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STOCHASTIC FINITE ELEMENT STRUCTURAL MODELS

H. CONTRERAS, R. E. SCHOLL
URS/John A. Blume & Associates, Engineers,
130 Jessie Street, San Francisco, California 94105, U.S.A.

ABSTRACT

In a new approach to accounting for the main sources of uncertainty in the analysis and design of structures, stochastic differential and difference equations are combined with the finite-element method. Loads for multidimensional structures are idealized as stochastic processes and incorporated into finite-element dynamic models with uncertainty in their parameters.

The theoretical basis of the stochastic differential and difference equations and of the finite-element method are presented. Stochastic finite elements are introduced as a means to identify or consider uncertainty in parameters. Seismic disturbances are used as an illustration of simulating loads with stochastic processes. Numerical examples show the capabilities and feasibility of the proposed methodology.

1. Introduction

Structural engineers must face two fundamental difficulties: representing external disturbances and building adequate mathematical models of structures. For those designing nuclear reactors, the severe consequences of structural failure make a probabilistic approach to design inescapable. This article discusses the application of stochastic processes to the representation of loads and disturbances, and the use of stochastic finite elements to model structures. We present, as a tool for multidimensional analysis of structures, the use of stochastic differential equations [1] with the finite-element method [2]. Solution of problems associated with time in dynamic models of structures is accomplished by a step-by-step integration procedure that can be implemented in any general-purpose, finite-element program. To account for uncertainty in structural parameters, we will propose a method for considering the a priori uncertain parameters [3] and a method for identifying the a posteriori uncertain parameters through measurements of the behavior of the real structure.

2. Stochastic Differential Equations

Stochastic dynamic systems whose representations are continuous mathematical expressions and whose state-spaces are finite, n-dimensional vectors can be modeled by finite-dimensional Markov processes that are the output of stochastic differential or difference equations. This mathematical tool, based on Itô calculus from a Bayesian point of view, gives the optimum estimate of a system's state for a certain a priori mathematical model and a posteriori data. Thus, measurements and uncertainty in the parameters of the model can be incorporated into a dynamic system.

Let the continuous stochastic dynamic system be described by the (generally nonlinear) stochastic differential equation:

$$\dot{x}_t = \mathrm{f}(x_t,t) + \mathrm{G}(x_t,t)w_t, \text{ when } t \geq t_0 \tag{1}$$

where f is a generally nonlinear real n-vector function; x_t is an n-vector; G is an nxr matrix; and $\{w_t, t > t_0\}$ is a white-noise r-vector process with $\mathrm{E}\{w_t w_\tau^T\} = Q\delta$, where E and δ are the expected-value and Dirac operators, respectively. This system can be measured continuously with the m-vector, z, such that:

$$z = \mathrm{h}(x_t,t) + v_t, \text{ when } t \geq t_0 \tag{2}$$

where h is an m-vector real function and $\{v_t\ t > t_0\}$ is an m-vector white-noise process with $\mathrm{E}\{v_t v_\tau^T\} = R\delta$, when $R > 0$. We suppose that w_t, v_t, and x_t are independent. If the white-noise processes w_t and v_t are Gaussian, eqs. (1) and (2) constitute a vector stochastic differential equation, as per Itô [1].

A computer solution to this problem necessitates the discretization in time of eqs. (1) and (2). The equivalent stochastic difference equations are:

$$x_{k+1} = \Psi(x_k,k+1,k) + \Gamma(x_k,k)w_{k+1}, \text{ when } k = 0,1,... \qquad (3)$$

where k is a parameter associated with time, t_k; x_k is a state-space n-vector; Ψ is an n-dimensional real function; Γ is an nxr matrix; and $\{w_k, k = 1,2,...\}$ is an r-vector sequence of white noise such that $E\{w_k w_l^T\} = Q\delta$.

