate as an integral over the initial configuration of a cracked body. Finally, two particular cases of three dimensional cracked bodies are presented. Using finite element approximation in order to obtain the strain and stress distribution over the initial cracked configuration and later using a post-processing technique, the energy release rate for uniform velocity fields at the front of the crack is evaluated by numerical integration in both cases. These values are compared with the available results from the literature in order to show the potentiality of the present approach.

2. SHAPE SENSITIVITY ANALYSIS

In the present section, we introduce the concept of shape change and we shall study the behavior of functionals when the shape of a body is modified. Proposed originally by Cèa (1981) and widely discussed by Haug, Choi and Komkov (1986) this approach simulates a change in shape by a *motion* from an *initial configuration* to a *deformed* configuration.

Let us identify the initial shape of the body with the three-dimensional domain $\Omega \subset R^3$ bounded by $\partial\Omega$ with outward unit normal vector $\mathbf{n}$. The body is under a given traction over part of the boundary denoted as $\partial\Omega^t$ and a prescribed displacement over the remaining part of the boundary $\partial\Omega^u$. ($\partial\Omega = \partial\Omega^t \cup \partial\Omega^u$). The change of the shape and more precisely, the change of the initial domain Ω is described by the scalar parameter τ and the **known** vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$, that defines the transformation from Ω to Ω_τ as:

$$\mathbf{x}_\tau = \mathbf{x}_\tau(\mathbf{x}) = \mathbf{x} + \tau\mathbf{v}(\mathbf{x}) \tag{1}$$

Thus, the shape change is a smooth one-parameter family of transformations, Gurtin (1981). In other words, $\mathbf{v}$ is the direction of the domain variation and this means that, for a given direction $\mathbf{v}$, the shape change of Ω is uniquely controlled by parameter $\tau \in R^+$.

The transformed domain Ω_τ might be considered as a *deformed configuration* of the initial domain Ω under the transformation from Ω to Ω_τ defined by Eq.(1). Furthermore, introducing continuum mechanics terminology, an analogy can be drawn between *change of shape* and *motion of a body.*

The domain Ω_τ for different values of the parameter τ, which characterizes the shape change, is known in continuum mechanics as the trajectory of the motion of a body. In this sense Ω_τ can be seen as the place occupied by Ω at time τ. When $\tau = 0$, the *deformed* domain Ω_τ reduces to the *initial domain* Ω_0, which we denote Ω for simplicity. Specifically,

$$\Omega_\tau = \left\{\mathbf{x}_\tau ; \mathbf{x}_\tau = \mathbf{x} + \tau\mathbf{v}(\mathbf{x}), \mathbf{x} \in \Omega, \tau \in R^+\right\} \tag{2}$$

Since at each τ, the shape change is a one to one transformation of Ω onto Ω_τ, there is an inverse transformation from Ω_τ to Ω:

$$\Omega_\tau \mapsto \Omega \quad ; \quad \partial\Omega_\tau \mapsto \partial\Omega \quad ; \quad \mathbf{x}_\tau \mapsto \mathbf{x} \tag{3}$$

Any scalar, vector or tensor field, can be defined over the initial domain Ω, or over the *deformed* domain Ω_τ. Within the continuum mechanics terminology, we call them *material* and *spatial descriptions*, respectively. For instance, in the particular case of the displacement field $\mathbf{u}$ we may write:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}(\mathbf{x},\tau);\ (\mathbf{x} \in \Omega; \tau \in R^+); && \text{for the material description} \\ \mathbf{u}_\tau &= \mathbf{u}_\tau(\mathbf{x}_\tau,\tau);\ (\mathbf{x}_\tau \in \Omega_\tau; \tau \in R^+); && \text{for the spatial description} \end{aligned} \tag{4}$$

We also introduce the concept of change of shape gradient equivalent to the deformation gradient of continuum mechanics, so that:

$$F = \nabla[\mathbf{x} + \tau\mathbf{v}(\mathbf{x})] = \mathbf{I} + \tau\nabla\mathbf{v} \tag{5}$$

Therefore, we define the material gradient (∇) as the gradient over the initial domain Ω and similarly (grad) designate the spatial gradient over the actual domain Ω_τ.

