# ASYMPTOTIC NON-NULL DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRIX UNDER LOCAL ALTERNATIVES

by

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1. Introduction. Asymptotic expansions of the distributions of the likelihood ratio (= LR) criteria based on a random sample from a multivariate normal population under fixed alternative hypothesis have been derived by Sugiura [11], (1) for the equality of covariance matrix to a given matrix, (2) for the equality of mean vector and covariance matrix to a given vector and a given matrix, and also by Sugiura and Fujikoshi [12], (3) for testing the hypothesis of independence between two sets of variates. The limiting non-null distribution of the LR criterion (4) for the equality of several covariance matrices has been obtained by Sugiura [11]. These limiting non-null distributions always degenerate at the null hypothesis so that the asymptotic formulas do not give good approximations when the alternative hypothesis is near to the null hypothesis, as we have experienced in calculating the approximate powers of Bartlett's test for homogeneity of variances in Sugiura and Nagao [14].

In this paper, we shall derive limiting non-null distributions of the LR criteria for the problems (1) and (2) under sequences of alternatives converging to the null hypothesis with the rate of convergence $N^{-\gamma}$, where $N$ means sample size, for arbitrary positive number $\gamma$ and then asymptotic expansions of the non-null distributions in the case of $\gamma = \frac{1}{2}$ and $\gamma = 1$ in the next two Sections. With the help of the hypergeometric function of matrix argument due to Constantine [2], we shall derive an asymptotic expansion of the distribution for the problem (3) in the case $\gamma = \frac{1}{2}$ in Section 4, and an

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asymptotic expansion of the distribution of the modified LR criterion, in Sugiura and Nagao [13], for the equality of two covariance matrices under the sequence of alternatives with $\gamma = 1$ in Section 5. The formulas in this paper can be applied to compute the approximate power, when the alternative hypothesis is near to the null hypothesis.

## 2. Asymptotic distributions of the modified LR criterion for $\Sigma = \Sigma_0$.

2.1. Moments of the criterion. Let the $p \times 1$ vectors $X_1, X_2, \ldots, X_N$ be a random sample from a p-variate normal distribution with unknown mean vector and covariance matrix $\Sigma$. We shall consider the modified LR criterion $\lambda_1^*$, instead of the LR criterion, for testing the hypothesis $H: \Sigma = \Sigma_0$ (a given positive definite matrix) against alternatives $K: \Sigma \neq \Sigma_0$, which is given by

$$(2.1) \qquad \lambda_1^* = \left(\frac{e}{n}\right)^{np/2} |S\Sigma_0^{-1}|^{n/2} \operatorname{etr}\{-\frac{1}{2} \Sigma_0^{-1} S\},$$

where the symbol etr means exp · tr and $S = \sum_{j=1}^{N} (X_j - \overline{X})(X_j - \overline{X})'$ with $\overline{X} = \sum_{j=1}^{N} X_j/N$ and $n = N-1$. The unbiasedness of this modified LR criterion was proved by Sugiura and Nagao [13], the monotonicity property of which was established by Nagao [6]. The h-th moment of the statistic $\lambda_1^*$ under K can be found in Anderson [1, p.266] as

$$(2.2) \qquad E[\lambda_1^{*h}|K] = \left(\frac{2e}{n}\right)^{nph/2} \frac{\Gamma_p(\frac{1}{2}n(1+h))}{\Gamma_p(\frac{1}{2}n)} \frac{|\Sigma\,\Sigma_0^{-1}|^{\frac{1}{2}nh}}{|I + h\Sigma\,\Sigma_0^{-1}|^{\frac{1}{2}n(1+h)}},$$

where the function $\Gamma_p(x)$ is defined by $\pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma(x - \frac{j-1}{2})$.

2.2. Asymptotic distributions when $\gamma < \frac{1}{2}$. First we shall consider the limiting distribution of the statistic $-2\log\lambda_1^*$ under the sequence of alternatives $K_\gamma$: $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + n^{-\gamma}\,\Theta$ with $\gamma > 0$ and positive definite (= pd) matrix $\Theta$. When $\gamma < \frac{1}{2}$, we can write the characteristic function of $(-\,2\log\lambda_1^*)/n^{1/2-\gamma}$ as

$$\left(\frac{2e}{n}\right)^{-itpn^{\gamma+\frac{1}{2}}} \frac{\Gamma_p(\frac{1}{2}n-itn^{\gamma+\frac{1}{2}})}{\Gamma_p(\frac{1}{2}n)} \left(1 - 2itn^{\gamma-\frac{1}{2}}\right)^{-\frac{1}{2}np+itpn^{\gamma+\frac{1}{2}}}$$

(2.3)

$$\cdot\ |\Sigma\,\Sigma_0^{-1}|^{-itn^{\gamma+\frac{1}{2}}}\ \left|I - \frac{2itn^{-1/2}}{1-2itn^{\gamma-1/2}}\,\Theta\right|^{-\frac{1}{2}n+itn^{\gamma+\frac{1}{2}}}.$$

Applying the asymptotic formula $\log\Gamma(x+h) = \log\sqrt{2\pi} + (x+h-\frac{1}{2})\log x - x + O(|x|^{-1})$ to each gamma function in the first factor of (2.3), we can see that the first factor is equal to $1 + O(n^{\gamma-1/2})$, which tends to one as $n$ tends to infinity. By the formula

(2.4) $$-\log|I - n^{-1}Z| = \sum_{j=1}^{\ell} n^{-j}\,\mathrm{tr}\,Z^j/j + O(n^{-\ell-1}),$$

which is valid for large $n$ such that all characteristic roots of the symmetric matrix $n^{-1}Z$ have absolute values less than one, the second factor of (2.3) can be evaluated as

(2.5) $$\exp[it\{\sqrt{n}\,\mathrm{tr}\,\Theta - n^{\gamma+\frac{1}{2}}\log|\Sigma\,\Sigma_0^{-1}|\} - t^2\,\mathrm{tr}\,\Theta^2] + O(n^{\gamma-\frac{1}{2}}).$$

Hence the characteristic function of $n^{\gamma-1/2}[-\,2\log\lambda_1^* - n^{1-\gamma}\,\mathrm{tr}\,\Theta + n\log|\Sigma\,\Sigma_0^{-1}|]$ tends to $\exp[-\,t^2\,\mathrm{tr}\,\Theta^2]$ as $n$ tends to infinity, which implies the following theorem.

Theorem 2.1. Let $\lambda_1^*$ be the modified LR statistic given by (2.1). Then under the sequence of alternatives $K_\gamma$: $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + n^{-\gamma}\,\Theta$ for $0 < \gamma < \frac{1}{2}$, the limiting distribution of the statistic

$$n^{\gamma-\frac{1}{2}}[-\,2\log\lambda_1^* - n\{\mathrm{tr}(\Sigma\,\Sigma_0^{-1} - I) - \log|\Sigma\,\Sigma_0^{-1}|\}] \tag{2.6}$$

is normal with mean zero and variance $2\mathrm{tr}\Theta^2$ as $n$ tends to infinity.

Noting that the asymptotic variance $2\mathrm{tr}\Theta^2$ is equal to $2\mathrm{tr}(\Sigma\,\Sigma_0^{-1} - I)^2 \cdot n^{2\gamma}$, we can see that the limiting distribution in Theorem 2.1 is of the same form as under fixed alternative hypothesis given by Sugiura [11]. It may be noted that Theorem 2.1 holds if we replace the modified LR statistic $\lambda_1^*$ to the LR statistic.

2.3. Asymptotic distributions when $\gamma \geqq \frac{1}{2}$. We shall now consider the sequence of alternative hypothesis $K_\gamma$ for $\gamma \geqq \frac{1}{2}$. From (2.2), we can get the characteristic function of $-\,2\log\lambda_1^*$ under $K_\gamma$ as

$$C(t) = \left(\frac{n}{2e}\right)^{npit} \frac{\Gamma_p(\frac{1}{2}n(1-2it))}{\Gamma_p(\frac{1}{2}n)(1-2it)^{\frac{1}{2}np(1-2it)}} \cdot \frac{|I + n^{-\gamma}\,\Theta|^{-nit}}{|I - \frac{2it}{1-2it}\,n^{-\gamma}\,\Theta|^{\frac{1}{2}n(1-2it)}}\,. \tag{2.7}$$

The first factor of $C(t)$ in (2.7) is simply the characteristic function of $-\,2\log\lambda_1^*$ under the null hypothesis, which was expanded asymptotically by Sugiura [11] as

(2.8)
$$(1-2it)^{-p(p+1)/4}[1 + \frac{B_2}{n}\{\frac{1}{1-2it} - 1\} + \frac{1}{6n^2}\{\frac{3B_2^2-4B_3}{(1-2it)^2} - \frac{6B_2^2}{1-2it} + 3B_2^2 + 4B_3\} + O(n^{-3})],$$

where

$$\begin{aligned} B_2 &= p(2p^2 + 3p - 1)/24 \\ B_3 &= -\,p(p-1)(p+1)(p+2)/32\,. \end{aligned} \tag{2.9}$$

Applying the asymptotic formula (2.4) to the second factor of the characteristic function (2.7), we can get

$$(2.10)\qquad \exp[-\,nit\,\log|I + n^{-\gamma}\,\Theta| - \frac{n}{2}\,(1-2it)\log|I - \frac{2it}{1-2it}\,n^{-\gamma}\,\Theta|]$$

$$= \exp\left[\sum_{k=2}^{\ell} \frac{\operatorname{tr}\Theta^k}{n^{\gamma k-1}}\left\{(-1)^k it + \frac{2^{k-1}\,(it)^k}{(1-2it)^{k-1}}\right\} + O(n^{(\ell+1)\gamma-1})\right]$$

$$= \exp[n^{1-2\gamma}\,\frac{it\,\operatorname{tr}\Theta^2}{2(1-2it)} + n^{1-3\gamma}\,\frac{\operatorname{tr}\Theta^3}{6}\left\{\frac{1}{(1-2it)^2} - \frac{3}{1-2it} + 2\right\}$$

$$+\, n^{1-4\gamma}\,\frac{\operatorname{tr}\Theta^4}{8}\left\{\frac{1}{(1-2it)^3} - \frac{4}{(1-2it)^2} + \frac{6}{1-2it} - 3\right\} + O(n^{1-5\gamma})].$$

