INELASTIC ANALYSIS OF PLANE STRESS PROBLEMS

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Abstract

This paper presents a simple method of inelastic analysis of plane stress problems which can be expanded to handle plane strain and three-dimensional problems. In this method, the plane stress continuum is discretized into a set of one-dimensional truss elements. The geometric and the elastic properties of these elements are evaluated using a set of empirical formulas. The relation between the force-deformation equation of the discrete truss elements and the constitutive equations of the plane stress problem is evaluated and presented. This approach reduces a two-dimensional plane stress problem into a much simpler one-dimensional problem where the nonlinear force-deformation relation of the one-dimensional truss element can be treated, very simply, by applying the static or dynamic loads to the discretized structure in such a way that the force in each of the truss members follows the one-dimensional force-deformation relation.

Many practical nonlinear materials problems have been solved using this approach. The cost of analysis has proved to be considerably less than that of conventional finite element methods. Comparisons between the results of this suggested method and the finite element method show excellent agreement. It has been possible with a relatively small effort to incorporate this method into already existing general purpose, linear analysis, programs as STRUDL and NASTRAN. This makes the use of this method an easy and simple procedure by all design engineers who are already familiar with such programs.

1. Introduction

The analysis of linear plane systems by the plane framework method has been investigated by many people. In particular by A. Hrennikoff [1], A. Yettram [2], R. Szilar [3], A. Ang [4] and C. W. McCormick [5]. This paper presents a simple and accurate method for nonlinear analysis of plane framework problems.

2. Approximate Approach

2.1 Mathematical Model

Consider the homogenous and isotropic plane stress (plane strain) problem. The mathematical model is shown in Figure 1. It consists of a set of one dimensional truss elements connected at the nodes by a set of no-moment hinges. The panels of this model could be either rectangular or square, as described by Hrennikoff [1]. However, he suggested that the closer the aspect ratios of the panels a/b to the unity, the more accurate the results, in case of linear analysis.

The first step in analysis is to assign properties to the mathematical model. It was found that [1] the areas of the members should be dependent on Poisson's ratio of the material in order to get good accuracy. The same reference suggested that for Poisson's ratio of 0.33, the best choices of areas are:

A_1 = Area of the diagonal members = $\frac{0.75}{\sqrt{2}}$ ah

A_2 = Area of the horizontal and vertical members = 0.75 ah

where

a = Length of the square panel.

h = thickness of the plate.

= 1.0 for plane strain problems.

2.2 Force-Deformation-Relation and Element Matrix

The nonlinear force-deformation relation of a typical truss element is shown in Figure 2. It can be expressed as

$$F = K_s \Delta U \qquad \text{(1a)}$$

and the incremental form is

$$\delta F = K_t \, \delta\Delta U \qquad \text{(1b)}$$

where

F = Force in the element

δF = Incremental force

ΔU = Elongation of the element

$\delta(\Delta U)$ = Incremental elongation

K_s = Secant stiffness

K_t = Tangent stiffness

The secant and tangent stiffness matrices of the truss element can be expressed as:

$$\underset{\sim}{K}_s = \frac{E_s A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2a}$$

$$\underset{\sim}{K}_t = \frac{E_t A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2b}$$

where A is the cross sectional area of the element, L the element length, E_s the secant modulus of elasticity and E_t the tangent modulus of elasticity.

2.3 Tangent Method of Analysis

The tangent force deformation relation, equation 1a, as well as the element tangent stiffness matrix of trusses, equation 2a, can be used to formulate the system general tangent equilibrium equation, in the ith increment

$$\left[\underset{\sim}{K}_t\right]_{i-1} \delta \underset{\sim}{U}_i = \delta \underset{\sim}{P}_i \tag{3}$$

where:

$\left[\underset{\sim}{K}_t\right]_{i-1}$ = System tangent stiffness matrix

$= \left[\underset{\sim}{K}_t \; (U_{i-1})\right]$

$\delta \underset{\sim}{U}_i$ = Incremental displacement

$\delta \underset{\sim}{P}_i$ = Incremental load

Equation 3 is a nonlinear equation in which the tangent stiffness matrix K_t is a function of the displacement level. If the total applied load vector is P and n is the number of increments,

$$\underset{\sim}{P} = \sum_{i=1}^{i=n} \delta \underset{\sim}{P}_i \tag{4}$$

The choice of the load increment depends on the nature of the problem [6 , 7].

It is possible to solve equation (3) for the first increment (i = 1) with $\underset{\sim}{K}_{t\,o}$ as the linear stiffness matrix. The total displacement $\underset{\sim}{U}_1$ can be obtained, then the new tangent matrix $[\underset{\sim}{K}_t]_o$, can be evaluated, and a new incremental loading can be applied. This method can be repeated until all the

(n) increments are applied. The expression of the total displacement after (i) increments is

$$\underset{\sim}{U}_i = \sum_{j=1}^{j=i} \underset{\sim}{U}_j \tag{5}$$

Figure 3 shows the details of the tangent method of analysis.

