LINEAR PROBLEMS IN P-TH ORDER AND STABLE PROCESSES*

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ABSTRACT

This work extends to processes with finite moments or order p, $1 < p < 2$ and to symmetric α-stable processes, $1 < \alpha < 2$, some of the basic linear theory known for processes with finite second moments ($p=2$) and for Gaussian processes ($\alpha=2$). Here the "covariation" plays a role analogous to the covariance. Specifically, stochastic integrals of two types are introduced and studied for p-th order processes and in particular for symmetric stable processes. Regression estimates and linear estimates on certain symmetric stable processes are evaluated, including regression and linear filtering of signal in noise. Also, for certain symmetric stable inputs, the identification of a linear system from the input covariation and the input-output cross covariation is considered, and the way the distribution of the output depends on the linear system is studied.

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1. INTRODUCTION

The linear theory of Gaussian processes, indeed of random processes with finite second moments (second order processes), has been fully developed. This includes linear estimation, and in particular prediction and filtering, of second order processes, and the analysis and identification of linear systems with Gaussian inputs (or second order random inputs).

It is very desirable to have analogs of these results for larger classes of random processes which include the Gaussian processes or the second order processes as special elements. The availability of such results would make it possible to handle certain random processes which are nearly, but not exactly, Gaussian or second order. In this paper an attempt is made to develop a linear theory for the class of stable processes, to which Gaussian processes belong, and for the class of p-th order processes, i.e., processes with finite p-th order moments.

The class of stable processes is very important as stable distributions form a natural generalization of the normal distribution, they satisfy the important stability property (linear combinations of jointly stable variables are stable), they have heavier tails than the normal distribution and thus are more appropriate for models with outliers, and they arise as limit laws of normed sums of independent identically distributed random variables. For simplicity only stable processes with symmetric distributions will be considered; and their characteristic exponent will be denoted by α, $0 < \alpha \leq 2$. Gaussian processes are stable processes with $\alpha=2$.

Even though stable processes can be thought of as a one step generalization of Gaussian processes, they constitute too rich a class of processes. For instance, while all finite dimensional distributions of a (general) symmetric Gaussian process are fully described by means of one parameter, its covariance function, in order to describe all finite dimensional distributions of a (general) symmetric stable process with $0 < \alpha < 2$ one needs an infinite number of parameters (the spectral measures of all orders). It would therefore be very difficult to study nonlinear problems for general stable processes. Linear problems should be easier, but in fact they also turn out to be quite intractable in general. The best way to obtain explicit results seems to be to restrict attention to special classes of stable processes with sufficiently simple parametric descriptions, such as sub-Gaussian processes, processes with independent stable increments, and moving averages or Fourier transforms of processes with independent stable increments.

A basic difficulty in developing the linear theory of stable processes is due to the fact that while the linear space of a Gaussian process is a Hilbert space, the linear space of a stable process is a Banach space when $1 \leq \alpha < 2$ and only a metric space when $0 < \alpha < 1$. Here we focus our attention on the case $1 < \alpha < 2$ (finite mean, infinite variance) where it is shown that in many cases the "covariation" plays a role analogous to the role played by the covariance in the Gaussian case $\alpha=2$. While the covariance of two random variables is linear in both arguments, the covariation of two random variables is linear only in its first argument, and this hinders substantially its usefulness. However, the covariation has a certain linearity property in its second argument for sums of independent stable variables and this turns out to be very useful.

Most of the general results developed are valid for p-th order processes with $1 < p < 2$ (if $p \geq 2$ the second order process theory is applicable) as well as for symmetric stable processes with $1 < \alpha < 2$; but more specific results are given only for stable processes (the crucial difference being the unavailability of a linearity-like property of the covariation in its second argument in the general case).

The basic structure is developed in Sections 3 and 4. An appropriate Banach space of functions is shown to be isometric, via a stochastic integral, to (a subspace of) the linear space of a p-th order or a symmetric α-stable process, $p > 1$, $\alpha > 1$, (Section 3). When applied to a symmetric stable process with independent increments, our results yield the stochastic integral defined in [13], and when applied to an "absolutely continuous" process (Section 4), they yield a stochastic integral of a more particular form which can be regarded also as a sample path integral. A Fubini-type result which allows the interchange of stochastic and usual integration is also established (Section 4). These results are used to evaluate regression estimates and linear estimates for certain classes of stable processes, including regression and linear filtering of signal in noise (Section 5), and also to identify and analyse linear systems with certain specific stable inputs (Section 6).

2. DEFINITIONS AND PRELIMINARY RESULTS

Let $\xi = \{\xi_t, t \in T\}$ be a stochastic process with underlying probability space (Ω, F, P) such that $\xi_t \in L_p(\Omega)$ for all $t \in T$, where $1 < p < \infty$, and let $\ell(\xi)$ be the space of all finite linear combinations of elements of $\{\xi_t, t \in T\}$. Then we call ξ a p-th order process and define a norm on $\ell(\xi)$ by

$$\|\zeta\|_p = (E|\zeta|^p)^{1/p}, \ \zeta \in \ell(\xi) .$$

The linear space $L(\xi)$ of the process ξ is the completion of $\ell(\xi)$ with respect to this norm, i.e., in $L_p(\Omega)$. Throughout this paper when raising a number u to a power $b > 0$, we shall use the convention $(u)^b = |u|^b \operatorname{sign}(u)$.

If M is a closed subspace of $L_p(\Omega)$ (such as $L(\xi)$), then for each fixed $\zeta \varepsilon M$ the expression

$$<\eta,\zeta>_p = E[\eta(\zeta)^{p-1}], \quad \eta \varepsilon M ,$$

defines a continuous linear functional on M, which by Hölder's inequality has norm $|\zeta|_p^{p-1}$. When $p=2$, i.e., for second order processes, $<\eta,\zeta>_2$ equals the usual inner product $E[\eta\zeta]$.

An important subclass of p-th order processes is the family of symmetric α-stable (SαS) stochastic processes with $1 < \alpha \leq 2$. When $\alpha=2$ these are the familiar zero mean Gaussian processes. For $1 < \alpha < 2$, the SαS processes are defined by consistent finite dimensional distributions with characteristic functions (ch.f.'s) of the form

$$\phi(y) = \exp\{-\int_S |<x,y>|^\alpha \, d\Gamma(x)\}, \ y \ \varepsilon \ R^n,$$

where Γ is a uniquely determined [7, p. 36] finite symmetric measure on the Borel subsets of the unite sphere $S = \{x \ \varepsilon \ R^n; <x,x> = 1\}$ [9, p. 264]. Following [12, p. 357], Γ is called the <u>spectral measure</u> of the distribution. In particular, for each SαS random variable ζ there exists a number $|\zeta|_\alpha \geq 0$ such that $E(e^{ir\zeta}) = \exp\{-|\zeta|_\alpha^\alpha \, |r|^\alpha\}$ for all $r \ \varepsilon \ R$.

It is well known that a SαS process is a p-th order process for any p satisfying $1 < p < \alpha$. For a linear space of SαS random variables, the function $\zeta \to |\zeta|_\alpha$ defines a norm [13, p. 413]. An application of Theorem 2 of [15, p. 862] shows that this norm is related to the usual $L_p(\Omega)$ norm by

$$||\zeta||_p = C(p,\alpha) \ |\zeta|_\alpha ,$$

where $C(p,\alpha)$ is the following constant depending on α and p, $1 \leq p < \alpha \leq 2$,

$$C(p,\alpha) = \left[\frac{2^{p-1} \int_0^\infty s^{-p/\alpha-1}(1-e^{-s})ds}{\alpha \int_0^\infty v^{-p-1} \sin^2 v \, dv} \right]^{1/p} .$$

So the linear space $L(\xi)$ of a SαS process ξ is the completion of $\ell(\xi)$ with respect to either norm, and it can be seen from the form of the multivariate ch.f. that $L(\xi)$ is a family of jointly SαS random variables.

In the sequel when a Banach space of random variables with finite p-th moments is considered, its norm will be the usual $\|\cdot\|_p$ norm, and when a Banach space of SαS random variables is considered, it will be normed either by $\|\cdot\|_p$, $1 < p < \alpha$, or by $|\cdot|_\alpha$. Because $|\zeta|_\alpha$ appears in the expression for the ch.f. of a SαS random variable ζ, it will often be the more natural choice for a norm in the SαS case.

From the form of the ch.f. of two jointly SαS random variables η and ζ, it follows that

$$-\log E[\exp\{i(r_1\eta+r_2\zeta)\}] = \int_S |r_1x_1 + r_2x_2|^\alpha \, d\Gamma(x),$$

where Γ is the spectral measure on $S = \{x \in R^2 : x_1^2 + x_2^2 = 1\}$. For each such pair of variables, we define

$$[\eta,\zeta]_\alpha = \int_S x_1(x_2)^{\alpha-1} \, d\Gamma(x) .$$

Notice that $[\eta,\zeta]_\alpha$ is the derivative with respect to r of $\frac{1}{\alpha}\int|r\, x_1+x_2|^\alpha \, \Gamma(dx)$ evaluated at $r=0$.

If η and ζ are jointly SαS random variables, then [6, Theorem 1.4] shows that

$$E(\eta|\zeta) = \frac{[\eta,\zeta]_\alpha}{[\zeta,\zeta]_\alpha}\zeta \qquad \text{a.s.}$$

Linearity of the map $\eta \to [\eta,\zeta]_\alpha$ therefore follows from the linearity of conditional expectation, and Hölder's inequality shows the map to be continuous with norm $|\zeta|_\alpha^{\alpha-1}$. When $\alpha=2$, i.e., for jointly Gaussian random variables with zero mean, it follows that $[\eta,\zeta]_2$ equals $E(\eta\zeta)$, the covariance of η and ζ.

The covariation of η with ζ will be defined by $\langle\eta,\zeta\rangle_p$ in the p-th order case and by $[\eta,\zeta]_\alpha$ in the SαS case. The following property of the covariation is analogous to the Riesz representation for continuous linear functionals on a Hilbert space.

PROPOSITION 2.1: *Let* M *be a Banach space of* SαS *random variables (of random variables with finite* p-th *moments). If* A *is a continuous linear functional on* M, *then there exists a unique* $\zeta \in M$ *such that* $A = [\cdot,\zeta]_\alpha$ $(A = \langle\cdot,\zeta\rangle_p)$.

PROOF: A proof is sketched for the SαS case; the p-th order case is identical. Consider $N = \{\eta \in M: A(\eta) = 0\}$, a subspace of M. If $N = M$, take $\zeta = 0$. Otherwise, choose $\eta_1 \in M - N$ and let η_2 be the best approximation to η_1 in N (see Theorem 1.11 and Corollary 3.5 of [14]). Define $\eta_3 = (\eta_1-\eta_2)/|\eta_1-\eta_2|$, $\zeta = \{A(\eta_3)\}^{1/(\alpha-1)}\eta_3$, and $\zeta_0 = \zeta/|A(\eta_3)|^{\alpha/(\alpha-1)}$. For every $\eta \in M$ write

$$\eta = \eta - A(\eta)\zeta_0 + A(\eta)\zeta_0.$$

Note that $\eta - A(\eta)\zeta_0 \in N$ and consequently that

$$[\eta,\zeta]_\alpha = A(\eta)\ [\zeta_0,\zeta]_\alpha = A(\eta)\ .$$

To see uniqueness, suppose that ζ^* also satisfies the required conditions, and let Γ be the spectral measure for (ζ,ζ^*). Then

$$[\frac{\zeta}{|\zeta|_\alpha}, \zeta^*]_\alpha = [\frac{\zeta}{|\zeta|_\alpha}, \zeta]_\alpha = |\zeta|_\alpha^{\alpha-1} = |\zeta^*|_\alpha^{\alpha-1} \quad ,$$

so that

$$\int_S x_1(x_2)^{\alpha-1}\, d\Gamma(x) = [\zeta,\zeta^*]_\alpha = |\zeta|_\alpha\ |\zeta^*|_\alpha^{\alpha-1}$$

$$= [\int_S |x_1|^\alpha\, d\Gamma(x)]^{\alpha^{-1}}\ [\int_S |x_2|^\alpha\, d\Gamma(x)]^{(\alpha-1)/\alpha}$$

It follows from Hölder's inequality that $x_1 = cx_2$ a.e. $[\Gamma]$, for some $c > 0$ and thus

$$|\zeta - c\zeta^*|_\alpha^\alpha = \int_S |x_1 - x_2|^\alpha\, d\Gamma(x) = 0\ .$$

Hence $\zeta = c\zeta^*$ and by $|\zeta|_\alpha = |\zeta^*|_\alpha$, $c = 1$ and thus $\zeta = \zeta^*$. []

In the SαS case the covariation $[\eta,\zeta]_\alpha$ possesses a certain linearity property with respect to ζ when ζ is a linear combination of <u>independent</u> SαS random variables. Specifically, if $\zeta = a_1\zeta_1 + a_2\zeta_2$ where ζ_1 and ζ_2 are independent and if ζ_1, ζ_2, and η are jointly SαS, then

$$[\eta,\zeta]_\alpha = (a_1)^{\alpha-1}\ [\eta,\zeta_1]_\alpha + (a_2)^{\alpha-1}\ [\eta,\zeta_2]_\alpha. \tag{2.1}$$

This property is an immediate result of the definition of covariation and [10, Theorem 1.2.1] (see also [11, Theorem 2.1]) which states that, if Γ is the spectral measure for (η,ζ_1,ζ_2), then $\Gamma\{x \in R^3 : x_2x_3 \neq 0\} = 0$. It is likewise immediate under the same conditions that

$$[\zeta_2,\zeta_1]_\alpha = 0 \qquad (2.2)$$

while the converse is not true in general, i.e., if two jointly SαS r.v.'s have covariation zero they are not necessarily independent.