2.1 *Transformations from Ω_τ to Ω*

Point, vector, volume, area, and grad operator defined in Ω_τ, transforms to Ω in the following form:

$$\mathbf{x}_\tau \to \mathbf{x} + \tau\mathbf{v}(\mathbf{x}); \quad d\mathbf{x}_\tau \to F d\mathbf{x} = (\mathbf{I} + \tau\nabla\mathbf{v})d\mathbf{x}; \quad \mathrm{grad}\,\mathbf{u}_\tau \to \nabla\mathbf{u}F^{-1};$$
$$d\Omega_\tau \to \det F\; d\Omega; \quad d\partial\Omega_\tau \to ||F^{-T}\mathbf{n}||\det F\; d\partial\Omega. \tag{6}$$

2.2 *Total Derivatives*

For shape sensitivity analysis it will be useful to introduce the total derivative with respect to τ, similar to the concept of the material time derivative of spatial fields used in continuum mechanics. For instance, in the case of the displacement field $\mathbf{u}_\tau$, the material description of its total derivative at $\tau = 0$, is given by:

$$\dot{\mathbf{u}}(\mathbf{x};\mathbf{v}) = \frac{d}{d\tau}\mathbf{u}_\tau(\mathbf{x}_\tau,\tau)\Big|_{\tau=0} = \frac{\partial}{\partial\tau}\mathbf{u}_\tau[\mathbf{x} + \tau\mathbf{v}(\mathbf{x}),\tau]\Big|_{\tau=0} \tag{7}$$

We identify the foregoing total derivative, as the directional derivative of $\mathbf{u}$ at point $\mathbf{x}$ in the direction $\mathbf{v}$. The following results will be useful

$$\dot{\mathbf{x}}_\tau = \mathbf{v}(\mathbf{x}); \quad \dot{\overline{d\mathbf{x}_\tau}} = \nabla\mathbf{v}d\mathbf{x}; \quad \dot{\overline{d\Omega_\tau}} = \mathrm{div}\mathbf{v}\; d\Omega = \nabla\mathbf{v}\cdot\mathbf{I}\; d\Omega;$$
$$\dot{\overline{d\partial\Omega_\tau}} = \mathrm{div}_s\mathbf{v}\; d\partial\Omega = \nabla\mathbf{v}\cdot(\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\; d\partial\Omega; \quad \dot{\overline{grad\,\mathbf{u}_\tau}} = \nabla\dot{\mathbf{u}} - \nabla\mathbf{u}\nabla\mathbf{v}. \tag{8}$$

3. RATE OF CHANGE OF POTENTIAL ENERGY

The purpose of this section is to study the derivative of the potential energy stored in an elastic body with respect to the change of shape. We shall consider a body submitted to the action of external loading and prescribed displacement along the boundary (for simplicity we shall not consider body forces). We shall also assume homogeneous, elastic material (not necessarily linear) with specific strain energy function, denoted by ϕ, which has the following property:

$$\phi : (E = \nabla\mathbf{u}^s) \in Sym \mapsto R; \qquad T = \frac{d\phi}{dE} = \phi_{,E} \tag{9}$$

To each shape of the body identified with the one parameter domain Ω_τ and boundary $\partial\Omega_\tau$, the potential strain energy may be expressed as:

$$\Pi_\tau(\mathbf{u}_\tau) = U_\tau - V_\tau = \int_{\Omega_\tau}\phi_\tau\; d\Omega_\tau - \int_{\partial\Omega_\tau^t}\bar{\mathbf{t}}_\tau\cdot\mathbf{u}_\tau\; d\partial\Omega_\tau \tag{10}$$

where ϕ_τ denotes the specific strain energy, $\bar{\mathbf{t}}_\tau$ the prescribed tractions on the boundary $\partial\Omega_\tau^t$ and $\mathbf{u}_\tau$ the kinematically admissible displacement field. Further, the domain integral in the right hand side of the above expression, represents the total strain energy stored in the body and the boundary integral, the external work.

