Dynamic Buckling of Circular Cylindrical Shells Under Step Loading

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Abstract

This paper aims at comparing and evaluating several solutions to the problem of dynamic stability of an imperfection-sensitive circular cylindrical shell under step loading with the author's solution.

The first phase of the paper is devoted to the development of a simple mechanical model having a nonlinear, polynomial spring which is a modification of the models by Budiansky and Hutchinson.

The second phase of the paper concerns the development of dynamic equations whichare modified forms of those obtained by Budiansky and Hutchinson by including the axial inertia similar to that by Tamura and Babcock, and yields a Mathieu-type equation. This is demonstrated analytically by the singular perturbation method using multiple time scales as has been done by Danielson. The results seek to relate the dynamic buckling strength directly to the static buckling strength. Various tables and graphs emphasize interesting comparisons with the different solutions of several authors.

1. Introduction

The dynamic buckling of circular cylindrical shells under step loading poses an interesting challenge to engineers and scientists because of the imperfection-sensitivity of the shells and the lack of general guidelines for their design. The present paper, in the first phase, makes a special study based on an imperfection-sensitive model which is a modification on that devised by Budiansky and Hutchinson[1] and brings out interesting features. The final phase of the analysis concerns itself with an end-loaded cylinder under a dynamic step load. The present analysis is the same as that by Budiansky and Hutchinson[2] but modifies the latter by the inclusion of axial inertia as has been done by Tamura and Babcock[3]. Both the symmetric and unsymmetric buckling modes (ξ_1, ξ_2) and the symmetric and unsymmetric imperfections ($\bar{\xi}_1$, $\bar{\xi}_2$) have been considered in this analysis and the dynamic equilibrium equations are based on the following assumptions:

1. The buckling modes are also natural vibration modes of the unloaded structure.
2. The buckled shape is composed of an axisymmetric mode and one with diamond-shaped square knuckles.
3. Circumferential inertia effects are negligible.
4. Donnell shell theory is used.
5. Boundary conditions are defined by the assumed buckling modes.
6. Prebuckling inertia is negligible.

2. Simple Model

The present paper is based on the simple dynamic buckling model shown in Fig. 1. The force in the nonlinear, polynomial spring is related to its shortening, X, by

$$F = KL\ (\xi - \alpha\xi^2 - \alpha^2\xi^3), \tag{1}$$

where $\xi = X/L$ and $L > 0$, $\alpha > 0$, α = constant.

Assuming initial displacement of $\bar{X} = L\,\bar{\xi}$, static equilibrium under the load, λ, may be written as

$$(1 - \lambda/\lambda_c)\ \xi - \alpha\xi^2 - \alpha^2\xi^3 = (\lambda/\lambda_c)\ \bar{\xi}, \tag{2}$$

where $\lambda_c = \frac{1}{2} KL$ represents the classical buckling load.

Transforming variables to $Z = \xi/\bar{\xi}$ Eq. (2) may be written

$$(1 - \lambda/\lambda_c)\ Z - (\alpha\,\bar{\xi})\ Z^2 - (\alpha^2\bar{\xi}^2)\ Z^3 = \lambda/\lambda_c \tag{3}$$

To find the critical static load, λ_s, we set $d\lambda / dz = 0$.

If λ is a dynamic load, λ (t), and the mass M is considered, Eq. (3) may be cast as the equation of dynamic equilibrium

$$\ddot{Z} + (1 - \lambda/\lambda_c)\ Z - (\alpha\,\bar{\xi})\ Z^2 - \left(\frac{\alpha^2\bar{\xi}^2}{2}\right) Z^3 = \lambda/\lambda_c \tag{4}$$

where the dots denote differentiation with respect to $t\sqrt{K/M}$.