Multiplying (2.8) and (2.10), we can see that

$$(2.11)\qquad \begin{aligned} C(t) &= (1-2it)^{-p(p+1)/4} + o(1) && \text{when}\quad \gamma > \tfrac{1}{2} \\ &= (1-2it)^{-p(p+1)/4}\,\exp\left\{\frac{it\,\operatorname{tr}\Theta^2}{2(1-2it)}\right\} + o(1) && \text{when}\quad \gamma = \tfrac{1}{2}\,, \end{aligned}$$

which implies that the limiting distribution of $-\,2\log\lambda_1^*$ is $\chi^2$ with $f_1 = p(p+1)/2$ degrees of freedom when $\gamma > \frac{1}{2}$ and noncentral $\chi^2$ with $f_1$ degrees of freedom and noncentrality parameter $\operatorname{tr}\Theta^2/4$ when $\gamma = \frac{1}{2}$. Further, we can get the asymptotic expansion of the characteristic function of $-\,2\log\lambda_1^*$ when $\gamma = 1$ as

$$(2.12)\qquad C(t) = (1-2it)^{-f_1/2}\,[1 + \frac{1}{n}\left\{\frac{1}{4}\operatorname{tr}\Theta^2 + B_2\right\}\left\{\frac{1}{1-2it} - 1\right\}$$

$$+\,\frac{1}{6n^2}\left\{\frac{3B_2^2-4B_3}{(1-2it)^2} - \frac{6B_2^2}{1-2it} + 3B_2^2 + 4B_3\right\} + \frac{1}{n^2}\sum_{\alpha=0}^{2} g_{2\alpha}(1-2it)^{-\alpha}] + O(n^{-3})\;,$$

where

$$(2.13)\qquad \begin{aligned} g_0 &= \frac{B_2}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{3}\,\mathrm{tr}\,\Theta^3 + \frac{1}{32}\,(\mathrm{tr}\,\Theta^2)^2 \\ g_2 &= -\frac{B_2}{2}\,\mathrm{tr}\,\Theta^2 - \frac{1}{2}\,\mathrm{tr}\,\Theta^3 - \frac{1}{16}\,(\mathrm{tr}\,\Theta^2)^2 \\ g_4 &= \frac{B_2}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{6}\,\mathrm{tr}\,\Theta^3 + \frac{1}{32}\,(\mathrm{tr}\,\Theta^2)^2 . \end{aligned}$$

When $\gamma = \frac{1}{2}$, we can get another asymptotic formula for $C(t)$ from (2.8) and (2.10) as

$$(2.14)\qquad C(t) = (1-2it)^{-f_1/2}\exp\left\{\frac{it\,\mathrm{tr}\,\Theta^2}{2(1-2it)}\right\}\cdot\Big[1 + \frac{\mathrm{tr}\,\Theta^3}{6\sqrt{n}}\Big\{\frac{1}{(1-2it)^2} - \frac{3}{1-2it} + 2\Big\} + \frac{1}{n}\sum_{\alpha=0}^{4} h_{2\alpha}(1-2it)^{-\alpha}\Big] + O(n^{-3/2}) ,$$

where the coefficients $h_{2\alpha}$ $(\alpha = 0, \ldots, 4)$ are given by

$$(2.15)\qquad \begin{aligned} h_0 &= -B_2 - \frac{3}{8}\,\mathrm{tr}\,\Theta^4 + \frac{1}{18}\,(\mathrm{tr}\,\Theta^3)^2 \\ h_2 &= B_2 + \frac{3}{4}\,\mathrm{tr}\,\Theta^4 - \frac{1}{6}\,(\mathrm{tr}\,\Theta^3)^2 \\ h_4 &= -\frac{1}{2}\,\mathrm{tr}\,\Theta^4 + \frac{13}{72}\,(\mathrm{tr}\,\Theta^3)^2 \\ h_6 &= \frac{1}{8}\,\mathrm{tr}\,\Theta^4 - \frac{1}{12}\,(\mathrm{tr}\,\Theta^3)^2 \\ h_8 &= \frac{1}{72}\,(\mathrm{tr}\,\Theta^3)^2 . \end{aligned}$$

Inverting the characterisitc function (2.8), we have

$$(2.16)\qquad P_H(-2\log\lambda_1^* < z) = P(\chi^2_{f_1} < z) + \frac{B_2}{n}\{P(\chi^2_{f_1+2} < z) - P(\chi^2_{f_1} < z)\}$$
$$+ \frac{1}{6n^2}\{(3B_2^2 - 4B_3)P(\chi^2_{f_1+4} < z) - 6B_2^2P(\chi^2_{f_1+2} < z) + (3B_2^2 + 4B_3)P(\chi^2_{f_1} < z)\}$$
$$+ O(n^{-3}),$$

where $\chi^2_{f_1}$ means a $\chi^2$ variate with $f_1$ degrees of freedom. It follows from (2.11) and the characteristic functions (2.12), (2.14) that the following theorem holds.

Theorem 2.2. Under the sequence of alternative hypothesis $K_\gamma$: $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + n^{-\gamma}\,\Theta$ ($\Theta$ is pd), the limiting distributions of the modified LR statistic $-2\log\lambda_1^*$ given by (2.1) is $\chi^2$ with $f_1 = p(p+1)/2$ degrees of freedom, when $\gamma > \frac{1}{2}$, and noncentral $\chi^2$ with $f_1$ degrees of freedom and noncentrality parameter $\operatorname{tr}\Theta^2/4$, when $\gamma = \frac{1}{2}$. When $\gamma = 1$,

$$(2.17)\quad P_\gamma(-2\log\lambda_1^* < z) = P_H(-2\log\lambda_1^* < z) + \frac{1}{4n}\operatorname{tr}\Theta^2\,\{P(\chi^2_{f_1+2} < z) - P(\chi^2_{f_1} < z)\} + \frac{1}{n^2}\sum_{\alpha=0}^{2} g_{2\alpha}\,P(\chi^2_{f_1+2\alpha} < z) + O(n^{-3}),$$

where $P_H(-2\log\lambda_1^* < z)$ is given by (2.16) and the coefficients $g_{2\alpha}$ ($\alpha = 0, 1, 2$) are given by (2.13) with $B_2$ and $B_3$ in (2.9). When $\gamma = \frac{1}{2}$, we have

$$(2.18)\quad P_\gamma(-2\log\lambda_1^* < z) = P(\chi^2_{f_1}(\delta^2) < z) + \frac{\operatorname{tr}\Theta^3}{6\sqrt{n}}\{P(\chi^2_{f_1+4}(\delta^2) < z) - 3P(\chi^2_{f_1+2}(\delta^2) < z) + 2P(\chi^2_{f_1}(\delta^2) < z)\} + \frac{1}{n}\sum_{\alpha=0}^{4} h_{2\alpha}P(\chi^2_{f_1+2\alpha}(\delta^2) < z) + O(n^{-3/2}),$$

where the symbol $\chi^2_{f_1}(\delta^2)$ means the noncentral $\chi^2$ variate with $f_1 = p(p+1)/2$ degrees of freedom and noncentrality parameter $\delta^2 = \operatorname{tr}\Theta^2/4$, and the $h_{2\alpha}$ are given by(2.15).

2.4. Numerical examples. It may be useful to note that applying the general inverse expansion formula of Hill and Davis [4] to the asymptotic null-distribution of $-2\log\lambda_1^*$ given by Theorem 2.1 in Sugiura [11], we can get an asymptotic formula of the $100\alpha$ % point of $-2\log\lambda_1^*$ in terms of the $100\alpha$ % point of the $\chi^2$ distribution with $f_1 = p(p+1)/2$ degrees of freedom as

$$(2.19)\quad u + \frac{2B_2}{nf_1}u + \frac{1}{3n^2}\left\{\frac{u^2}{f_1^2(f_1+2)}(-6B_2^2 - 4f_1B_3) + \frac{u}{f_1^2}(6B_2^2 - 4f_1B_3)\right\}$$

$$+ \frac{1}{n^3}(u^3g_3 + u^2g_2 + ug_1) + O(n^{-4}),$$

where $u$ is so chosen that $P(\chi^2_{f_1} > u) = \alpha$ and

$$(2.20)\quad \begin{aligned} g_1 &= \frac{4}{3f_1^3}(f_1^2B_4 - 2f_1B_2B_3 + B_2^3) \\ g_2 &= \frac{4}{3f_1^3(f_1+2)}(f_1^2B_4 - 2f_1B_2B_3 - 5B_2^3) \\ g_3 &= \frac{4}{3f_1^3(f_1+2)(f_1+4)}(f_1^2B_4 + 4f_1B_2B_3 + 4B_2^3) \end{aligned}$$

with $B_2$, $B_3$ in (2.9) and $B_4 = p(6p^4 + 15p^3 - 10p^2 - 30p + 3)/480$.

We shall examine the effectiveness of these formulas in the following simple examples.

Example 2.1. When $p = 1$ and $n = 10$, the exact 5 % point of $-2\log\lambda_1^*$ can be obtained from Table I by Pachares [7] as 3.9682. The asymptotic formula (2.19) gives

5 % points of $-2\log\lambda_1^*$

| | $p = 1$ and $n = 10$ | $p = 2$ and $n = 100$ |
|---|---|---|
| first term | 3.84146 | 7.81473 |
| term of order $n^{-1}$ | 0.12805 | 0.05644 |
| term of order $n^{-2}$ | -0.00060 | 0.00055 |
| term of order $n^{-3}$ | -0.00059 | 0.00001 |
| approx. value | 3.9683 | 7.87173 |
| exact value | 3.9682 | |

Our asymptotic formula (2.19) gives good approximations for the percentage points.