To obtain the rate of change of the potential energy when the domain is modified, we derive both sides of the expression (10) with respect to the parameter τ. In order to do this, we must first express the integrals over Ω_τ and $\partial\Omega_\tau$ as integrals over Ω and $\partial\Omega$, using the change of shape gradient, $F = I + \tau\nabla\mathbf{v}$, equation (5). Then, the material derivative of the total strain energy U_τ, becomes:

$$\frac{d}{d\tau}U_\tau = \frac{d}{d\tau}\left(\int_{\Omega_\tau}\phi_\tau[(\mathrm{grad}\;\mathbf{u}_\tau)^s]d\Omega_\tau\right) = \int_\Omega\frac{\partial}{\partial\tau}(\phi\;\det\;F)\;d\Omega \tag{11}$$

At $\tau = 0$ this derivative results:

$$\dot{U} = \frac{d}{d\tau} U_\tau \Big|_{\tau=0} = \int_\Omega (\dot{\phi} + \phi \text{ div } \mathbf{v}) \, d\Omega \tag{12}$$

Since the spatial gradient transforms into the material gradient as follows: $\text{grad}\,\mathbf{u}_\tau = \nabla\mathbf{u}F^{-1}$, equation (6.c), by the chain rule and from equation (6.e), we have:

$$\dot{\phi} = \frac{d}{d\tau}\phi_\tau[(\text{grad } \mathbf{u}_\tau)^s]\Big|_{\tau=0} = T \cdot [\nabla\dot{\mathbf{u}}^s - (\nabla\mathbf{u}\nabla\mathbf{v})^s] = T \cdot \nabla\dot{\mathbf{u}} - \nabla\mathbf{u}^T T \cdot \nabla\mathbf{v} \tag{13}$$

Using the above result and the definition of divergence of a vector field, div $\mathbf{v} = \mathbf{I} \cdot \nabla\mathbf{v}$, the material derivative $\dot{U}$ may be written as:

$$\dot{U} = \int_\Omega \left[T \cdot \nabla\dot{\mathbf{u}} + (\phi\mathbf{I} - \nabla\mathbf{u}^T T) \cdot \nabla\mathbf{v}\right] \, d\Omega \tag{14}$$

Proceeding as before, the material derivative of the external work may be expressed as:

$$\frac{d}{d\tau}V_\tau = \frac{d}{d\tau}\left(\int_{\partial\Omega^t_\tau} \bar{\mathbf{t}}_\tau \cdot \mathbf{u}_\tau \, d\partial\Omega_\tau\right) = \int_{\partial\Omega^t} \frac{\partial}{\partial\tau}\left(\bar{\mathbf{t}} \cdot \mathbf{u}||F^{-T}n||\text{det } F\right) d\partial\Omega \tag{15}$$

Assuming that the prescribed load at the boundary remains unchanged, $\dot{\bar{\mathbf{t}}} = 0$, then, the above expression at $\tau = 0$ takes the form:

$$\dot{V} = \frac{d}{d\tau}V_\tau\Big|_{\tau=0} = \int_{\partial\Omega^t} (\bar{\mathbf{t}} \cdot \dot{\mathbf{u}} + \bar{\mathbf{t}} \cdot \mathbf{u} \;\; \text{div}_s\mathbf{v}) \, d\partial\Omega \tag{16}$$