Considering a step loading with homogeneous initial conditions at t = 0, the first integral of Eq. (4) is

$$\dot{Z}^2 + (1 - \lambda/\lambda_c)\ Z^2 - 2/3\ (\alpha\bar{\xi})\ Z^3 - (\alpha^2\bar{\xi}^2/2)\ Z^4 = 2(\lambda/\lambda_c)\ Z. \tag{5}$$

Assuming periodic motion (λ = small) when the amplitude Z max is reached, $\dot{Z}$ max = 0, and Eq. (6) reduces to

$$(1 - \lambda/\lambda_c)\ Z\text{ max} - 2/3\ (\alpha\bar{\xi})\ Z^2\text{max} - (\alpha^2\bar{\xi}^2)\ Z^3\text{max} = 2\ (\lambda/\lambda_c) \tag{6}$$

Using $d\lambda/dz$ max = 0, we finally get

$$\frac{\lambda_D}{\lambda_S} = \frac{1}{2}\left[\frac{\frac{1}{3}\left(1-\frac{\lambda D}{\lambda C}\right)\left[\sqrt{q}-\frac{4}{3}\right]-\frac{2}{27}\left[q-\frac{8}{3}\sqrt{q}+\frac{16}{9}\right]-\frac{1}{54}\left[q^{3/2}-4q+\frac{48}{9}\sqrt{q}-\frac{64}{27}\right]}{\frac{1}{3}\left(1-\frac{\lambda S}{\lambda C}\right)\left[\sqrt{p}-1\right]-\frac{1}{9}\left[p-2\sqrt{p}+1\right]-\frac{1}{54}\left[2p^{3/2}-6p+6\sqrt{p}-2\right]}\right] \tag{7}$$

where $p = \left\{1 + 3(1-\lambda_S/\lambda_C)\right\}$

$q = \left\{\frac{16}{9} + 6(1-\lambda_D/\lambda_C)\right\}$

λ_S = the critical static load

λ_D = the critical dynamic load

λ_C = the classical static load

3. Circular Cylindrical Shell Model

The radial component of displacement is assumed of the form

$$W = \xi_1 h \cos\frac{P_o X}{R} + \xi_2 h \sin\frac{P_o X}{2R} \cos\frac{P_o Y}{2R}, \tag{8}$$

where X and Y are the meridional and circumferential distances, respectively and h=the thickness of the shell. The following dynamic equilibrium equations may be obtained similar to those given by Budiansky and Hutchinson[2] with the exception that the present paper considers also axial inertia :

$$\frac{1}{\left[\omega^{(1)}\right]^2}\frac{d^2\xi_1}{d_t^2} + (1-\lambda'/\lambda_c)\xi_1 + \frac{1}{\left\{S_c^{(1)}, e_c^{(1)}\right\}}\left[\frac{3}{2}\xi_1^2\left\{S_c^{(1)}, L_2\left(u_c^{(1)}\right)\right\}\right.$$

$$+ \xi_1\xi_2\left(\left\{2S_c^{(1)}, L_{11}\left(u_c^{(1)}, u_c^{(2)}\right)\right\} + \left\{S_c^{(2)}, L_2\left(u_c^{(1)}\right)\right\}\right)$$

$$\left.+ \xi_2^2\left(\left\{S_c^{(2)}, L_{11}\left(u_c^{(1)}, u_c^{(2)}\right)\right\} + \frac{1}{2}\left\{S_c^{(1)}, L_2\left(u_c^{(2)}\right)\right\}\right)\right] = (\lambda'/\lambda_c)\bar{\xi}_1 \tag{9}$$

$$\frac{1}{\left[\omega^{(2)}\right]^2}\frac{d^2\xi_2}{d_t^2} + (1-\lambda'/\lambda_c)\xi_2 + \frac{1}{\left\{S_c^{(2)}, e_c^{(2)}\right\}}\left[\frac{3}{2}\xi_2^2\left\{S_c^{(2)}, L_2\left(u_c^{(2)}\right)\right\}\right.$$

$$+ \xi_1\xi_2\left(\left\{2S_c^{(2)}, L_{11}\left(u_c^{(1)}, u_c^{(2)}\right)\right\} + \left\{S_c^{(1)}, L_2\left(u_c^{(2)}\right)\right\}\right)$$