Example 2.2. When $p = 2$ and $n = 100$, the approximate 5 % point of $-2\log\lambda_1^*$ is given by Example 2.1 as 7.87173. The asymptotic powers should be computed by the different formulas in Theorem 2.2 according to the departure of the alternative hypothesis K from the null hypothesis. The non-centrality parameter $\delta^2 = \mathrm{tr}\,\Theta^2/4$ in Theorem 2.2 may be used as a measure of the "distance" of K from the null hypothesis. Let us specify the alternatives K as $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + \Delta I$. Then $\delta^2 = n\Delta^2/2$ and the following case 1 ($\Delta = 0.5$) is computed by the formula (2.16) in Sugiura [11] with the normal distribution function and its derivatives, as well as the cases 2 ($\Delta = 0.1$) and 3 ($\Delta = 0.02$) by the formulas (2.18) and (2.17) respectively.

approximate powers, when $p = 2$ and $n = 100$

| | $\Delta = 0.5$ | $\Delta = 0.1$ | $\Delta = 0.02$ |
|---|---|---|---|
| $\delta^2$ | 12.5 | 0.5 | 0.02 |
| first term | 0.8651 | 0.1134 | 0.05 |
| second term | 0.0714 | -0.0033 | 0.00229 |
| third term | 0.0052 | 0.0018 | 0.00001 |
| approx. power | 0.942 | 0.112 | 0.0523 |

## 3. Asymptotic distributions of the LR criterion for $\Sigma = \Sigma_0$ and $\mu = \mu_0$.

3.1. Moments of the criterion. Let a random sample of size N from a p-variate normal population with mean vector $\mu$ ($p\times 1$) and covariance matrix $\Sigma$ ($p\times p$, pd) be given. We wish to test the hypothesis H: $\mu = \mu_0$ (a given vector) and $\Sigma = \Sigma_0$ (a given pd matrix) against alternatives K: $\mu \neq \mu_0$ and/or $\Sigma \neq \Sigma_0$. The h-th moment of the LR criterion $\lambda_2$ for this problem is given by Sugiura [11] as

$$(3.1)\qquad E[\lambda_2^h|K] = \left(\frac{2e}{N}\right)^{Nph/2} \frac{\Gamma_p((n+Nh)/2)}{\Gamma_p(n/2)} \frac{|\Sigma\,\Sigma_0^{-1}|^{Nh/2}}{|I + h\,\Sigma\,\Sigma_0^{-1}|^{N(1+h)/2}}$$

$$\times\ \mathrm{etr}[-\frac{Nh}{2}\,\Sigma_0^{-1}(\mu-\mu_0)(\mu-\mu_0)'\ \{I - h\,\Sigma_0^{-1}(\Sigma^{-1} + h\,\Sigma_0^{-1})^{-1}\}],$$

where $n = N-1$. The unbiasedness of the LR criterion $\lambda_2$ without modification was proved by Sugiura and Nagao [13]. The asymptotic expansions of the distributions of $-2\log\lambda_2$ both under the null hypothesis and a fixed alternative hypothesis have been derived by Sugiura [11].

3.2. Asymptotic distributions when $\gamma < \frac{1}{2}$. Now we shall specify the sequence of alternative hypotheses as $K_\gamma$: $\mu - \mu_0 = N^{-\gamma}\,\Sigma_0^{1/2}\,\nu$ and $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + N^{-\gamma}\,\Theta$, and consider the case $0 < \gamma < \frac{1}{2}$. From (3.1) we can express the characteristic function of $(-2\log\lambda_2)N^{\gamma-1/2}$ as

$$(3.2)\qquad \left(\frac{2e}{N}\right)^{-pitN^{\gamma+\frac{1}{2}}} \frac{\Gamma_p(\frac{1}{2}N - itN^{\gamma+\frac{1}{2}} - \frac{1}{2})}{\Gamma_p(\frac{1}{2}N - \frac{1}{2})(1 - 2itN^{\gamma-\frac{1}{2}})^{\frac{1}{2}Np-itpN^{\gamma+\frac{1}{2}}}}$$

$$\times\ |\Sigma\,\Sigma_0^{-1}|^{-itN^{\gamma+\frac{1}{2}}}\ \left|I - \frac{2itN^{-\frac{1}{2}}}{1-2itN^{\gamma-\frac{1}{2}}}\,\Theta\right|^{-\frac{1}{2}N+itN^{\gamma+\frac{1}{2}}}$$

$$\times\ \mathrm{etr}[itN^{\frac{1}{2}-\gamma}\,\nu'\nu - 2t^2\,\nu'(I - 2itN^{\gamma-\frac{1}{2}}\,\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2})^{-1}(I + N^{-\gamma}\,\Theta)\nu].$$

The first factor and the second factor have the same form as the characteristic function (2.3). Hence we can see that the first factor is equal to $1 + O(N^{\gamma-1/2})$, which tends to one as $N \to \infty$, and the second factor is given by (2.5) after substituting $n$ for $N$. The third factor is easily evaluated as $\mathrm{etr}[itN^{1/2-\gamma}\, \nu'\nu - 2t^2\, \nu'\nu] + o(1)$. It follows that the characteristic function of $(-2\log\lambda_2)N^{\gamma-1/2}$ can be expressed as

$$(3.3)\quad \exp[it\{\sqrt{N}\mathrm{tr}\Theta - N^{\gamma+\frac{1}{2}}\log|\Sigma\, \Sigma_0^{-1}| + N^{\frac{1}{2}-\gamma}\, \nu'\nu\} - t^2(\mathrm{tr}\Theta^2 + 2\nu'\nu)] + o(1)\ ,$$

which implies that the characteristic function of the statistic

$$(3.4)\quad N^{\gamma-\frac{1}{2}}[-2\log\lambda_2 - N\{\mathrm{tr}(\Sigma\, \Sigma_0^{-1} - I) - \log|\Sigma\, \Sigma_0^{-1}| + (\mu - \mu_0)'\, \Sigma_0^{-1}(\mu - \mu_0)\}]$$

tends to $\exp[-t^2(\mathrm{tr}\ \Theta^2 + 2\nu'\nu)]$ as $N \to \infty$. Thus we can get the following theorem.

<u>Theorem</u> <u>3.1</u>. Under the sequence of alternatives $K_\gamma$: $\mu - \mu_0 = N^{-\gamma}\, \Sigma_0^{1/2}\nu$ and $\Sigma_0^{-1/2}\, \Sigma\, \Sigma_0^{-1/2} = I + N^{-\gamma}\, \Theta$, for $0 < \gamma < \frac{1}{2}$, the limiting distribution of the statistic (3.4) is normal with mean zero and variance $2\mathrm{tr}\ \Theta^2 + 4\nu'\nu$ as $N$ tends to infinity.

The limiting distribution in the above theorem is of the same form as under fixed alternative hypothesis given by Sugiura [11].

3.3. <u>Asymptotic distributions when</u> $\gamma \geqq \frac{1}{2}$. We shall now consider the asymptotic distribution of $-2\log\lambda_2$ under $K_\gamma$ when $\gamma \geqq \frac{1}{2}$. From (3.1), the characteristic function of $-2\log\lambda_2$ can be expressed as

$$(3.5)\qquad \left(\frac{2e}{N}\right)^{-Npit} \frac{\Gamma_p(\frac{1}{2}N(1-2it) - \frac{1}{2})}{\Gamma_p(\frac{1}{2}N - \frac{1}{2})(1-2it)^{Np(1-2it)/2}} \cdot \frac{|I + N^{-\gamma}\, \Theta|^{-Nit}}{|I - \frac{2it}{1-2it}N^{-\gamma}\, \Theta|^{N(1-2it)/2}}$$

$$\cdot \operatorname{etr}\{itN^{1-2\gamma}\nu'\nu - \frac{2t^2}{1-2it}N^{1-2\gamma}\nu'(I - \frac{2it}{1-2it}\frac{\Theta}{N^\gamma})^{-1}(I + \frac{\Theta}{N^\gamma})\nu\} .$$

The first factor is equal to the characteristic function of $-2\log\lambda_2$ under the null hypothesis, which was expanded asymptotically by Sugiura [11] as

(3.6)

$$(1-2it)^{-\frac{1}{2}p-\frac{p}{4}(p+1)}[1 + \frac{B_2'}{N}\{\frac{1}{1-2it} - 1\} + \frac{1}{6N^2}\{\frac{3B_2'^2-4B_3'}{(1-2it)^2} - \frac{6B_2'^2}{1-2it} + 3B_2'^2 + 4B_3'\} + O(N^{-3})],$$

where

$$B_2' = p(2p^2+9p+11)/24$$

(3.7)
$$B_3' = -p(p+1)(p+2)(p+3)/32 .$$

The second factor in (3.5) can be expanded by (2.10) after substituting $n$ for $N$. The third factor can be expanded asymptotically as

(3.8)

$$\operatorname{etr}[\frac{it}{1-2it}N^{1-2\gamma}\nu'\nu + \frac{2(it)^2}{(1-2it)^2}N^{1-3\gamma}\nu'\Theta\nu + \frac{4(it)^3}{(1-2it)^3}N^{1-4\gamma}\nu'\Theta^2\nu + O(N^{1-5\gamma})].$$

When $\gamma > \frac{1}{2}$, both the second and the third factors tend to one as $N \to \infty$. It follows from (3.6) that the characteristic function of $-2\log\lambda_2$ tends to $(1-2it)^{-f_2/2}$ where $f_2 = p + p(p+1)/2$ as $N \to \infty$. Thus the limiting distribution of $-2\log\lambda_2$ is $\chi^2$ with $f_2$ degrees of freedom. Moreover, we can get the asymptotic formula of the characteristic function when $\gamma = 1$ as

(3.9)
$$(1-2it)^{-f_2/2}[1 + \frac{1}{N}\{\frac{1}{4}\operatorname{tr}\Theta^2 + \frac{\nu'\nu}{2} + B_2'\}\{\frac{1}{1-2it} - 1\}$$
$$+ \frac{1}{6N^2}\{\frac{3B_2'^2-4B_3'}{(1-2it)^2} - \frac{6B_2'^2}{1-2it} + 3B_2'^2 + 4B_3'\} + \frac{1}{N^2}\sum_{\alpha=0}^{2} g_{2\alpha}'(1-2it)^{-\alpha} + O(N^{-3}),$$

where

$$g_0' = \frac{B_2'}{4}\,\mathrm{tr}\Theta^2 + \frac{1}{3}\,\mathrm{tr}\Theta^3 + \frac{1}{32}(\mathrm{tr}\Theta^2)^2 + \frac{\nu'\nu}{2}(\frac{1}{4}\mathrm{tr}\Theta^2+B_2') + \frac{1}{2}\nu'\Theta\nu + \frac{(\nu'\nu)^2}{8}$$

$$(3.10)\qquad g_2' = -\frac{B_2'}{2}\,\mathrm{tr}\Theta^2 - \frac{1}{2}\,\mathrm{tr}\Theta^3 - \frac{1}{16}(\mathrm{tr}\Theta^2)^2 - \nu'\nu(\frac{1}{4}\mathrm{tr}\Theta^2+B_2') - \nu'\Theta\nu - \frac{(\nu'\nu)^2}{4}$$

$$g_4' = \frac{B_2'}{4}\,\mathrm{tr}\Theta^2 + \frac{1}{6}\,\mathrm{tr}\Theta^3 + \frac{1}{32}(\mathrm{tr}\Theta^2)^2 + \frac{\nu'\nu}{2}(\frac{1}{4}\mathrm{tr}\Theta^2+B_2') + \frac{1}{2}\,\nu'\Theta\nu + \frac{(\nu'\nu)^2}{8}.$$