From equations (14,16) and interchanging terms, the material derivative of the potential energy, yields:

$$\begin{aligned} \dot{\Pi} = \dot{U} - \dot{V} \;\; &= \;\; \int_\Omega T \cdot \nabla\dot{\mathbf{u}} \, d\Omega - \int_{\partial\Omega^t} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, d\partial\Omega + \\ &+ \;\; \int_\Omega \left(\phi\mathbf{I} - \nabla\mathbf{u}^T T\right) \cdot \nabla\mathbf{v} d\Omega - \int_{\partial\Omega^t} \bar{\mathbf{t}} \cdot \mathbf{u} \;\; \text{div}_s\mathbf{v} d\partial\Omega \end{aligned} \tag{17}$$

Since the body is held in equilibrium under mixed boundary conditions, the expression of the virtual work vanishes:

$$\int_\Omega T \cdot \nabla\dot{\mathbf{u}} \, d\Omega - \int_{\partial\Omega^t} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, d\partial\Omega = 0 \qquad \forall\dot{\mathbf{u}} \in Var_{\dot{\mathbf{u}}} \tag{18}$$

Thus, from equations (17)-(18) the total derivative of the potential energy, results:

$$\dot{\Pi}(\mathbf{u}; \mathbf{v}) = \int_\Omega (\phi\mathbf{I} - \nabla\mathbf{u}^T T) \cdot \nabla\mathbf{v} d\Omega - \int_{\partial\Omega^t} \bar{\mathbf{t}} \cdot \mathbf{u} \;\; \text{div}_s\mathbf{v} d\partial\Omega \tag{19}$$

From the above expression and since $\mathbf{v}$ is a known vector field, it is interesting to note that, in order to evaluate the potential energy rate, we only need to carry out the stress analysis in the initial shape in order to find the value of the fields $\mathbf{u}, \nabla\mathbf{u}, \phi,$ and T. Moreover, with an inspection of the above equation, we recognize in the domain integral on the right hand side, the expression in brackets of the energy momentum tensor:

$$\Sigma = \phi\mathbf{I} - \nabla\mathbf{u}^T T \tag{20}$$

This tensor, was first introduced by Eshelby (1975) into elastostatics, in the context of infinitesimal deformations and subsequently given central place in the same author's development of continuum approach, when studying defects in solid media.

4. FRACTURE MECHANICS: POTENTIAL ENERGY RELEASE RATE

Let us consider a three dimensional body containing a planar crack. We assume this cracked body is in equilibrium with a given traction on its boundary. For simplicity we suppose null traction along the crack faces. As was aforemention, to simulate crack advance we can see this fenomenum as a modification of the shape of the body. Then, using an appropriate smooth velocity field over the whole body on equation (19), we can obtain the corresponding decrease of the stored potential energy due to the crack advance, in other words, $\dot{\Pi}$ becomes the potential energy release rate G, Taroco and Feijóo (1997).

Regarding to the velocity field $\mathbf{v}$, we assume it takes null values over the boundary of the body and its distribution along the front of the crack is on the same plane of the crack. Since $\dot{\Pi}$ means the specific energy release rate, we also assume for simplicity, that the integral of the velocity along the front of the crack takes unit value:

$$\int_{\Gamma} \mathbf{v} \cdot \mathbf{n} d\Gamma = 1 \tag{21}$$

where Γ is the front of the crack, and $\mathbf{n}$ is the unit normal to Γ.

From the above consideration, the second integral of the equation (19) vanishes and the expression of the decrease of stored potential energy with respect to unit crack advance yields:

$$\dot{\Pi} = G = \int_{\Omega} (\phi \mathbf{I} - \nabla \mathbf{u}^T T) \cdot \nabla \mathbf{v} d\Omega \tag{22}$$

We remark that, after obtaining $\phi, \mathbf{u}$ and T, eq (22) gives the energy release rate for different crack profile advances represented by the associated velocity field $\mathbf{v}$. The evaluation of the equation (22) could be performed by using, for example, the Finite Element Method for the stress analysis obtaining $\phi, \nabla \mathbf{u}$ and T. On the other hand, to evalute the velocity distribution over the whole body, we need to solve by the same method a Poisson type problem, obtaining $\nabla \mathbf{v}$. Due to the singularity of the stress in the front of the crack, it is convenient to adopt a velocity field so that its $\nabla \mathbf{v}$ is null in a neighborhood of this region. The Poisson type equation to be solved to perform the properly velocity field distribution must take into account the above consideration. Finally, $\dot{\Pi}$ can be evaluated by numerical integration over each element of the finite element mesh.