An interesting subclass of SαS processes is the family of sub-Gaussian processes [1, p. 251] which have an especially simple parametric description. Specifically, if $R: T \times T \to R^1$ is of nonnegative definite type, $1 < \alpha \leq 2$, and $t_1,\ldots,t_N \varepsilon T$, then

$$\phi(r_1,\ldots,r_N) = \exp\{-2^{-\alpha/2}(\sum_{m,n=1}^{N} r_m r_n(t_m,t_n))^{\alpha/2}\}$$

is the ch.f. of a multivariate stable distribution [12, p. 359]. The family of such ch.f.'s generated by varying N and $t_1,\ldots,t_N$ is clearly consistent, and a stochastic process $\{\xi_t, t \varepsilon T\}$ with finite-dimensional distributions so defined is called α-sub-Gaussian with parameter R or, more briefly, α-SG(R). The zero mean Gaussian case 2-SG(R) will be denoted simply G(R). Following are some properties of α-sub-Gaussian processes for later use.

PROPOSITION 2.2. *If* $\xi = \{\xi_t, t \varepsilon T\}$ *is* α-SG(R), *then* $L(\xi)$ *is an* α*-sub-Gaussian family of random variables, i.e. for all* N *and* $X_1,\ldots,X_N \varepsilon L(\xi)$ *the vector* $(X_1,\ldots,X_N)$ *is* α-SG.

PROOF. Given any n, any $t_1,\ldots,t_n$ in T, and real numbers $r_1,\ldots,r_n$, we have

$$| r_1\xi_{t_1}+\ldots+r_n\xi_{t_n}|_\alpha = 2^{-1/2}[\sum_{j,k=1}^{n} r_j r_k R(t_j,t_k)]^{1/2} = 2^{-1/2} \|r_1\eta_{t_1}+\ldots+r_n\eta_{t_n}\|_2,$$

where $\eta = \{\eta_t, t \varepsilon T\}$ is a G(R) process. From the continuity of norms it follows that $L(\xi)$ is α-SG(R). []

COROLLARY 2.3. In the notation of Proposition 2.2, let X_1 and X_2 be linear operations on ξ, say $X_i = \ell_i(\xi)$, and let $\eta = \{\eta_t, t \varepsilon T\}$ be G(R). If $Y_i = \ell_i(\eta)$, $i = 1,2$, then

$$[X_1,X_2]_\alpha = \frac{E(Y_1Y_2)}{2^{\alpha/2}[E(Y_2^2)]^{1-\alpha/2}}$$

and, by symmetry,

$$[X_2,X_1]_\alpha = \frac{E(Y_1Y_2)}{2^{\alpha/2}[E(Y_1^2)]^{1-\alpha/2}} \quad .$$

In particular, $[X_1,X_2]_\alpha = 0$ if and only if $[X_2,X_1]_\alpha = 0$. Moreover, if $[X_1,X_1]_\alpha = [X_2,X_2]_\alpha$, then $[X_1,X_2]_\alpha = [X_2,X_1]_\alpha$.

PROOF: The first statement follows by evaluating at $r=0$ the derivative with respect to r of

$$-\frac{1}{\alpha} \log E[\exp\{i(rX_1+X_2)\}] = \exp\{-2^{-\alpha/2}[r^2E(Y_1^2) + 2rE(Y_1Y_2) + E(Y_2^2)]^{\alpha/2}\} \ .$$

The rest is clear from

$$[X_1,X_1]_\alpha = |X_1|_\alpha^\alpha = 2^{-\alpha/2}\ [E(Y_1)^2]^{\alpha/2} \ . \qquad []$$

In fact, sub-Gaussian distributions are (variance) mixtures of Gaussian distributions. Specifically, if $\xi = \{\xi_t, t \varepsilon T\}$ is α-SG(R) and $\eta = \{\eta_t, t \varepsilon T\}$ is G(R), then ξ has the same distribution as $\zeta^{1/2}\eta = \{\zeta^{1/2}\eta_t, t \varepsilon T\}$ where the random variable ζ is independent of η and has Laplace transform

$W(\lambda) = \exp\{-\lambda^{\alpha/2}\}$ (i.e., is a positive stable random variable of index $\alpha/2$), as is seen from

$$E\exp\{\, i\sum_{n=1}^{N} r_n\, \zeta^{1/2}\eta_{t_n}\} = W[2^{-1}\sum_{m,n=1}^{N} r_m r_n\, R(t_m,t_n)]$$

$$= \exp\{-2^{-\alpha/2}[\sum_{m,n=1}^{N} r_n r_m\, R(t_m,t_n)]^{\alpha/2}\} \quad .$$

3. THE INTEGRAL $\int_T f(t)d\xi_t$.

A stochastic integral $\int_T f(t)\, d\xi_t$ is defined for appropriate (nonrandom) "functions" $f \varepsilon \Lambda_\alpha(\xi)$, and the function space $\Lambda_\alpha(\xi)$ is examined when ξ satisfies certain "smoothness" conditions (Proposition 3.1), when ξ is α-sub-Gaussian (Proposition 3.2), and when ξ is SαS with independent increments. In the latter case a separating family of continuous linear functionals on $L(\xi)$ is also obtained (Corollary 3.4).

Let $\xi = \{\xi_t, t \varepsilon T\}$ be a p-th order or SαS process, where T is taken for simplicity to be an interval. The stochastic integral is defined in a similar fashion for both the p-th order and the SαS cases, but we shall use the notation for the SαS case in our development.

The following assumptions are used to define the integral $\int_T f d\xi$; they are quite natural and they reduce to the standard assumptions used to define a Lebesgue-Stieltjes integral when ξ is nonrandom. Assume that ξ possesses weak right limits, i.e., assume by Proposition 2.1 that $[\xi_{t+0}, \zeta]_\alpha$ exists for every $\zeta \varepsilon L(\xi)$ and every $t \varepsilon T$, and denote the weak right limit of ξ at t by ξ_{t+0}. Assume in addition that $[\xi_t,\zeta]_\alpha$ is of bounded variation on T whenever ζ is of the form $\zeta = \Sigma_{k=1}^n a_k(\xi_{t_k+0} - \xi_{t_{k-1}+0})$, where $n \geq 1$, $a_k \varepsilon R$, and $t_0 < t_1 <\ldots< t_n$ all in T. This latter assumption will be used to define a norm on the function space in terms of Lebesgue-Stieltjes integral. The use of weak right limits ξ_{t+0} will be compatible with our definition of a measure from a function of bounded variation.

Let S be the linear space of all step functions of the form $f(t) = \Sigma_{k=1}^{n} a_k \chi_{(t_{k-1}, t_k]}(t)$, and for each such f define $\int_T f\, d\xi$ to be $\Sigma_{k=1}^{n} a_k(\xi_{t_k+0} - \xi_{t_{k-1}+0})$. We introduce a norm on S by

$$\|f\|_S^\alpha = \int_T f(t)\, d_t[\xi_t, \int f d\xi]_\alpha = \sum_{k=1}^{n} a_k \int_T \chi_{(t_{k-1},t_k]}(t)\, d_t[\xi_t, \int f d\xi]_\alpha$$

$$= \sum_{k=1}^{n} a_k[\xi_{t_k+0} - \xi_{t_{k-1}+0}, \int f d\xi]_\alpha = [\int f d\xi, \int f d\xi]_\alpha = |\int f d\xi|_\alpha^\alpha \ .$$

Let $\Lambda_\alpha(\xi)$ be the completion of S with respect to this norm. Every element $f \in \Lambda_\alpha(\xi)$ can be represented as $f = \{f_n\}$, a Cauchy sequence in S. It follows that $\{\int_T f_n\, d\xi\}$ is a Cauchy sequence in $L(\xi)$, and we shall denote its limit by $\int_T f\, d\xi$, which is easily seen to depend only on f and not on the specific Cauchy sequence $\{f_n\}$. Then the man $\Lambda_\alpha(\xi) \to L(\xi)$ defined by $f \to \int_T f\, d\xi$ is an isometry from $\Lambda_\alpha(\xi)$ onto a closed subspace of $L(\xi)$. If we assume that $\xi_{t'} = 0$ for some $t' \in T$ and that the process ξ is weakly continuous from the right, then this isometry will be onto $L(\xi)$.

If the process ξ is of weak bounded variation, i.e., if $[\xi_t, \zeta]_\alpha$ is of bounded variation on T for every $\zeta \in L(\xi)$, then it is clear that ξ has weak right limits and consequently that the integral $\int_T f\, d\xi$ is defined. Under this stronger (smoothness) condition on ξ, continuous functions can be regarded as members of $\Lambda_\alpha(\xi)$ in a natural way.

<u>PROPOSITION 3.1</u>: <u>If</u> $\xi = \{\xi_t,\ a \le t \le b\}$ <u>is a SαS (or p-th order) process of weak bounded variation, then all continuous functions on</u> $[a,b]$ <u>belong to</u> $\Lambda_\alpha(\xi)$.

PROOF: If f is a function on T and π is a partition of $[a,b]$ defined by $a = t_0 < t < \ldots < t_m = b$, let $f_\pi(t) = \Sigma_{k=1}^{m} f(t'_k)\chi_{(t_{k-1},t_k]}(t)$, where t'_k is an arbitrary point in $(t_{k-1},t_k]$, and $\Delta(\pi) = \max_{1<k<m} (t_k - t_{k-1})$. If f is a continuous function and $\{\pi_n\}$ is a sequence of partitions of $[a,b]$ with $\Delta(\pi_n) \to 0$, then $f_{\pi_n} \varepsilon S$ for every n and

$$\int_T f(t)\, d_t[\xi_t,\zeta]_\alpha = \lim_{n\to\infty} \int_T f_{\pi_n}(t)\, d_t\, [\xi_t,\zeta]_\alpha = \lim_{n\to\infty} [\int_T f_{\pi_n}\, d\xi,\, \zeta]_\alpha$$

for all $\zeta \varepsilon L(\xi)$. Hence the sequence $\{\int_T f_{\pi_n} d\xi\}_{n=1}^{\infty}$ converges weakly to some $\eta \varepsilon L(\xi)$. Define $\int_T f(t)d\xi_t = \eta$, and note that this definition does not depend on the particular partition of $[a,b]$ chosen. Thus we may regard the continuous function f as an element of $\Lambda_\alpha(\xi)$. []

If additional conditions are placed on ξ, it can be shown that other classes of functions belong to $\Lambda_\alpha(\xi)$. For instance, if ξ is also assumed to be weakly continuous (i.e., if $[\xi_t, \zeta]_\alpha$ is a continuous function on $[a,b]$ for all $\zeta \varepsilon L(\xi)$), then $\Lambda_\alpha(\xi)$ "contains" all functions of bounded variation on $[a,b]$.

For an α-SG(R) process ξ, it is clear from Corollary 2.3 that the function $[\xi_t, \zeta]_\alpha$, $\zeta \varepsilon L(\xi)$, has right limits or is of bounded variation if and only if the same properties hold for $E(\eta_t\zeta')$, where η is a G(R) process and $\zeta' \varepsilon L(\eta)$ corresponds to $\zeta \varepsilon L(\xi)$. Thus the integral $\int_T f(t)d\xi_t$ is defined if and only if all functions in the reproducing kernel Hilbert space of R (RKHS(R)) have right limits and all functions of the form $R(\cdot, u+0) - R(\cdot, v+0)$, $u,v \varepsilon T$, have bounded variation.

PROPOSITION 3.2: *Let* $\xi = \{\xi_t, t \varepsilon T\}$ *be* α-SG(R), *where* R *is such that the stochastic integral* $\int_T f(t)d\xi_t$ *is defined for* $f \varepsilon \Lambda_\alpha(\xi)$. *Then*

$\Lambda_\alpha(\xi) = \Lambda_2(\eta)$, where $\eta = \{\eta_t,\ t \varepsilon T\}$ is a $G(R)$ process, and for every $f \varepsilon \Lambda_\alpha(\xi)$,

$$||f||_{\Lambda_\alpha(\xi)} = 2^{1/2}\ ||f||_{\Lambda_2(\eta)} .$$

PROOF: The result is immediate from Corollary 2.3, since the step functions f are dense in both function spaces and

$$||f||^\alpha_{\Lambda_\alpha(\xi)} = [\int fd\xi, \int fd\xi]_\alpha = 2^{-\alpha/2}\ [E(\int fd\eta)^2]^{-\alpha/2}||f||^\alpha_{\Lambda_2(\eta)} . \qquad []$$

If the α-SG(R) process ξ is weakly continuous from the right (or equivalently, if all functions in RKHS(R) are right continuous), then Proposition 3.2 shows that $\Lambda_\alpha(\xi)$ coincides with the space $\Lambda_2(R)$ defined in [5]. When R is the covariance of a (second order) process with orthogonal increments, then the conditions under which the stochastic integral and the space $\Lambda_\alpha(\xi)$ have been defined are satisfied and $\Lambda_\alpha(\xi) = L_2(dF)$ where F is a nondecreasing function such that for all $t \leq s$, $F(s)-F(t) = R(s,s)-2R(t,s)+R(t,t)$.

