Abstract

WANG, SHUNMIN. Interval computations for fuzzy relational equations and cooperative game theory. (Under the direction of Dr. Shu-Cherng Fang and Dr. Henry L.W. Nuttle.)

This dissertation introduces the concepts of the tolerable solution set, united solution set, and controllable solution set of interval-valued fuzzy relational equations. Given a continuous *t-norm*, it is proved that each of the three types of the solution sets of interval-valued fuzzy relational equations with a *max-t-norm* composition, if nonempty, is composed of one maximum solution and a finite number of minimal solutions. Necessary and sufficient conditions for the existence of solutions are given. Computational procedures based on the constructive proofs are proposed to generate the complete solution sets and examples are given to illustrate the procedures. Similarly, it is proved that each type of solution set of interval-valued fuzzy relational equations with a *min-s-norm* composition, if nonempty, is composed of one minimum solution and a finite number of maximal solutions.

For interval-valued games, a new method for ranking interval numbers is introduced. Interval-valued cooperative games are defined based on this method. It is proved that a unique payoff function, which is similar to the Shapley value function, exists and satisfies certain desired properties of an interval-valued cooperative game. Furthermore, this payoff function can be applied to non-superadditive games.

INTERVAL COMPUTATIONS FOR FUZZY RELATIONAL EQUATIONS AND COOPERATIVE GAME THEORY

by

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Biography

WANG, SHUNMIN was born on December 26, 1965 in China. He received his B.S. degree from Nanjing University of Science and Technology, China, in 1987, and his M.S. degree from Tianjin University, China, in 1990, both in Computer Engineering. From 1990 to 1997, he worked for the Sixth Research Institute of China Ministry of Electrical Industry where he integrated several distributed control systems and data acquisition systems for 300 MW power plants. In the fall 1997, he enrolled in the Ph.D. program of the Graduate Program in Operations Research at North Carolina State University, Raleigh, NC. He is married to Li, Na. His daughter, Sylvia, was born in 2000.

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Chapter 1

Introduction

Interval arithmetic is an arithmetic defined on real intervals, rather than on real numbers. Modern development of interval arithmetic began with Moore's dissertation [30]. Since then, thousands of research articles and numerous books have appeared on this subject. Interval computation has been used to estimate the range of functions, roundoff errors, and truncation errors. Many successful applications can be found in [17].

1.1 Interval computations

Given $\underline{a}$, $\overline{a} \in \Re$ and $\underline{a} \leq \overline{a}$, the closed interval $[\underline{a}, \overline{a}]$ defines an *interval number* $a^I = [\underline{a}, \overline{a}] \triangleq \{a \in \Re \mid \underline{a} \leq a \leq \overline{a}\}$. Obviously, when $\underline{a} = \overline{a}$, the interval number a^I reduces to a real number $\underline{a}$. We say that a real number a is a *member* of an interval number a^I, written as $a \in a^I$, if $\underline{a} \leq a \leq \overline{a}$. Let $I(\Re)$ be the set of all interval numbers defined on $\Re$.

Let $\underline{\mathbf{a}} = (\underline{a}_i)_m$ and $\overline{\mathbf{a}} = (\overline{a}_i)_m$ be two m-dimensional real vectors with $\underline{a}_i \leq \overline{a}_i$, for $i = 1, ..., m$. In this case, we say $\underline{\mathbf{a}} \leq \overline{\mathbf{a}}$ and define an *interval vector* $\mathbf{a}^I = [\underline{\mathbf{a}}, \overline{\mathbf{a}}] \triangleq \{\mathbf{a} = (a_i)_m \mid \underline{\mathbf{a}} \leq \mathbf{a} \leq \overline{\mathbf{a}}\}$. We say that a real vector $\mathbf{a}$ is a *member* of an interval vector $\mathbf{a}^I$, written as $\mathbf{a} \in \mathbf{a}^I$, if $\underline{\mathbf{a}} \leq \mathbf{a} \leq \overline{\mathbf{a}}$. We say $\mathbf{a}^I$ is an interval row vector if its members are row vectors. Similarly, we say $\mathbf{a}^I$ is an interval column vector if its members are column vectors. For simplicity, we let $I(\Re^m)$ be the set of all interval column vectors whose members are defined on $\Re^m$.

Let $\underline{A} = (\underline{a}_{ij})_{m\times n}$ and $\overline{A} = (\overline{a}_{ij})_{m\times n}$ be two $m\times n$ real matrices with $\underline{a}_{ij} \leq \overline{a}_{ij}$, for $i = 1, ..., m$ and $j = 1, ..., n$. In this case, we say $\underline{A} \leq \overline{A}$ and define an *interval matrix* $A^I = [\underline{A}, \overline{A}] \triangleq \{A =$

$(a_{ij})_{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}$. We say that a real matrix A is a member of an interval matrix A^I, written as $A \in A^I$, if $\underline{A} \leq A \leq \overline{A}$. For simplicity, we denote $I(\Re^{m\times n})$ to be the set of all interval matrices whose members are defined on $\Re^{m\times n}$.

An *interval-valued linear equation* is an equation of the form $A^I\mathbf{x} = \mathbf{b}^I$, where $A^I \in I(\Re^{m\times m})$ and $\mathbf{b}^I \in I(\Re^m)$. One active research area in interval computations is to find the solution set of a given interval linear equation. Assume all the members of A^I are nonsingular, three types of solution sets are defined as follows:

1) The *tolerable solution set*: $\{\mathbf{x} \in \Re^m \mid$ for each $A \in A^I$, there exists $\mathbf{b} \in \mathbf{b}^I$ such that $A\mathbf{x} = \mathbf{b}\}$ [31, 53, 20]. It is also called the restricted solution set.

2) The *united solution set*: $\{\mathbf{x} \in \Re^m \mid$ there exist $A \in A^I$ and $\mathbf{b} \in \mathbf{b}^I$ such that $A\mathbf{x} = \mathbf{b}\}$ [34, 13, 31, 16, 33].

3) The *controllable solution set*: $\{\mathbf{x} \in \Re^m \mid$ for each $\mathbf{b} \in \mathbf{b}^I$, there exists $A \in A^I$ such that $A\mathbf{x} = \mathbf{b}\}$ [54, 20].