Observations (measurements related to the state of the system) will be expressed by the m-vector, z, such that:

$$z_k = h(x_k,k) + v_k, \text{ when } k = 1,2,... \qquad (4)$$

where h is an m-vector real function and $\{v_k, k = 1,2,...\}$ is an m-vector white-noise sequence with $E\{v_k v_l^T\} = R\delta$, where $R > 0$. As before, w_k, v_k, and x_k are assumed to be independent.

The linear formulation of this equation can be solved by Kalman filtering or other appropriate algorithms [4]. To solve the generally nonlinear problem, linearization around the mean (extended Kalman filtering) [5] or Monte Carlo with reduction of variance [6] can be used.

The state-space vector can be augmented with any desired parameters, p, of the system:

$$x_t = \begin{Bmatrix} x_t \\ p \end{Bmatrix} = \begin{Bmatrix} \text{time varying state} \\ \text{system parameters} \end{Bmatrix} \qquad (5)$$

In this augmented state-space vector, the parameters, p, can be estimated (in the presence of measured observations) or considered uncertain and discarded at each integration step.

3. Finite-Element Method

The finite-element method has developed rapidly with the increasing use of digital computers. It is a valuable tool to solve boundary problems. At first, engineers used the method intuitively in structural analysis [7]. Mathematicians, however, soon recognized the underlying principles to be a form of the Rayleigh-Ritz-Galerkin method [8], thus extending the application of the finite-element method to all physical sciences.

Mathematical models of physical problems can generally be solved in a variational form, that is by finding a function that minimizes an integral equation. We will present the procedure without proof.

Let $J(a)$ be the functional associated with a physical problem that is defined as:

$$J(a) = \langle La,a \rangle - 2 \langle b,a \rangle \qquad (6)$$

where b and a are functions and L is a linear operator. If we define the internal product $\langle \alpha,\beta \rangle$ as $\int_V \alpha\beta dV$, eq. (6) becomes an integral equation. The function u extremizes the functional given by eq. (6) if and only if the first variation is zero, such that:

$$\delta J(a) = \left.\frac{\partial J(a)}{\partial a}\right|_{a = u} = 0 \tag{7}$$

or, operating in eq. (6), $\delta J(a) = 2(Lu - b) = 0 \rightarrow Lu = b$, where $Lu = b$ is the corresponding Euler-Lagrange equation.

The finite-element method is based on the Rayleigh-Ritz-Galerkin method of choosing a finite number of trial functions $\{\phi_1,\ldots\phi_n\}$ and finding, among all linear combinations, $u^* = \sum d_i\phi_i$, the minimizing one. The coefficients, d_i, are found by solving an algebraic system of n simultaneous equations that is given by the minimization of eq. (6). Choosing these trial functions is difficult, but the finite-element method is both convenient and practical in its computations. The idea is simple: the structure or region of interest is divided into easily identified parts, or finite elements, e.g., bars, triangles, or rectangles. Simple trial functions, generally algebraic polynomials, are chosen for each element. Boundary conditions are imposed locally along the edge of each element, rather than globally along a complicated boundary. The accuracy of this method can be increased by refining the subdivision.

4. Stochastic Finite Elements

Partial differential equations, representing a system at a certain time, t, can be discretized with the finite-element method and solved to yield the initial state of the system at instant $t = t_0$. Integration in time can be accomplished through the use of the stochastic differential equation method. Combination of these methods is one of the most powerful tools for analysis available.

Let the linear dynamic equation of a structure modeled with finite elements be given by:

$$M\ddot{x} + C\dot{x} + Kx = \zeta \tag{8}$$

where M, C, and K are mass, damping, and stiffness matrices, respectively, of the structure; x is the generalized displacement vector, and ζ is an external forces vector. In general, M, C, K, and ζ are functions of time.