When $\gamma = \frac{1}{2}$, the second factor in (3.5) tends to $\exp[\frac{1}{2}\,it\,\mathrm{tr}\,\Theta^2/(1-2it)]$ as $N \to \infty$ and the third factor tends to $\exp[it\nu'\nu/(1-2it)]$ as $N \to \infty$. Hence from (3.6) the characteristic function of $-2\log\lambda_2$ tends to $(1-2it)^{-f_2/2} \cdot \exp[it\,\{\frac{1}{2}\,\mathrm{tr}\,\Theta^2 + \nu'\nu\}/(1-2it)]$ as $N \to \infty$, which implies that the limiting distribution is noncentral $\chi^2$ with $f_2$ degrees of freedom and noncentrality parameter $\delta^2 = \frac{1}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{2}\,\nu'\nu$. Further, we can get an asymptotic formula for the characteristic function of $-2\log\lambda_2$ as

$$(3.11)\qquad (1-2it)^{-f_2/2}\exp[\frac{it}{1-2it}(\nu'\nu + \frac{1}{2}\,\mathrm{tr}\,\Theta^2)]$$

$$\cdot\,[1 + \frac{1}{6\sqrt{N}}\{\frac{\mathrm{tr}\Theta^3+3\nu'\Theta\nu}{(1-2it)^2} - \frac{3\mathrm{tr}\Theta^3+6\nu'\Theta\nu}{1-2it} + 2\,\mathrm{tr}\,\Theta^3 + 3\nu'\Theta\nu\}$$

$$+ N^{-1}\sum_{\alpha=0}^{4} h_{2\alpha}'(1-2it)^{-\alpha} + O(N^{-3/2})],$$

where

$$h_0' = -B_2' - \frac{3}{8}\,\mathrm{tr}\,\Theta^4 + \frac{1}{18}(\mathrm{tr}\Theta^3)^2 - \frac{\nu'\Theta^2\nu}{2} + \frac{(\nu'\Theta\nu)^2}{8} + \frac{1}{6}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$h_2' = B_2' + \frac{3}{4}\,\mathrm{tr}\,\Theta^4 - \frac{1}{6}(\mathrm{tr}\Theta^3)^2 + \frac{3}{2}\,\nu'\Theta^2\nu - \frac{1}{2}(\nu'\Theta\nu)^2 - \frac{7}{12}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$(3.12)\qquad h_4' = -\frac{1}{2}\,\mathrm{tr}\,\Theta^4 + \frac{13}{72}(\mathrm{tr}\Theta^3)^2 - \frac{3}{2}\,\nu'\Theta^2\nu + \frac{3}{4}(\nu'\Theta\nu)^2 + \frac{3}{4}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$h_6' = \frac{1}{8}\,\mathrm{tr}\,\Theta^4 - \frac{1}{12}(\mathrm{tr}\Theta^3)^2 + \frac{1}{2}\,\nu'\Theta^2\nu - \frac{1}{2}(\nu'\Theta\nu)^2 - \frac{5}{12}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$h_8' = \frac{1}{72}(\mathrm{tr}\Theta^3)^2 + \frac{1}{8}(\nu'\Theta\nu)^2 + \frac{1}{12}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3\,.$$

Inverting the characteristic function (3.6) under the null hypothesis $H$, we have

(3.13) $$P_H(-2\log\lambda_2 < z) = P(\chi^2_{f_2} < z) + \frac{B'_2}{N}\{P(\chi^2_{f_2+2} < z) - P(\chi^2_{f_2} < z)\}$$
$$+ \frac{1}{6N^2}\{(3B'^2_2 - 4B'_3)P(\chi^2_{f_2+4} < z) - 6B'^2_2 P(\chi^2_{f_2+2} < z) + (3B'^2_2 + 4B'_3)P(\chi^2_{f_2} < z)\} + O(N^{-3}).$$

From the asymptotic formulas (3.9) and (3.11) for the characteristic function, we can get the following theorem.

Theorem 3.2. Under the sequence of alternatives $K_\gamma$: $\Sigma_0^{-1/2}(\mu - \mu_0) = N^{-\gamma}\nu$ and $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2} = I + N^{-\gamma}\Theta$, the limiting distribution of the LR statistic $-2\log\lambda_2$ is $\chi^2$ with $f_2 = p + p(p+1)/2$ degrees of freedom, when $\gamma > \frac{1}{2}$ and noncentral $\chi^2$ with $f_2 = p + p(p+1)/2$ degrees of freedom and noncentrality parameter $\delta^2 = \frac{1}{4}\operatorname{tr}\Theta^2 + \frac{1}{2}\nu'\nu$, when $\gamma = \frac{1}{2}$. When $\gamma = 1$, we have

(3.14)
$$P_\gamma(-2\log\lambda_2 < z) = P_H(\chi^2_{f_2} < z) + \frac{1}{N}\{\frac{1}{4}\operatorname{tr}\Theta^2 + \frac{1}{2}\nu'\nu\}\{P(\chi^2_{f_2+2} < z) - P(\chi^2_{f_2} < z)\}$$
$$+ \frac{1}{N^2}\sum_{\alpha=0}^{2} g'_{2\alpha} P(\chi^2_{f_2+2\alpha} < z) + O(N^{-3}),$$

where $g_0$, $g_2$ and $g_4$ are given by (3.10) and $P_H(\chi^2_{f_2} < z)$ is given by (3.13). When $\gamma = \frac{1}{2}$, we have

(3.15) $$P_\gamma(-2\log\lambda_2 < z) = P(\chi^2_{f_2}(\delta^2) < z) + \frac{1}{6\sqrt{N}}\{(\operatorname{tr}\Theta^3 + 3\nu'\Theta\nu)P(\chi^2_{f_2+4}(\delta^2) < z)$$
$$- (3\operatorname{tr}\Theta^3 + 6\nu'\Theta\nu)P(\chi^2_{f_2+2}(\delta^2) < z) + (2\operatorname{tr}\Theta^3 + 3\nu'\Theta\nu)P(\chi^2_{f_2}(\delta^2) < z)\}$$
$$+ N^{-1}\sum_{\alpha=0}^{4} h'_{2\alpha} P(\chi^2_{f_2+2\alpha}(\delta^2) < z) + O(N^{-3/2}),$$

where $h_{2\alpha}$ $(\alpha = 0, \ldots, 4)$ are given by (3.12).

3.4. <u>More general sequences of alternatives</u>. If we consider the sequences of alternatives $K_{\gamma_1\gamma_2}$ : $\Sigma_0^{-1/2}(\mu - \mu_0) = N^{-\gamma_1}\nu$ and $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + N^{-\gamma_2}\Theta$, the asymptotic distribution of $-2\log\lambda_2$ can be obtained by the same argument as above.

<u>Theorem 3.3</u>. Under the sequences of alternatives $K_{\gamma_1\gamma_2}(\gamma_1 \neq \gamma_2)$, the limiting distributions of the LR statistic $-2\log\lambda_2$ are given in the following.

(1) When $\gamma_2 < \frac{1}{2}$ and $\gamma_2 < \gamma_1$, the statistic

$$(3.16)\quad \frac{1}{N^{\frac{1}{2}-\gamma_2}}[-2\log\lambda_2 - N\{\mathrm{tr}(\Sigma\,\Sigma_0^{-1} - I) - \log|\Sigma\,\Sigma_0^{-1}| + (\mu - \mu_o)'\,\Sigma_0^{-1}(\mu - \mu_0)\}]$$

is distributed asymptotically according to the normal distribution with mean zero and variance $2\mathrm{tr}\,\Theta^2$. If further $\frac{1}{2} \leq \gamma_1$, the term $(\mu - \mu_0)'\,\Sigma_0^{-1}(\mu - \mu_0)$ in (3.16) may be omitted.

(2) When $\gamma_2 = \frac{1}{2} < \gamma_1$, $-2\log\lambda_2$ has asymptotically the noncentral $\chi^2$ distribution with $f_2 = p + p(p+1)/2$ degrees of freedom and noncentrality parameter $\mathrm{tr}\,\Theta^2/4$.

(3) When $\frac{1}{2} < \gamma_2, \gamma_1$, $-2\log\lambda_2$ has asymptotically the $\chi^2$ distribution with $f_2 = p + p(p+1)/2$ degrees of freedom.

(4) When $\gamma_1 < \frac{1}{2}$ and $\gamma_1 < \gamma_2$, the statistic

$$(3.17)\quad \frac{1}{N^{\frac{1}{2}-\gamma_1}}[-2\log\lambda_2 - N\{\mathrm{tr}(\Sigma\,\Sigma_0^{-1} - I) - \log|\Sigma\,\Sigma_0^{-1}| + (\mu - \mu_0)'\,\Sigma_0^{-1}(\mu - \mu_0)\}]$$

has asymptotically the normal distribution with mean zero and variance $4\nu'\nu$. If further $\frac{1}{2} \leq \gamma_2$, the term $\mathrm{tr}(\Sigma\,\Sigma_0^{-1} - I) - \log|\Sigma\,\Sigma_0^{-1}|$ in (3.17) may be omitted.

(5) When $\gamma_1 = \frac{1}{2} < \gamma_2$, $-2\log\lambda_2$ has asymptotically the noncentral $\chi^2$ distribution with $f_2$ degrees of freedom and noncentrality parameter $\nu'\nu/2$.

The above theorem shows that the limiting distributions are normal when $\min(\gamma_1, \gamma_2) < \frac{1}{2}$, noncentral $\chi^2$ when $\min(\gamma_1, \gamma_2) = \frac{1}{2}$ and $\chi^2$ when $\min(\gamma_1, \gamma_2) > \frac{1}{2}$.

3.5. Numerical examples. Since the asymptotic null-distribution of $-2\log\lambda_2$ has the same form as that in the previous section as was shown by Sugiura [11], the asymptotic formula (2.19) for the percentage point can be used, also, in this case after changing $n$ to $N$ and $B_\alpha$ to $B'_\alpha$ ($\alpha = 2, 3, 4$) in (3.7) with $B'_4 = p(6p^4 + 45p^3 + 110p^2 + 90p + 3)/480$ and putting $f_1 \to f_2 = p(p+1)/2 + p$.

