A Dynamic Problem of Thermoelasticity with Finite Wave Speeds for a Layer

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SUMMARY

In this paper the exact solution to a one-dimensional initial-boundary value problem for a homogeneous isotropic thermoelastic layer in a linear dynamic thermoelasticity with finite wave speeds is presented. The initial data are assumed to be homogeneous, and thermoelastic disturbances inside the layer and for time $t>0$ are produced by a time dependent normal load and temperature applied on its boundary. The method of finding the solution is based on a decomposition theorem which asserts that a thermoelastic process obeying a central fourth order partial differential equation can be decomposed into two simpler processes satisfying two wave equations with a convolution. The two simple processes correspond to two initial-boundary value problems of Dirichlet type with two unknown boundary data which are uniquely determined from the data of the original problem . The solution is given in the form of an expansion in space eigenfunctions with coefficients depending parametrically on time and on the data.

1. Formulation of the Problem

Let L be a layer made of the homogeneous isotropic thermoelastic material of Green-Lindsay type (see [1]) that occupies the region

$$0 \leq x_1 \leq l, \quad |x_2| < \infty \ , \ |x_3| < \infty \tag{1}$$

where x_i ($i = 1,2,3$) are the dimensionless cartesian coordinates and l is the dimensionless thickness of the layer introduced in [2]. Also suppose that the layer has a quiescent past, i.e. the displacement vector $\underline{u}$, the displacement velocity $\dot{\underline{u}}$, the temperature θ and the temperature velocity $\dot{\theta}$ vanish for $t = 0$ and for every point of $\bar{L}$. The dot over a function indicates partial derivative of the function with respect to t while $\bar{L} = L + \partial L$, where ∂L is the boundary of L. Moreover, assume that the layer is loaded on the planes $x_1 = 0$ and $x_1 = l$ by a smooth normal pressure and temperature which do not depend of x_2 and x_3, and have a quiscent past, i.e. they vanish together with their first time derivatives for $t = 0$. To such a time dependent thermo-mechanical load there corresponds a thermoelastic process inside the layer which is described by a scalar potential $\phi = \phi(x,t)$ that depends on x and t only ($x = x_1$). The potential ϕ generates the displacement vector $\underline{u}$, stress tensor $\underline{S}$ and temperature θ according to the formulae (see eqs. (2)-(4) of [2]):

$$u_1 = \frac{\partial \phi}{\partial x} \ , \quad u_2 = u_3 = 0 \ , \tag{2}$$

$$S_{11} = \frac{\partial^2 \phi}{\partial t^2} \ , \quad S_{22} = S_{33} = \frac{\partial^2 \phi}{\partial t^2} - 2\left(\frac{c_2}{c_0}\right)^2 \frac{\partial^2 \phi}{\partial x^2} \ , \tag{3}$$

$$S_{12} = S_{23} = S_{31} = 0 \ ,$$

$$\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\phi \ , \tag{4}$$

and ϕ satisfies the equation

$$\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} - \tau_0 \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right) - \varepsilon \frac{\partial^3}{\partial x^2 \partial t}\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\right]\phi = 0 \tag{5}$$

$$\text{for } (x,t) \in [0,l] \times [0,\infty) .$$

Here c_2 is the shear velocity for the solid and $c_0 = \hat{x}_0/\hat{t}_0$, where $\hat{x}_0$ and $\hat{t}_0$ stand for the units of length and time introduced in [1]. Moreover ε is the dimensionless coupling parameter, while τ_1 and τ_2 are the dimensionless relaxation times [1] ($\tau_1 \geq \tau_0 \geq 0$). If $\tau_1 = \tau_0 = 0$, eqs. (2)-(5) reduce to those of the classical thermoelasticity (see [3] and [4]).
The boundary conditions take the form

$$S_{11}(0,t) = -\sigma_0(t), \quad S_{11}(1,t) = -\sigma_1(t) \tag{6}$$

$$\theta(0,t) = \theta_0(t), \quad \theta(1,t) = \theta_1(t) \tag{7}$$

where $\sigma_0(t)$, $\sigma_1(t)$, $\theta_0(t)$ and $\theta_1(t)$ are prescribed functions on $[0,\infty)$. The quiescent past of the surface thermo-mechanical load means that

$$\sigma_0(0) = \dot{\sigma}_0(0) = 0, \quad \sigma_1(0) = \dot{\sigma}_1(0) = 0 \tag{8}$$

$$\theta_0(0) = \dot{\theta}_0(0) = 0, \quad \theta_1(0) = \dot{\theta}_1(0) = 0 \tag{9}$$