2.4 Accelerated Method of Analysis

A major problem of the tangent method of analysis is that the number of increments can be very large. Another approach, used to accelerate the convergence of the problem, uses a combination of the tangent equilibrium equation (3) and the secant equilibrium equation

$$\left[\underset{\sim}{K}_s\right]_i \underset{\sim}{U}_i = \underset{\sim}{P} \tag{6}$$

where $\left[\underset{\sim}{K}_s\right]_i = \left[\underset{\sim}{K}_s \quad (\underset{\sim}{U}_i)\right]$

= Secant stiffness matrix.

The secant stiffness matrix can be obtained using the secant force-deformation relation (1b) and the secant element matrix (2b). In the ith increment of the accelerated approach, the total displacement U_i and the accompanied load, P_i, are known. The tangent equation

$$\left[\underset{\sim}{K}_t\right]_i \quad \delta\underset{\sim}{U}_{i+1} = \underset{\sim}{P} - \underset{\sim}{P}_i \tag{7}$$

is solved for δU_{i+1}, the new total displacements U_{i+1} is obtained by

$$\underset{\sim}{U}_{i+1} = \underset{\sim}{U}_i + \delta\underset{\sim}{U}_{i+1} \tag{8}$$

The secant equation (6) is used then to obtain the new load $\underset{\sim}{P}_{i+1}$

$$\underset{\sim}{P}_{i+1} = \left[\underset{\sim}{K}_s\right]_{i+1} \underset{\sim}{U}_{i+1} \tag{9}$$

Equations (7) through (9) are repeated until the problemconverges within any required accuracy.

2.5 State of Stresses

Evaluation of the nonlinear state of stresses in the system can be of importance in many engineering problems. The necessary equations to evaluate the stresses will be presented in this section. The plane secant stress-strain relation is

$$\underset{\sim}{\sigma} = \underset{\sim}{D}_s \underset{\sim}{\epsilon} \tag{10}$$

where $\underset{\sim}{\sigma}$ = plane stress vector

$\underset{\sim}{D}_s$ = secant plane rigidity matrix

$$= F_s \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/_2 \end{bmatrix}$$

$F_s = E_s/(1-\nu^2)$ for plane stress problems

ν = Poisson's ratio

$\underset{\sim}{\epsilon}$ = strain vector

Similar expression can be obtained for plane strain problems [8].

For a rectangular panel, Figure 5, the displacements can be approximated by linear polynomials [9], so that

$$[u \vdots v] = \underset{\sim}{\phi}\, \underset{\sim}{U}^n$$

U^n = vector of displacements at the corners of the panel

$\underset{\sim}{\phi}$ = linear polynomials

u, v = horizontal and vertical displacements in the panel

The strain vector could be expressed as

$$\underset{\sim}{\epsilon} = \underset{\sim}{B}\underset{\sim}{U}^n$$

$$\underset{\sim}{\sigma} = \underset{\sim}{D}_s \underset{\sim}{B}\underset{\sim}{U}^n \qquad (11)$$

Equation (11) gives the relation between the displacements of a panel and the nonlinear state of stresses in the panel.

3. Practical Applications of the Method

This method of analysis can be applied to plane problems using any existing structural analysis computer program with truss elements. The only requirement is that the program be flexible enough to accept a restart type command to account for the incremental approach. Most general purpose programs, such as STRUDL and NASTRAN, do have this capability. These two computer codes were used successfully for implementing this method for nonlinear analysis of some practical problems, such as reanalysis and redesign of the steam generator supports in a PWR plant due to the overstressing resulted from a new definition of loading conditions.

4. Example

To demonstrate the accuracy of this method, a plane stress problem will be solved and the results compared with those obtained from a finite element solution of the same problem. Figure 6 shows a slitted specimen subjected to in-plane loading. The load-displacement relation resulting from the use of the present method is shown in Figure 7. The finite element solution for the same problem, as obtained in reference [10] is shown in the same figure. The agreement between the two approaches is excellent.

5. Conclusions

A method of nonlinear inelastic analysis of plane problems was presented in this paper. The method is simple, economical and can be adapted to almost any general purpose program. A comparison between this method and the finite elements method showed excellent agreement.

