

*This research was partially supported by the National Science Foundation
under Grant GU-2059.*

**BASES IN L_2 SPACES WITH APPLICATIONS TO STOCHASTIC
PROCESSES WITH ORTHOGONAL INCREMENTS**

STAMATIS CAMBANIS
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 711

September, 1970

Bases in L_2 Spaces with Applications to Stochastic
Processes with Orthogonal Increments*

STAMATIS CAMBANIS

ABSTRACT. Orthonormal bases are explicitly given for L_2 spaces on the real line. They are applied in obtaining orthogonal expansions for certain classes of stochastic processes and in constructing stochastic processes with orthogonal increments.

*

This research was partially supported by the National Science Foundation under Grant GU-2059.

1. *Orthonormal bases in L_2 spaces on the real line.* The following notation is used. R is the real line; B is the σ -algebra of Lebesgue measurable subsets of R ; m is the Lebesgue measure on (R, B) ; $E = (a, b)$ with $-\infty \leq a < b \leq +\infty$; B_E is the σ -algebra of Lebesgue measurable subsets of E ; m_E is the Lebesgue measure on (E, B_E) ; μ is a σ -finite measure on (E, B_E) with $\mu(B)$ finite for every bounded Lebesgue measurable set in $(a+\epsilon, b-\epsilon)$ for all $\epsilon > 0$.

Explicit orthonormal bases in $L_2(E, B_E, m_E)$ are known: the Hermite functions for $E = R$; the Laguerre functions (with an appropriate linear change of variable) for $E = (a, +\infty)$, a finite, and for $E = (-\infty, b)$, b finite; the Legendre polynomials and the trigonometric functions for $E = (a, b)$, a and b finite. The problem of constructing orthonormal bases in $L_2(E, B_E, \mu)$ with μ a finite measure is considered in [1] and a complete answer is given in [7]. In this section, orthonormal bases are given explicitly for every $L_2(E, B_E, \mu)$ with the set E and the measure μ as determined above.

Fix $t_0 \in E$ and define $F: E \rightarrow R$ by

$$(1.1) \quad F(t) = \begin{cases} \mu((t_0, t]) & \text{for } t_0 < t < b \\ 0 & \text{for } t = t_0 \\ -\mu((t, t_0]) & \text{for } a < t < t_0 \end{cases}$$

F is a right continuous, nondecreasing function and has at most countable points of jumps, at say, $D = \{d_m\} \subset E$. Define the measures λ and ν on (E, B_E) by $\lambda(B) = \mu(B \cap D)$ and $\nu(B) = \mu(B \cap \{E \setminus D\})$ for all $B \in B_E$. Then $\mu = \lambda + \nu$ and $\nu(\{t\}) = 0$ for all $t \in E$. Define $G: E \rightarrow R$ by (1.1) with μ replaced by ν . G is a continuous, nondecreasing function. Let $G(a)$ and $G(b)$ denote the limits of G on the right at a and on the left at b respectively.

THEOREM 1. The family $\{\phi_m(t)\}_m \cup \{f_n(t)\}_n$ is an orthonormal basis in $L_2(E, \mathcal{B}_E, \mu)$, where $\phi_m(t) = \mu^{-\frac{1}{2}}(\{d_m\})I_{\{d_m\}}(t)$ (I is the indicator function) and the f_n 's are determined as follows:

1. If $G(a) = -\infty$ and $G(b) = +\infty$ then $f_n(t) = h_n(G(t))I_{E \sim D}(t)$, $n = 0, 1, 2, \dots$, where $h_n(u)$ is the Hermite function $h_n(u) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} (-1)^n \exp(u^2/2) d^n[\exp(-u^2)]/du^n$, $u \in \mathbb{R}$, $n = 0, 1, 2, \dots$