Example 3.1. When $N = 100$ and $p = 2$, we have the following approximations to the 5 % point.

| | |
|---|---|
| first term | 11.0705 |
| term of order $N^{-1}$ | 0.1365 |
| term of order $N^{-2}$ | 0.0024 |
| term of order $N^{-3}$ | 0.0000 |
| approx. value | 11.2094 |

For the alternative hypothesis $K$: $\Sigma_0^{-1/2}\, \Sigma\, \Sigma_0^{-1/2} = \mathrm{diag}(1.1, 0.9)$ and $\Sigma_0^{-1/2}(\mu - \mu_0) = (0.05, 0.05)'$, the following approximate powers are computed by the formulas (3.15) and (3.14), based on the above 5 % point.

| approximate power | (3.15) | (3.14) |
|---|---|---|
| first term | 0.1217 | 0.0500 |
| second term | 0 | 0.0610 |
| third term | 0.0036 | 0.0150 |
| approx. power | 0.125 | 0.127 |

## 4. Asymptotic distributions of the LR criterion for independence.

4.1. Preliminaries. Sugiura and Fujikoshi [12] have proved that the moments of the LR statistic $\lambda_3$ for testing the independence between $p_1$ and $p_2$ sets of variates $(p_1 \leqq p_2)$ in a p-variate $(p = p_1 + p_2)$ normal population, based on a random sample of size $N$, could be expressed, under alternative hypothesis, by using the hypergeometric function with matrix argument due to Constantine [2] as

$$(4.1) \qquad E[\lambda_3^{\frac{2}{N}h}] = \frac{\Gamma_{p_1}(h+\frac{1}{2}(N-p_2-1))\Gamma_{p_1}(\frac{1}{2}(N-1))}{\Gamma_{p_1}(\frac{1}{2}(N-p_2-1))\Gamma_{p_1}(\frac{1}{2}(N-1)+h)} |I - P^2|^{\frac{1}{2}(N-1)}$$

$$\cdot \; {}_2F_1(\tfrac{1}{2}(N-1),\ \tfrac{1}{2}(N-1);\ h + \tfrac{1}{2}(N-1);\ P^2),$$

where $P^2 = \text{diag}(\rho_1^2, \rho_2^2, \ldots, \rho_{p_1}^2)$ and $\rho_\alpha$ $(\alpha = 1, 2, \ldots, p_1)$ are the population canonical correlations. The following lemma for zonal polynomials due to Fujikoshi [3] and Sugiura and Fujikoshi [12] will be used later.

Lemma 4.1. Let $C_\kappa(Z)$ be a zonal polynomial corresponding to the partition $\kappa = \{k_1, k_2, \ldots, k_p\}$ with $k_1 + k_2 + \ldots + k_p = k$ and $k_1 \geqq k_2 \geqq \ldots \geqq k_p \geqq 0$. Putting

$$a_1(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(k_\alpha - \alpha)$$

(4.2)

$$a_2(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(4k_\alpha^2 - 6\alpha k_\alpha + 3\alpha^2) ,$$

then the following equalities hold.

$$\sum_{k=\ell}^{\infty} \sum_{(\kappa)} C_\kappa(Z)/(k-\ell)! = (\mathrm{tr}Z)^\ell \,\mathrm{etr}\, Z, \quad \text{for } \ell = 0, 1, 2, \ldots \tag{4.3}$$

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} a_1(\kappa)\, C_\kappa(Z)/k! = (\mathrm{tr}Z^2)\mathrm{etr}\, Z \tag{4.4}$$

$$\sum_{k=1}^{\infty} \sum_{(\kappa)} a_1(\kappa)\, C_\kappa(Z)/(k-1)! = (2\mathrm{tr}Z^2 + \mathrm{tr}Z^2\mathrm{tr}Z)\ \mathrm{etr}\, Z \tag{4.5}$$

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} a_1(\kappa)^2 C_\kappa(Z)/k! = \{(\mathrm{tr}Z^2)^2 + 4\mathrm{tr}Z^3 + \mathrm{tr}Z^2 + (\mathrm{tr}Z)^2\}\mathrm{etr}\, Z \tag{4.6}$$

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} a_2(\kappa) C_\kappa(Z)/k! = \{4\mathrm{tr}Z^3 + 3\mathrm{tr}Z^2 + 3(\mathrm{tr}Z)^2 + \mathrm{tr}Z\}\ \mathrm{etr}\, Z. \tag{4.7}$$

It is also convenient to note the following lemma, the proof of which comes by considering $\log(a)_\kappa$, where $\kappa$ is a partition $\{k_1, k_2, \ldots, k_p\}$ of $k$ and

$$(a)_\kappa = \prod_{\alpha=1}^{p} \prod_{j=1}^{k_\alpha} (a - \tfrac{1}{2}(\alpha+1) + j) . \tag{4.8}$$

<u>Lemma</u> <u>4.2</u>.

$$(am + b)_\kappa = (am)^k \,[1 + \frac{1}{am}\{kb + \frac{1}{2}a_1(\kappa)\} \tag{4.9}$$

$$+ \frac{1}{24a^2m^2}\{12b^2k(k-1) + 12(k-1)ba_1(\kappa) + 3a_1(\kappa)^2 - a_2(\kappa) + k\} + 0(m^{-3})]$$

holds, where $a_1(\kappa)$ and $a_2(\kappa)$ are defined by (4.2).

4.2. Asymptotic distribution when $\gamma = \frac{1}{2}$. We shall now consider the asymptotic distribution of the LR statistic $-2\rho\log\lambda_3$ under the sequence of alternatives $K_\gamma$: $P = m^{-1/2}\,\Theta$, where the correction factor $\rho$ is given by $\rho = 1 - (p_1 + p_2 + 3)/2N$ and $m = \rho N$ (Anderson [1, p.239]). The characteristic function of $-2\rho\log\lambda_3$ under $K_\gamma$ can be written from (4.1) as

$$\frac{\Gamma_{p_1}(\frac{1}{2}m(1-2it) + \frac{1}{4}(p_1-p_2+1))\Gamma_{p_1}(\frac{1}{2}m + \Delta)}{\Gamma_{p_1}(\frac{1}{2}m + \frac{1}{4}(p_1-p_2+1))\Gamma_{p_1}(\frac{1}{2}m(1-2it) + \Delta)}$$

$$(4.10)\qquad \cdot\ |I - \frac{1}{m}\,\Theta^2|^{\frac{1}{2}m+\Delta} \cdot \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{\{(\frac{1}{2}m + \Delta)_\kappa\}^2}{(\frac{1}{2}m(1-2it) + \Delta)_\kappa m^k} \cdot \frac{C_\kappa(\Theta^2)}{k!}\ ,$$

with the abbreviated notation $\Delta = (p_1 + p_2 + 1)/4$. The first factor gives the characteristic function of $-2\rho\log\lambda_3$ under the null hypothesis $(\Theta = 0)$, which can be expanded asymptotically as

$$(4.11)\qquad (1-2it)^{-p_1p_2/2}\,[1 + \frac{p_1p_2}{48m^2}\,(p_1^2 + p_2^2 - 5)\{\frac{1}{(1-2it)^2} - 1\} + O(m^{-3})].$$

The second factor in (4.10) can be expanded by the formula (2.4) as

$$(4.12)\qquad |I - \frac{1}{m}\,\Theta^2|^{\frac{1}{2}m+\Delta} = \mathrm{etr}(-\frac{1}{2}\,\Theta^2) \cdot [1 - \frac{1}{m}(\Delta\mathrm{tr}\Theta^2 + \frac{1}{4}\mathrm{tr}\Theta^4)$$

$$+ \frac{1}{m^2}\,\{\frac{1}{2}(\Delta\mathrm{tr}\Theta^2 + \frac{1}{4}\mathrm{tr}\Theta^4)^2 - \frac{1}{2}\,\Delta\,\mathrm{tr}\,\Theta^4 - \frac{1}{6}\,\mathrm{tr}\,\Theta^6\} + O(m^{-3})]\ .$$

The third factor in (4.10) can be expressed by Lemma 4.2 as

$$(4.13)\quad \sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{C_\kappa(\Theta^2)}{\{2(1-2it)\}^k}\cdot\frac{1}{k!}\,[1+\frac{1}{m}\{2k\Delta+a_1(\kappa)\}\,(2-\frac{1}{1-2it})$$

$$+\frac{1}{m^2}\{\frac{1}{2}(2-\frac{1}{1-2it})^2(2k\Delta+a_1(\kappa))^2+\frac{1}{3}(1-\frac{1}{2(1-2it)^2})(k-12\Delta^2k-12\Delta a_1(\kappa)-a_2(\kappa))\}$$

$$+\,O(m^{-3})\,],$$

which can be simplified by Lemma 4.1 as

$$(4.14)\quad \exp[\frac{tr\Theta^2}{2(1-2it)}]\cdot[1+\frac{1}{m}\{\frac{2\Delta tr\Theta^2}{1-2it}+\frac{tr\Theta^4-2\Delta tr\Theta^2}{2(1-2it)^2}-\frac{tr\Theta^4}{4(1-2it)^3}\}$$

$$+\frac{1}{m^2}\sum_{\alpha=1}^{6}g_{2\alpha}(1-2it)^{-\alpha}+O(m^{-3})\,],$$

where

$$(4.15)\quad \begin{aligned}
g_2 &= 2\Delta^2\ tr\ \Theta^2\\
g_4 &= (3\Delta+\tfrac{1}{4})tr\ \Theta^4+(2\Delta^2+\tfrac{1}{4})\ (tr\Theta^2)^2-4\Delta^2\ tr\ \Theta^2\\
g_6 &= \tfrac{5}{6}\ tr\Theta^6+\Delta tr\Theta^4\ tr\Theta^2-(4\Delta+\tfrac{1}{2})tr\Theta^4-(2\Delta^2+\tfrac{1}{2})(tr\Theta^2)^2+2\Delta^2 tr\Theta^2\\
g_8 &= \tfrac{1}{8}(tr\Theta^4)^2-tr\Theta^6-\Delta tr\Theta^4 tr\Theta^2+(\tfrac{3}{2}\Delta+\tfrac{1}{4})tr\Theta^4+(\tfrac{1}{2}\Delta^2+\tfrac{1}{4})(tr\Theta^2)^2\\
g_{10} &= -\tfrac{1}{8}(tr\Theta^4)^2+\tfrac{1}{3}\ tr\ \Theta^6+\tfrac{1}{4}\ \Delta\ tr\ \Theta^4\ tr\ \Theta^2\\
g_{12} &= \tfrac{1}{32}\ (tr\Theta^4)^2.
\end{aligned}$$