We present eq. (8) in state-space format by making $x \equiv \{x,\dot{x}\}^T$ such that:

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}C & -M^{-1}C \end{bmatrix} \begin{Bmatrix} x \\ x \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{Bmatrix} 0 \\ \zeta \end{Bmatrix} \tag{9}$$

or, in a simplified form,

$$\dot{x} = Ax + By \tag{10}$$

To discretize in time, we have to find the state transition matrix $\Phi(k + 1,k)$ defined as:

$$\Phi(k+1,k) = e^{A(t_{k+1}-t_k)} = \sum_{j=0}^{\infty} \frac{1}{j!} A^j (t_{k+1} - t_k)^j \tag{11}$$

The discretized system is then $x_{k+1} = \Phi(k+1,k)x_k + \Lambda(k+1,k)y_k$, where

$$\Lambda(k+1,k) = \int_{t_k}^{t_{k+1}} e^{A\xi} B d\xi \tag{12}$$

If we add to eq. (10) a term accounting for white noise, we have:

$$x_{k+1} = \Phi(k+1,k)x_k + \Lambda(k+1,k)y_k + \Gamma(k)w_{k+1} \tag{13}$$

which is a linear stochastic difference equation as defined above. Here w_k is a vector white-noise sequence, and $\Gamma(k)$ a scaling matrix.

5. Earthquake Simulation

The generation of stochastic processes to simulate loads will be illustrated with examples of seismic disturbances. It is usually assumed [9] that seismic waves arrive at the underlying bedrock in accordance with a nonhomogeneous Poisson process, which can be represented as a white-noise process multiplied by a deterministic envelope function [10]. Up to their second moments, the two processes are equivalent.

Stochastic differential equations are specially suited to this task. The white noise, w, once filtered through a series of second-order filters, can be subjected to the deterministic envelope function in order to obtain the final earthquake acceleration process. This process is easily introduced into the formulation of stochastic differential equations. The series of second-order filters will increase the dimension of the problem while retaining its base characteristics.

The series of second-order filters can be represented in matrix notation as:

$$\dot{x}_g = \begin{bmatrix} \Omega_1 & & & & \\ \Delta & \Omega_2 & & \underline{0} & \\ & \Delta & \cdot & & \\ & & \cdot & & \\ & & & \cdot & \\ \underline{0} & & & \Delta & \Omega_n \end{bmatrix} x_g + \begin{Bmatrix} 0 \\ -w \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

where,

$$\Omega_i = \begin{bmatrix} 0 & 1 \\ -\gamma_i^2 & -2\gamma_i\theta_i \end{bmatrix} \tag{15}$$

$$\Delta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{16}$$

and γ_i, θ_i are parameters associated with the seismicity of the site. Eq. (14) augments the system as the vector x_g augments the state x.

6. Uncertain Parameters

Frequently a dynamic system has some parameters whose values are imprecisely known. Such parameters can be regarded as random variables with known a priori statistics. In order to model this type of system with stochastic difference equations, we must augment the state-space, x, with the uncertain parameters, p, as follows:

$$x_{p_k} = \begin{Bmatrix} x_k \\ p \end{Bmatrix}, \text{ where } p = p_k = p_{k+1} \tag{17}$$

The augmented system will be

$$x_{p_{k+1}} = \begin{bmatrix} \Phi(k+1,k) & \Psi(k+1,k) \\ \underline{0} & I \end{bmatrix} x_{p_k} + \begin{bmatrix} \Gamma_k \\ 0 \end{bmatrix} w_{k+1} \tag{18}$$

with these observations:

$$z_k = \begin{bmatrix} M_k, \underline{0} \end{bmatrix} x_{p_k} + v_k \tag{19}$$

Kalman filtering or other solution methods can be applied to this augmented system to produce estimates of x_p. If the parameters are not required to be estimated, they can be considered and their estimation discarded. This can be done even when observations are not available.

For structures modeled with finite elements, the terms of the matrices $M^{-1}K$ and $M^{-1}C$ can be considered as uncertain. With a suitable arrangement of these terms in the augmented state, x_p, the terms of $\Psi(k+1,k)$ can be computed at each stage, k, by using the values of the estimate, x_k, arranged in a matrix with a stair form [11, 12].