2. If $-\infty < G(a)$ and $G(b) = +\infty$ then $f_n(t) = \ell_n(G(t)+G(a))I_{E \sim D}(t)$, $n = 0, 1, 2, \dots$, and if $G(a) = -\infty$ and $G(b) < +\infty$ then $f_n(t) = \ell_n(G(b)-G(b+a-t))I_{E \sim D}(t)$, $n = 0, 1, 2, \dots$, where $\ell_n(u)$ is the Laguerre function $\ell_n(u) = (n!)^{-1} e^{t/2} d^n(u^n e^{-u})/du^n$, $u \in (0, +\infty)$, $n = 0, 1, 2, \dots$

3. If $-\infty < G(a) < G(b) < +\infty$ then $f_n(t) = p_n(G(t))I_{E \sim D}(t)$, $n = 0, 1, 2, \dots$, where $p_n(u)$ is the Legendre polynomial $p_n(u) = C_n d^n[(u-G(a))(u-G(b))]/du^n$, $G(a) < u < G(b)$, $n = 0, 1, 2, \dots$, or $f_n(t) = [G(b)-G(a)]^{-\frac{1}{2}} \exp[i2\pi n(G(t)-G(a))/(G(b)-G(a))]I_{E \sim D}(t)$, $n = 0, \pm 1, \pm 2, \dots$

PROOF. Since $L_2(E, \mathcal{B}_E, \mu) = L_2(E, \mathcal{B}_E, \lambda) \oplus L_2(E, \mathcal{B}_E, \nu)$, if $\{\psi_m(t)\}_m$ and $\{g_n(t)\}_n$ are orthonormal bases in $L_2(E, \mathcal{B}_E, \lambda)$ and $L_2(E, \mathcal{B}_E, \nu)$ respectively, then $\{\phi_m(t)\}_m \cup \{f_n(t)\}_n$ is an orthonormal basis in $L_2(E, \mathcal{B}_E, \mu)$, where $\phi_m(t) = \psi_m(t)I_D(t)$ and $f_n(t) = g_n(t)I_{E \sim D}(t)$. An obvious choice for $\psi_m(t)$ and $\phi_m(t)$ is $\phi_m(t) = \psi_m(t) = \mu^{-\frac{1}{2}}(\{d_m\})I_{\{d_m\}}(t)$. The choice of bases in $L_2(E, \mathcal{B}_E, \nu)$ depends on the finiteness or not of $G(a)$ and $G(b)$. There are three distinct cases and the proof will be given for case 1, the remaining cases being similar. In case 1, $G(a) = -\infty$ and $G(b) = +\infty$, and G maps E onto \mathbb{R} . Let q be the measure induced on $(\mathbb{R}, \mathcal{B})$ by G and ν , $q(B) = \nu(G^{-1}(B))$ for every $B \in \mathcal{B}$. Since for every $u, v \in \mathbb{R}$ with $u < v$, $G^{-1}\{(u, v)\} = \{t \in E, u < G(t) < v\}$, it follows that $q\{(u, v)\} = v - u$ and hence $q = m$, the Lebesgue measure. If $G^{-1}: \mathbb{R} \rightarrow E$ is defined by $G^{-1}(u) = \text{sup}\{t \in E, G(t) \leq u\}$ then [4, p.163] $\int_E f(t) \nu(dt) = \int_{\mathbb{R}} f(G^{-1}(u)) m(du)$ and

$\int_E g(G(t)) \nu(dt) = \int_R g(u) m(du)$. Let now $\{e_n(u)\}_n$ be any orthonormal basis in $L_2(R, \mathcal{B}, m)$ and let $\{g_n(t)\}_n$ be defined by $g_n(t) = e_n(G(t))$. The set $\{g_n(t)\}_n$ is orthonormal in $L_2(E, \mathcal{B}_E, \nu)$ since

$$\int_E g_n(t) \overline{g_k(t)} \nu(dt) = \int_R e_n(u) \overline{e_k(u)} m(du) = \delta_{nk}$$

and complete since for every $f \in L_2(E, \mathcal{B}_E, \nu)$

$$\begin{aligned} \int_E |f(t)|^2 \nu(dt) &= \int_R |f(G^{-1}(u))|^2 m(du) \\ &= \sum_n \left| \int_R f(G^{-1}(u)) \overline{e_n(u)} m(du) \right|^2 \\ &= \sum_n \left| \int_E f(t) \overline{g_n(t)} \nu(dt) \right|^2. \end{aligned}$$

The result follows by taking $e_n = h_n$, the Hermite function ($n = 0, 1, 2, \dots$).