It follows from (4.11), (4.12) and (4.14) that the characteristic function (4.10) of $-2\rho\log\lambda_3$ under $K_\gamma$ can be expanded asymptotically as

$$(4.16)\quad (1-2it)^{-p_1p_2/2}\ etr[\frac{it\ tr\Theta^2}{1-2it}]\cdot[1+\frac{1}{m}\{-\frac{1}{4}tr\Theta^4-\Delta tr\Theta^2+\frac{2\Delta tr\Theta^2}{1-2it}$$

$$+\frac{tr\Theta^4-2\Delta tr\Theta^2}{2(1-2it)^2}-\frac{tr\Theta^4}{4(1-2it)^3}\}+\frac{1}{m^2}\{\frac{p_1p_2}{48}(p_1^2+p_2^2-5)(\frac{1}{(1-2it)^2}-1)+\sum_{\alpha=0}^{6}h_{2\alpha}(1-2it)^{-\alpha}\}$$

$$+\,O(m^{-3})\,],$$

where the coefficients $h_{2\alpha}$ $(\alpha = 1, 2, \ldots, 6)$ are given by

$$
(4.17)\quad
\begin{aligned}
h_0 &= \frac{1}{2}\left(\frac{1}{4}\mathrm{tr}\Theta^4 + \Delta\mathrm{tr}\Theta^2\right)^2 - \frac{1}{6}\mathrm{tr}\Theta^6 - \frac{1}{2}\Delta\mathrm{tr}\Theta^4\\
h_2 &= -\frac{1}{2}\Delta\mathrm{tr}\Theta^4\mathrm{tr}\Theta^2 - 2\Delta^2(\mathrm{tr}\Theta^2)^2 + 2\Delta^2\mathrm{tr}\Theta^2\\
h_4 &= -\frac{1}{8}(\mathrm{tr}\Theta^4)^2 - \frac{\Delta}{4}\mathrm{tr}\Theta^4\mathrm{tr}\Theta^2 + (3\Delta + \frac{1}{4})\mathrm{tr}\Theta^4 + (3\Delta^2 + \frac{1}{4})(\mathrm{tr}\Theta^2)^2 - 4\Delta^2\mathrm{tr}\Theta^2\\
h_6 &= \frac{1}{16}(\mathrm{tr}\Theta^4)^2 + \frac{5}{4}\Delta\mathrm{tr}\Theta^4\mathrm{tr}\Theta^2 + \frac{5}{6}\mathrm{tr}\Theta^6 - (4\Delta + \frac{1}{2})\mathrm{tr}\Theta^4 - (2\Delta^2 + \frac{1}{2})(\mathrm{tr}\Theta^2)^2\\
&\quad + 2\Delta^2\mathrm{tr}\Theta^2\\
h_\alpha &= g_\alpha \quad (\alpha = 8, 10, 12).
\end{aligned}
$$

Inverting this characterisitc function, we can conclude the following theorem.

Theorem 4.1. Under the sequence of alternatives $K_\gamma$: $P^2 = m^{-1}\,\Theta^2$, the distribution of the LR statistic $-2\rho\log\lambda_3$ for testing the independence between $p_1$ and $p_2$ sets of variates $(p_1 \leqq p_2)$ can be expressed asymptotically for large $m = \rho N = N - (p_1+p_2+3)/2N$ as

$$
(4.18)\quad
\begin{aligned}
P(-2\rho\log\lambda_3 < z) &= P(\chi^2_{f_3}(\delta^2) < z) + \frac{1}{m}\{-(\frac{1}{4}\mathrm{tr}\Theta^4 + \Delta\mathrm{tr}\Theta^2)P(\chi^2_{f_3}(\delta^2) < z)\\
&+ 2\Delta\mathrm{tr}\Theta^2 \cdot P(\chi^2_{f_3+2}(\delta^2) < z) + \frac{1}{2}(\mathrm{tr}\Theta^4 - 2\Delta\mathrm{tr}\Theta^2)P(\chi^2_{f_3+4}(\delta^2) < z)\\
&- \frac{1}{4}\mathrm{tr}\Theta^4 P(\chi^2_{f_3+6}(\delta^2) < z)\} + \frac{1}{m^2}[\frac{p_1p_2}{48}(p_1^2+p_2^2-5)\{P(\chi^2_{f_3+4}(\delta^2) < z)\\
&- P(\chi^2_{f_3}(\delta^2) < z)\} + \sum_{\alpha=0}^{6} h_{2\alpha}P(\chi^2_{f_3+2\alpha}(\delta^2) < z)] + O(m^{-3}),
\end{aligned}
$$

where the symbol $\chi^2_{f_3}(\delta^2)$ means the noncentral $\chi^2$ variate with $f_3 = p_1p_2$ degrees of freedom and noncentrality parameter $\delta^2 = \frac{1}{2}\mathrm{tr}\Theta^2$ and the coefficients $h_{2\alpha}$ $(\alpha = 0, 1, \ldots, 6)$ are given by (4.17) with $\Delta = (p_1+p_2+1)/4$.

4.3. Limiting distributions when $\gamma \neq \frac{1}{2}$. We shall now consider the limiting distributions of the LR statistic $-2\rho\log\lambda_3$ under the sequences of alternatives $K_\gamma$: $P = m^{-\gamma}\,\Theta$ $(\gamma < \frac{1}{2})$. Noting that the moments (4.1) can be rewritten by the Kummer transformation formula as in Sugiura and Fujikoshi [12],

$$(4.19)\qquad E[\lambda_3^{\frac{2}{N}h}] = \frac{\Gamma_{p_1}(h + \frac{1}{2}(N-p_2-1))\Gamma_{p_1}(\frac{1}{2}(N-1))}{\Gamma_{p_1}(\frac{1}{2}(N-p_2-1))\Gamma_{p_1}(h + \frac{1}{2}(N-1))} \cdot |I - P^2|^h$$

$$\cdot\ {}_2F_1(h, h;\ h + \frac{1}{2}(N-1);\ P^2)$$

and considering the characteristic function of $(-2\rho\log\lambda_3)/m^{\frac{1}{2}-\gamma}$ based on the above moments, we can easily see that the first four gamma products tend to one and the hypergeometric function ${}_2F_1$ tends to $\exp[-2t^2 \mathrm{tr}\Theta^2]$ as $m \to \infty$. This gives the first part of the following theorem. Also, when $\gamma > \frac{1}{2}$, we can see from (4.1) that the characteristic function of $-2\rho\log\lambda_3$ tends to $(1-2it)^{-p_1p_2/2}$ as $m \to \infty$. Hence, we can imply the following theorem.

Theorem 4.2. Under the sequence of alternatives $K_\gamma$: $P = m^{-\gamma}\,\Theta$ the limiting distributions of the LR statistic $-2\rho\log\lambda_3$ for the hypothesis of independence are given in the following.

(1) When $0 < \gamma < \frac{1}{2}$, the statistic

$$(4.20)\qquad \frac{1}{m^{1/2-\gamma}}\,[-2\rho\log\lambda_3 - m\log|I - P^2|]$$

has asymptotically the normal distribution with mean zero and variance $4\mathrm{tr}\Theta^2$.

(2) When $\gamma > \frac{1}{2}$, the statistic $-2\rho\log\lambda_3$ has asymptotically the $\chi^2$ distribution with $p_1p_2$ degrees of freedom.

4.4. Numerical example. Computing the one more term in the asymptotic formula of the characteristic function in the null case given in (4.11), we can get

$$(4.21)\qquad P_H(-2\rho\log\lambda_3 < z) = P(\chi^2_{f_3} < z) + \frac{\omega_2}{m^2}\{P(\chi^2_{f_3+4} < z) - P(\chi^2_{f_3} < z)\}$$

$$+ \frac{1}{m^4}[\omega_4\{P(\chi^2_{f_3+8} < z) - P(\chi^2_{f_3} < z)\} - \omega_2^2\ \{P(\chi^2_{f_3+4} < z) - P(\chi^2_{f_3} < z)\}] + 0(m^{-5}),$$

where

$$\omega_2 = \frac{p_1p_2(p_1^2+p_2^2-5)}{48}$$

$$(4.22)\qquad \omega_4 = \frac{1}{2}\ \omega_2^2 + \frac{p_1p_2}{1920}\{3p_1^4 + 3p_2^4 - 50\ p_1^2 - 50\ p_2^2 + 10\ p_1^2p_2^2 + 159\}\ .$$

Since the above formula has the same form as in the case of multivariate linear hypothesis given in Anderson [1, p.208], we can use the asymptotic formula for the percentage point of the LR statistic for multivariate linear hypothesis given by Hill and Davis [4] for our present purpose, that is, the $100\alpha$ % point of $-2\rho\log\lambda_3$ is given by

$$u + \frac{1}{m^2}\frac{2u\ \omega_2}{f(f+2)}(u + f + 2) + \frac{1}{m^4}[\frac{2u\ \omega_4}{f(f+2)(f+4)(f+6)}\{u^3 + (f+6)u^2$$

$$(4.23)\qquad + (f+6)(f+4)u + (f+6)(f+4)(f+2)\} - \frac{\omega_2^2\ u}{f^2(f+2)^2}\{u^3 + (f-2)u^2$$

$$+ (f-6)(f+2)u + (f-2)(f+2)^2\}] + 0(m^{-5}),$$

where $f = p_1p_2$ and $u$ is determined such that $P(\chi^2_f > u) = \alpha$.

Example 4.1. When $N = 87$ and $p_1 = 2$, $p_2 = 3$, the asymptotic formula (4.22) gives the following approximate 5 % point.

| | |
|---|---|
| first term | 12.5916 |
| term of order $m^{-2}$ | 0.0016 |
| term of order $m^{-4}$ | 0.0000 |
| approx. value | 12.5932 |

Pillai and Jayachandran [8] gave the exact 5 % point as 12.593156 (computed from their Table 8. Lower 5% points of $\{W^{(2)}\}^{1/2}$ $m = 0$ and $n = 40$). For the alternative hypotheses $K_1$: $\rho_1^2 = 0.001$, $\rho_2^2 = 0.05$ and $K_2$: $\rho_1^2 = 0.05$, $\rho_2^2 = 0.1$, the following approximate powers are computed by the formula (4.18) in this section and (2.12) in Sugiura and Fujikoshi [12] respectively.

approximate power

| | $K_1$ | $K_2$ |
|---|---|---|
| first term | 0.28505 | 0.5231 |
| second term | 0.00680 | 0.2871 |
| third term | 0.00023 | -0.0038 |
| approx. power | 0.2921 | 0.806 |

Pillai and Jayachandran [8] gave the exact power when the alternative hypothesis is $K_1$ as 0.2919 in their Table 9. Hence our approximate power 0.2921 shows good approximation to the exact value.