$$\left.+ \xi_1^2\left(\left\{S_c^{(1)}, L_{11}\left(u_c^{(1)}, u_c^{(2)}\right)\right\} + \frac{1}{2}\left\{S_c^{(2)}, L_2\left(u_c^{(1)}\right)\right\}\right)\right] = (\lambda'/\lambda_c)\bar{\xi}_2 \tag{10}$$

where $\lambda' = \lambda + \frac{CR\xi_o(t)}{L}$; $C = \sqrt{3(1-\nu^2)}$

R = the radius of the shell

L = the length of the shell

$$\frac{1}{[\omega^{(0)}]^2}\frac{d^2\xi_o^*}{dt^2} + \xi_o^* = \lambda\, H\,(t), \tag{11}$$

where $\xi_o^* = \frac{CR}{L}\xi_o$

The only axial inertia term considered results from the variable $\xi_o(t)$, while the circumferential inertia terms will be completely neglected. The introduction of this time-dependent variable in the problem gives rise to an approximation of the first axial mode of vibration. Higher axial modes are not considered and their influence is thought to be unimportant for the axial mode coupling. Hence we get

$$\frac{1}{[\omega^{(1)}]^2}\frac{d^2\xi_1}{dt^2} + (1 - \lambda'/\lambda_c)\,\xi_1 - (3C/32)\,\xi_2^2 = (\lambda'/\lambda_c)\,\bar{\xi}_1 \tag{9a}$$

$$\frac{1}{[\omega^{(2)}]^2}\frac{d^2\xi_2}{dt^2} + \left[(1 - \lambda'/\lambda_c) - (3C/2)\,\xi_1\right]\xi_2 = (\lambda'/\lambda_c)\,\bar{\xi}_2 \tag{10a}$$

Using Eq. (9a) and dropping the inertia term as well as setting $\bar{\xi}_1 = 0$, we get

$$(1 - \lambda'/\lambda_c)\,\xi_1 - (3C/32)\,\xi_2^2 = 0 \tag{12}$$

From (10a) and (12) we get

$$\boxed{\frac{1}{[\omega^{(2)}]^2}\frac{d^2\xi_2}{dt^2} + (1-\lambda'/\lambda_c)\,\xi_2 - \left[\frac{9C^2}{64(1-\lambda'/\lambda_c)}\right]\xi_2^3 = (\lambda'/\lambda_c)\,\bar{\xi}_2} \tag{13}$$

Now a solution of Eq. (11) gives

$$\xi_o^* = \lambda\,(1 - \cos\bar{\tau}) \tag{14}$$

where $\bar{\tau} = \omega_o t$

Substituting Eq. (14) into Eq. (13) and neglecting the cubic term ξ_2^3, we arrive at the Mathieu - type equation :

$$\boxed{\frac{d^2\xi_2}{d\bar{\tau}^2} + \left[\left(1 - \frac{2\lambda}{\lambda_c}\right)\left(\frac{\omega^{(2)}}{\omega^{(0)}}\right)^2 + \frac{\lambda}{\lambda_c}\left(\frac{\omega^{(2)}}{\omega^{(0)}}\right)^2\cos\bar{\tau}\right]\xi_2 = \frac{\lambda}{\lambda_c}\left(2 - \cos\bar{\tau}\right)\bar{\xi}_2} \tag{15}$$

where $\bar{\tau} = \omega^{(0)}t$

Set

$$\left.\begin{aligned} \varepsilon &= (2\lambda/\lambda_c) \quad (\omega^{(2)}/\omega^{(0)})^2 \\ \delta &= (1-2\lambda/\lambda_c) \quad (\omega^{(2)}/\omega^{(0)})^2 \\ K &= (\lambda/\lambda_c) \ \bar{\xi}_2 \end{aligned}\right\} \tag{16}$$

Then (25) becomes

$$\frac{d^2\xi_2}{d\bar{\tau}^2} + \left[\delta + \varepsilon/2 \quad \cos \bar{\tau} \right] \xi_2 = K \ (2-\cos\bar{\tau}) \tag{17}$$

The initial conditions are

$$y \ (0) = 0 \ , \frac{dy}{d\bar{\tau}}(0) = 0 \tag{18}$$

A uniform approximation to Eq. (17) as $\varepsilon \to 0$ is desired. A Singular Perturbation technique using multiple scales[4] is taken recourse to.