For the remainder of this section we will consider ξ to be a SαS process such that, given points $t_1<t_2<\ldots<t_n$ in T, the random variables $\xi_{t_1}, \xi_{t_2}-\xi_{t_1}, \ldots, \xi_{t_n}-\xi_{t_{n-1}}$ are independent. Under this assumption of independent increments, it can be shown that when T is a finite interval the conditions for defining $\int_T f(t)\, d\xi_t$ are satisfied [10, pp. 58-59], and we obtain the same stochastic integral as in [13, p. 146]. To make clear this latter point, note that the process $\{\xi_{t+0},\ t \varepsilon T\}$ also has independent increments, and let $F(t) = |\xi_{t+0}|^\alpha_\alpha$. By [13, p. 414] the nondecreasing function F satisfies

$$F(t_2) - F(t_1) = |\xi_{t_2+0} - \xi_{t_1+0}|_\alpha^\alpha$$

for $t_1 < t_2$. Let f and g be two step functions on T:

$$f(t) = \sum_{j=1}^{n} f_j \chi_{(t_{j-1},t_j]}(t),\quad g(t) = \sum_{j=1}^{n} g_j \chi_{(t_{j-1},t_j]}(t)\ .$$

Write X_j for $\xi_{t_j+0} - \xi_{t_{j-1}+0}$, $1 \leq j \leq n$, and recall properties (2.1) and (2.2). Then

$$[\int f d\xi, \int g d\xi]_\alpha = \sum_{j=1}^{n} f_j\, [X_j, \int g d\xi]_\alpha = \sum_{j=1}^{n} f_j (g_j)^{\alpha-1}\, [X_j, X_j]_\alpha$$

$$= \int \sum_{j=1}^{n} f_j (g_j)^{\alpha-1} \chi_{(t_{j-1},t_j]}\, dF = \int f(g)^{\alpha-1}\, dF\ .$$

In particular, $||f||^\alpha_{\Lambda_\alpha(\xi)} = \int |f|^\alpha\, dF$, and therefore $\Lambda_\alpha(\xi) = L_\alpha(dF)$ since the step functions are dense in both spaces. This definition of the integral $\int_T f d\xi$ easily extends to the case where T is an infinite interval. The norm on the set S of all step functions that are zero outside a compact subset of T is defined as before, and the completion $\Lambda_\alpha(\xi)$ of S with respect to this norm is $L_\alpha(T, B_T, dF)$ where for $T = (-\infty,\infty)$, say, F is defined by $F(t) = \operatorname{sgn}(t)\, ||\xi_{t+0}||_\alpha^\alpha$.

<u>PROPOSITION 3.3</u>: <u>Suppose that</u> ξ <u>is a SαS process with independent increments.</u> <u>and let</u> $\eta = \int f\, d\xi$ <u>and</u> $\zeta = \int g\, d\xi$, <u>where</u> f <u>and</u> g <u>belong to</u> $L_\alpha(dF)$. <u>Then</u>

$$[\eta,\zeta]_\alpha = \int_T f(g)^{\alpha-1}\, dF\ .$$

<u>PROOF</u>: If Γ is the spectral measure for (η,ζ), then

$$\int |rx_1+x_2|^\alpha \, d\Gamma(x) = |r\eta+\zeta|_\alpha^\alpha = \|rf+g\|^\alpha_{L_\alpha(dF)} = \int |rf+g|^\alpha \, dF .$$

Thus $[\eta,\zeta]_\alpha$ is the derivative with respect to r of $\frac{1}{\alpha}\int |rf+g|^\alpha \, dF$ evaluated at $r = 0$. (Note that the function t^α is differentiable when $\alpha > 1$.) []

COROLLARY 3.4: Let $T = (-\infty,\infty)$, $\xi_0 = 0$, and M be the closed subspace of $L(\xi)$ which is the image of $L_\alpha(dF)$ under the isometry $f \to \int_T f(t)d\xi_t$. Then the set of continuous linear functionals $\{[\cdot,\xi_{t+0}]_\alpha, \ t \in T\}$ separates points on M.

PROOF: Given $\eta \in M$, choose $g \in L_\alpha(dF)$ such that $\eta = \int_T g(s)\, d\xi_s$. If $[\eta,\zeta_{t+0}]_\alpha = 0$ for all $t \in T$, then by Proposition 3.3,

$$\int_{(-t,0]} g(s)dF(s) = 0 = \int_{(a,t]} g(s)\, dF(s)$$

for all $t \in T$, which implies that $g = 0$ a.e. [dF] and hence $\eta = 0$ in M. []

If the process ξ is weakly continuous from the right, then $L_\alpha(dF)$ is isometric to $L(\xi)$ and Corollary 3.4 implies that the set of continuous linear functionals $\{[\cdot,\xi_t]_\alpha, \ t \in T\}$ separates points on $L(\xi)$.

4. THE INTEGRAL $\int_T f(t)\xi_t \, d\nu(t)$

In spite of its general form, the stochastic integral of the previous section is defined under rather mild assumptions which are usually satisfied in real-world applications. The integral discussed in the present section has a more particular form, but is defined under even less stringent conditions and can often be interpreted as a sample path integral.

We shall consider ξ to be a p-th order process and q to be such that $p^{-1}+q^{-1} = 1$. A stochastic process $\{\xi_t, t \in T\}$ on a probability space (Ω, F, P) is called measurable if $(t,\omega) \to \xi_t(\omega)$ is a product measurable map from $T \times \Omega$ into R. The following result can be established by an argument analogous to the discussion in [2, pp. 280-281].

LEMMA 4.1. Let $\xi = \{\xi_t, t \in T\}$ be a measurable p-th order process with index set T an arbitrary interval of the real line, and let $\alpha > p$ be given. Then there exists a finite measure ν on $(T,\mathcal{B}_T)$ such that ν is equivalent to Lebesgue measure on T and

$$\int_T ||\xi_t||_p^\alpha \, d\nu(t) < \infty .$$

Under the conditions of Lemma 4.1,

$$E \int_T |\xi_t(\omega)|^p \, d\nu(t) = \int_T ||\xi_t||_p^p \, d\nu(t) < \infty ,$$

since $p < \alpha$ and ν is a finite measure; so the sample paths $\xi_t(\omega)$ belong to $L_p(T,\mathcal{B}_T,\nu)$ with probability one, by the measurability of ξ and Fubini's theorem. We can therefore define a stochastic process $\eta = \{\eta_t, t \in T\}$ by

$$\eta_t(\omega) = \int_{(-\infty,t)\cap T} \xi_s(\omega) d\nu(s) \quad \text{for each } t \in T, \text{ a.s.}$$

and observe that $\eta_t \in L_p(\Omega)$, since

$$E|\eta_t|^p \leq \{\nu[(-\infty,t)\cap T]\}^{p/q} \, E \int_T |\xi_s(\omega)|^p \, d\nu(s) < \infty .$$

PROPOSITION 4.2: The stochastic process η is of strong bounded variation.

PROOF: For every partition $t_0<t_1<\ldots<t_n$ of T,

$$\sum_{k=1}^{n} \|\eta_{t_k} - \eta_{t_{k-1}}\|_p \le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\xi_s\|_p |dv(s) \le \int_T \|\xi_s\|_p \, dv(s) < \infty \quad ,$$

since if $a<b$ and $\zeta(\omega) = \int_{t_1}^{t_2} \xi_s(\omega)\, dv(s)$ we have

$$\|\zeta\|_p^p = \langle \int_{t_1}^{t_2} \xi_s(\omega)\, dv(s), \zeta \rangle_p = \int_{t_1}^{t_2} \langle \xi_s, \zeta \rangle_p \, dv(s) \le \|\zeta\|_p^{p-1} \int_{t_1}^{t_2} \|\xi_s\|_p \, dv(s),$$

and thus

$$\|\int_{t_1}^{t_2} \xi_s(\omega)\, dv(s)\|_p \le \int_{t_1}^{t_2} \|\xi_s\|_p \, dv(s) \; . \qquad []$$

From Proposition 4.2 it easily follows that for every $\zeta \varepsilon L(\xi)$, $\langle \eta_t, \zeta \rangle_p$ is of bounded variation on T. Therefore the stochastic integral $\int f(t) d\eta_t$ is defined for $f \varepsilon \Lambda_p(\eta)$. We further define $\lambda_p(\xi) \equiv \Lambda_p(\eta)$ and for every $f \varepsilon \lambda_p(\xi)$

$$\int_T f(t)\xi_t \, dv(t) = \int_T f(t) d\eta_t \; .$$

We shall see that the stochastic integral $\int f(t)\xi_t \, dv(t)$ can be expressed as a sample path integral for all step functions $f \varepsilon \mathcal{S}$ (Lemma 4.3) and all $f \varepsilon L_q(v)$ (Theorem 4.4) and that the sample path integrals of the form $\int f(t)\xi_t(\omega)\, dv(t)$ belong to $L(\eta)$ for all $f \varepsilon L_q(T, \mathcal{B}_T, v)$ (Theorem 4.4). In addition, these sample path integrals are dense in $L(\xi)$ when ξ is a weakly continuous process (Theorem 4.5).

LEMMA 4.3. For every $f \varepsilon \mathcal{S}$ the sample path integral $\int f(t)\xi_t(\omega)\, dv(t)$ equals $\int f(t)\xi_t \, dv(t)$ with probability one.

PROOF: It is clear that the sample path integral exists since f is a bounded function and $\xi_t(\omega) \varepsilon L_p(v)$ a.s. Any given $\zeta \varepsilon L_p(\Omega)$ determines the continuous linear functional $\langle\cdot,\zeta\rangle_p$ on $L_p(\Omega)$, which can be restricted to $L(\eta)$. Therefore Proposition 2.1 yields a unique $\zeta_1 \in L(\eta)$ such that $\langle\cdot,\zeta_1\rangle_p = \langle\cdot,\zeta\rangle_p$ on $L(\eta)$. Note that for $t_1 \le t_2$,

$$\int_{t_1}^{t_2} d_t\langle\eta_t,\zeta_1\rangle_p = \langle\eta_{t_2}-\eta_{t_1},\zeta_1\rangle_p = \langle\eta_{t_2}-\eta_{t_1},\zeta\rangle_p$$

$$= \left\langle\int_{t_1}^{t_2} \xi_t(\omega)\, dv(t),\ \zeta\right\rangle_p = \int_{t_1}^{t_2} \langle\xi_t,\zeta\rangle_p\, dv(t) \quad .$$

Thus

$$\left\langle\int_T f(t)\,\xi_t\, dv(t) - \int_T f(t)\,\xi_t(\omega)\, d\times(t),\ \zeta\right\rangle_p$$

$$= \left\langle\int_T f(t)\,\xi_t\, dv(t),\ \zeta\right\rangle_p - E[(\zeta)^{p-1} \int_T f(t)\,\xi_t(\omega)\, dv(t)]$$

$$= \left\langle\int_T f(t)\,\xi_t\, dv(t),\ \zeta_1\right\rangle_p - \int_T f(t)\, E[(\zeta)^{p-1}\, \xi_t]\, dv(t)$$

$$= \int_T f(t)\, d_t\, \langle\eta_t,\xi_1\rangle_p - \int_T f(t)\, \langle\xi_t,\zeta\rangle_p\, dv(t) = 0 \qquad []$$

We have seen that $\xi_t(\omega) \in L_p(T,\mathcal{B}_T,v)$ with probability one, so that for all $f \in L_q(T,\mathcal{B}_T,v)$ the sample path integral $\int_T f(t)\xi_t(\omega)\, dv(t)$ is defined a.s. and is easily seen to belong to $L_p(\Omega)$. In fact, it belongs to $L(\eta)$, and the function space $\lambda_p(\xi)$ contains $L_q(v)$ in a sense which we now make precise.

THEOREM 4.4: Every function $f \in L_q(v)$ determines (uniquely) an element $\tilde{f} \in \lambda_p(\xi)$ such that

$$\int_T f(t)\xi_t(\omega)\, dv(t) = \int_T \tilde{f}(t)\xi_t\, dv(t) \text{ a.s.,}$$

where the left-hand side is a sample path integral and the right-hand side a stochastic integral.

PROOF: Given any $g \in L_q(v)$, let $\zeta = \int_T g(t)\xi_t(\omega)\, dv(t)$, and observe that

$$\|\zeta\|_p^p = \langle \int_T g(t)\, \xi_t(\omega)\, dv(t), \zeta \rangle_p = \int_T g(t) \langle \xi_t, \zeta \rangle_p \, dv(t)$$

$$\leq \|g\|_{L_q(v)} \{ |\int_T \langle \xi_t, \zeta \rangle_p|^p \, dv(t)\}^{1/p}$$

$$\leq \|g\|_{L_q(v)} \|\zeta\|_p^{p-1} \{ \int_T \|\xi_t\|_p^p \, dv(t)\}^{1/p} .$$

Thus for all $g \in L_q(v)$ we have

$$\|\int_T g(t)\xi_t(\omega)\, dv(t)\|_p \leq \|g\|_{L_q(v)} \{ \int_T \|\xi_t\|_p^p \, dv(t)\}^{1/p} .$$

Given any $f \in L_q(v)$ let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in S converging to f in $L_q(v)$. For each n the sample path integral $\int_T f_n(t)\xi_t(\omega)\, dv(t)$ belongs to $L(\eta)$ by Lemma 4.3, and

$$\|\int_T [f(t)-f_n(t)]\xi_t(\omega)\, dv(t)\|_p \leq \|f-f_n\|_{L_q(v)} \{ \int_T \|\xi_t\|_p^p \, dv(t)\}^{1/p} \to 0 .$$

as $n \to \infty$, so that the sample path integral $\int_T f(t)\xi_t(\omega)\, dv(t)$ belongs to $L(\eta)$. Note that $\{f_n\}$ is a Cauchy sequence in , since

$$\|f_m - f_n\|_S = \|\int_T [f_m(t)-f_n(t)]\xi_t(\omega)\, dv(t)\|_p \to 0$$

as $m,n \to \infty$, and denote its limit in $\lambda_p(\xi)$ by $\tilde{f}$. Then

$$\|\int_T f(t)\xi_t(\omega)\,dv(t) - \int_T \tilde{f}(t)\xi_t\,dv(t)\|_p = \lim_{m,n\to\infty} \|\int_T [f_m(t)-f_n(t)]\xi_t(\omega)\,dv(t)\|_p = 0,$$

so that $\int_T f(t)\xi_t(\omega)\,dv(t) = \int_T \tilde{f}(t)\xi_t\,dv(t)$ a.s. []

Identifying f with $\tilde{f}$ we can consider $L_q(v)$ as a subset of $\lambda_p(\xi)$. It is straightforward to check that the process η is continuous in p-th mean and consequently that $\lambda_p(\xi)$ is isometric to all of $L(\eta)$. Since $S \subset L_q(v)$, it is clear that $L_q(v)$ is dense in $\lambda_p(\xi)$ and hence that $L_q(v)$ is isometric to a dense subset of $L(\eta)$. The following result shows that $L_q(v)$ is isometric to a dense subset of $L(\xi)$ when ξ is weakly continuous.