## 5. Asymptotic distribution of the LR criterion for $\Sigma_1 = \Sigma_2$.

5.1. Moments of the criterion under local alternatives. Let the $p \times 1$ vectors $X_{\alpha 1}, X_{\alpha 2}, \ldots, X_{\alpha N_\alpha}$ be a random sample form a normal population with mean vector $\mu_\alpha$ and covariance matrix $\Sigma_\alpha$ for $\alpha = 1, 2$. The modified LR criterion for testing the hypothesis H: $\Sigma_1 = \Sigma_2$ against alternatives K: $\Sigma_1 \neq \Sigma_2$ with unknown mean vectors, is given by

$$\lambda_4^* = \left\{\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right\}^{p/2} \frac{|S_1|^{n_1/2} |S_2|^{n_2/2}}{|S_1 + S_2|^{n/2}}, \tag{5.1}$$

where $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \overline{X}_j)(X_{j\alpha} - \overline{X}_j)'$ and $\overline{X}_j = N_j^{-1} \sum_{\alpha=1}^{N_j} X_{j\alpha}$ with $n_j = N_j - 1$ and $n = n_1 + n_2$. It is easy to see that the non-null distribution of $\lambda_4^*$ depends only on p characteristic roots of $\Sigma_1\Sigma_2^{-1}$. Unbiasedness of this modified LR criterion was proved by Sugiura and Nagao [13]. The limiting distribution of $-2\log\lambda_4^*$ under fixed alternative hypothesis K was obtained by Sugiura [11]. We shall now consider the moments of the statistic $\lambda_4^*$ under the sequence of alternatives $K_n$: $\Sigma_2^{-1/2}\,\Sigma_1\,\Sigma_2^{-1/2} = I + m^{-1}\,\Theta$, where the $p \times p$ matrix $\Theta$ is pd and $m = \rho n$ with the correction factor

$$\rho \;=\; 1 - \frac{2p^2+3p-1}{6(p+1)}\left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n}\right) . \tag{5.2}$$

Without loss of generality, we may assume that the statistic $S_1$ has the Wishart distribution $W_p(n_1, \Gamma)$ and $S_2$ has $W_p(n_2, I)$, where $\Gamma = \Sigma_2^{-1/2}\,\Sigma_1\,\Sigma_2^{-1/2}$. Since $S_1$ and $S_2$ are independent, we can write the h-th moment of $\lambda_4^*$ as

$$E[\lambda_4^{*h}\,|K_n] = \left(\frac{n^n}{n_1^{n_1}\,n_2^{n_2}}\right)^{ph/2} \frac{1}{2^{np/2}\Gamma_p(n_1/2)\Gamma_p(n_2/2)} \int \frac{|S_1|^{\frac{n_1(1+h)-p-1}{2}}\,|S_2|^{\frac{n_2(1+h)-p-1}{2}}}{|\Gamma|^{n_1/2}\,|S_1+S_2|^{nh/2}}$$

$$\cdot\ \mathrm{etr}\{-\tfrac{1}{2}\,(S_1+S_2) + \tfrac{1}{2}(I-\Gamma^{-1})S_1\}\; dS_1\, dS_2, \tag{5.3}$$

where the range of integration is such that the two $p \times p$ symmetric matrices $S_1$ and $S_2$ are pd. We can expand the last part $\mathrm{etr}\{\frac{1}{2}(I-\Gamma^{-1})S_1\}$ in the above integration to infinite series by zonal polynomials as

$$\sum_{k=0}^{\infty}\ \sum_{(\kappa)} C_\kappa(\tfrac{1}{2}(I-\Gamma^{-1})S_1)/k!\ .$$

Transforming the variables $(S_1, S_2)$ to $(U_1, U_2)$ by $U_1 = S_1$ and $U_2 = U_1^{-1/2}\,S_2\,U_1^{-1/2}$ ($U_1^{1/2}$ is chosen pd) with the Jacobian $|\partial(S_1,S_2)/\partial(U_1,U_2)|$

$= |U_1|^{(p+1)/2}$ and integrating out with respect to $U_1$ by the formula due to Constantine [2]

$$(5.4) \qquad \int_{S>0} \{\mathrm{etr}(-RS)\}|S|^{t-(p+1)/2}\, C_\kappa(ST)\,dS = \Gamma_p(t)(t)_\kappa |R|^{-t}\, C_\kappa(TR^{-1}) ,$$

which holds for any $p \times p$ pd matrix $R$, symmetric matrix $T$ and complex number $t$ satisfying $R(t) > (p-1)/2$, we can rewrite the moment (5.3) as

$$(5.5) \qquad \left(\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right)^{ph/2} \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \sum_{k=0}^{\infty} \sum_{(\kappa)} \int \frac{|U_2|^{\frac{n_2(1+h)-p-1}{2}} (\frac{1}{2}n)_\kappa}{|\Gamma|^{n_1/2} |I+U_2|^{n(1+h)/2}}$$

$$\cdot\, C_\kappa((I-\Gamma^{-1})(I+U_2)^{-1})/k!\; d\,U_2 .$$

Putting $V = (I + U_2)^{-1}$ with the Jacobian $|\partial U_2/\partial V| = |V|^{-p-1}$ and integrating out with respect to $V$ by the formula due to Constantine [2]

$$(5.6) \qquad \int_0^I |S|^{t-(p+1)/2}\, |I - S|^{u-(p+1)/2}\, C_\kappa(RS)\,dS = \frac{\Gamma_p(t)\Gamma_p(u)(t)_\kappa}{\Gamma_p(t+u)(t+u)_\kappa} C_\kappa(R) ,$$

which holds for any $p \times p$ pd matrix $R$, we can express the $h$-th moment (5.3) as

$$(5.7) \qquad \left(\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right)^{ph/2} \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \cdot \frac{\Gamma_p(\frac{1}{2}n_1(1+h))\Gamma_p(\frac{1}{2}n_2(1+h))}{\Gamma_p(\frac{1}{2}n(1+h))\; |\Gamma|^{n_1/2}}$$

$$\cdot\, {}_2F_1(\tfrac{1}{2}n,\ \tfrac{1}{2}n_1(1+h);\ \tfrac{1}{2}n(1+h);\ \ I - \Gamma^{-1}).$$

5.2. Asymptotic distribution under $K_n$. Applying the Kummer transformation formula ${}_2F_1(a_1, a_2; b; Z) = |I - Z|^{-a_2} \cdot {}_2F_1(b-a_1, a_2; b; -Z(I-Z)^{-1})$ in James [5] to (5.7), we can write the characteristic function of $-2\rho\log\lambda_4^*$ under $K_n$ as

$$\left(\frac{m^m}{m_1^{m_1} m_2^{m_2}}\right)^{-\rho it} \frac{\Gamma_p(\frac{1}{2}m+\Delta)\Gamma_p(\frac{1}{2}m_1(1-2it)+\Delta_1)\Gamma_p(\frac{1}{2}m_2(1-2it)+\Delta_2)}{\Gamma_p(\frac{1}{2}m_1+\Delta_1)\Gamma_p(\frac{1}{2}m_2+\Delta_2)\Gamma_p(\frac{1}{2}m(1-2it)+\Delta)}$$

(5.8)

$$\cdot \left|I + \frac{1}{m}\Theta\right|^{-m_1 it} \cdot \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(-mit)_\kappa (\frac{1}{2}m_1(1-2it)+\Delta_1)_\kappa}{(\frac{1}{2}m(1-2it)+\Delta)_\kappa\, m^k} \cdot \frac{C_\kappa(-\Theta)}{k!} ,$$

where $m_\alpha = \rho\, n_\alpha$ and $\Delta_\alpha = \frac{1}{2}(n_\alpha - m_\alpha) = O(1)$ with $m = m_1 + m_2$ and $\Delta = \Delta_1 + \Delta_2$. The first factor in the above expression gives the characteristic function of $-2\rho\log\lambda_4^*$ under the null hypothesis, which can be expanded asymptotically for large $m$ with fixed $\rho_\alpha = m_\alpha/m$ $(\alpha = 1, 2)$ in the usual way as in the previous sections, giving (Anderson [1,p.255])

$$(5.9)\qquad (1-2it)^{-f_4/2}\left[1 + \frac{\omega_2}{m^2}\left\{\frac{1}{(1-2it)^2} - 1\right\} + O\left(\frac{1}{m^3}\right)\right] ,$$

where $f_4 = p(p+1)/2$ and

$$(5.10)\qquad \omega_2 = \frac{1}{48}p(p^2-1)(p+2)\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - 1\right) - \frac{1}{2}p(p+1)\Delta^2 .$$

The second factor in (5.8) can be expanded easily by the formula (2.4) as

$$\left|I + \frac{1}{m}\Theta\right|^{-m_1 it} = \operatorname{etr}(-\rho_1 it\Theta) \cdot \left[1 + \frac{1}{2m}\rho_1 it \cdot \operatorname{tr}\Theta^2\right.$$

(5.11)

$$\left. + \frac{1}{m^2}\left\{-\frac{1}{3}\rho_1 it\cdot\operatorname{tr}\Theta^3 + \frac{1}{8}(\rho_1 it\cdot\operatorname{tr}\Theta^2)^2\right\} + O\left(\frac{1}{m^3}\right)\right] .$$