The following equations of the stability lines result :

$$\begin{aligned} \delta_c &= 0 : & \delta &= -\ 1/8 \ \varepsilon^2 + 0 \ (\varepsilon^4) \\ \delta_c &= 1/4 : & \delta &= 1/4 + 1/4 \ \varepsilon + 0 \ (\varepsilon^2) \\ & & \delta &= 1/4 - 1/4 \ \varepsilon + 0 \ (\varepsilon^2) \\ \delta_c &= 1 : & \delta &= 1 - 1/48 \ \varepsilon^2 + 0 \ (\varepsilon^4) \\ & & \delta &= 1 + 5/48 \ \varepsilon^2 + 0 \ (\varepsilon^4) \end{aligned}$$

Coincise expressions are finally obtained relating the dynamic buckling load directly to λ_S , λ_c , and $\omega^{(2)}/\ \omega^{(0)}$:

$$\left.\begin{aligned} \frac{\lambda_D}{\lambda_S} &= \frac{\left(1 - 2\lambda_D/\lambda_C\right)}{\left(1 - \lambda_S \ \lambda_C\right)^2} \left(\frac{27}{16} \ |\ \bar{\xi}_2 |\right), \quad 0 \leq \frac{\omega^{(2)}}{\omega^{(0)}} < \frac{1}{2} \\ \frac{\lambda_D}{\lambda_S} &= \frac{\frac{1}{10}\left[4 - \left(\frac{\omega^{(0)}}{\omega^{(2)}}\right)^2\right]}{\left(1 - \lambda_S/\lambda_C\right)^2} \left(\frac{27}{16} \ |\ \bar{\xi}_2 |\right), \quad \frac{\omega^{(2)}}{\omega^{(0)}} \geq \frac{1}{2} \end{aligned}\right\} \tag{20}$$

4. Conclusion

The results for the initial phase of the paper are presented in the form of correlations between the dynamic buckling strength and the static buckling strength of the initially imperfect simple model. The comparison of results with those by Budiansky and Hutchinson[1] (Table I) reveals a qualitative and quantitative agreement pattern.

The results for the final phase of the paper are in the form of graphs of (λ_D / λ_S) versus ($\omega^{(2)}$ / $\omega^{(0)}$) (Fig. 2). The comparison of results are made with those by Budiansky and Hutchinson[2] and those by Danielson[5]. As expected, the values are much lower than those from the analysis neglecting axial inertia due to the fact that the coupling between radial and axial motion is obtained. The elimination of the axial inertia term, ξ_0, from the present analysis results in comparable results in profound agreement with the corresponding analysis by Budiansky and Hutchinson[2], Danielson[5] and by Tamura and Babcock[3].

References

(1) J. Hutchinson and B. Budiansky, AIAA J., 4 (3) (1966) 525.

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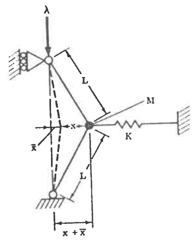
(3) Y.S. Tamura and C.D. Babcock, "Dynamic Stability Of Cylindrical Shells Under Step Loading," Journal of Applied Mechanics, March 1975.

(4) J. Cole, Perturbation Methods in Applied Mechanics (Blaisdell, Waltham, Mass. 1968).