It is clear from the proof that if $D = \emptyset$ then $L_2(E, \mathcal{B}_E, \lambda) = \{0\}$ and $\{f_n(t)\}_n$ is an orthonormal basis in $L_2(E, \mathcal{B}_E, \mu)$, while if $G(a) = G(b)$ then $L_2(E, \mathcal{B}_E, \nu) = \{0\}$ and $\{\phi_m(t)\}_m$ is an orthonormal basis in $L_2(E, \mathcal{B}_E, \mu)$. It should be noted that if $\mu = \mu_1 + \mu_2 + \mu_3$ is the decomposition of μ with respect to the Lebesgue measure m_E , where μ_1 is absolutely continuous with respect to m_E , μ_2 is purely atomic and μ_3 is singular with respect to m_E with $\mu_3(\{t\}) = 0$ for all $t \in E$ [4, p.182], then $\lambda = \mu_2$ and $\nu = \mu_1 + \mu_3$.

2. *Application to the representation of certain classes of stochastic processes.* Let $(\Omega, \mathcal{F}, P; x(t, \omega), t \in E)$ be a stochastic process with orthogonal increments. Fix $t_0 \in E$ and define $F: E \rightarrow \mathbb{R}$ by

$$(2.1) \quad F(t) = \begin{cases} E|x(t^+, \omega) - x(t_0^+, \omega)|^2 & \text{for } t_0 < t < b \\ 0 & \text{for } t = t_0 \\ -E|x(t_0^+, \omega) - x(t^+, \omega)|^2 & \text{for } a < t < t_0. \end{cases}$$

Then F is right continuous, nondecreasing function with jumps at the at most countable set of points $D = \{d_m\} \subset E$. Let X be the orthogonal $L_2(\Omega, F, P)$ -valued measure which is the extension of the measure defined on the prering $\{(s, t], a < s < t < b\}$ by $X((s, t], \omega) = x(t^+, \omega) - x(s^+, \omega)$ [6, Theorem 8.6]. Let also μ be the extension to \mathcal{B}_E of the measure defined on the same prering by $\mu\{(s, t]\} = E|x(t^+, \omega) - x(s^+, \omega)|^2 = E|X((s, t], \omega)|^2 = F(t) - F(s)$. Then μ and F correspond as in (1.1). Let ν and G be defined as in Section 1.

Let $H(x)$ and $H(X)$ be the subspaces of $L_2(\Omega, F, P)$ spanned by the increments of the stochastic process $x(t, \omega)$ and by the values of the $L_2(\Omega, F, P)$ -measure X respectively. Then $H(x) = H(X)$ and the Hilbert spaces $L_2(E, \mathcal{B}_E, \mu)$ and $H(X)$ are isomorphic [6, Theorem 5.9]. This isomorphism will be denoted by \leftrightarrow . If $f(u) \leftrightarrow \xi(\omega)$ then $\xi(\omega) = \int_E f(u) X(du, \omega) = \int_E f(u) dx(u, \omega)$. It follows that if $\phi_m(u) \leftrightarrow \eta_m(\omega)$ and $f_n(u) \leftrightarrow \xi_n(\omega)$, where ϕ_m and f_n are as in Theorem 1, then the set of random variables $\{\eta_m(\omega)\}_m \cup \{\xi_n(\omega)\}_n$ is an orthonormal basis in $H(X) = H(x)$ and

$$(2.2) \quad \eta_m(\omega) = \mu^{-\frac{1}{2}}(\{d_m\}) [x(d_m^+, \omega) - x(d_m^-, \omega)]$$

$$(2.3) \quad \xi_n(\omega) = \int_E f_n(u) dx(u, \omega).$$

If the stochastic process $(\Omega, F, P; y(t, \omega), t \in T, T \in \mathcal{B})$ admits the integral representation