5. NUMERICAL RESULTS

Now, in order to assess the accuracy of the proposed methodology, two well known examples are considered in this section. The first example is a plate with uniform thickness containing a through central crack and the second example, is an embedded penny-shaped crack in a cylindrical bar. The former case is subjected to plane strain state and the latter is exposed to symmetric revolution strain condition. We consider, in both cases, elastic material, uniformly applied remote traction in the longitudinal direction, and crack advance remaining on the same plane of the crack.

The characteristic dimension of the plate (figure 1) has been chosen such that $t =$ 1mm, $h = 60$ mm, $w = 20$ mm and $a = 2$ mm, where t is the thickness, h is the height,

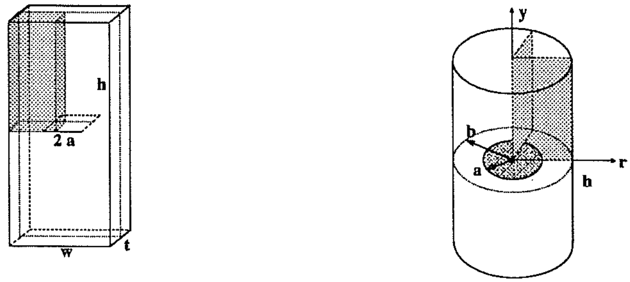

Figure 1: Geometry of the plate and the bar problems

w is the width and $2a$ is the length of the straight crack. In the penny-shaped crack bar (figure 1), we have used the following values $h = 64$ mm, $b = 16$ mm and $a = 2$ mm, where h is the height, b is the radius of the bar and a is the radius of the circular crack located in the center of the bar.

A uniform remote tensile stress of magnitude of 100N/mm^2 is applied on the ends of the plate and the bar and elastic isotropic material with Young's moduls $E = 2.1{\cdot}10^5\text{N/mm}^2$ and Poisson's radius $\nu = 0.3$ are adopted in both examples.

Concerning the appropriate velocity to simulate crack advance, we assume uniform normal component and null tangential distributions along the front of the crack in both cases. Taking into account that the thickness of the plate is 1 mm and the radius of the circular crack embedded in the bar is 2 mm, a suitable value of the normal velocity has been chosen for each case:

$$\mathbf{v}\cdot\mathbf{n} = 0.5\text{mm, for plate problem,} \qquad \mathbf{v}\cdot\mathbf{n} = 0.078054\text{mm, for bar problem} \tag{23}$$

It is easy to verify that with these values of the normal velocity, the integral (eq. 21) takes unit value in both examples.

In the finite element simulations, to obtain $\phi, \mathbf{u}, \nabla\mathbf{u}, T$ and the velocity distribution, only an eigth part of the plate and the bar were modelled because of symmetry. The finite element meshes consisting of tetrahedral elements using linear shape function with three degrees of freedom (three displacements/velocities) in every node, are presented in figures 2 and 3 for the two problems. In order to capture the high stress gradient in the region near the front of the crack, the finite element meshes were refined as it is shown in the same figures, Vénere (1996).