THEOREM 4.5: Suppose that ξ is weakly continuous from the right and that $T = (-\infty,\infty)$. Then the closure of $\{\int_T f(t)\,\xi_t(\omega)\,dv(t),\ f \in S\}$ in $L_p(\Omega)$ is $L(\xi)$.

PROOF: Fix $t \in T$, and for every integer $n \geq 1$ define

$$g_n(s) = v^{-1}\{(t, t+n^{-1})\}\,\chi_{(t,t+n^{-1})}(s) ,$$

so that $\int_T g_n\,dv = 1$. Given any $\varepsilon > 0$ and any $\zeta \in L_p(\Omega)$, use weak right continuity to choose an N such that $n \geq N$ implies that $|\langle\xi_t-\xi_s,\zeta\rangle_p| < \varepsilon$ for $t < s < t + 1/n$. Then for $n \geq N$,

$$|\langle\xi_t - \int_T g_n(s)\xi_t(\omega)\,dv(t),\zeta\rangle_p| = |\int_T g_n(s)\,\langle\xi_t-\xi_s,\zeta\rangle_p\,dv(s)|$$

$$\leq \int_T g_n(s)\,|\langle\xi_t-\xi_s,\zeta\rangle_p|\,dv(s) \leq \varepsilon \int_T g_n(s)\,dv(s) = \varepsilon .$$

Hence $\int_T g_n(s)\xi_t(\omega)\,dv(s)$ converges weakly to ξ_t and therefore ξ_t belongs to the closure of $\{\int_T f(t)\xi_t(\omega)\,dv(t): f \in S\}$. []

Thus, when ξ is weakly continuous, every element of $L(\xi)$ can be expressed as a limit in L_p (and hence also a.s.) of sample path integrals.

Specifically, if $\zeta \in L(\xi)$, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset L_q(\nu)$ such that

$$\zeta(\omega) = \lim_{n\to\infty} \int_T f_n(t)\xi_t(\omega)\, d\nu(t) \text{ a.s.}$$

It is frequently desirable (especially in applications, such as those considered in Section 6) to write the stochastic integral $\int g(t)\xi_t\, d\nu(t)$ in the form $\int f(t)\, \xi_t\, dt$. Since n and Lebesgue measure are equivalent, f and g are related by $f(x) = h(x)g(x)$ where $h(x) = d\nu(t)/dt$. A condition such as $g \in L_q(\nu)$ is therefore equivalent to $\int|f(x)|^q[h(x)]^{1-q}\, dt < \infty$ which we will write as $f \in L_q(h^{1-q})$ in the following (sometimes with no further reference to the definition of h through ν and Lemma 4.1).

The following theorem establishes a Fubini-type result which allows the interchange of stochastic and usual integration, and which is used in Section 6.

THEOREM 4.6: Let $\zeta = \{\zeta_t, -\infty<t<\infty\}$ be a weakly continuous SαS process with independent increments, $\zeta_0 = 0$, and $F(t) = \text{sgn}(t)\, |\zeta_t|_\alpha^\alpha$. Fix p and q such that $1 < p < \alpha$ and $p^{-1}+q^{-1} = 1$, and define $\xi_s = \int_{-\infty}^{\infty} a(s,t)\, d\zeta_t$ where $a \in L_\alpha\, (d\nu \times dF)$ and ν corresponds to ξ as in Lemma 4.1. If $f \in L_q\, (h^{1-q})$, $h(t) = d\nu(t)/dt$, then

$$\int_{-\infty}^{\infty} f(s)\, \xi_s\, ds = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(s)\, a(s,t)\, ds\}\, d\zeta_t \quad .$$

PROOF: The right hand integral is well defined since $\int_{-\infty}^{\infty} f(s)\, a(s,t)\, ds$ can be shown to belong to $L_\alpha(dF(t))$. For any bounded Borel set B, write $\zeta(B) = \int_B d\zeta_t$ and observe that

$$[\int_{-\infty}^{\infty} f(s)\, \xi_s\, ds,\ \zeta(B)]_\alpha = \int_{-\infty}^{\infty} f(s)[\xi_s, \zeta(B)]_\alpha\, ds$$

$$= \int_{-\infty}^{\infty} f(s) \int_B a(s,t)\, dF(t)\, ds = \int_B \int_{-\infty}^{\infty} f(s)\, a(s,t)\, ds\, dF(t)$$

$$= [\int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(s)\, a(s,t)\, ds\}\, d\zeta_t,\ \zeta(B)]_\alpha\ .$$

The conclusion now follows from Corollary 3.4. []

If the function $a(s,t)$ in Theorem 4.6 is defined to be 1 when $0<t\leq s$ and 0 otherwise, then the conclusion may be written

$$\int_0^{\infty} f(s)\, \zeta_s\, ds = \int_0^{\infty} \{ \int_t^{\infty} f(s)\, ds\}\, d\zeta_t\ .$$

If $a(s,t)$ is defined to be 1 when $0<t\leq s$, -1 when $s<t\leq 0$, and 0 otherwise, then $\xi_s = \zeta_s$ and the conclusion of Theorem 4.6 may be written

$$\int_{-\infty}^{\infty} f(s)\, \zeta_s\, ds = \int_{-\infty}^{0} \{-\int_{-\infty}^{t} f(s)\, ds\}\, d\zeta_s + \int_0^{\infty} \{ \int_t^{\infty} f(s)\, ds\}\, d\zeta_t$$

and when $\int_{-\infty}^{\infty} f(s)\, ds = 0$,

$$\int_{-\infty}^{\infty} f(s)\, \zeta_s\, ds = \int_{-\infty}^{\infty} \{ \int_t^{\infty} f(s)\, ds\}\, d\zeta_t\ .$$

All these are integration by parts formulae, and the general integration by parts formula over a finite interval is given in the following theorem.

THEOREM 4.7: *Let* $\xi = \{\xi_t, -\infty<t<\infty\}$ *be a weakly continuous* $S\alpha S$ *process with independent increments. If* $-\infty<a<b<\infty$ *and all integrals are well defined,*

$$\int_a^b f(t)\,\xi_t\,dt = \xi_a \int_a^b f(s)\,ds + \int_a^b \{ \int_t^b f(s)\,ds\}\,d\xi_t$$

$$= \xi_b \int_a^b f(s)\,ds - \int_a^b \{ \int_a^t f(s)\,ds\}\,d\xi_t$$

PROOF: For all $a \leq s \leq b$, using usual integration by parts, we have

$$[\xi_a \int_a^b f + \int_a^b (\int_t^b f)\,d\xi_t,\ \xi_s]_\alpha = [\xi_a,\xi_s]_\alpha \int_a^b f + \int_a^b (\int_t^b f)\,d_t\,[\xi_t,\xi_s]_\alpha$$

$$= \int_a^b f(t)\,[\xi_t,\xi_s]_\alpha\,dt = [\int_a^b f(t)\,\xi_t\,dt,\ \xi_s]_\alpha \ .$$

The first expression then follows by Corollary 3.4, and the second is established similarly. []

5. LINEAR ESTIMATION AND REGRESSION

In this section we consider the evaluation of regression estimates and of linear estimates in SαS processes.

Let $\{\eta, \xi_t, t \in T\}$ be SαS, $1<\alpha<2$, and $L(\xi)$ be the linear space of the SαS process $\xi = \{\xi_t, t \in T\}$. It is well known that the regression estimate of η based on ξ, $E(\eta|\xi)$, is not in general linear, i.e., it does not necessarily belong to $L(\xi)$ (in sharp contrast with the Gaussian case $\alpha=2$). (see for instance [11]) When T consists of one point, or when T is a finite set and the random variables ξ_t, $t \in T$, are independent, then $E(\eta|\xi)$ is linear [6]. Further cases where $E(\eta|\xi)$ is linear are shown below (Theorem 5.1 and Corollaries 5.2 to 5.5).

The linear estimate of η based on ξ is defined as the best approximation to η in $L(\xi)$, i.e., as the random variable $\hat{\eta}$ in $L(\xi)$ with minimum distance from ξ

$$|\eta-\hat{\eta}|_{\alpha} = \inf_{\zeta\varepsilon L(\xi)} \eta-\zeta\ {}_{\alpha}$$

and is denoted by $\ell(\eta|\xi)$. Thus $\hat{\eta}$ is uniquely determined by either of the following

$$\begin{aligned} &[\zeta,\ \eta-\hat{\eta}]_{\alpha} = 0 \qquad \text{for all } \zeta\varepsilon L(\xi)\ , \\ &[\xi_t,\ \eta-\hat{\eta}]_{\alpha} = 0 \qquad \text{for all } t\ \varepsilon\ T\ . \end{aligned} \tag{5.1}$$

When $\alpha=2$ the regression estimate is linear and equal to the linear estimate. When $1<\alpha<2$, even when the regression estimate is linear it may not be equal to the linear estimate. In this section we give examples of regression estimates that are linear and coincide with the linear estimates (Theorem 5.6 and Corollary 5.7) and examples of regression estimates that are linear but differ from the linear estimates (Theorem 5.8 and Corollary 5.9).

We first evaluate regression estimates in certain cases where they are linear. Nonlinear regression estimates seem very hard to evaluate (see [11]).

THEOREM 5.1. *Let* $\{\eta,\xi_t,\ t\varepsilon T\}$, T *a possibly infinite interval, be* $S\alpha S$ *such that the process* $\xi=\{\xi_t,\ t\varepsilon T\}$ *has independent increments, is weakly continuous from the right*, $\xi_\tau = 0$ *for some* $\tau\varepsilon T$, *and the function* $F(t) = \operatorname{sgn}(t-\tau)\,|\xi_t-\xi_\tau|_\alpha^\alpha$, $t\varepsilon T$, *is bounded*. *For every Borel set* B *of* T *define*

$$X(B) = \int_T \chi_B(t)\, d\xi_t\ , \quad \mu(B) = [\eta,\ X(B)]_\alpha\ .$$

Then μ is a finite signed measure which is absolutely continuous with respect to the measure induced by F, the Radon-Nikodym derivative $d\mu/dF$ belongs to $L_\alpha(dF)$, and

$$E(\eta|\xi) = \int_T (\frac{d\mu}{dF})(t)\, d\xi_t \quad \text{a.s.}$$

PROOF: To see that μ is countably additive, let $B = \bigcup_{i=1}^{\infty} B_i$ where the B_i's are disjoint measurable subsets of $[a,b]$, and observe that with $C_n = \cup_{i=n}^{\infty} B_i$

$$|[\eta, X(C_n)]_\alpha \leq |\eta|_\alpha\, |X(C_n)|_\alpha^{\alpha-1} = |\eta|_\alpha\, (\int_{C_n} dF)^{\alpha-1/\alpha} \underset{n\to\infty}{\to} 0,$$

by Hölder's inequality and dominated convergence. For every $n > 1$, $X(B_1),\ldots,X(B_n)$ are independent random variables by [13, p. 418], and property (2.1) yields

$$\mu(B) = \sum_{i=1}^{n-1} \mu(B_i) + [\eta, X(C_n)]_\alpha \quad ,$$

which as n increases to infinity becomes $\mu(B) = \sum_{i=1}^{\infty} \mu(B_i)$. For absolute continuity, $\int_T \chi_B(t)\, dF(t) = 0$ implies that $X(B) = 0$ a.s., whence $\mu(B) = 0$.

By [3, p. 604] we can choose a countable dense subset T_∞ of T such that

$$E(\eta|\, \xi_t,\ t\varepsilon T) = E(\eta|\, \xi_t,\ t\varepsilon T_\infty) \quad .$$

Enumerate the points in T_∞, and let $T_n = \{t_o,t_1,\ldots,t_n\colon t_o<t_1<t_2<\ldots<t_n\}$ be the set containing the first n points of T_∞ and $\tau(\xi_\tau = 0)$. Then

$$E(\eta | \xi_t,\ t\varepsilon T_n) = E(\eta\ \xi_{t_1} - \xi_{t_o},\ \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}})$$

$$= \sum_{k=1}^{n} \frac{\mu((t_{k-1},t_k])}{F(t_k) - F(t_{k-1})} (\xi_{t_k} - \xi_{t_{k-1}}) \qquad (5.2)$$

$$= \int \sum_{k=1}^{n} \frac{\mu((t_{k-1},t_k])}{F(t_k) - F(t_{k-1})}\ \chi_{(t_{k-1},t_k]}(t) d\xi_t ,$$

by Corollary 3.4 of [11]. Now $E(\eta | \xi_t,\ t\varepsilon T_n)$ converges to $E(\eta | \xi_t,\ t\varepsilon T_\infty)$ in $L(\xi)$ by [3, p. 319], and therefore the "integrands" in the final step of (5.1) form a Cauchy sequence in $L_\alpha(dF)$ which converges to $d\mu/dF$ in $L_\alpha(dF)$ and a.e. [dF] by standard martingale convergence theorems [4, p. 369]. Thus taking limits in (5.2) completes the proof. []

A more suggestive notation for $(d\mu/dF)(t)$ is $d_t[\eta,\ \xi_t]_\alpha/dF(t)$ so that

$$E(\eta\ \xi_t,\ t\varepsilon T) = \int_T \frac{d_t[\eta,\ \xi_t]_\alpha}{dF(t)}\ d\xi_t\ .$$

The following elementary properties may be considered as corollaries of Theorem 5.1 or appropriate modifications of it, but also follow immediately from elementary properties of conditional expectation and are stated here only for the purpose of comparison with the linear estimates.