$$(2.4) \quad y(t, \omega) = \int_E f(t, u) dx(u, \omega)$$

for all $t \in T$, where $f(t, \cdot) \in L_2(E, \mathcal{B}_E, \mu)$ for all $t \in T$, then it is also represented by the orthogonal expansion

$$(2.5) \quad y(t, \omega) = \sum_n a_n(t) \xi_n(\omega) + \sum_m b_m(t) \eta_m(\omega)$$

for all $t \in T$ in $L_2(\Omega, F, P)$, where by the isomorphism

$$(2.6) \quad a_n(t) = \int_E f(t,u) \overline{f_n(u)} \mu(du)$$

$$(2.7) \quad b_m(t) = \mu^{\frac{1}{2}}(\{d_m\}) f(t,d_m).$$

Also, if $R_y(t,s)$ is the autocorrelation function of $y(t,\omega)$ we have, for all $t,s \in T$,

$$\begin{aligned} R_y(t,s) &= \int_E f(t,u) \overline{f(s,u)} \mu(du) \\ &= \sum_n a_n(t) \overline{a_n(s)} + \sum_m b_m(t) \overline{b_m(s)}. \end{aligned}$$

Stochastic processes which admit integral representations of the form (2.4), and hence orthogonal expansions of the form (2.5), include all mean square continuous, wide sense stationary processes [7], all linear operations on such processes, and in general all processes in the linear span of the increments of a stochastic process with orthogonal increments. Also the purely nondeterministic processes, under a mean square continuity condition, admit the Cramèr-Hida canonical representation into an orthogonal sum of N terms of the form (2.4), where N is the spectral multiplicity of the process.

In particular, every stochastic process $\{x(t,\omega), t \in E\}$ with orthogonal increments admits the representation (2.4) with $y(t,\omega) = x(t^+, \omega) - x(t_0^+, \omega)$, $T = E$, $t, t_0 \in E$, and $f(t,u) = I_{(t_0, t]}(u)$. In this case, (2.5) gives

$$(2.8) \quad x(t^+, \omega) = x(t_0^+, \omega) + \sum_n a_n(t) \xi_n(\omega) + \sum_{d_m \in (t_0, t]} [x(d_m^+, \omega) - x(d_m^-, \omega)]$$

for all $t \in E$ in $L_2(\Omega, \mathcal{F}, P)$, where now

$$(2.9) \quad a_n(t) = \int_{(t_0, t]} \overline{f_n(u)} \mu(du) = \int_0^{G(t)} \overline{e_n(v)} m(dv)$$

and by Theorem 1 $\{e_n\}_n$ is an orthonormal basis in $L_2(A=(G(a), G(b)), \mathcal{B}_A, m_A)$.

3. *Application to stochastic processes with orthogonal Gaussian increments.* In this section $\{x(t, \omega), t \in E\}$ will be a stochastic process with zero mean and orthogonal Gaussian increments (the extension to the non-zero mean case is obvious). In this case $\{\xi_n(\omega)\}_n$ is a sequence of independent $N(0,1)$ random variables and the convergence of the series representations (2.5) and (2.8) is also almost sure (a.s.) for all $t \in T$.

The representation (2.5) provides a way of constructing every Gaussian stochastic process $\{y(t, \omega), t \in T\}$ which admits an integral representation of the form (2.4) with respect to stochastic process with orthogonal Gaussian increments, given $f(t, u)$ and $F(u)$. This way of construction applies to Gaussian mean square continuous, wide sense stationary processes, to linear operations on them, as well as to Gaussian purely nondeterministic processes.

The representation (2.8) provides a way of constructing every stochastic process $\{x(t, \omega), t \in E\}$ with orthogonal Gaussian increments on every interval E of the real line, bounded or unbounded, given $F(t)$ and $x(t_0^+, \omega)$ for some $t_0 \in E$. The third term in (2.8) clearly corresponds to the jumps of the process, while the second term is shown in Theorem 2 to correspond to the "continuous" part of the process; continuous not only in the mean square sense but also in the sense of the a.s. sample path continuity. This kind of construction, by means of a representation of the form (2.8), is known for the Wiener process; two particular representations, resulting from specific choices of the orthogonal bases $\{e_{nn}\}_n$, are due to Wiener and Lévy, and the general representation is due to Shepp [8].