These conditions are compatible with those for the layer with a quiescent past which, in terms of ϕ, have the form

$$\frac{\partial^k \phi}{\partial t^k}(x,0) = 0 \quad \text{for} \quad k = 0,1,2,3 \quad \text{and} \quad x \in [0,1] \tag{10}$$

The boundary conditions (6)-(7), written in terms of ϕ, take the form

$$\begin{aligned} &\phi(0,t) = \hat{f}_0(t), \quad \phi(1,t) = \hat{f}_1(t) \\ &\frac{\partial^2 \phi}{\partial x^2}(0,t) = \hat{g}_0(t), \quad \frac{\partial^2 \phi}{\partial x^2}(1,t) = \hat{g}_1(t) \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\hat{f}_0(t) = -t * \sigma_0, \quad \hat{f}_1(t) = -t * \sigma_1 \\ &\hat{g}_0(t) = \theta_0 + \tau_1 \dot{\theta}_0 - \sigma_0, \quad \hat{g}_1(t) = \theta_1 + \tau_1 \dot{\theta}_1 - \sigma_1 \end{aligned} \tag{12}$$

and $*$ stands for the convolution product on the time-axis.
Therefore, the layer problem is equivalent to finding function $\phi = \phi(x,t)$ that satisfies eq. (5), the initial conditions (10) and the boundary conditions (11).

2. Solution of the Problem

A key role in solving the problem is played by the following decomposition theorem [1]:
"If ϕ satisfies eqs. (5) and (10), then

$$\phi(x,t) = \phi_1(x,t) + \phi_2(x,t) \quad \text{on} \quad L \times [0,\infty) \tag{13}$$

where ϕ_1 and ϕ_2 meet the eqs.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v_1^2}\frac{\partial^2}{\partial t^2} - k_1 \frac{\partial}{\partial t} - \lambda - \lambda K*\right)\phi_1 = 0 \quad \text{on} \quad L \times [0,\infty) \tag{14}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v_2^2}\frac{\partial^2}{\partial t^2} - k_2 \frac{\partial}{\partial t} + \lambda + \lambda K*\right)\phi_2 = 0 \quad \text{on} \quad L \times [0,\infty) \tag{15}$$

Here $K = K(t)$ is a function on $[0,\infty)$ defined by the relation

$$K(t) = 2\frac{d}{dt}\left[\exp(\alpha t)\frac{J_1(\beta t)}{\beta t}\right], \quad t \in [0,\infty) , \tag{16}$$

where $J_1 = J_1(t)$ is the Bessel function of order 1 and the first kind, and

$$\begin{aligned} \alpha &= \left[1-\epsilon-(1+\epsilon)(\tau_0+\epsilon\tau_1)\right]\Delta^{-1} \\ \beta &= 2\sqrt{\epsilon}\left[1+(1+\epsilon)(\tau_1-\tau_0)\right]^{1/2}\Delta^{-1} \end{aligned} \tag{17}$$

with

$$\Delta = (1-\tau_0+\epsilon\tau_1)^2 + 4\epsilon\tau_0\tau_1 > 0 \tag{18}$$

Moreover

$$v_{1.2}^{-2} = \frac{1}{2}(1+\tau_0+\epsilon\tau_1 \pm \Delta^{\frac{1}{2}}) \tag{19}$$

$$k_{1.2} = \frac{1}{2}(1+\epsilon \mp \alpha\Delta^{\frac{1}{2}}) \quad , \quad \lambda = \frac{1}{4}\beta^2\Delta^{\frac{1}{2}} \tag{20}$$

(21)

$$K * \phi_k = \int_0^t K(t-\tau)\,\phi_k(x,\tau)\,d\tau \quad , \quad (k = 1, 2) . \tag{22}$$