<u>COROLLARY 5.2</u>. <u>If</u> ξ <u>is as in Theorem 5.1,</u> $a<b<c$, $a<\tau\leq b$, <u>and</u> $f\varepsilon L_\alpha([a,c],dF)$ <u>then</u>

$$E(\int_a^c f(t)d\xi_t | \xi_s,\ a<s\leq b) = \int_a^b f(t)\ d\xi_t\ .$$

COROLLARY 5.3. Let

$$X_t = \int_{-\infty}^{t} f(t,s)\, d\xi_s , \qquad -\infty<t<\infty ,$$

where ξ is $S\alpha S$, weakly continuous from the right and has independent increments, $F(s) - F(s') = \|\xi_s - \xi_{s'}\|_\alpha^\alpha$, for all $s>s'$, and for all t $f(t,\cdot) \in L_\alpha((-\infty,t], dF)$ and $L\{X, (-\infty,t]\} = L\{\Delta\xi, (-\infty,t]\}$ where $\Delta\xi$ denotes the increments of ξ. Then for all $\tau>0$ and t,

$$E(X_{t+\tau} \mid X_u, u\leq t) = \int_{-\infty}^{t} f(t+\tau,s)\, d\xi_s$$

$$\| X_{t+\tau} - E(X_{t+\tau} \mid X_u, u\leq s)\|_\alpha^\alpha = \int_t^{t+\tau} |f(t+\tau,s)|^\alpha\, dF(s) .$$

When ξ is an α-stable motion, i.e., when $dF(t) = dt$, (Brownian motion when $\alpha=2$), then $X_t = \int_{-\infty}^{t} f(t-s)d\xi_s$ is a stationary $S\alpha S$ process and under the conditions of Corollary 5.3

$$\| X_{t+\tau} - E(X_{t+\tau} \mid X_u, u\leq t)\|_\alpha^\alpha = \int_0^\tau |f(us)|^\alpha\, du .$$

It should be pointed out that for $1<\alpha<2$ it is not known whether all purely nondeterministic (i.e., $\cap_t L\{X, (-\infty,t]\}=\{0\}$) stationary $S\alpha S$ processes are moving averages of a stable motion (this is, of course, well known for $\alpha=2$).

In following corollary we evaluate the regression estimate of a functional of a signal based on signal plus noise, when the signal and noise are $s\alpha S$ processes of special form.

COROLLARY 5.4. Let

$$s_t = \int_{-\infty}^{\infty} f(t,\lambda)\, dS_\lambda, \quad n_t = \int_{-\infty}^{\infty} f(t,\lambda)\, dN_\lambda, \quad -\infty<t<\infty ,$$

<u>where</u> S,N <u>are independent, weakly right continuous,</u> $S\alpha S$ <u>processes with independent increments</u>, $S_0=0=N_0$, <u>and</u> $\{f(t,\cdot), -\infty<t<\infty\}$ <u>is complete in</u> $L_\alpha[d(F_s+F_n)]$ <u>where the functions</u> $F_s(\lambda) = \text{sgn}(\lambda)\,|S_\lambda|_\alpha^\alpha$, $F_n(\lambda) = \text{sgn}(\lambda)\,|N_\lambda|_\alpha^\alpha$, $-\infty<\lambda<\infty$, <u>are bounded. If</u> $g\varepsilon L_\alpha(dF_s)$ <u>then the regression estimate of</u> $\int_{-\infty}^{\infty} g(\lambda)dS_\lambda$ <u>based on</u> s_t+n_t, $-\infty<t<\infty$, <u>and the regression error are given by</u>

$$E(\int_{-\infty}^{\infty} g(\lambda)dS_\lambda \mid s_t+n_t, -\infty<t<\infty) = \int_{-\infty}^{\infty} g(\lambda)\, \phi_s(\lambda)\, d(S+N)_\lambda ,$$

$$e_r^\alpha = |\int_{-\infty}^{\infty} g(\lambda)dS_\lambda - E(\int_{-\infty}^{\infty} g(\lambda)dS_\lambda \mid s_t+n_t, -\infty<t<\infty)|_\alpha^\alpha$$

$$= \int_{-\infty}^{\infty} |g|^\alpha\, (\phi_s^{\alpha-1} + \phi_n^{\alpha-1})\, \phi_s\phi_n\, d(F_s+F_n)$$

<u>where</u> $\phi_s = dF_s/d(F_s+F_n)$ <u>and</u> $\phi_n = dF_n/d(F_s+F_n) = 1-\phi_s$.

<u>PROOF</u>: Put $\eta = \int_{-\infty}^{\infty} g\, dS$, $\xi_t = s_t+n_t$, $\zeta_\lambda = S_\lambda+N_\lambda$. Then $\xi_t = \int_{-\infty}^{\infty} f(t,\lambda)d\zeta_\lambda$ and $F(\lambda) = \text{sgn}(\lambda)\,|\zeta_\lambda|_\alpha^\alpha$ can be evaluated by using the independence of S and N, properties (2.1) and (2.2), and the linearity of $[\cdot,\cdot]_\alpha$ in its first argument, as follows

$$|\zeta_\lambda|_\alpha^\alpha = [\zeta_\lambda,\zeta_\lambda]_\alpha = [\zeta_\lambda, S_\lambda+N_\lambda]_\alpha = [\zeta_\lambda, S_\lambda]_\alpha + [\zeta_\lambda, N_\lambda]_\alpha$$

$$= [S_\lambda,S_\lambda]_\alpha + [N_\lambda,S_\lambda]_\alpha + [S_\lambda,N_\lambda]_\alpha + [N_\lambda,N_\lambda]_\alpha$$

$$= |S_\lambda|_\alpha^\alpha + |N_\lambda|_\alpha^\alpha = \text{sgn}(\lambda)\,\{F_s(\lambda) + F_n(\lambda)\} .$$

Thus $F(\lambda) = F_s(\lambda)+F_n(\lambda)$. Since $\{f(t,\cdot), -\infty<t<\infty\}$ is complete in $L_\alpha(dF)$, $\{\xi_t,-\infty<t<\infty\}$ is complete in $L(\zeta)$. Hence the (completed) σ-fields generated by $\{\xi_t, -\infty<t<\infty\}$ and $\{\zeta_\lambda, -\infty<\lambda<\infty\}$ are equal and $E(\eta|\xi_t,-\infty<t<\infty) = E(\eta|\zeta_\lambda,-\infty<\lambda<\infty)$ a.s. It then follows by Theorem 5.1 that

$$E(\eta|\xi_t,-\infty<t<\infty) = \int_{-\infty}^{\infty} \frac{d_\lambda[\eta,\zeta_\lambda]_\alpha}{dF(\lambda)} d\zeta_\lambda .$$

Using again the independence of S and N, and properties (2.1) and (2.2) we have

$$[\eta,\zeta_\lambda]_\alpha = [\eta, S_\lambda+N_\lambda]_\alpha = [\eta, S_\lambda]_\alpha + [\eta, N_\lambda]_\alpha = [\eta, S_\lambda]_\alpha .$$

If we put $\phi(\lambda,u) = 1$ when $0<u\leq\lambda$, $= -1$ when $\lambda<u\leq 0$, and $=0$ otherwise, then $S_\lambda = \int_{-\infty}^{\infty} \phi(\lambda,u)\, dS_u$ and by Proposition 3.3,

$$[\eta, \zeta_\lambda]_\alpha = [\int_{-\infty}^{\infty} g(u)dS_u, \int_{-\infty}^{\infty} \phi(\lambda,u)S_u]_\alpha = \int_{-\infty}^{\infty} g(u)(\phi(\lambda,u))^{\alpha-1}\, dF_s(u)$$

$$= \int_{(0,\lambda]} gdF, \quad \lambda>0; =0, \ \lambda=0; \quad = -\int_{(\lambda,0]} gdF_s, \ \lambda<0 .$$

The expression for the regression estimate follows from

$$d_\lambda[\eta,\zeta_\lambda]_\alpha/dF(\lambda) = g(\lambda)dF_s(\lambda)/d(F_s+F_n)(\lambda) = g(\lambda)\ \phi_s(\lambda) .$$

For the regression error we have, similarly,

$$e_r^\alpha = |\int_{-\infty}^{\infty} gdS - \int_{-\infty}^{\infty} g\ \phi_s\ d(S+N)|_\alpha^\alpha = |\int_{-\infty}^{\infty} g(1-\phi_s)dS - \int_{-\infty}^{\infty} g\phi_s\ dN|_\alpha^\alpha$$

$$= \int_{-\infty}^{\infty} |g|^\alpha\ \phi_n^\alpha\ dF_s + \int_{-\infty}^{\infty} |g|^\alpha\ \phi_s^\alpha\ dF_n$$

$$= \int_{-\infty}^{\infty} |g|^\alpha\ (\phi_s^{\alpha-1} + \phi_n^{\alpha-1})\ \phi_s\phi_n\ d(F_s + F_n) .$$

[]

When the spectra of signal and noise do not overlap, i.e., when the spectral measures dF_s and dF_n are singular, then $\phi_s\phi_n = 0$ a.e. $[d(F_s+F_n)]$ and the error of the regression estimate is zero. When F_s and F_n are absolutely continuous with spectral densities f_s and f_n then

$\phi_s = f_s(f_s+f_n)^{-1}$, $\phi_n = f_n(f_s+f_n)^{-1}$, and the regression error can be expressed as

$$e_r^\alpha = \int_{-\infty}^{\infty} |g(\lambda)|^\alpha \frac{f_s^{\alpha-1}(\lambda) + f_n^{\alpha-1}(\lambda)}{[f_s(\lambda) + f_n(\lambda)]^\alpha} f_s(\lambda)\, f_n(\lambda)\, d\lambda .$$

By putting $g(\lambda) = f(t,\lambda)$ in Corollary 5.4 we obtain expressions for the regression estimate of s_t based on signal plus noise, and of the resulting error. In the special case where both signal and noise are SαS processes on the positive real line with independent increments these expressions simplify to

$$E(S_t | S_u+N_u,\ 0\le u\le t') = \int_0^{\min(t,t')} \phi_s(u)\, d(S+N)_u$$

$$\| S_t-E(S_t| S_u+N_u,\ 0\le u\le t')\|_\alpha^\alpha = \int_{(0,\min(t,t')]} \{\phi_s^{\alpha-1}+\phi_n^{\alpha-1}\}\, \phi_s\phi_n\, d(F_s+F_n) \quad .$$

Another special case of interest is when the signal and the noise are harmonizable SαS processes with representations analogous to those of stationary Gaussian processes:

$$\eta_t = \int_0^\infty \cos(t\lambda)\, dH'_\lambda + \int_0^\infty \sin(t\lambda)\, dH''_\lambda \ , \quad -\infty<t<\infty, \qquad (5.3)$$

where H',H'' are independent, weakly right continuous, SαS processes with independent increments, $H'_0 = 0 = H''_0$ and $\|H'_\lambda\|_\alpha^\alpha = F_\eta(\lambda) = \|H''_\lambda\|_\alpha^\alpha$, $\lambda\ge 0$, with F_η bounded. Unlike the Gaussian (α=2) case however, such stable processes with $1<\alpha<2$ are not stationary.

<u>COROLLARY 5.5</u>. <u>Let both the signal</u> s <u>and the noise</u> n <u>have spectral representation of the type (5.3) and assume they are independent</u>. <u>Let</u> $x_t = s_t+n_t$, $X'_\lambda = S'_\lambda+N'_\lambda$, $X''_\lambda = S''_\lambda+N''_\lambda$. <u>If</u> $f,g \in L_\alpha(dF_s)$ <u>then</u>

$$E\left(\int_0^\infty f(\lambda)dS'_\lambda + \int_0^\infty g(\lambda)dS''_\lambda \mid x_t, -\infty<t<\infty\right) = \int_0^\infty f(\lambda)\ \phi_s(\lambda)dX'_\lambda + \int_0^\infty g(\lambda)\ \phi_s(\lambda)dX''_\lambda$$

<u>and the error of the regression estimate is given by</u>

$$e_r^\alpha = \int_0^\infty (|f|^\alpha + |g|^\alpha)(\phi_s^{\alpha-1} + \phi_n^{\alpha-1})\ \phi_s\phi_n d(F_s + F_n)$$

<u>where</u> $\phi_s = dF_s/d(F_s+F_n)$ <u>and</u> $\phi_n = dF_n/d(F_s+F_n) = 1-\phi_s$.

<u>PROOF</u>: Inversion formulae of (5.3) expressing H'_λ, H''_λ, $\lambda\geq 0$ in terms of η_t, $-\infty<t<\infty$, are identical to those valid when $\alpha=2$, the only difference being that the convergence is now with respect to the $|\cdot|_\alpha$ norm rather than in quadratic mean. These inversion formulae imply that $L(x) = L(X',X'')$ and, since S',S'' are independent and N',N'' are independent, that X' and X'' are also independent. A straightforward extension of Theorem 5.1 then gives

$$E(\eta\ x) = \int_0^\infty \frac{d_\lambda\ [\eta,\ X'_\lambda]_\alpha}{dF_x(\lambda)}\ dX'_\lambda + \int_0^\infty \frac{d_\lambda[\eta,\ X''_\lambda]_\alpha}{dF_x(\lambda)}\ dX''_\lambda$$

from which the results follow by putting $\eta = \int_0^\infty fdS' + \int_0^\infty gdS'$ and noting that $F_x = F_s+F_n$. []

We now turn our attention to the evaluation of linear estimates in certain cases. It should be pointed out that linear estimates are harder to evaluate than regression estimates. For instance, the regression estimate of η based on a simple random variable ξ is given by $E(\eta|\xi) = a\xi$ where $a = [\eta,\xi]_\alpha/[\xi,\xi]_\alpha$, while the linear estimate of η based on ξ is of the form $\ell(\eta|\xi) = b\xi$ where b cannot be found in general; if the representation $\eta = \int_0^1 fd\zeta$, $\xi = \int_0^1 gd\zeta$, is used where ζ is a SαS process with independent increments and $f,g \in L_\alpha(dF)$, $F(t) = |\zeta_t-\zeta_0|_\alpha^\alpha$, [13] then b satisfies

$$0 = [\xi,\eta-b\xi]_\alpha = \int_0^1 g(f-bg)^{\alpha-1}\ dF$$

which in general, when $1<\alpha<2$, cannot be solved for b (when $\alpha=2$ the solution is of course straightforward). It may therefore seem somewhat surprising that in certain specific cases the linear estimate of η based on $\xi = \{\xi_t, t\varepsilon T\}$ with T an interval, can be evaluated; this is feasible only because in these cases η is "appropriately" related with ξ.