Remark 1.1 *Notice that when $\underline{A} \neq \overline{A}$ and $\underline{\mathbf{b}} \neq \overline{\mathbf{b}}$, the following sets are empty:*

- $\{\mathbf{x} \in \Re^m \mid$ *there exists* $\mathbf{b} \in \mathbf{b}^I$ *such that* $A\mathbf{x} = \mathbf{b}$, *for all* $A \in A^I\}$.
- $\{\mathbf{x} \in \Re^m \mid$ *there exists* $A \in A^I$ *such that* $A\mathbf{x} = \mathbf{b}$, *for all* $\mathbf{b} \in \mathbf{b}^I\}$.
- $\{\mathbf{x} \in \Re^m \mid A\mathbf{x} = \mathbf{b}$, *for each* $A \in A^I$ *and each* $\mathbf{b} \in \mathbf{b}^I\}$.

Therefore, they are not discussed in this dissertation.

The most commonly used solution set is the united solution set. The united solution set usually is not an interval vector, and need not be convex. In general, it has a very complicated structure [31]. In particular, in every orthant, the united solution set forms a convex polytope. Oettli [35] showed that the united solution set of

$$\begin{cases} A^I\mathbf{x} = \mathbf{b}^I \\ \mathbf{x} \geq 0 \end{cases}$$

is equivalent to the solution set of

$$\begin{cases} \underline{A}\mathbf{x} \leq \overline{\mathbf{b}} \\ \overline{A}\mathbf{x} \geq \underline{\mathbf{b}} \\ \mathbf{x} \geq 0 \end{cases} .$$

1.2 Fuzzy sets and fuzzy relational equations

Fuzzy set theory was proposed three decades ago by Zadeh [62]. Fuzzy relational equations is an important research topic within fuzzy set theory. Applications of the fuzzy relational equations in medical diagnosis, expert systems, pattern recognition, and control engineering have been reported. For most such systems the input and output variables are dependent in some sense, but the "relationship" between the input and output variables often cannot be represented simply by analytical functions. In addition, the input and output values may be obtained by imprecise measurement [32]. In these cases, we may consider using interval values to describe the input-output relations. One of the objectives of this research is to solve interval-valued fuzzy relational equations.

A fuzzy set is an extension of the traditional set. For a traditional set, an element either belongs to or does not belong to it. In fuzzy set theory, each element is assigned a value between 0 and 1 to represent the degree or membership of the element in the given fuzzy set. More formally, a fuzzy set $\tilde{X}$ which defined on a universal discourse U is a set of ordered pairs $\{(u, \mu_{\tilde{X}}(u)) \mid u \in U, \text{ and } \mu_{\tilde{X}}(u) \in [0,1]\}$.

In order to mimic human beings' logical thinking, we have to identify some set operators which capture the meaning of conjunction (AND), disjunction (OR), and implication (If...Then). In fuzzy set theory, triangular norms, triangular co-norms and fuzzy relations are introduced to deal with conjunction, disjunction and implication.

The name "triangular norm" was originally related to the problem of extending geometric triangular inequalities to the statistical metric in probabilistic metric spaces [29]. As defined by D. Nola, et.al. [8] and Schweizer and Sklar [50], the triangular norm (*t-norm* for short) is a real function mapping from $[0,1] \times [0,1]$ to $[0,1]$ which satisfies the following conditions:

(a) $t(x, y) = t(y, x)$ (commutative),

(b) $t(x, t(y, z)) = t(t(x, y), z)$ (associative),

(c) $t(x, y) \leq t(x, z)$, if $y \leq z$ (monotonically nondecreasing),

(d) $t(x, 0) = 0$ and $t(x, 1) = x$ (boundary condition), for any $x \in [0, 1]$.

The triangular co-norm (s-norm for short), associated with a given t-norm t, is a real function mapping from $[0,1] \times [0,1]$ to $[0,1]$ such that $s(x,y) = 1 - t(1-x, 1-y)$, for any $x,\ y \in [0,1]$. More specifically, s-norm satisfies the following conditions:

(a) $s(x,y) = s(y,x)$ (commutative),

(b) $s(x, s(y,z)) = s(s(x,y), z)$ (associative),

(c) $s(x,y) \leq s(x,z)$, if $y \leq z$ (monotonically nondecreasing),

(d) $s(x,1) = 1$ and $s(x,0) = x$ (boundary condition), for any $x \in [0,1]$.

The commonly used *t-norms* include:

(a) minimum operation $t(x,y) = \min\{x,\ y\}$,

(b) algebraic product $t(x,y) = x\ y$,

(c) bounded difference $t(x,y) = \max\{0,\ x+y-1\}$,

(d) Einstein product $t(x,y) = \frac{xy}{2-[x+y-xy]}$,

(e) Hamacher product $t(x,y) = \frac{xy}{x+y-xy}$.

It is well known that the minimum operation *min* is the "largest" *t-norm* and the conjugate *s-norm* maximum *max* is the "smallest" *s-norm* [63], that is, $t(x,y) \leq \min\{x,y\}$ and $s(x,y) \geq \max\{x,y\}$ for any $x,\ y \in [0,1]$.

Through the use of *t-norm* and *s-norm*, the operations of intersection and union of ordinary sets can be extended to fuzzy sets. But the selection of a particular *t-norm* for intersection and *s-norm* for union is strongly application dependent. Even though there is no universal rule for selecting an appropriate set operator, Yager provides some guidelines for selecting an appropriate *t-norm* and *s-norm* [61].

Notation 1.1 *Analogous to* $[0,1]^2 = [0,1] \times [0,1]$*, let us denote* $[0,1]^n = [0,1]^{n-1} \times [0,1]$, $n = 2, 3, \ldots.$ *Also we denote* $[0,1]^{m\times n}$ *as the set of all* $m \times n$ *matrices whose members are between* 0 *and* 1*. Further we denote* $I([0,1]^{m\times n})$ *to be the set of all interval matrices whose members are defined on* $[0,1]^{m\times n}$*.*

While *t-norms* and *s-norms* are defined to map from $[0,1]^2$ to $[0,1]$, they can be extended to map from $[0,1]^n$ to $[0,1]$ by using the associative property. More specifically, $t(a_3, a_2, a_1) = t(a_3, t(a_2, a_1))$, $t(a_4, a_3, a_2, a_1) = t(a_4, t(a_3, a_2, a_1))$, and $t(a_n, a_{n-1}, ..., a_1) = t(a_n, t(a_{n-1}, ..., a_1))$.

Since Sanchez proposed the concept of fuzzy relational equation [47], many researchers have been attracted to work in this area. A fuzzy relation is a fuzzy subset of a Cartesian product of two sets as defined below.