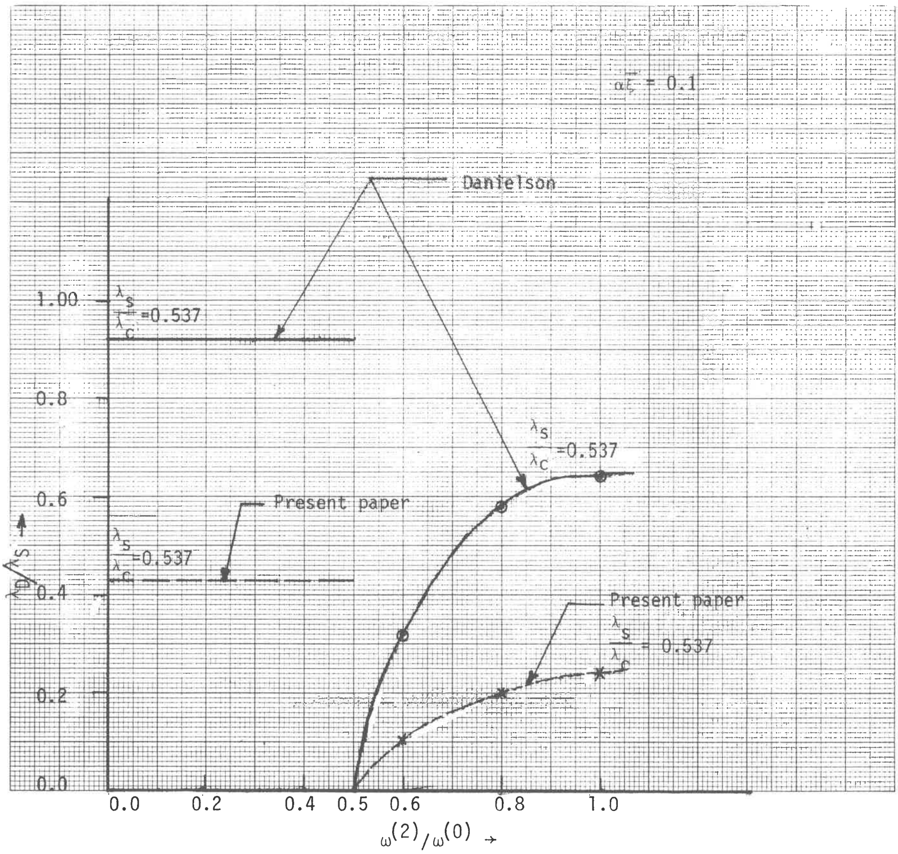
(5) D.A. Danielson, AIAA J. 7(8) (1969) 1506.

TABLE I. Comparison of dynamic buckling loads of simple models and an axially loaded cylinder *

λ_S/λ_C	λ_D/λ_S					
	$\bar{\xi}_1/\bar{\xi}_2 = 0$		Quadratic Spring	Cubic Spring	$\left(\frac{1-\lambda_D/\lambda_C}{1-\lambda_S/\lambda_C}\right)^2 = \sqrt{2}\left(\lambda_D/\lambda_S\right)$ *	Polynomial Spring
	Hutchinson & Budianski	Present Paper	Hutchinson & Budiansky	Hutchinson & Budiansky	Hutchinson & Budiansky	Present Paper
0.1	0.760	0.775	0.780	0.730	0.740	0.778
0.2	0.790	0.812	0.820	0.760	0.780	0.819
0.3	0.820	0.830	0.850	0.790	0.810	0.837
0.4	0.840	0.868	0.870	0.820	0.830	0.896
0.5	0.875	0.875	0.890	0.840	0.865	0.933
0.6	0.910	0.900	0.920	0.880	0.900	0.994
0.7	0.925	0.925	0.930	0.910	0.915	1.000
0.8	0.960	0.945	0.960	0.930	0.950	1.000
0.9	0.970	1.000	0.970	0.960	0.965	1.000
0.99	0.990	1.000	1.000	0.990	0.990	1.000



Simple Model



Plot of (λ_D / λ_S) Versus ($\omega^{(2)}$/ $\omega^{(0)}$)