The numerical results obtained for the potential energy release rate in both examples were:

$$G_{\text{plate}} = 0.27432\text{N/mm}; \qquad G_{\text{bar}} = 0.1028\text{N/mm} \tag{24}$$

The comparison of these values is in agreement with the results reported in the literature. In particular, Isida (1971), using expansions of complex stress potentials combined with boundary collocation method, obtains for the same cracked plate the value $G_{\text{plate}} = 0.28605\text{N/mm}$. On the other hand, Benthem and Koiter (1973) using

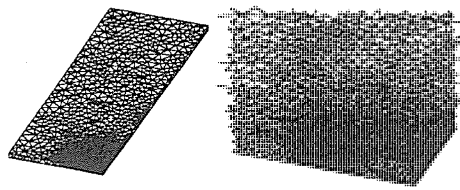

Figure 2: Finite element mesh for the plate problem

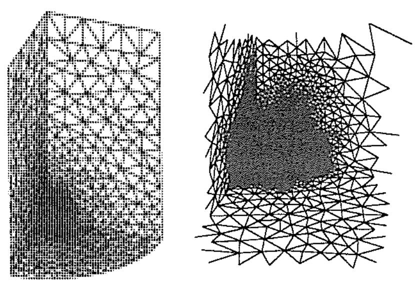

Figure 3: Finite element mesh for the bar problem

asymptotic approximation, obtain for the infinite cracked bar problem the value $G_{\text{bar}} = 0.1106\text{N/mm}$.

6. CONCLUSIONS

In spite of the fact that Shape Sensitivity Analysis has been strongly related and traditionally applied to structural optimization problems, this paper shows how this approach allows us to obtain the expression for the energy release rate in a three-dimensional cracked body. Moreover, this approach can be extended in a straightforward manner to more complex geometry, using velocity fields appropriately representing the desired crack advance.

Therefore, the Shape Sensitivity Analysis could be used in Fracture Mechanics as an alternative method to establish general expression for G. Also, Shape Sensitivity Analysis, together with numerical methods, such as the Finite Element Method, are able to perform reliable numerical results of this important parameter in real fracture mechanical problems.

Finally, it may be concluded that the application of Shape Sensitivity Analysis to Fracture Mechanics emerges as a promising method to analyze three-dimensional cracked bodies.

ACKNOWLEDGEMENTS

This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, and by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina. The authors gratefully acknowledge for the software facilities provided by the TACSOM Group (**www.lncc.br/~tacsom**).

REFERENCES

1. Benthem, J.P. and Koiter, W.T., 1973. Asymptotic approximation to crack problems. *Method Analysis and Solutions of Crack Problems*, Sih, G.C., ed. Noordhoff International Publishing, Holland.

2. Céa, J., (1981), "Problems of shape optimal design", in *Optimization of Distributed Parameters Structures*, Ed. E. J. Haug and J. Céa, Sijthoff and Noordhoff, Alphen aan den Rijn.

3. Eshelby, J.D., 1975. The elastic energy-momentum tensor. *Journal of Elasticity*, Vol. 5, 3/4, 321-335.

4. Gurtin, M.E., 1981. *An introduction to continuum mechanics*, Mathematics in Science and Engineering, Academic Press, New York.

5. Griffiths, A.A., 1921. The phenomena of ruptur and flow in solids, *Phil. Trans. Roy. Soc. London*, A. 221, 163-198.

6. Haug, E. J., Choi, K. K., Komkov, V., 1986. *Design Sensitivity Analysis of Structural Systems*, Academic Press.

7. Isida, M., 1971. Effect of width and tength on stress intensity factors of internally cracked plates under various boundary conditions. *Int. J. Frac.*, Vol. 7, 3, 301-316.

8. Vénere, M. J., 1996. Adaptive techniques for the Finite Element Method in two and three-dimensional domains, D. Sc. these, Instituto Balseiro, Universidad Nacional de Cuyo, Argentina.

9. Taroco, E., Feijóo, R.A., 1997. Rate of Change of the J-Rice Integral in Elastic Fractured Membranes,*Proceedings of the PACAM V - Pan American Congress of Applied Mechanics - Applied Mechanics in the Americas, Vol. 4 - Mechanics and Dynamics of Solids*, 298-301, Porto Rico, January.