While $\ell(\eta|\xi)$ cannot be evaluated under the general assumptions of Theorem 5.1, it can be evaluated under the more special assumptions of Corollaries 5.2 to 5.5.

<u>THEOREM 5.6</u>. <u>Under the assumptions of Corollary 5.2</u>,

$$\ell(\int_a^c f(t)\, d\xi_t \mid \xi_s,\ a<s\leq b) = \int_a^b f(t)\, d\xi_t$$

<u>PROOF</u>. Let $\eta = \int_a^c f d\xi$. Since ξ is weakly right continuous and $\xi_\tau = 0$ for some $a<\tau\leq b$, the map $L_\alpha((a,b],dF) \to L(\xi,(a,b])$ defined by $g \to \int_a^b g d\xi$ is onto. Thus $\hat{\eta} = \ell(\eta|\xi_s,\ a<s\leq b)\ \varepsilon\ L(\xi,\ (a,b])$ is of the form $\hat{\eta} = \int_a^b g d\xi$ for some $g\ \varepsilon\ L_\alpha((a,b],\ dF)$. Similarly every $\zeta\ \varepsilon\ L(\xi,\ [a,b])$ is of the form $\zeta = \int_a^b h d\xi$, $h\ \varepsilon\ L_\alpha((a,b],\ dF)$, and thus condition (5.1) is equivalent to

$$0 = [\zeta, \eta-\hat{\eta}]_\alpha = [\int_a^c h\chi_{(a,b]} d\xi,\ \int_a^c (f-g\chi_{(a,b]}) d\xi]_\alpha$$

$$= \int_a^b h(f-g)^{\alpha-1}\, dF$$

for all $h\ \varepsilon\ L_\alpha((a,b],\ dF)$, which implies $f = g$ a.e. $[dF]$ on $(a,b]$. []

<u>COROLLARY 5.7</u>. <u>Under the assumptions of Corollary 5.3</u>

$$\ell(X_{t+\tau} \mid X_u,\ u\leq t) = \int_{-\infty}^t f(t+\tau,s) d\xi_s \quad .$$

<u>PROOF</u>: We have $Y = X_{t+\tau} - \int_{-\infty}^t f(t+\tau,s)\, d\xi_s = \int_t^{t+\tau} f(t+\tau,s)\ \ d\xi_s\ \varepsilon\ L(\Delta\xi,(t,t+\tau])$.

Since ξ has independent increments, every ζ in $L(\Delta\xi, (-\infty,t])$ is independent of Y and by (2.2), $[\zeta,Y]_\alpha = 0$ so that (5.1) is satisfied. The result then follows from $\int_{-\infty}^{t} f(t+\tau,s)d\xi_s \in L(\Delta\xi, (-\infty,t])$. []

Thus under the assumptions of Corollaries 5.2 and 5.3, the regression estimates are linear and equal to the linear estimates. We now show that under the assumptions of Corollaries 5.4 and 5.5 the regression estimates, which are linear, differ from the linear estimates.

In the remaining examples $\xi=\{\xi_t = s_t + n_t, t\varepsilon T\}$ where the signal s and the noise n are indepedent SαS processes, and $\eta \varepsilon L(s)$. The linear estimate of $\hat{\eta}=\ell(\eta|\xi)$ is the limit (with respect to the norm $|\cdot|_\alpha$) of a sequence of finite linear combinations of random variables from $\{\xi_t, t\varepsilon T\}$, and we denote this linear map by A: $\hat{\eta} = A(\xi)$. Since s and n are independent, their corresponding sequences of finite linear combinations converge (with respect to $|\cdot|_\alpha$), to $A(s) = \hat{\eta}_s$ and $A(n) = \hat{\eta}_n$ respectively (i.e., the same linear operation of s and n) so that $\hat{\eta} = A(\xi) = A(s) + A(n) = \hat{\eta}_s + \hat{\eta}_n$. The characterizing equation (5.1) for $\hat{\eta}$ is then equivalent to

$$
\begin{aligned}
0 &= [\xi_t, \eta-\hat{\eta}]_\alpha = [\xi_t, (\eta-\hat{\eta}_s) - \hat{\eta}_n]_\alpha = [\xi_t, \eta-\hat{\eta}_s]_\alpha - [\xi_t, \hat{\eta}_n]_\alpha \\
&= [s_t, \eta-\hat{\eta}_s]_\alpha - [n_t, \hat{\eta}_n]_\alpha \quad \text{for all } t \varepsilon T. \qquad (5.4)
\end{aligned}
$$

A similar calculation gives the linear estimation error

$$e_\ell^\alpha = |\eta-\hat{\eta}|_\alpha^\alpha = |(\eta-\hat{\eta}_s) - \hat{\eta}_n|_\alpha^\alpha = |\eta-\hat{\eta}_s|_\alpha^\alpha + |\hat{\eta}_n|_\alpha^\alpha \qquad (5.5)$$

<u>THEOREM 5.8</u>. <u>Under the assumptions of Corollary 5.4</u>,

$$\ell(\int_{-\infty}^{\infty} g dS \mid s_t+n_t,\ -\infty<t<\infty) = \int_{-\infty}^{\infty} g \frac{\phi_s^{1/(\alpha-1)}}{\phi_s^{1/(\alpha-1)} + \phi_n^{1/(\alpha-1)}} d(S+N) ,$$

$$e_\ell^\alpha = \int_{-\infty}^{\infty} |g|^\alpha \frac{\phi_s \phi_n}{(\phi_s^{1/(\alpha-1)} + \phi_n^{1/(\alpha-1)})^{\alpha-1}} d(F_s+F_n) .$$

PROOF: Put $\eta = \int g dS$. Since $\hat{\eta} = \ell(\eta\ \xi) \varepsilon L(\xi) = L(S+N)$ we have $\hat{\eta} = \int h\, d(S+N)$ for some $h \varepsilon L_\alpha[d(F_s+F_n)]$ so that $\hat{\eta}_s = \int h dS$ and $\hat{\eta}_n = \int h\, dN$. By (5.4) we have for all $-\infty<t<\infty$,

$$0 = [\int_{-\infty}^{\infty} f(t,\lambda)\, dS_\lambda, \int_{-\infty}^{\infty}(g-h)dS]_\alpha - [\int_{-\infty}^{\infty} f(t,\lambda)\, dN_\lambda, \int_{-\infty}^{\infty} h\, dN]_\alpha$$

$$= \int_{-\infty}^{\infty} f(t,\lambda)\, (g(\lambda)\ h(\lambda))^{\alpha-1}\, dF_s(\lambda) - \int_{-\infty}^{\infty} f(t,\lambda)\, (h(\lambda))^{\alpha-1}\, dF_n(\lambda)$$

$$= \int_{-\infty}^{\infty} f(t,\lambda)\, \{(g(\lambda)-h(\lambda))^{\alpha-1}\, \phi_s(\lambda) - (h(\lambda))^{\alpha-1}\, \phi_n(\lambda)\}\, d(F_s+F_n)(\lambda) .$$

Since $\{f(t,\cdot),\ -\infty<t<\infty\}$ is complete in $L_\alpha[d(F_s+F_n)]$, this is equivalent to

$$(g-h)^{\alpha-1}\, \phi_s - (h)^{\alpha-1}\, \phi_n = 0 \quad \text{a.e.} \quad d(F_s+F_n)$$

and thus to

$$h = g[1 + (\phi_n/\phi_s)^{1/(\alpha-1)}]^{-1} \quad \text{a.e.} \quad d(F_s+F_n) .$$

For the linear estimation error we obtain from (5.5),

$$e_\ell^\alpha = |\int(g-h)dS|_\alpha^\alpha + |\int h\, dN|_\alpha^\alpha = \int |g-h|^\alpha\, dF_s + \int |h|^\alpha\, dF_n$$

$$= \int(|g-h|^\alpha\, \phi_s + |h|^\alpha\, \phi_n)\, d(F_s+F_n)$$

and the final expression follows by substituting h. []

By putting $g(\lambda) = f(t,\lambda)$ in Theorem 5.8 we obtain expressions for the linear estimate of s_t based on signal plus noise, and of the resulting error. In the special case where both signal and noise are SαS processes on the positive real line with independent increments, these expressions simplify to (under the assumptions of Corollary 5.4)

$$\ell(S_t | S_u+N_u, 0\leq u\leq t') = \int_0^{\min(t,t')} \frac{\phi_s^{1/(\alpha-1)}}{\phi_s^{1/(\alpha-1)} + \phi_s^{1/(\alpha-1)}} d(S+N)$$

$$|S_t-E(S_t | S_u+N_u, 0\leq u\leq t')|_\alpha^\alpha = \int_{(0,\min(t,t')]} \frac{\phi_s\phi_n}{(\phi_s^{1/\alpha-1} + \phi_n^{1/\alpha-1})^{\alpha-1}} d(F_s+F_n) .$$

COROLLARY 5.9. Under the assumptions of Corollary 5.5,

$$\ell(\int_0^\infty f dS' + \int_0^\infty g dS'' | x_t, -\infty<t<\infty) = \int_0^\infty \frac{\phi_s^{1/(\alpha-1)}}{(\phi_s^{1/(\alpha-1)} + \phi_n^{1/(\alpha-1)})^{\alpha-1}} (f dX' + g dX'')$$

$$e^\alpha = \int_0^\infty (f^\alpha+g^\alpha) \frac{\phi_s\phi_n}{(\phi_s^{1/(\alpha-1)} + \phi_n^{1/(\alpha-1)})^{\alpha-1}} d(F_s+F_n)$$

PROOF: Putting $\eta = \int_0^\infty f dS' + \int_0^\infty g dS''$ and $\hat{\eta} = \ell(\eta\ x) = \int_0^\infty \hat{f}\, dX' + \int_0^\infty \hat{g}\, dX''$ we have $\hat{\eta}_s = \int_0^\infty \hat{f}\, dS' + \int_0^\infty \hat{g}\, dS'$, $\hat{\eta}_n = \int_0^\infty \hat{f}\, dN' + \int_0^\infty \hat{g}\, dN''$ and by (5.4) for all $-\infty<t<\infty$,

$$0 = [s_t, \eta-\hat{\eta}]_\alpha - [n_t, \hat{\eta}_n]_\alpha$$

$$= \int_0^\infty \cos(t\lambda)(f-\hat{f})^{\alpha-1} dF_s + \int_0^\infty \sin(t\lambda)(g-\hat{g})^{\alpha-1} dF_s$$

$$-\int_0^\infty \cos(t\lambda)\ (\hat{f})^{\alpha-1} dF_n - \int_0^\infty \sin(t\lambda)\ (\hat{g})^{\alpha-1} dF_n$$

$$= \int_0^\infty \cos(t\lambda)\ \{(f-\hat{f})^{\alpha-1} \phi_s - (\hat{f})^{\alpha-1} \phi_n\}\ d(F_s+F_n)$$

$$+ \int_0^\infty \sin(t\lambda)\ \{(g-\hat{g})^{\alpha-1} \phi_s - (\hat{g})^{\alpha-1} \phi_n\}\ d(F_s+F_n) .$$

It follows that a.e. $d(F_s{}_-F_n)$

$$(f-\hat{f})^{\alpha-1} \phi_s = (\hat{f})^{\alpha-1} \phi_n, \quad (g-\hat{g})^{\alpha-1}\phi_s = (\hat{g})^{\alpha-1} \phi_s$$

and the expressions are derived as in Theorem 5.8. []

Corollary 5.9 solves the nonrealizable (if we think of t as time) linear filtering problem for harmonizable SαS signal and noise of the type

(5.3). The solution of the realizable linear filtering problem will be considered elsewhere.

6. LINEAR SYSTEM ANALYSIS AND IDENTIFICATION

We consider a linear system with input the SαS process ξ, output X, and input-output relationship described by one of the following:

(I) $$X_s = \int f(s,t)\, d\xi_t$$

(II) $$X_s = \int f(s,t)\, \xi_t\, dt$$

where f, ξ and the index sets are such that the indicated integrals are well defined (specific conditions will be stated in each case to be considered). The impulse response of the system is f and we will frequently focus attention on time invariant systems: $f(s,t) = f(t-s)$. We will be concerned here with analyzing and identifying linear systems of type I or II.

The system identification problem is the determination of the impulse response f of the system from the joint distribution of the input and output. It turns out that in many cases of interest, knowledge of the joint distribution of the input and the output determines uniquely the impulse response. We then concentrate on the more significant (from an application viewpoint) problem of actually expressing the impulse response function f explicitly in terms of the input covariation function and the input-output cross covariation function. The advantage of this approach is that the system is identified by using not the full joint distribution of the input and output SαS processes, but only a portion of it which can be estimated in special cases.