6. Acknowledgement

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7. References

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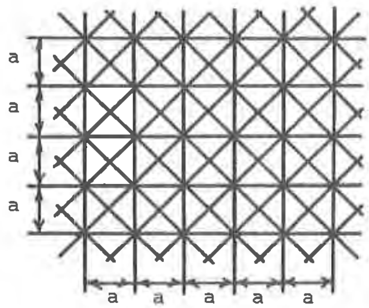

Figure 1
Mathematical Model

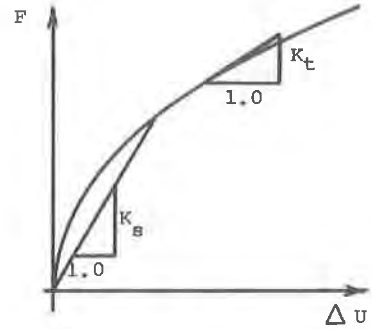

Figure 2
Nonlinear Force-Deformation Curve

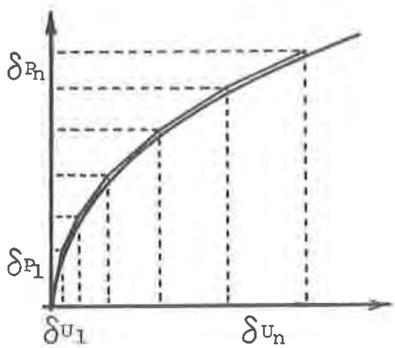

Figure 3
Tangent Approach

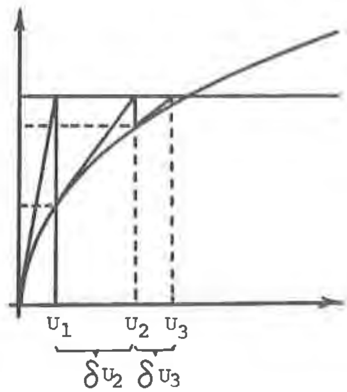

Figure 4
Accelerated Approach

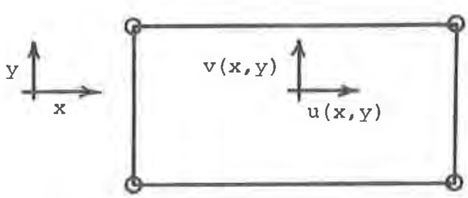

Figure 5
Coordinate System

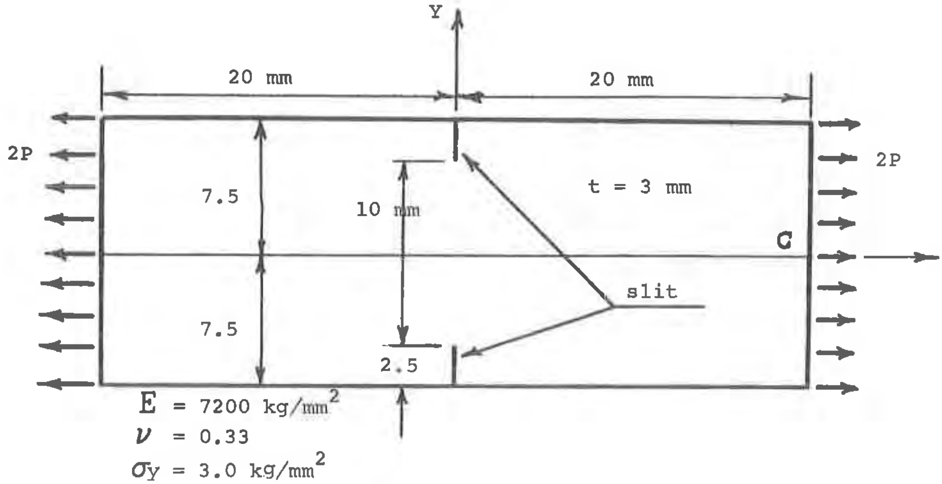

Figure 6
Slitted Specimen

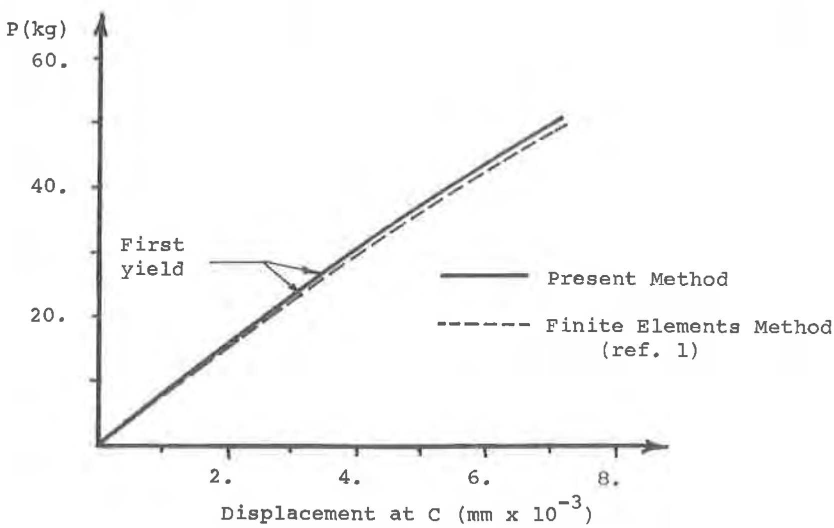

Figure 7
Force-Displacement Curve