Definition 1.1 *Let U and W be universal sets, then* $\tilde{R} \triangleq \{((u,w), \mu_{\tilde{R}}(u,w)) \mid (u,w) \in U \times W$ *and* $\mu_{\tilde{R}}(u,w) \in [0,1]\}$ *is a fuzzy relation on $U \times W$.*

Note that a fuzzy relation itself is a fuzzy set and the membership $\mu_{\tilde{R}}(u,w)$ can be viewed as the strength of the link between $u \in U$ and $w \in W$. Given a relation $\tilde{R}_1$ on $U \times V$ and a relation $\tilde{R}_2$ on $V \times W$, it is possible to define a composition $\tilde{R}_1 \circ \tilde{R}_2$ of these two relations that links U and W directly. The most commonly used composition is the *max-min* composition. Let $\tilde{R} = \tilde{R}_1 \circ \tilde{R}_2$, for each $u \in U$ and $w \in W$, then the composition can be represented by $\mu_{\tilde{R}}(u,w) = max_{v \in V} \; min\{\mu_{\tilde{R}_1}(u,v), \mu_{\tilde{R}_2}(v,w)\}$, which is called fuzzy relational equation. Depending on the applications, in general, the *max* operation can be replaced by any *s-norm* and the *min* operation can be replaced by any *t-norm*. For example, by *max-t-norm* composition, we mean $\mu_{\tilde{R}}(u,w) = max_{v \in V} \; t(\mu_{\tilde{R}_1}(u,v), \mu_{\tilde{R}_2}(v,w))$, and we denote it by $\tilde{R} = \tilde{R}_1 \vartriangle \tilde{R}_2$. Similarly, by *min-s-norm* composition, we mean $\mu_{\tilde{R}}(u,w) = min_{v \in V} \; s(\mu_{\tilde{R}_1}(u,v), \mu_{\tilde{R}_2}(v,w))$, and we denote it by $\tilde{R} = \tilde{R}_1 \circledS \tilde{R}_2$.

When U, V and W are finite sets, the fuzzy relations can be expressed in terms of memberships. Let $U = \{u_k \mid k = 1, 2, \ldots, p\}, V = \{v_i \mid i = 1, 2, \ldots, m\}$, and $W = \{w_j \mid j = 1, 2, \ldots, n\}$, then $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}$ can be rewritten as:

$\tilde{R}_1 = \{((u_k, v_i), c_{ki}) \mid k = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, m, \; c_{ki} \in [0,1]\}$,

$\tilde{R}_2 = \{((v_i, w_j), a_{ij}) \mid i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n, \; a_{ij} \in [0,1]\}$,

$\tilde{R} = \{((u_k, w_j), b_{kj}) \mid k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n, \; b_{kj} \in [0,1]\}$,

where c_{ki}, a_{ij} and b_{kj} are the grades of membership of (u_k, v_i), (v_i, w_j) and (u_k, w_j), respectively. Furthermore, if we denote $C = (c_{ki})_{p \times m}$, $A = (a_{ij})_{m \times n}$, $B = (b_{kj})_{p \times n}$, then the composition $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}$ can be expressed in terms of matrix operation $C \circ A = B$ with

$b_{kj} = \max_{1\leq i\leq m} \min\{c_{ki},\ a_{ij}\}$, $\forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n$. Basically there are three types of problems related to fuzzy relational equations:

1) Direct problem: given C and A, determine B such that $C \circ A = B$. Basically this problem is used for prediction. The solution is straight forward.

2) Constructive problem: given C and B, determine A such that $C \circ A = B$. This is also called an identification problem.

3) Inverse problem: given A and B, determine C such that $C \circ A = B$. This is also called a diagnosis problem.

Let us take a close look at the constructive problem and the inverse problem. Due to the commutative property of *s-norms* and *t-norms*, we suspect that by solving one of these problems, we also solve the other, which is true, as shown in the following two lemmas.

Lemma 1.1 *Let $C = (c_{ki})_{p\times m} \in [0,1]^{p\times m}$, $A = (a_{ij})_{m\times n} \in [0,1]^{m\times n}$, $B = (b_{kj})_{p\times n} \in [0,1]^{p\times n}$, then the equation $C \vartriangle A = B$ is equivalent to the equation $A^T \vartriangle C^T = B^T$ where T means matrix transpose and $\vartriangle$ denotes the max-t-norm composition.*

Proof. Let $\dot{a}_{ij}$, $\dot{c}_{ij}$ and $\dot{b}_{ij}$ denote the element of row i and column j of A^T, C^T, and B^T, respectively.

$C \vartriangle A = B \Longleftrightarrow \max_{1\leq i\leq m} t(c_{ki}, a_{ij}) = b_{kj}$, $\forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n \Longleftrightarrow \max_{1\leq i\leq m} t(a_{ij}, c_{ki}) = b_{kj}, \forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n \Longleftrightarrow \max_{1\leq i\leq m} t(\dot{a}_{ji}, \dot{c}_{ik}) = \dot{b}_{jk}, \forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n \Longleftrightarrow A^T \vartriangle C^T = B^T$. ∎

Lemma 1.2 *Let $C = (c_{ki})_{p\times m} \in [0,1]^{p\times m}$, $A = (a_{ij})_{m\times n} \in [0,1]^{m\times n}$, $B = (b_{kj})_{p\times n} \in [0,1]^{p\times n}$, then the equation $C \circledS A = B$ is equivalent to the equation $A^T \circledS C^T = B^T$ where T means matrix transpose and $\circledS$ denotes the min-s-norm composition.*

Proof. Let $\dot{a}_{ij}$, $\dot{c}_{ij}$ and $\dot{b}_{ij}$ denote the element of row i and column j of A^T, C^T, and B^T, respectively.

$C \circledS A = B \Longleftrightarrow \min_{1\leq i\leq m} s(c_{ki}, a_{ij}) = b_{kj}$, $\forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n \Longleftrightarrow \min_{1\leq i\leq m} s(a_{ij}, c_{ki}) = b_{kj}, \forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n \Longleftrightarrow \min_{1\leq i\leq m} s(\dot{a}_{ji}, \dot{c}_{ik}) = \dot{b}_{jk}, \forall\ k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n \Longleftrightarrow A^T \circledS C^T = B^T$. ∎

These two lemmas imply that we can convert the constructive problem to the inverse problem and vice versa.