Let ϕ_1 be a solution of eq. (14) that satisfies the conditions

$$\phi_1(\cdot,0) = \dot{\phi}_1(\cdot,0) = 0 \quad \text{on } L \tag{23}$$

$$\phi_1 = f_1 \quad \text{on} \quad \partial L \times [0,\infty) , \tag{24}$$

where f_1 is a given function on $\partial L \times [0,\infty)$. Similarly, let ϕ_2 be a solution of eq. (15) such that

$$\phi_2(\cdot,0) = \dot{\phi}_2(\cdot,0) = 0 \quad \text{on } L \tag{25}$$

$$\phi_2 = f_2 \quad \text{on} \quad \partial L \times [0,\infty) , \tag{26}$$

where f_2 is a given function on $\partial L \times [0,\infty)$.
If we look for ϕ_k ($k = 1, 2$) in the form

$$\phi_k(x,t) = \sum_i T_{ki}(t)\,\varphi_i(x) \quad \text{on} \quad L \times [0,\infty) \tag{27}$$

where φ_i ($i = 1,2,3,\ldots,\infty$) stand for the space eigenfunctions for the operator $-d^2/dx^2$, while $T_{ki}(t)$ are the solutions to associate integro-differential eqs. on the time-axis, we arrive at the formulae (see [2])

$$\phi_1(x,t) =$$

$$-\sum_{i=1}^{\infty}\left\{\sum_{k=0}^{\infty}(-)^k\,\lambda_1^k\,G_k * H_{ik}^{(1)} * \left[f_1(\xi,t) * \frac{d\varphi_i}{d\xi}\right]_{\xi=0}^{\xi=1}\right\}\varphi_i(x) \tag{28}$$

$$\phi_2(x,t) =$$

$$- \sum_{i=1}^{\infty} \left\{ \sum_{k=0}^{\infty} \hat{\lambda}_2^k G_k * H_{ik}^{(2)} * \left[f_2(\xi,t) * \frac{d\varphi_i}{d\xi} \right]_{\xi=0}^{\xi=1} \right\} \varphi_i(x) \qquad (29)$$

The functions G_k in (28) and (29) are defined by

$$G_0(t) = \delta(t), \quad G_k(t) = k\left(\frac{2}{\beta}\right)^k \exp(\alpha t) \frac{J_k(\beta t)}{t}, \quad k \geq 1, \qquad (30)$$

where $\delta = \delta(t)$ is the Dirac delta function and $J_k = J_k(t)$ is the Bessel function of order k and of the first kind. The functions $H_{ik}^{(a)}$ $(a = 1, 2)$ are given by

$$H_{ik}^{(a)}(t) = \frac{v_a^2 \exp(-\mu_a t)}{(2\lambda_{ai})^k k!} \left[\hat{A}_{k+1}(y_{ai}; \zeta_{ai}) \cos y_{ai} + \hat{B}_{k+1}(y_{ai}; \zeta_{ai}) \sin y_{ai} \right] \qquad (31)$$

where $\hat{A}_{k+1} = \hat{A}_{k+1}(y;\zeta)$ and $\hat{B}_{k+1} = \hat{B}_{k+1}(y;\zeta)$ are the polynomials in y of the k-th degree satisfying the recurrence relations

$$\begin{aligned} \hat{A}'_{k+1} + \hat{B}_{k+1} &= (k+1)\hat{A}_k + (\hat{A}'_k - \zeta\hat{A}_k + \hat{B}_k)y \\ \hat{B}'_{k+1} - \hat{A}_{k+1} &= (k+1)\hat{B}_k + (\hat{B}'_k - \zeta\hat{B}_k - \hat{A}_k)y \end{aligned} \qquad (32)$$

where prime indicates derivative with respect to y.
In particular, for $k = 1,2,3$ we obtain

$$\begin{aligned} &\hat{A}_1 = 1, \quad \hat{B}_1 = -\zeta \\ &\hat{A}_2 = (1 - \zeta^2)y, \quad \hat{B}_2 = (1 + \zeta^2) - 2\zeta y \\ &\hat{A}_3 = 3\zeta(1 + \zeta^2)y + (1 - 3\zeta^2)y^2 \\ &\hat{B}_3 = -3\zeta(1 + \zeta^2) + 3(1 + \zeta^2)y - \zeta(3 - \zeta^2)y^2 \end{aligned} \qquad (33)$$