The <u>covariation function</u> $C_{\xi\xi}(s,t)$ of ξ is defined as the covariation of ξ_s with ξ_t and the <u>cross covariation function</u> $C_{X\xi}(s,t)$ of X with ξ is defined as the covariation of X_s with ξ_t. For systems I and II respectively we have

(I') $$C_{X\xi}(s,t) = \int f(s,u)\, d_u\, C_{\xi\xi}(u,t)$$

(II') $$C_{X\xi}(s,t) = \int f(s,u)\, C_{\xi\xi}(u,t)\, du\ .$$

For a discussion of the estimation of $C_{X\xi}$ and $C_{\xi\xi}$ in special situations, see [8]. Of course when $\alpha = 2$, covariations and cross covariations become the usual covariances and cross covariances, and hence the approach taken here is the analogue for stable processes to the approach taken for Gaussian or second order porcesses. (The present set up, including (I') and (II'), is applicable to p-th order processes as well, but we are concentrating on SαS processes because they arise naturally in applications.) This problem is considered in detail for the following classes of SαS inputs ξ: stable processes with independent increments (Propositions 6.1 and 6.2), sub-Gaussian processes (Case 6.3), and moving averages (Case 6.3 and Case 6.4) and Fourier transforms (Case 6.5) of processes with independent increments.

The system analysis problem is the study of the statistical properties of the output when the statistics of the input and the system are known. In our case the output X is SαS and we study, for certain specific SαS inputs, the dependence of the distribution of the output on the linear system. Specifically, we consider time invariant linear systems, and for two such systems with impulse response f and g we find necessary and sufficient conditions on f,g so that with a specific SαS input ξ, the outputs of the two systems have the same distribution. Kanter [7] has solved this problem for system (I) and a stable motion input, and we consider here both systems (I) and (II) and inputs that are stable motions (Proposition 6.6), moving averages of stable motions (Case 6.7), and Fourier transforms of SαS processes with independent increments (Case 6.8). This can also be considered as a kind of system identification problem: for certain specific SαS inputs, what part of the impulse response can be determined from the distribution of the output?

Whenever type (II) systems are considered we shall assume without further notice that $f(s,\cdot) \leftarrow L_q(h^{1-q})$ (see discussion preceding Theorem 4.6).

System Identification

The question here is to investigate what can be determined about the system function f from knowledge of $C_{X\xi}$ and $C_{\xi\xi}$ by suing (I') or (II') as appropriate. We first consider SαS inputs ξ with independent increments. The main results are in Propositions 6.1 and 6.2.

PROPOSITION 6.1. *If* $\xi = \{\xi_t, -\infty<t<\infty\}$ *is a weakly continuous* SαS *process with independent increments* $\xi_0 = 0$, *and* $F(t) = \text{sgn}(t)\ \|\xi_t\|_\alpha^\alpha$, *then each impulse response function* $f(s,\cdot) \varepsilon L_\alpha(dF)$ *for system (I) is determined in* $L_\alpha(dF)$ *by the cross covariation function* $C_{X\xi}(s,t)$, *all real* t. *Explicitly,*

$$f(s,t) = \left[\frac{dC_{X\xi}(s,\cdot)}{dF}\right](t) = \lim_{n\to\infty} \frac{C_{X\xi}(s,t^{(n)}_{k(n,t)+1}) - C_{X\xi}(s,t^{(n)}_{k(n,t)})}{F(t^{(n)}_{k(n,t)+1}) - F(t^{(n)}_{k(n,t)})} \quad \text{a.e. } [dF]$$

where $\{(t_k^{(n)}, t_{k+1}^{(n)}]\}_{k=-\infty}^{\infty}$ *is a partition of* $(-\infty,\infty)$ *which becomes finer as* n *increases,* $\sup_k (t_{k+1}^{(n)} - t_k^{(n)}) \to 0$ *as* $n \to \infty$, *and* $k(n,t)$ *is the unique* k *such that* $t \,\varepsilon\, (t_k^{(n)}, t_{k+1}^{(n)}]$.

PROOF: The first part is immediate from Corollary 3.4. Since $C_{X\xi}(s,t) = \int_{-\infty}^t f(s,u)dF(u)$ by Proposition 3.3, we have $f(s,t) = d_tC_{X\xi}(s,t)/dF(t)$ a.e. [dF] and the second expression follows by an exercise similar to (20.61)(b) of [4]. []

If $dF(t) = dt$ in Proposition 6.1, then ξ is called α-stable motion (Brownian mortion when $\alpha = 2$). For such an input ξ and a time invariant system of type (I),

$$C_{X\xi}(s,t) = \int_{-\infty}^{t} f(u-s)du = \int_{-\infty}^{t-s} f(v)dv,$$

so that $C_{X\xi}(s,t) = C_{X\xi}(t-s)$ and $f(\tau) = C'_{X\xi}(\tau)$ a.e.

PROPOSITION 6.2: *If* $\xi = \{\xi_t, -\infty<t<\infty\}$ *is a weakly continuous SαS process with independent increments,* $\xi_0 = 0$ *and* $F(t) = \text{sgn}(t)\,\|\xi_t\|_\alpha^\alpha$, *then for each fixed* s *the cross covariation function* $C_{X\xi}(s)$ *determines* $f(s,\cdot)$ *for system* (II) *by*

$$f(s,t) = -\frac{d}{dt}\left[\frac{dC_{X\xi}(s,\cdot)}{dF}\right](t) \quad \text{a.e. [Leb]}.$$

PROOF: If

$$\phi(t,u) = \begin{cases} 1 & \text{for } 0<u\leq t \\ -1 & \text{for } t\leq u\leq 0 \\ 0 & \text{otherwise} \end{cases}$$

then $\xi_t = \int_{-\infty}^{\infty} \phi(t,u)\, d\xi_t$ for all t and by Theorem 4.6

$$X_s = \int_{-\infty}^{\infty} f(s,t)\, \xi_t\, dt = \int_{-\infty}^{\infty} a(s,u)\, d\xi_u$$

where

$$a(s,u) = \int_{-\infty}^{\infty} f(s,t)\, \phi(t,u)dt = \begin{cases} \int_u^{\infty} f(s,v)\, dv & \text{for } u\geq 0 \\ -\int_{-\infty}^{u} f(s,v)\, dv & \text{for } u<0. \end{cases}$$

Hence

$$C_{X\xi}(s,t) = [X_s,\xi_t]_\alpha = \int_{-\infty}^{\infty} a(s,u)(\phi(t,u))^{\alpha-1}\, dF(u)$$

$$= \int_0^t a(s,u)\, dF(u) \quad \text{for } t\geq 0 \quad \text{and} \quad = -\int_t^0 a(s,u)\, dF(u) \quad \text{for } t<0,$$

from which the results follows. []

When ξ is α-stable motion and f is time invariant in Proposition 6.2, then we have for any fixed s,

$$f(t) = \frac{\partial^2}{\partial t^2} C_{X\xi}(s,t) \quad \text{a.e. [Leb]}.$$

We now consider sub-Gaussian inputs and inputs that are moving averages of stable motions. The cases we shall examine are special cases of the following example.

Case 6.3: Suppose that ξ is a SαS process with covariation function of the form

$$C_{\xi\xi}(s,t) = \int_{-\infty}^{\infty} e^{i(t-s)\lambda} \phi(\lambda)\, d\lambda$$

where $\phi \in L_1(R^1, \text{Leb})$ and $\phi \neq 0$ a.e. [Leb] on R^1. Assuming integrability of $f(s,\cdot)$ and $\lambda\phi(\lambda)$, we obtain for system (I)

$$C_{X\xi}(s,t) = \int_{-\infty}^{\infty} f(s,u)\left(\int_{-\infty}^{\infty} (-i\lambda)\, e^{i(t-u)\lambda} \phi(\lambda)\, d\lambda\right) du$$

$$= i \int_{-\infty}^{\infty} e^{-it\lambda} \lambda\phi(-\lambda)\, \hat{f}(s,\lambda)\, d\lambda \, .$$

Thus knowledge of ϕ and $C_{X\xi}(s,t)$, all t, determines $\hat{f}(s,\lambda)$ and hence $f(s,t)$ a.e. t. If moreover $C_{X\xi}(s,\cdot) \in L_1$ we have

$$\hat{C}_{X\xi}(s,\lambda) = 2\pi i\, \lambda\, \phi(-\lambda)\, \hat{f}(s,\lambda)$$

from which $f(s,t)$ can be expressed in terms of $\hat{C}_{X\xi}(s,\lambda)$ and ϕ via inverse Fourier transform provided $\hat{f}(s,\lambda)$ is integrable in λ. A similar calculation for system (II) yields

$$C_{X\xi}(s,t) = \int_{-\infty}^{\infty} e^{-it\lambda} \phi(-\lambda)\, \hat{f}(s,\lambda)\, d\lambda$$

when $f(s,\cdot)$ is integrable, and thus similar conclusions, including

$$\hat{C}_{X\xi}(s,\lambda) = 2\pi\, \phi(-\lambda)\, \hat{f}(s,\lambda)$$

when $C_{X\xi}(s,\cdot)$ is integrable. For time invariant systems $f(s,t) = f(t-s)$ we have $C_{X\xi}(s,t) = C_{X\xi}(t-s)$ for both systems (I) and (II) and the resulting expressions are

$$\text{(I): } \hat{C}_{X\xi}(\lambda) = i\,\lambda\phi(-\lambda)\,\hat{f}(\lambda), \qquad \text{(II): } \hat{C}_{X\xi}(\lambda) = 2\pi\,\phi(-\lambda)\,\hat{f}(\lambda)\,.$$

If ξ is α-SG(R) with R a stationary covariance function, then by Corollary 2.3

$$C_{\xi\xi}(s,t) = \frac{R(t-s)}{2^{\alpha/2}\,R(0)^{1-\alpha/2}}$$

and this $C_{\xi\xi}$ is a stationary covariance function as well. Assuming that the spectral distribution of R is absolutely continuous, $C_{\xi\xi}$ is as in Example 6.3 with $\phi \geq 0$ a.e. and the results of Example 6.3 apply to obtain expressions for the system f.

If ξ is a moving average of an α-stable motion ζ:

$$\xi_t = \int_{-\infty}^{\infty} a(t-u)\,d\,\zeta_u\,, \quad a \in L_\alpha \quad ,$$

then

$$C_{\xi\xi}(s,t) = \int_{-\infty}^{\infty} a(s-t+v)\,(a(v))^{\alpha-1}\,dv$$

and if in addition $(a)^{\alpha-1}$, $\hat{a} \in L_1$, then $C_{\xi\xi}$ is of the form of Example 6.3 with $\phi(\lambda) = (2\pi)^{-1}\,\hat{\alpha}(-\lambda)\,(\hat{\alpha}^{\alpha-1})(\lambda)$, and the results of Example 6.3 apply.

One can also handle certain cases where ξ is a moving average of a SαS process with orthogonal increments, which is not a stable motion.

<u>CASE 6.4</u>. Let $\zeta = \{\zeta_t,\ -\infty<t<\infty\}$ be a SαS process with independent increments and $dF \ll dLeb$ with $dF/dLeb = \psi \in L_1$, and let $\xi_t = \int_{-\infty}^{\infty} a(t-u)\,d\zeta_u$, $a(t-\cdot) \in L_\alpha(dF)$, $-\infty<t<\infty$. Then

$$C_{\xi\xi}(s,t) = \int_{-\infty}^{\infty} a(s-u)(a(t-u))^{\alpha-1}\,\psi(u)\,du\,.$$

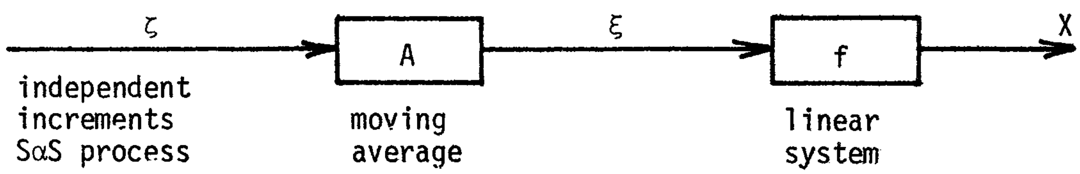
If the system (II) is time invariant with $f \in L_1$ and if $a \in L_{\alpha-1}$ then a calculation similar to Case 6.3 gives

$$\hat{C}_{X\xi}(-\lambda,\mu) = \hat{a}(-\lambda)(a^{\alpha-1})^{\wedge}(\mu)\ \hat{\psi}(\mu-\lambda)\ \hat{f}(\lambda)$$

from which f is uniquely determined provided $\hat{a}$, $(a^{\alpha-1})^{\wedge}$, $\hat{\psi} \neq 0$ a.e. If the system (I) is time invariant with $f \in L_1$, and if a is absoluately continuous, and for each t, $C_{\xi\xi}(s,t)$ is absolutely continuous is s with derivative $\int_{-\infty}^{\infty} a'(s-u)(a(t-u))^{\alpha-1}\ \psi(u)\ du$, and if $a' \in L_1$, $\alpha \in L_{\alpha-1}$, then

$$\hat{C}_{X\xi}(-\lambda,\mu) = (\hat{\alpha}')(-\lambda)(\alpha^{\alpha-1})^{\wedge}(\mu)\ \hat{\psi}(\mu-\lambda)\ \hat{f}(\lambda)\ .$$

We can diagram this case as follows:



If ζ were available along with X, one could use $C_{X\zeta}$ to determine $A^\circ f$ and then attempt to untangle f from it. Equivalently, if $A: \zeta \to \xi$ is invertible and if one can find A^{-1}, one would apply A^{-1} to the input ξ to generate ζ and then use $C_{X\zeta}$, to find $A \circ f$.

We finally consider the inputs that are Fourier transforms of SαS processes with independent increments and time invariant systems (I) and (II).