THEOREM 2. For every real valued, separable stochastic process $\{x(t, \omega), t \in E\}$ with orthogonal Gaussian increments the term $x_c(t, \omega) = \sum_n a_n(t) \xi_n(\omega)$ in the representation (2.8) has almost surely continuous paths.

PROOF. It suffices to show that $x_c(t, \omega)$ is continuous a.s. on every compact subinterval of E . Hence it suffices to prove the theorem for E a compact interval, and without loss of generality we take $E = [0, 1]$. If we choose in particular the orthonormal basis

$$\{e_0(v) = G^{-\frac{1}{2}}(1), e_n(v) = [2/G(1)]^{\frac{1}{2}} \cos[n\pi v/G(1)], n = 1, 2, \dots\}$$

we obtain from (2.9)

$$\{a_0(t) = G(t)/G(1), a_n(t) = ([2G(1)]^{\frac{1}{2}}/n\pi) \sin[n\pi G(t)/G(1)], n = 1, 2, \dots\}$$

Then it is shown as in the classical proof for the Wiener process (see for example [5, Theorem 2.1]) that the series

$$a_0(t)\xi_0(\omega) + \sum_{k=0}^{+\infty} \sum_{n=2^k+1}^{2^{k+1}} a_n(t)\xi_n(\omega)$$

converges uniformly in $t \in [0, 1]$ a.s. It follows [2, p.146 or p.210] that the series $\sum_{n=0}^{+\infty} a_n(t)\xi_n(\omega)$ converges uniformly in $t \in [0, 1]$ a.s. and thus the sample paths of $x_c(t, \omega)$ are continuous a.s.

A straightforward consequence of Theorem 2 is the following well known result, for which a different proof is thus provided.

COROLLARY [3, p.173]. Every real valued, mean square continuous, separable stochastic process with orthogonal increments has almost surely continuous paths.

It also follows from (2.8) and (2.9) that

$$(3.1) \quad x_c(t, \omega) = \sum_n a_n(t) \xi_n(\omega) = W(G(t), \omega)$$

for all $t \in E$ a.s., where W is the Wiener process. Use of (3.1) and of the sample path continuity of the Wiener process provides another way of obtaining

the conclusion of Theorem 2. The proof given in Theorem 2 merely shows that the same techniques employed in the study of the Wiener process can be used in studying all processes with orthogonal Gaussian increments.

REFERENCES

1. N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space*, vol. 1, Ungar, New York, 1961.
2. J. Delporte, *Fonctions aléatoires presque sûrement continues sur un intervalle fermé*, *Ann. Inst. H. Poincaré*, B.1 (1964/65), 111-215.
3. I.I. Gikhman and A.V. Skorokhod, *Introduction to the Theory of Random Processes*, Saunders, Philadelphia, 1969.
4. P.R. Halmos, *Measure Theory*, Van Nostrand, Princeton, N.J., 1950.
5. T. Hida, *Stationary Stochastic Processes*, Princeton University Press, Princeton, N.J., 1970.
6. P. Masani, *Orthogonally scattered measures*, *Advances in Math.* 2(1968), 61-117.
7. E. Masry, B. Liu and K. Steiglitz, *Series expansion of wide-sense stationary random processes*, *IEEE Trans. Information Theory*, vol. IT-14 (1968), 792-796.
8. L.A. Shepp, *Radon-Nikodym derivatives of Gaussian measures*, *Ann. Math. Stat.* 37(1966), 321-354.

UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NORTH CAROLINA 27514

AMS Subject Classifications. Primary 4217, 6069.

Key Words and Phrases. Bases in L_2 spaces, orthogonal expansion of processes, processes with orthogonal increments.