The parameters μ_a, λ_{ai} and ζ_{ai} in (31) are defined by

$$\begin{aligned} &\mu_a = v_a h_a, \quad h_a = v_a k_a/2, \quad \lambda_{ai} = v_a(\lambda_i - h_a^2)^{1/2} \\ &\lambda_i = (i\pi/l)^2, \quad \zeta_{ai} = \mu_a \lambda_{ai}^{-1} \end{aligned} \qquad (34)$$

while the variable y_{ai} in (31) is given by

$$y_{ai} = \lambda_{ai} t \,. \qquad (35)$$

The parameters $\hat{\lambda}_1$ and $\hat{\lambda}_2$ in (28) and (29), respectively, are defined by

$$\hat{\lambda}_a = \lambda v_a^2 \,. \qquad (36)$$

The functions $\varphi_i = \varphi_i(x)$ in (28) and (29) take the form

$$\varphi_i(x) = (2/l)^{1/2} \sin(\pi i x/l) \quad . \tag{37}$$

Finally, the symbol $[q(\xi)]_{\xi=0}^{\xi=l}$ in (28) and (29) is defined by

$$[q(\xi)]_{\xi=0}^{\xi=l} = q(l) - q(0) \quad , \tag{38}$$

where $q = q(\xi)$ is a function on $[0,l]$.
One can show that the function ϕ given by (13), where ϕ_1 and ϕ_2 are defined by (28) and (29), respectively, satisfies the field eq. (5) and the homogeneous initial conditions (10).

To meet the boundary conditions (11) we proceed in the following way. We extend eqs. (14) and (15) to the boundary $\partial L \times [0,\infty)$, add them side by side and transform (11) to the system of two ordinary integro-differential eqs. for f_1 and f_2:

$$\begin{aligned} &f_1 + f_2 = \hat{f} \\ &L_1 f_1 + L_2 f_2 = \hat{g} \quad \text{on} \quad \partial L \times [0,\infty) \end{aligned} \tag{39}$$

where

$$L_{1.2} = \frac{1}{v_{1.2}^2} \frac{d^2}{dt^2} + k_{1.2} \frac{d}{dt} \pm \lambda \pm \lambda K * \tag{40}$$

and

$$\begin{aligned} &\hat{f} = \hat{f}_0, \quad \hat{g} = \hat{g}_0 \quad \text{for} \quad \xi = 0 \\ &\hat{f} = \hat{f}_1, \quad \hat{g} = \hat{g}_1 \quad \text{for} \quad \xi = l \end{aligned} \tag{41}$$

A solution of eqs. (39) subject to the homogeneous initial conditions $f_k(\cdot,0) = \dot{f}_k(\cdot,0) = 0$, $k = 1, 2$, takes the form

$$\begin{aligned} f_{1.2} = \frac{1}{2}\hat{f} \mp \frac{1}{2}\left(\frac{1}{v_1^2} - \frac{1}{v_2^2}\right)^{-1}\Big[&(k_1 + k_2)H * \hat{f} \\ &+ \left(\frac{1}{v_1^2} + \frac{1}{v_2^2}\right)H * \dot{\hat{f}}\Big] \pm \left(\frac{1}{v_1^2} - \frac{1}{v_2^2}\right)^{-1} H * \hat{g} \quad , \end{aligned} \tag{42}$$

where $H(t) = \exp(\alpha t) J_0(\beta t)$ and $J_0 = J_0(t)$ is the Bessel function of order zero and of the first kind.

Therefore, the function ϕ given by (13), where ϕ_1 and ϕ_2 are defined by (28) and (29), respectively, and f_k, $k = 1, 2$, are given by (42), constitutes the exact solution of the initial-boundary value problem for the layer. The corresponding formulae for the displacements, stresses and temperature are obtained from eqs. (2),(3),(4),(13),(28),(29) and (42).

Closure

The paper outlines a method of finding the exact series solution to a one-dimensional initial-boundary value problem for the layer obeying a generalized

dynamic thermoelasticity. The formulae obtained can be used for a numerical analysis of thermoelastic disturbances in the layer subject to an arbitrary aperiodic time-dependent surface thermo-mechanical load. The solution includes as a particular case a solution of the classical coupled thermoelasticity ($\tau_0 = \tau_1 = 0$) which has not been obtained so far either. A one-dimensional solution of the classical thermoelasticity for a semi-space obtained by a similar method is due to Brun [5].

References

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