7. Parameter Estimation

In the preceding section we presented the general problem of uncertain parameters. When desired, it is possible to estimate the parameters and the state simultaneously, if observations are available. As knowledge of the system improves, the covariance of the parameters will be reduced and the parameters will converge to their true values. The finite-element model is only an idealization of reality, and as such the parameters will not converge to deterministic values.

8. Numerical Examples

We will illustrate the stochastic and finite-element methods by applying them to a shear-frame structure with two degrees of freedom. Assume that the base of the structure is excited by modulated white noise, $\ddot{x}_0$, given by:

$$\ddot{x}_0 = m_t w_t \tag{20}$$

where w_t = Gaussian white noise, and m_t = a deterministic modulation function, such that:

$$m_t = 3.06\left(e^{-0.25t} - e^{-0.63t}\right)H_t \tag{21}$$

with H_t = Heavyside function. The structural values at levels 1 and 2 are: mass, $m_1 = m_2 =$ 40,000; stiffness, $k_1 = 4k_2 = 640{,}000$; and damping, $c_1 = 2c_2 = 6{,}400$. Variation coefficients of 0.2 and 0.3 are used for k and c, respectively. The results of solving the stochastic system are shown in Figure 1, where the value of $\sigma_i = \sqrt{\mathrm{E}x_i^2}$ (i for levels 1 and 2) is compared with deterministic values for k and c.

In order to identify parameters, we use the same data, but substitute a base excitation given by the following equation: $\ddot{x}_0 = \sin \omega t$, when $0 \le t \le 10$ sec, and zero when $0 > t > 10$ sec. Assume $\omega = \pi$ rad sec^{-1}. Starting at rest with $t = 0$, we use the Runge-Kutta method to integrate the deterministic dynamic equation at intervals of 0.05 sec for a total of 2,000 points. To this time history, white noise with 10% variance is added. The second-story accelerations given by this are taken as the uncertain measurements. Other values used are $x_0 = [0,0,0,0,10^3,10^3,10^5,10^5]^T$, $P_0 = 10^4 I$, $Q = 0.04I$, and $R = 1$.

Figure 2 shows the convergence of the parameters to their true values.

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10. Figures

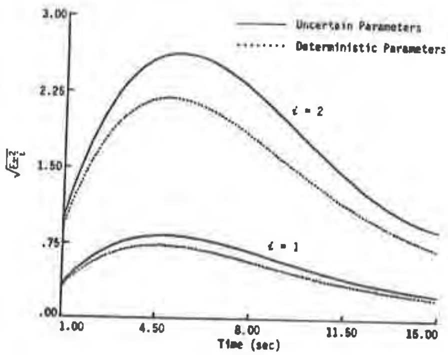

FIGURE 1 MEAN SQUARE RESPONSE, LEVELS 1 AND 2

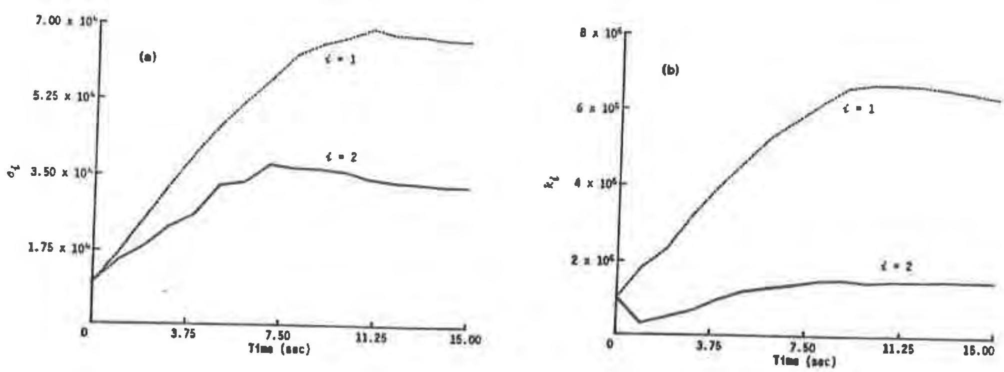

FIGURE 2 PARAMETER CONVERGENCE, LEVELS 1 AND 2