Let C_k and B_k be the *k-th* row vectors of C and B, for $k = 1, 2, \ldots, p$, respectively, we know that

$$C \circ A = B$$

is equivalent to

$$\begin{cases} C_1 \circ A = B_1 \\ C_2 \circ A = B_2 \\ \qquad \vdots \\ C_p \circ A = B_p \end{cases}.$$

Moreover, for the inverse problem, the solution of $C_{k_1} \circ A = B_{k_1}$ for C_{k_1} is independent of the solution of $C_{k_2} \circ A = B_{k_2}$ for C_{k_2}, for $k_1 \neq k_2$. This implies that we can solve $C_k \circ A = B_k$, for $k = 1, 2, \ldots, p$, separately for the inverse problem. Therefore, it is reasonable for us to focus on the inverse problem with C and B being row vectors only.

1.3 N-person cooperative games

The theory of cooperative games can be used to analyze interactive decision-making processes occurring in economics and other social sciences. In a cooperative game, players may form "coalitions" in order to increase their respective payoffs. The reward of each player is determined by the power of that player and with whom that player cooperates. Let $N = \{1, 2, \ldots, n\}$ be a set of players in an n-person game. A nonempty subset of N is called a coalition. We denote the "value" of the coalition S by $v(S)$, which is the total payoff that the coalition earns (collectively) where $v(\cdot)$ is defined on every subset of N. This function is also called the characteristic function, because N and $v(\cdot)$ uniquely determine a game, $\Gamma = (N, v)$. It is important to know what coalitions should be formed and how the payoffs should be distributed among players of each of these coalitions when a game is played. An answer to these questions is called a "*solution concept*" for cooperative games. Currently several solution concepts are available, but none of them is perfect [55]. We are interested in finding some solution concepts for n-person cooperative games whose characteristic functions are interval-valued.

1.4 Research objectives

In this dissertation our objectives are:

- Defining and finding solution sets of interval-valued *max-t-norm* fuzzy relational equations.
- Defining and finding solution sets of interval-valued *min-s-norm* fuzzy relational equations.
- Finding solution sets of n-person cooperative games whose characteristic functions are interval-valued.

1.5 Dissertation outline

The dissertation is organized as follows. Literature surveys on fuzzy relational equations and cooperative games are given in Chapter 2. Results related to fuzzy relational equations with *max-t-norm* compositions are presented in Chapter 3. Some results related to fuzzy relational equations with *min-s-norm* compositions are presented in Chapter 4. Interval-valued cooperative games are discussed in Chapter 5. Finally, a summary and discussion of future research are given in Chapter 6.

Chapter 2

Literature Review

2.1 Fuzzy relational equations

From Lemma 1.1 and Lemma 1.2 of Chapter 1, we know that the constructive problem can be converted to the inverse problem and vice verse. Hence, this dissertation will focus on the inverse problem.

2.1.1 Constant-valued fuzzy relational equations

Given a fuzzy matrix $A = (a_{ij})_{m\times n} \in [0,1]^{m\times n}$, a vector $\mathbf{b} = (b_1, b_2, ..., b_n) \in [0,1]^n$, and a composition $\circ \in \{\triangle, Ⓢ\}$, where $\triangle$ denotes *max-t-norm* composition and Ⓢ denotes *min-s-norm* composition, respectively, we define a constant-valued fuzzy relational equation as follows:

$$\mathbf{x} \circ A = \mathbf{b}. \tag{2.1}$$

Solving (2.1) means finding all vectors $\mathbf{x} = (x_1, x_2, ..., x_m) \in [0,1]^m$ such that $\max_{1\leq i\leq m} t(x_i, a_{ij}) = b_j$, if $\circ$ represents $\triangle$, or $\min_{1\leq i\leq m} s(x_i, a_{ij}) = b_j$, if $\circ$ represents Ⓢ, for $j = 1, ..., n$. Let $\Sigma(A, \mathbf{b})$ denote the solution set of (2.1). In order to characterize $\Sigma(A, \mathbf{b})$, we give the following definitions.

Definition 2.1 *Given $\acute{x}$, $\acute{a} \in [0,1]$, a t-norm t is called lower semicontinuous if $t(\acute{x}, \acute{a})$ is left-hand continuous on $[0,1]$ with respect to $\acute{x}$ for any fixed $\acute{a}$. Similarly, t is called upper semicontinuous if $t(\acute{x}, \acute{a})$ is right-hand continuous on $[0,1]$ with respect to $\acute{x}$ for any fixed $\acute{a}$.*

Moreover, t is called continuous if $t(\acute{x}, \acute{a})$ *is continuous on* $[0,1]$ *with respect to* $\acute{x}$ *for any fixed* $\acute{a}$.

Definition 2.2 *Given* $\acute{x}$, $\acute{a} \in [0,1]$, *an s-norm s is called lower semicontinuous if* $s(\acute{x}, \acute{a})$ *is left-hand continuous on* $[0,1]$ *with respect to* $\acute{x}$ *for any fixed* $\acute{a}$. *Similarly, s is called upper semicontinuous if* $s(\acute{x}, \acute{a})$ *is right-hand continuous on* $[0,1]$ *with respect to* $\acute{x}$ *for any fixed* $\acute{a}$. *Moreover, s is called continuous if* $s(\acute{x}, \acute{a})$ *is continuous on* $[0,1]$ *with respect to* $\acute{x}$ *for any fixed* $\acute{a}$.

Definition 2.3 *For* $\mathbf{x}^1$, $\mathbf{x}^2 \in \Re^m$, *we say* $\mathbf{x}^1 \leq \mathbf{x}^2$ *if and only if* $x_i^1 \leq x_i^2$, $\forall\ i = 1, 2, \cdots, m$. *Similarly, for* A^1, $A^2 \in \Re^{m \times n}$, $A^1 \leq A^2$ *if and only if* $a_{ij}^1 \leq a_{ij}^2$, $\forall\ i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$.

Definition 2.4 $\hat{\mathbf{x}} \in \Sigma(A, \mathbf{b})$ *is a maximum solution if* $\mathbf{x} \leq \hat{\mathbf{x}}$, $\forall\ \mathbf{x} \in \Sigma(A, \mathbf{b})$. *Similarly,* $\check{\mathbf{x}} \in \Sigma(A, \mathbf{b})$ *is a minimum solution if* $\mathbf{x} \geq \check{\mathbf{x}}$, $\forall\ \mathbf{x} \in \Sigma(A, \mathbf{b})$.