<u>CASE 6.5</u>. We assume that $\xi_t = \int_{-\infty}^{\infty} \cos(t\lambda)\ d\zeta_\lambda$ where ζ has independent SαS increments with $F(\lambda) = \text{sgn}\ (\lambda)\ \|\zeta_\lambda\|_\alpha^\alpha$ bounded, and for system (I) also continuous at zero. Then

$$C_{\xi\xi}(u,0) = \int_{-\infty}^{\infty} \cos\ (\xi\lambda)\ dF(\lambda)$$

For system (I) we assume that $\int_{-\infty}^{\infty} |\lambda|\ dF(\lambda) < \infty$ and $f \varepsilon L_1(R',\text{Leb})$. Then we have from (I'),

$$C_{X\xi}(s,0) = \int_{\infty}^{\infty} f(s-u)\ d_u\ C_{\xi\xi}(u,0) = - \int_{\infty}^{\infty} \int f(s-u)\ \lambda\sin(u\lambda)\ dF(\lambda)\ du$$

$$= - \int_{-\infty}^{\infty} \lambda\{\int_{-\infty}^{\infty} f(v)\ \sin[(s-v)\lambda]dv\}\ dF(\lambda)$$

$$= - \int_{-\infty}^{\infty} \lambda\{\sin(s\lambda) \int_{-\infty}^{\infty} f(v)\cos(v\lambda)dv - \cos(s\lambda) \int_{-\infty}^{\infty} f(v)\sin(v\lambda)d\lambda\}\ dF(\lambda)\ .$$

Denoting by f_e, f_o the even, odd parts of f ($f=f_e+f_o$, $f_e(t) = (1/2)\ [f(t)+f(-t)]$, $f_o(t)=(1/2)[f(t)-f(-t)]$) and by $\hat{f}_e$, $\hat{f}_o$ their (real) Fourier transforms we have

$$C_{X\xi}(s,0) = - \int_{-\infty}^{\infty} -\lambda\ \hat{f}_e(\lambda)\ \sin(s\lambda)\ dF(\lambda) + \int_{-\infty}^{\infty} \lambda\ \hat{f}_o(\lambda)\ \cos(s\lambda)\ dF(\lambda)$$

and hence

$$C_{X\xi,e}(s,0) = \int_{-\infty}^{\infty} \lambda\ \hat{f}_o(\lambda)\ \cos(s\lambda)\ dF(\lambda)$$

$$C_{X\xi,o}(s,0) = - \int_{-\infty}^{\infty} \lambda\ \hat{f}_e(\lambda)\ \sin(s\lambda)\ dF(\lambda)\ .$$

Since $\lambda\ \hat{f}_o(\lambda)$ is even and $\lambda\ \hat{f}_e(\lambda)$ is odd, these two integrals determine $\hat{f}_e$ and $\hat{f}_o$ uniquely a.e. [dF]. If in addition dF is absolutely continuous with respect to Lebesgue measure then $\hat{g}_e(\lambda)\ \hat{f}_o(\lambda)$ are determined for all λ, and thus $f_e(t)$, $f_o(t)$, and hence $f(t)$ also, are determined a.e. System (II) is treated similarly as

$$C_{X\xi}(s,0) = \int_{-\infty}^{\infty} \hat{f}_e(\lambda)\ \cos(s\lambda)\ dF(\lambda) + \int_{-\infty}^{\infty} \hat{f}_o(\lambda)\ \sin(s\lambda)\ dF(\lambda)\ .$$

<u>System Analysis</u>

For certain known SαS inputs, we wish to specify the part of the system function f which is uniquely determined from the distribution of the output. Equivalently, if linear systems with impulse response functions f and g produce outputs X^f and X^g, respectively, to the same SαS input ξ, we find necessary and sufficient conditions on f and g for the outputs of the two systems to have the same distribution, i.e., $X^f \overset{d}{=} X^g$.

Kanter [7] showed that if the input ξ is a stable motion, $0<\alpha<2$, and the system (I) is time invariant:

$$X_t^f = \int_{-\infty}^{\infty} f(t-s)\, d\xi_s, \qquad -\infty<t<\infty ,$$

with $f \in L_\alpha(R', \text{Leb})$, then the distribution of the output X^f determines f up to translation and a global sign. Equivalently, $X^f \overset{d}{=} X^g$ if and only if $f(t) = g(t-a)$ a.e. or $f(t) = -g(t-a)$ a.e. for some real a.

For time invariant system (II) we have the following result.

PROPOSITION 6.6. Let the input ξ be a stable motion with $\xi_0 = 0$ and $1<\alpha<2$, and let the system (II) be time invariant:

$$X_t^f = \int_{-\infty}^{\infty} f(t-s)\, \xi_t\, ds , \qquad -\infty<t<\infty .$$

(a) If $\int_{-\infty}^{\infty} f(u)du = 0$, then the distribution of the output X^f determines f a.e. up to translation and a global sign.

(b) If $\int_{-\infty}^{\infty} f(u)du \neq 0$, then the univariate distribution of the output X^f determines f a.e. up to a global sign.

PROOF: From the proof of Proposition 6.2 we have $X_t^f = \int_{-\infty}^{\infty} a(t,u)\, d\xi_u$ with

$$a(t,u) = \int_{-\infty}^{t-u} f, \; u\geq 0; \quad = -\int_{t-u}^{\infty} f, \; u<0 .$$

(a) If $\int_{-\infty}^{\infty} f = 0$ then $a(t,u) = \text{sgn}(u) \int_{-\infty}^{t-u} f$ and

$$\sum_{n=1}^{N} a_n X_{tn}^f = \int_{-\infty}^{\infty} \text{sgn}(u) \left[\sum_{n=1}^{N} a_n \int_{-\infty}^{t_n-u} f \right] d\xi_u .$$

The condition $X^f \overset{d}{=} X^g$ holds if and only if for all choices of N, a_n and t_n, $|\Sigma_{n=1}^N a_n X_{tn}^f|_\alpha = |\Sigma_{n=1}^N a_n X_{tn}^g|_\alpha$ or equivalently

$$\int_{-\infty}^{\infty}\left|\sum_{n=1}^{N} a_n \int_{-\infty}^{t_n-u} f\right|^{\alpha} du = \int_{-\infty}^{\infty}\left|\sum_{n=1}^{N} a_n \int_{-\infty}^{t_n-u} g\right|^{\alpha} du \; .$$

By Kanter's result this latter condition holds if and only if for some real a, $\int_{-\infty}^{t} f = \pm \int_{-\infty}^{t-a} g$ for all t, or equivalently $f(t) = \pm g(t-a)$ a.e. [Leb].

(b) Assume now $\int_{-\infty}^{\infty} f \neq 0$. Then the univariate distributions of X^f determine for all t,

$$\int_{-\infty}^{\infty} |a(t,u)|^{\alpha} du = \int_{-\infty}^{0} \left|\int_{t-u}^{\infty} f\right|^{\alpha} du + \int_{0}^{\infty} \left|\int_{-\infty}^{t-u} f\right|^{\alpha} du$$

$$= \int_{t}^{\infty} \left|\int_{v}^{\infty} f\right|^{\alpha} dv + \int_{-\infty}^{t} \left|\int_{-\infty}^{v} f\right|^{\alpha} dv$$

and by differentiation

$$A(t) = \left|\int_{-\infty}^{t} f\right|^{\alpha} - \left|\int_{t}^{\infty} f\right|^{\alpha} \; .$$

It is clear that X^f and X^{-f} have identical univariate distributions, so that the univariate distributions of X^f would, at best, determine f a.e. up to a global sign. Putting

$$\sigma(t) = \int_{-\infty}^{t} f \quad (\sigma(\infty) \neq 0), \qquad B_c(x) = |x|^{\alpha} - |x-c|^{\alpha} \; ,$$

we have $A(t) = B_{\sigma(\infty)}[\sigma(t)]$. Also, using $d|x|^{\alpha}/dx = \alpha(x)^{\alpha-1}$, $-\infty<x<\infty$ $(\alpha>1)$, we obtain $B_c'(x) = \alpha[(x)^{\alpha-1} - (x-c)^{\alpha-1}]$ and it is easily checked that for all x, $B_c'(x) > 0$ when $c > 0$ and $B_c'(x) < 0$ when $c < 0$. Thus B_c is strictly monotonic when $c \neq 0$. The modulus of $\sigma(\infty)$ is determined by $|\sigma(\infty)| = [A(\infty)]^{1/\alpha} = c > 0$ say. If we take $\sigma(\infty) = c$ then $\sigma(t)$ is uniquely determined from $A(t) = B_c[\sigma(t)]$ for each t and thus f is determined uniquely a.e., and let us denote it by f_1. Similarly if we take $\sigma(\infty) = -c$, f is determined uniquely a.e., and let us denote it by f_2. Since $B_{-c}(x) = B_c(-x)$, it follows that $f_2 = -f_1$ a.e., and thus f is determined uniquely a.e. up to a global sign. []

Kanter's result can also be applied when the input ξ is a moving average of stable motion.

CASE 6.7. Let ζ be an α-stable motion and $\xi_t = \int_{-\infty}^{\infty} a(t-u)\, d\zeta_u$, where $a \in L_\alpha(R', \text{Leb})$. Further conditions needed for system (I) are the absolute continuity of a and $f, g, a' \in L_1(R', \text{Leb})$, and $f_* a' \in L_\alpha(R', \text{Leb})$. Then for any $t' < t$,

$$\xi_t - \xi_{t'} \quad \int_{-\infty}^{\infty} [a(t-u)-a(t'-u)] d\zeta_u = \int_{-\infty}^{\infty} \{\int_{t'}^{t} a'(w-u)dw\}\, d\zeta_u ,$$

so that for system (I),

$$X_t = \int_{-\infty}^{\infty} f(t-s)\, d\xi_s = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(t-w)\, a'(w-u)dw\}\, d\zeta_u = \int_{-\infty}^{\infty} (f_* a')(t-u)\, d\zeta_u$$

By Kanter's result, $X^f \overset{d}{=} X^g$ if and only if $(f_* a')(t) = \pm\, (g_* a')(t-\tau)$ a.e. (t) for some τ, or equivalently $\hat{f}(\lambda)(a')^\wedge(\lambda) = \pm\, \hat{g}(\lambda)(a')^\wedge(\lambda)e^{-\tau\lambda}$ a.e. If we assume now that $(a')^\wedge \neq 0$ a.e., the latter condition is $f(t) = \pm\, g(t-\tau)$ a.e.

The case of system (II) is similar since, by Theorem 4.7,

$$X_t = \int_{-\infty}^{\infty} f(t-s)\xi_s\, ds = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(t-s)a(s-u)ds\}\, d\zeta_u = \int_{-\infty}^{\infty} (f_* a)(t-u)\, d\zeta_u ,$$

and a now plays the role of a'.

Finally, we consider inputs that are Fourier transforms of SαS processes with independent increments and time invariant systems (I) and (II).

CASE 6.8. The assumptions and notation are the same as in Case 6.5. For system (I) we have for all $t' < t$,

$$\xi_t - \xi_{t'} = \int_{-\infty}^{\infty} \{\int_{t'}^{t} [-\lambda \sin(s\lambda)]ds\}\, d\zeta_\lambda$$

so that

$$X_t^f = \int_{-\infty}^{\infty} f(t-s)d\xi_s = \int_{-\infty}^{\infty} \{-\int_{-\infty}^{\infty} f(t-s)\lambda \sin(s\lambda)ds\}d\zeta_\lambda$$

$$= \int_{-\infty}^{\infty} \lambda\{\hat{f}_o(\lambda) \cos(t\lambda) - \hat{f}_e(\lambda) \sin(t\lambda)\}\, d\zeta_\lambda .$$

It follows that

$$C_{XX}(t,0) = \int_{-\infty}^{\infty} \lambda\{\hat{f}_o(\lambda)\cos(t\lambda) - \hat{f}_e(\lambda)\sin(t\lambda)\}\,[\lambda\,\hat{f}_o(\lambda)]^{\alpha-1}\,dF(\lambda)$$

$$= \int_{-\infty}^{\infty} |\lambda\,\hat{f}_o(\lambda)|^{\alpha}\cos(t\lambda)\,dF(\lambda) - \int_{-\infty}^{\infty} |\lambda|^{\alpha}\,\hat{f}_e(\lambda)(\hat{f}_o(\lambda))^{\alpha-1}\sin(t\lambda)\,dF(\lambda)$$

the first term being its even part and the second its odd part. Hence $C_{XX}(t,0)$, all t, determines uniquely a.e. [dF], $|\hat{f}_o|$ and $\hat{f}_e(f_o)^{\alpha-1}$, and certainly $|\hat{f}_o|$ and $|\hat{f}_e|$ (and thus also $|\hat{f}|$).

When f is even $(f=f_e,\ f_o=0)$ then $C_{XX}(t,0)$ determines $\hat{f}$ uniquely, and the distribution of X^f depends on f only through $\hat{f}$. This follows immedaitely from

$$|\sum_{n=1}^{N} a_n X_{tn}^f|_{\alpha}^{\alpha} = \int_{-\infty}^{\infty} |\lambda\,\hat{f}(\lambda)|^{\alpha}\,|\sum_{n=1}^{N} a_n \sin(t_n\lambda)|^{\alpha}\,dF(\lambda)\ .$$

We have the same results of course when f is odd.

For system (II) a similar calculation yields

$$X_t^f = \int_{-\infty}^{\infty} \{\hat{f}_e(\lambda)\cos(t\lambda) + \hat{f}_o(\lambda)\sin(t\lambda)\}\,d\zeta_{\lambda}$$

from which we obtain results identical to those for system (I).

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This work extends to processes with finite moments or order p, $1 < p < 2$ and to symmetric α-stable processes, $1 < \alpha < 2$, some of the basic linear theory known for processes with finite second moments ($p=2$) and for Gaussian processes ($\alpha=2$). Here the "covariation" plays a role analogous to the covariance. Specifically, stochastic integrals of two types are introduced and studied for p-th order processes and in particular for symmetric stable processes. Regression estimates and linear estimates on certain symmetric

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20. stable processes are evaluated, including regression and linear filtering of signal in noise. Also, for certain symmetric stable inputs, the identification of a linear system from the input covariation and the input-output cross covariation is considered, and the way the distribution of the output depends on the linear system is studied.