The third factor in (5.8) can be written, by Lemma 4.2 , as

(5.12)
$$\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(\rho_1 it)^k C_\kappa(\Theta)}{k!}[1+\frac{a_1(\kappa)}{m}\{-\frac{1}{2it}+\frac{\rho_2/\rho_1}{1-2it}\}+\frac{1}{m^2}\{-\frac{2\Delta a_1(\kappa)}{(1-2it)^2}\cdot\frac{\rho_2}{\rho_1}$$
$$+(k-a_2(\kappa))(\frac{1}{24(it)^2}+\frac{\rho_1^{-2}-1}{(1-2it)^2})+\frac{1}{2}a_1(\kappa)^2(\frac{\rho_2/\rho_1}{1-2it}-\frac{1}{2it})^2\}+0(\frac{1}{m^3})],$$

which can be simplified, by Lemma 4.1 , as

(5.13)
$$\mathrm{etr}(\rho_1 it\Theta)\cdot[1+\frac{1}{m}\mathrm{tr}\Theta^2\{-\frac{\rho_1^2 it}{2}-\frac{\rho_1\rho_2(it)^2}{1-2it}\}+\frac{1}{m^2}\{-\frac{\rho_1\rho_2}{2}\Delta\mathrm{tr}\Theta^2(1-\frac{1}{1-2it})^2$$
$$-\frac{1}{24}(1-\frac{2(1-\rho_1^2)}{1-2it}+\frac{1-\rho_1^2}{(1-2it)^2})(4\rho_1 it\cdot\mathrm{tr}\Theta^3+3\mathrm{tr}\Theta^2+3[\mathrm{tr}\Theta]^2)$$
$$+\frac{1}{8}(-1+\frac{1-\rho_1}{1-2it})^2([\rho_1 it\cdot\mathrm{tr}\Theta^2]^2+4\rho_1 it\cdot\mathrm{tr}\Theta^3+\mathrm{tr}\Theta^2+[\mathrm{tr}\Theta]^2)\}+0(\frac{1}{m^3})].$$

Multiplying the three factors (5.9), (5.11), (5.13) together and arranging the terms according to the power of $(1-2it)^{-1}$, we can get the asymptotic formula for the characteristic function of $-2\rho\log\lambda_4^*$ under $K_n$ as

(5.14)
$$(1-2it)^{-f_4/2}[1+\frac{\rho_1\rho_2}{4m}\mathrm{tr}\Theta^2\{\frac{1}{1-2it}-1\}+\frac{1}{m^2}\sum_{\alpha=0}^{2}g_{2\alpha}(1-2it)^{-\alpha}$$
$$+\frac{\omega_2}{m^2}\{\frac{1}{(1-2it)^2}-1\}+0(\frac{1}{m^3}),$$

where

(5.15)
$$g_0=\rho_1\rho_2\{\frac{\rho_1\rho_2}{32}(\mathrm{tr}\Theta^2)^2-\frac{\rho_1^{-2}}{6}\mathrm{tr}\Theta^3-\frac{\Delta}{2}\mathrm{tr}\Theta^2\}$$
$$g_2=\rho_1\rho_2\{-\frac{\rho_1\rho_2}{16}(\mathrm{tr}\Theta^2)^2-\frac{\rho_2}{2}\mathrm{tr}\Theta^3+(\Delta+\frac{1}{4})\mathrm{tr}\Theta^2+\frac{1}{4}(\mathrm{tr}\Theta)^2\}$$
$$g_4=\rho_1\rho_2\{\frac{\rho_1\rho_2}{32}(\mathrm{tr}\Theta^2)^2+\frac{1-2\rho_1}{6}\mathrm{tr}\Theta^3-(\frac{\Delta}{2}+\frac{1}{4})\mathrm{tr}\Theta^2-\frac{1}{4}(\mathrm{tr}\Theta)^2\}.$$

Inverting the characteristic function (5.14), we can get the following theorem.

Theorem 5.1. Under the sequence of alternatives $K_n: \Sigma_2^{-1/2} \Sigma_1 \Sigma_2^{-1/2} = I + m^{-1} \Theta$, the distribution of the modified LR criterion given by (5.1) can be expanded asymptotically for large $m = \rho n$ with fixed $\rho_\alpha = n_\alpha/n$ $(\alpha = 1, 2)$ as

$$
\begin{aligned}
P(-2\rho\log\lambda_4^* < z) &= P(\chi^2_{f_4} < z) + \frac{\rho_1\rho_2}{4m}\mathrm{tr}\Theta^2 \{P(\chi^2_{f_4+2} < z) - P(\chi^2_{f_4} < z)\} \\
&+ \frac{1}{m^2} [\omega_2 \{P(\chi^2_{f_4+4} < z) - P(\chi^2_{f_4} < z)\} + \sum_{\alpha=0}^{2} g_{2\alpha} P(\chi^2_{f_4+2\alpha} < z)] + O(\frac{1}{m^3}),
\end{aligned} \tag{5.16}
$$

where $f_4 = p(p+1)/2$, and the correction factor $\rho$ is given by (5.9). The coefficients $g_{2\alpha}$ $(\alpha = 0, 1, 2)$ are given by (5.15) with $\Delta = (n-m)/2$ and $\omega_2$ is given by (5.10).

5.3. Numerical examples. Evaluating the asymptotic formula (5.9) for the characteristic function under $H$ more precisely, we can get

$$
\begin{aligned}
P(-2\rho\log\lambda_4^* < z) &= P(\chi^2_{f_4} < z) + \frac{\omega_2}{m^2} \{P(\chi^2_{f_4+4} < z) - P(\chi^2_{f_4} < z)\} \\
&+ \frac{\omega_3}{m^3} \{P(\chi^2_{f_4+6} < z) - P(\chi^2_{f_4} < z)\} + \frac{1}{m^4} [\omega_4 \{P(\chi^2_{f_4+8} < z) - P(\chi^2_{f_4} < z)\} \\
&- \omega_2^2\{P(\chi^2_{f_4+4} < z) - P(\chi^2_{f_4} < z)\}] + O(m^{-5}) ,
\end{aligned} \tag{5.17}
$$

where $\omega_2$ is defined by (5.10) and

$$
\begin{aligned}
\omega_3 &= \frac{p}{720}(6p^4 + 15p^3 - 10p^2 - 30p + 3)(\rho_1^{-3} + \rho_2^{-3} - 1) \\
&- \frac{p}{12}(p^2 - 1)(p + 2)\Delta(\rho_1^{-2} + \rho_2^{-2} - 1) + \frac{4}{3}p(p + 1)\Delta^3
\end{aligned}
$$

$$(5.18) \qquad \omega_4 = \frac{1}{2}\omega_2^2 + \frac{p}{480}(p-1)(2p^4 + 8p^3 + 3p^2 - 17p - 14)(\rho_1^{-4} + \rho_2^{-4} - 1)$$
$$- \frac{p}{120}(6p^4 + 15p^3 - 10p^2 - 30p + 3)\Delta(\rho_1^{-3} + \rho_2^{-3} - 1)$$
$$+ \frac{p}{4}(p^2 - 1)(p+2)\Delta^2(\rho_1^{-2} + \rho_2^{-2} - 1) - 3p(p+1)\Delta^4.$$

Applying the inverse expansion formula due to Hill and Davis [4] to the above expression, we get the following asymptotic formula for the $100\alpha\%$ point of $-2\rho\log\lambda_4^*$ :

$$u + \frac{2\omega_2 u(u+f+2)}{m^2 f(f+2)} + \frac{2\omega_3 u}{m^3 f(f+2)(f+4)}\{u^2 + (f+4)u + (f+4)(f+2)\}$$
$$(5.19) \qquad + \frac{1}{m^4}\left[\frac{2\omega_4 u}{f(f+2)(f+4)(f+6)}\{u^3 + (f+6)u^2 + (f+6)(f+4)u + (f+6)(f+4)(f+2)\}\right.$$
$$\left. - \frac{\omega_2^2 u}{f^2(f+2)^2}\{u^3 + (f-2)u^2 + (f+2)(f-6)u + (f+2)^2(f-2)\}\right] + O(\frac{1}{m^5}) \ ,$$

where $u$ is defined such that $P(\chi_f^2 > u) = \alpha$ and $f = p(p+1)/2$.

Example 5.1. The approximate 5% points of the statistic $-2\rho\log\lambda_4^*$ given by (5.1) are computed by the formula (5.19) in the following two cases.

| | $p = 1$ | $p = 2$ |
|---|---|---|
| | $n_1 = 4,\ n_2 = 20$ | $n_1 = 13,\ n_2 = 63$ |
| first term | 3.8415 | 7.81473 |
| term of order $m^{-2}$ | -0.0389 | 0.00777 |
| term of order $m^{-3}$ | -0.0045 | -0.00010 |
| term of order $m^{-4}$ | 0.0030 | 0.00001 |
| approx. value | 3.801 | 7.82241 |

The exact 5% point in the first case according to Table 743 in Ramachandran [10] is 3.80. Thus our approximate value is accurate to 2 decimal palces. In the univariate case, an asymptotic expansion of the distribution of $-2\rho\log\lambda_4^*$

under the fixed alternative hypothesis has been derived by Sugiura and Nagao [14], by which approximate power, when alternative hypothesis is not near to the null hypothesis, can be computed.

Example 5.2. Specifying the alternative hypothesis $K$ as $\Sigma_2^{-1/2}\ \Sigma_1\ \Sigma_2^{-1/2} = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_p)$ and using the 5% points obtained in Example 5.1, we can get the approximate values of the power of the modified LR test from the formula (5.16).

| | $p = 1$ | $p = 2$ |
|---|---|---|
| | $n_1 = 4,\ n_2 = 20$ | $n_1 = 13,\ n_2 = 63$ |
| | $\delta_1 = 0.5$ | $\delta_1 = \delta_2 = 1.05$ |
| first term | 0.0512 | 0.049828 |
| second term | 0.0443 | 0.001479 |
| third term | 0.0083 | -0.000008 |
| approx. power | 0.104 | 0.05130 |

From Table 744a in Ramachandran [10], we can see that the exact power in the first case is 0.113. In the second case, we can see from Table 2 in Pillai and Jayachandran [9] $(m = 5,\ n = 30)$ that the powers of the other tests are given by

| | |
|---|---|
| Roy's largest root criterion (root of $S_1\ S_2^{-1}$) | 0.067021 |
| Lawley-Hotelling's trace criterion $(\mathrm{tr}\ S_1\ S_2^{-1})$ | 0.070087 |
| Pillai's criterion $(\mathrm{tr}\ S_1(S_1 + S_2)^{-1})$ | 0.070338 |
| Wilks' criterion $(\lvert I + S_1\ S_2^{-1}\rvert)$ | 0.070273. |

However, this does not mean that the modified LR criterion is worse. Because the above four test criteria are to test the null hypothesis $H$: $\Sigma_1 = \Sigma_2$ against the alternatives $K$: $\delta_\alpha \geq 1$ for $\alpha = 1, 2, \ldots, p$ and $\sum_{\alpha=1}^{p} \delta_\alpha > 1$, which is an extension of the one-sided test and the modified LR criterion is against all alternatives K: $\Sigma_1 \neq \Sigma_2$, which is an extension of the two-sided test.

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