Definition 2.5 $\hat{\mathbf{x}} \in \Sigma(A, \mathbf{b})$ *is a maximal solution if* $\mathbf{x} \geq \hat{\mathbf{x}}$ *implies* $\mathbf{x} = \hat{\mathbf{x}}$, $\forall\ \mathbf{x} \in \Sigma(A, \mathbf{b})$. *Similarly,* $\check{\mathbf{x}} \in \Sigma(A, \mathbf{b})$ *is a minimal solution if* $\mathbf{x} \leq \check{\mathbf{x}}$ *implies* $\mathbf{x} = \check{\mathbf{x}}, \forall\ \mathbf{x} \in \Sigma(A, \mathbf{b})$.

For *max-t-norm* composition,

$$\mathbf{x} \bigtriangleup A = \mathbf{b}, \tag{2.2}$$

Pedrycz introduced a φ operator [40].

Definition 2.6 *Given* $\acute{a}$, $\acute{b} \in [0,1]$, *the* φ *operator associated with a given t-norm is defined by*

$$\acute{a}\ \varphi\ \acute{b} = \sup\{x \in [0,1] \mid t(x, \acute{a}) \leq \acute{b}\}.$$

Given $\acute{a}$, $\acute{b}$, $\acute{x} \in [0,1]$, it can be verified that (Di Nola et al. (1989)):

(a) $t((\acute{a}\ \varphi\ \acute{b}), \acute{a}) \leq \acute{b}$,

(b) $t(\acute{x}, \acute{a}) \leq \acute{b}$ if and only if $\acute{x} \leq \acute{a}\ \varphi\ \acute{b}$.

Definition 2.7 *Given* $\acute{a}$, $\acute{b} \in [0,1]$ *and* $\acute{a} \geq \acute{b}$, *the* γ *operator associated with a given t-norm is defined by*

$$\acute{a}\ \gamma\ \acute{b} = \inf\{x \in [0,1] \mid t(x, \acute{a}) \geq \acute{b}\}.$$

$t(\acute{x}, \acute{a})$	$\acute{a}\ \varphi\ \acute{b}$	$\acute{a}\ \gamma\ \acute{b}$	$\acute{a}\ \ell\ \acute{b}$
	$\acute{a} > \acute{b}$	$(\acute{a} \geq \acute{b})$	$\acute{a} \geq \acute{b}$
$\min(\acute{x}, \acute{a})$	$\acute{b}$	$\acute{b}$	$\acute{b}$
$\acute{a}\ \acute{x}$	$\frac{\acute{b}}{\acute{a}}$	$\frac{\acute{b}}{\acute{a}}$	$\frac{\acute{b}}{\acute{a}}$
$\max\{0,\ \acute{a} + \acute{x} - 1\}$	$1 + \acute{b} - \acute{a}$	$\begin{cases} 0 & , if\ \acute{b} = 0 \\ 1 + \acute{b} - \acute{a} & , otherwise \end{cases}$	$\begin{cases} 0, & \acute{b} = 0,\ \acute{a} \leq 1 \\ 1 + \acute{b} - \acute{a} & , otherwise \end{cases}$
$\frac{\acute{a}\acute{x}}{2-[\acute{a}+\acute{x}-\acute{a}\acute{x}]}$	$\frac{2\acute{b}-\acute{b}\acute{a}}{\acute{a}+\acute{b}-\acute{b}\acute{a}}$	$\frac{2\acute{b}-\acute{b}\acute{a}}{\acute{a}+\acute{b}-\acute{b}\acute{a}}$	$\frac{2\acute{b}-\acute{b}\acute{a}}{\acute{a}+\acute{b}-\acute{b}\acute{a}}$
$\frac{\acute{a}\acute{x}}{\acute{a}+\acute{x}-\acute{a}\acute{x}}$	$\frac{\acute{b}\acute{a}}{\acute{a}+\acute{b}\acute{a}-\acute{b}}$	$\frac{\acute{b}\acute{a}}{\acute{a}+\acute{b}\acute{a}-\acute{b}}$	$\frac{\acute{b}\acute{a}}{\acute{a}+\acute{b}\acute{a}-\acute{b}}$

Table 2.1: φ and ℓ operators of t-norms

Lemma 2.1 *Given* $\acute{a}$, $\acute{b}$, $\acute{x} \in [0,1]$ *and* $\acute{a} \geq \acute{b}$, *for a continuous t-norm,*
(a) $t((\acute{a}\ \gamma\ \acute{b}), \acute{a}) \geq \acute{b}$.
(b) $t(\acute{x}, \acute{a}) \geq \acute{b}$ *if and only if* $\acute{x} \geq \acute{a}\ \gamma\ \acute{b}$.

Proof. (a) Let $\bar{x} = \acute{a}\ \gamma\ \acute{b}$. Since t is continuous and $\bar{x} = \inf\{x \in [0,1] \mid t(x, \acute{a}) \geq \acute{b}\}$, we have $t(\bar{x}, \acute{a}) \geq \acute{b}$.
(b) If $t(\acute{x}, \acute{a}) \geq \acute{b}$, then $\acute{a}\ \gamma\ \acute{b} = \inf\{x \in [0,1] \mid t(x, \acute{a}) \geq \acute{b}\} \leq \acute{x}$. On the other hand, if $\acute{x} \geq \acute{a}\ \gamma\ \acute{b}$, then $\acute{x} \geq \inf\{x \in [0,1] \mid t(x, \acute{a}) \geq \acute{b}\}$. Since *t-norms* are non-decreasing, we have $t(\acute{x}, \acute{a}) \geq \acute{b}$. ■

We see that $\acute{a}\ \varphi\ \acute{b} \in \{x \in [0,1] \mid t(x, \acute{a}) \leq \acute{b}\}$ if and only if the *t-norm* is lower semicontinuous. Obviously, $\acute{a}\ \varphi\ \acute{b} = 1$, for $\acute{a} \leq \acute{b}$, since $t(\acute{x}, \acute{a}) \leq \min(\acute{x}, \acute{a})$. Table 2.1 lists examples of commonly used *t-norms* and their associated φ and γ operators. For more examples of φ operators, refer to [8, 40].

Theorem 2.1 *If* (2.2) *has a solution, then* $\hat{\mathbf{x}} = \left(\min_{j=1}^{n}(a_{ij}\ \varphi\ b_j)\right)_{i=1,2,\ldots,m}$ *is the (unique) maximum solution* [8].

Finding all the minimal solutions is more complicated. In order to characterize the minimal solutions, De Baets introduced the ℓ operator [4].

Definition 2.8 *Given* $\acute{a}$, $\acute{b} \in [0,1]$, *the* ℓ *operator associated with a given t-norm is defined by*

$$\acute{a}\ \ell\ \acute{b} = \begin{cases} 1 & , if\ \acute{a} < \acute{b} \\ \inf\{\acute{x} \in [0,1] \mid t(\acute{x}, \acute{a}) \geq \acute{b}\} & , Otherwise \end{cases}.$$

Table 2.1 also lists several examples of the ℓ operator. Also notice that $\acute{a}\ \ell\ \acute{b} \in \{\acute{x} \in [0,1] \mid t(\acute{x}, \acute{a}) \geq \acute{b}\}$ if and only if the *t-norm* is lower semicontinuous.

Definition 2.9 *Given $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, we define $\mathbf{y}^{i,j} \in [0,1]^m$ as*

$$y_k^{i,j} = \begin{cases} a_{ij}\ \ell\ b_j & , \text{ if } k = i \\ 0 & , \text{ otherwise} \end{cases} .$$

Definition 2.10 *Let $\hat{\mathbf{x}} = \left(\min_{j=1}^{n}(a_{ij}\ \varphi\ b_j)\right)_{i=1,2,\ldots,m}$ be the maximum solution of* (2.2)*. We define the set $\Omega^j = \{\ \mathbf{y}^{i,j} \in [0,1]^m \mid a_{ij} \geq b_j$ and $\mathbf{y}^{i,j} \leq \hat{\mathbf{x}}$, for $i = 1, 2, \ldots, m.\}$ for every $j = 1, 2, \ldots, n$.*

Theorem 2.2 *The minimal solutions of* (2.2) *are in the set $\{\ \max_{j=1}^{n} \mathbf{f}^j \mid \mathbf{f}^j \in \Omega^j\}$ where max is understood as a component-wise operation* [4]*.*

Di Nola et.al. [8] showed that, if (2.2) has a solution and the *t-norm* is a lower semicontinuous (LSC) function, then there is a maximum solution, but minimal solutions may not exist. Moreover, if (2.2) has a solution and the *t-norm* is an upper semicontinuous (USC) function, then there exist minimal solutions, but the maximum solution may not exist. For continuous *t-norms*, such as the minimum operator and the algebraic product operator, since it is both LSC and USC, if (2.2) has a solution, the solution set can be completely determined by one maximum and a finite number of minimal solutions.

The commonly seen *max-min* and *max-product* compositions are special cases of the *max-t-norm* composition. Many results have been obtained for the *max-min* and *max-product* compositions. Bourke and Fisher [3] reviewed the algorithms for determining the complete solution set of fuzzy relational equations with *max-product* composition. They also listed references for algorithms with the *max-min* composition. Loetamonphong and Fang [24] provided an efficient procedure to find the whole solution set for fuzzy relational equations with *max-product* composition. In real applications, knowing the solution structure is important. Even though the solution set is non-convex, by taking the advantage of the special solution structure, Fang and Li [11] solved a linear optimization problem with *max-min* fuzzy relational equation constraints by using a branch-and-bound solution technique. The method was extended by Loetamonphong and Fang [25] to solve linear optimization problems with *max-product* fuzzy relational equation constraints. In addition, Lu and Fang [27] designed a genetic algorithm to solve nonlinear optimization problems with *max-min* fuzzy relational equation constraints. Recently, Loetamonphong, Fang and Young [26] solved multi-objective optimization problems with *max-min* fuzzy relational equation constraints by using genetic algorithms.

Analogous to that for *max-t-norm* composition, for *min-s-norm* composition,

$$\mathbf{x} \textcircled{s} A = \mathbf{b}, \tag{2.3}$$

Pedrycz introduced a β operator [40].

Definition 2.11 *Given $\acute{a}$, $\acute{b} \in [0,1]$, the β operator associated with a given s-norm is defined by*

$$\acute{a}\ \beta\ \acute{b} = \inf\{x \in [0,1] \mid s(x, \acute{a}) \geq \acute{b}\}.$$

Given $\acute{a}$, $\acute{b}$, $\acute{x} \in [0,1]$, it is easy to verify that:

(a) $s((\acute{a}\ \beta\ \acute{b}), \acute{a}) \geq \acute{b}$,

(b) $s(\acute{x}, \acute{a}) \geq \acute{b}$ if and only if $\acute{x} \geq \acute{a}\ \beta\ \acute{b}$.

We see that $\acute{a}\ \beta\ \acute{b} \in \{\acute{x} \in [0,1] \mid s(\acute{x}, \acute{a}) \geq \acute{b}\}$ if and only if the *s-norm* is upper semicontinuous. Obviously, $\acute{a}\ \beta\ \acute{b} = 0$, for $\acute{a} \geq \acute{b}$, since $s(\acute{x}, \acute{a}) \geq \max(\acute{x}, \acute{a})$.

Definition 2.12 *Given $\acute{a}$, $\acute{b} \in [0,1]$ and $\acute{a} \leq \acute{b}$, the ω operator associated with a given s-norm is defined by*

$$\acute{a}\ \omega\ \acute{b} = \sup\{x \in [0,1] \mid s(x, \acute{a}) \leq \acute{b}\}.$$

Lemma 2.2 *Given $\acute{a}$, $\acute{b}$, $\acute{x} \in [0,1]$ and $\acute{a} \leq \acute{b}$, for a continuous s-norm,*
(a) *$s((\acute{a}\ \omega\ \acute{b}), \acute{a}) \leq \acute{b}$.*
(b) *$s(\acute{x}, \acute{a}) \leq \acute{b}$ if and only if $\acute{x} \leq \acute{a}\ \omega\ \acute{b}$.*

Proof. (a) Let $\bar{x} = \acute{a}\ \omega\ \acute{b}$. Since s is continuous and $\bar{x} = \sup\{x \in [0,1] \mid s(x, \acute{a}) \leq \acute{b}\}$, we have $s(\bar{x}, \acute{a}) \leq \acute{b}$.
(b) If $s(\acute{x}, \acute{a}) \leq \acute{b}$, then $\acute{a}\ \omega\ \acute{b} = \sup\{x \in [0,1] \mid s(x, \acute{a}) \leq \acute{b}\} \geq \acute{x}$. On the other hand, if $\acute{x} \leq \acute{a}\ \omega\ \acute{b}$, then $\acute{x} \leq \sup\{x \in [0,1] \mid s(x, \acute{a}) \leq \acute{b}\}$. Since s-norms are non-decreasing, we have $s(\acute{x}, \acute{a}) \leq \acute{b}$.
∎

Table 2.2 lists examples of commonly used *s-norms* and their associated β and ω operators. For more examples of β operator, refer to [8, 40].

$s(\acute{x}, \acute{a})$	$\acute{a}\ \omega\ \acute{b}$	$\acute{a}\ \beta\ \acute{b}$	
	$(\acute{a} \leq \acute{b})$	$\acute{a} < \acute{b}$	$\acute{a} \geq \acute{b}$
$\max(\acute{x}, \acute{a})$	$\acute{b}$	$\acute{b}$	0
$\acute{a} + \acute{x} - \acute{a}\ \acute{x}$	$\frac{\acute{b}-\acute{a}}{1-\acute{a}}$	$\frac{\acute{b}-\acute{a}}{1-\acute{a}}$	0
$\min\{1,\ \acute{a}+\acute{x}\}$	$\begin{cases} 1 & , if\ \acute{b}=1 \\ \acute{b}-\acute{a} & , otherwise \end{cases}$	$\acute{b}-\acute{a}$	0
$\frac{\acute{a}+\acute{x}}{1+\acute{a}\acute{x}}$	$\frac{\acute{b}-\acute{a}}{1-\acute{b}\acute{a}}$	$\frac{\acute{b}-\acute{a}}{1-\acute{b}\acute{a}}$	0
$\frac{\acute{a}+\acute{x}-2\acute{a}\acute{x}}{1-\acute{a}\acute{x}}$	$\frac{\acute{b}-\acute{a}}{1+\acute{b}\acute{a}-2\acute{a}}$	$\frac{\acute{b}-\acute{a}}{1+\acute{b}\acute{a}-2\acute{a}}$	0

Table 2.2: β and ω operators of s-norms

Most papers in the literature deal with the minimum solution of (2.3).

Theorem 2.3 *If* (2.3) *has a solution, then* $\check{\mathbf{x}} = \left(\max_{j=1}^{n}(a_{ij}\ \beta\ b_j)\right)_{i=1,2,\ldots,m}$ *is the (unique) minimum solution* [8]*.*

As pointed out by De Baets [4], the *min-s-norm* composition is “equally interesting and important, but less known” compared to commonly known *max-t*-composition. Pedrycz solved the *min-s* fuzzy relational equations by using a method similar to that for solving *max-t* fuzzy relational equations [39, 42]. Di Nola et.al. further showed that, for continuous *s-norms*, such as the maximum operator, if (2.3) has a solution, the solution set can be completely determined by one minimum and a finite number of maximal solutions [8]. In this dissertation, we will deal with continuous *s-norms* only.

For applications, Kundu demonstrated the superiority of the $min - max$ (special case of *min-s*-composition) rule in terms of the properties of fuzzy equivalence relation [19].

When the composition “$\circ$” is taken to be a general “*s-norm-t-norm*” (*s-t* for short) composition, a general structure for the solution set of (2.1) is not available [6]. However, it is still possible to find one solution vector in $\Sigma(A, \mathbf{b})$. For example, for a continuous *s-norm*, Levrat et al. showed a way to find one analytical solution [22]. Pedrycz used an induced optimization scheme to solve *s-t* fuzzy relational equations [43]. The underlying idea is to optimize the so-called performance index (e.g., minimize the mean square error between $\mathbf{b}$ and $\mathbf{x} \circ A$) by iteratively modifying $\mathbf{x}$ according to the direction expressed by the gradient of the performance index.

In case there is no solution for (2.1), we are interested in getting approximate solutions [8, 57, 14]. Generally there are three approaches to handle this situation. The first one is to omit some equations. The second one is to use numerical methods or soft computing techniques. The third approach is to modify the fuzzy relational equation by, for example, allowing the vector $\mathbf{b}$ and/or the fuzzy matrix A to have interval-valued elements.

2.1.2 Interval-valued fuzzy relational equations

In this dissertation we consider both A and $\mathbf{b}$ to be interval-valued; that is, A is of the form $A^I = [\underline{A}, \overline{A}]$ where $\underline{A} = \left(\underline{a}_{ij}\right)_{m\times n}$, $\overline{A} = \left(\overline{a}_{ij}\right)_{m\times n}$, with $0 \le \underline{a}_{ij} \le \overline{a}_{ij} \le 1$, and $\mathbf{b}$ is of the form $\mathbf{b}^I = [\underline{\mathbf{b}}, \overline{\mathbf{b}}]$ where $\underline{\mathbf{b}} = (\underline{b}_j)_n$, $\overline{\mathbf{b}} = (\overline{b}_j)_n$, with $0 \le \underline{b}_j \le \overline{b}_j \le 1$. First let us introduce the interval-valued fuzzy relational equations.

Definition 2.13 *An interval-valued fuzzy relational equation is an equation of the form* $\mathbf{x} \circ A^I = \mathbf{b}^I$,

$$\mathbf{x} \circ A^I = \mathbf{b}^I, \tag{2.4}$$

where $A^I \in I([0,1]^{m\times n})$, $\mathbf{b}^I \in I([0,1]^n)$, *and* $\circ$ *denotes the max-t-norm or min-s-norm composition.*

In this research we restrict our discussion to continuous *t-norms* and *s-norms* for reasons to be discussed in the next chapters.

Analogous to the interval linear equations, we define three solution sets for interval-valued fuzzy relational equations (2.4), namely the united solution set, the tolerable solution set, and the controllable solution set.

Definition 2.14 *The solution set* $\Sigma_T(A^I, \mathbf{b}^I) \triangleq \{\mathbf{x} \in [0,1]^m \mid$ *for each* $A \in A^I$, *there exists* $\mathbf{b} \in \mathbf{b}^I$ *such that* $\mathbf{x} \circ A = \mathbf{b}\}$ *is called the tolerable solution set of* (2.4).

Definition 2.15 *The solution set* $\Sigma_U(A^I, \mathbf{b}^I) \triangleq \{\mathbf{x} \in [0,1]^m \mid$ *there exist* $A \in A^I$ *and* $\mathbf{b} \in \mathbf{b}^I$ *such that* $\mathbf{x} \circ A = \mathbf{b}\}$ *is called the united solution set of* (2.4).

Definition 2.16 *The solution set* $\Sigma_C(A^I,\ \mathbf{b}^I) \triangleq \{\mathbf{x} \in [0,1]^m \mid$ *for each* $\mathbf{b} \in \mathbf{b}^I$*, there exists* $A \in A^I$ *such that* $\mathbf{x} \circ A = \mathbf{b}\}$ *is called the controllable solution set of* (2.4).

The definitions of maximum, minimum, maximal, and minimal solutions for $\Sigma_T(A^I,\ \mathbf{b}^I)$, $\Sigma_U(A^I,\ \mathbf{b}^I)$, and $\Sigma_C(A^I,\ \mathbf{b}^I)$ are similar to their crisp counterparts (Definitions 2.4-2.5.)

Example 1 *Consider* $(x_1\ x_2\ x_3) \diamond \begin{pmatrix} [0.3, 0.7] & [0.5, 0.6] \\ [0.4, 0.9] & [0.1, 0.7] \\ [0.2, 0.5] & [0.8, 1.0] \end{pmatrix} = \begin{pmatrix} [0.3, 0.8] & [0.5, 0.7] \end{pmatrix}$*, where* $\diamond$ *stands for the max-min composition. We have* $\underline{A} = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$, $\overline{A} = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.7 \\ 0.5 & 1.0 \end{pmatrix}$, $\underline{\mathbf{b}} = (0.3\ \ 0.5)$*, and* $\overline{\mathbf{b}} = (0.8\ \ 0.7)$*. The tolerable solution set is* $\{([0.5, 1]\ \ [0, 0.8]\ \ [0, 0.7]),$

$([0.3, 1]\ \ [0, 0.8]\ \ [0.5, 0.7]),\ ([0,\ 1]\ \ [0.3, 0.8]\ \ [0.5, 0.7])\}$.

Obviously, a tolerable solution is a united solution. A controllable solution is also a united solution. Also notice that for conventional fuzzy relational equations, where A^I reduces to a single matrix and $\mathbf{b}^I$ reduces to a single vector, the three solution sets coincide with each other.

All of the three solution concepts for fuzzy relational equations are useful. For instance, in fuzzy control, we want to find all input states ($\mathbf{x}$) which guarantee that the output states ($\mathbf{b}$) are within a tolerable range ($\mathbf{b}^I$) for ALL input-output relations ($A \in A^I$). This is essentially the tolerable solution set case. For fuzzy diagnosis, we want to find all sets of symptoms ($\mathbf{x}$) which can cause the set of diseases ($\mathbf{b} \in \mathbf{b}^I$). This means that for any $\mathbf{x}$ we only require one $A \in A^I$ such that $\mathbf{x} \circ A = \mathbf{b}$, for some $\mathbf{b} \in \mathbf{b}^I$, which is essentially a united solution. Another question naturally arises in fuzzy control is whether there exist some inputs ($\mathbf{x}$) which can lead to any specified outputs ($\mathbf{b} \in \mathbf{b}^I$) by the appropriate choice (control) of the relation matrix ($A \in A^I$). This is the controllable solution set.

Our objective is to find the united solution set, the tolerable solution set, and the controllable solution set of (2.4). Here we restrict our discussion to continuous *t-norms* and *s-norms*.

There are papers that deal with *max-min* interval-valued problems. Wang and Chang [58] investigated the properties of the tolerable solution set and proposed a solution procedure for

Composition	Value	Solution	Reference	Comment
max-min	crisp	analytical	[47, 38, 8]	
		approximate	[8]	
	interval	tolerable	[58, 23]	
		united/controllable		no work known
max-product	crisp	analytical	[39, 24]	
		approximate	[8]	
	interval	tolerable		no work known
		united/controllable		no work known
max-t-norm	crisp	analytical	[8, 4]	
		approximate	[8]	
	interval	tolerable		no work known
		united/controllable		no work known
s-t-norm	crisp	analytical	[22]	find one solution
		approximate	[43]	find one solution
	interval	tolerable		no work known
		united/controllable		no work known

Table 2.3: Fuzzy relational equations references

max-min interval-valued fuzzy relational equations. They proved that if tolerable solutions exist, then the solution set can be determined by one maximum and several minimal solutions. Li and Fang [23] provided a more efficient approach to find the tolerable solution set for the *max-min* interval-valued fuzzy relational equations. Our method is inspired by Li and Fang's approach. Table 2.3 classifies the literature related to the solution of fuzzy relational equations. From this table we see that there are no published papers related to the tolerable solution set, united solution set, and controllable solution set, for general *max-t-norm* composition.

For the *min-s-norm* composition, to our knowledge, there are no published papers related to the tolerable solution set, united solution set, and controllable solution set, for any *s-norm*.

2.2 Cooperative games

Recall that $N = \{1, 2, \ldots, n\}$ denotes a set of players in an n-person game, while $v(S)$ denotes the value of the coalition S, $S \subseteq N$. N and $v(\cdot)$ determine a cooperative game $\Gamma = (N, v)$. A "*solution concept*" describes how the payoffs should be distributed among players. Before discussing solution concepts, we need some basic definitions [55].

2.2.1 Basic definitions

A game $\Gamma = (N, v)$ is called

- superadditive, if $v(S \cup S^{'}) \geq v(S) + v(S')$ for S, $S' \subset N$ and $S \cap S' = \phi$.
- weakly superadditive, if $v(N) \geq v(S) + \sum\limits_{i \in N-S} v(\{i\})$.
- monotonic, if $v(S) \geq v(T)$ for $S \supset T$.
- constant-sum, if $v(S) + v(N - S) = v(N)$ for any $S \subset N$.
- normalized, if $v(\{i\}) = 0$ for $i \in N$.
- inessential, if $v(N) = \sum\limits_{i \in N} v(\{i\})$.
- essential, if $v(N) > \sum\limits_{i \in N} v(\{i\})$.
- rational, if $v(N) \geq \sum\limits_{i \in N} v(\{i\})$.
- strategically equivalent with $\Gamma' = (N, v')$, if there exist α and $\boldsymbol{\beta}$ such that $v(S) = \alpha v'(S) + \sum\limits_{i \in S} \beta_i$ for all $S \subset N$.

Definition 2.17 *Let x_i be the payoff for player i, for i=$1, 2, \ldots, n$, then $\mathbf{x} = (x_1, x_2, ..., x_n)$ is called a "payoff vector". Moreover, $\mathbf{x}$ is called an "imputation" if it satisfies*

(i) $v(N) = \sum\limits_{i \in N} x_i$ *(group rationality)*

(ii) $x_i \geq v(\{i\})$ *(individual rationality).*

We further denote the set of all imputations by $X(\Gamma)$.

Definition 2.18 *Let S be a coalition, and let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be two imputations. We say $\mathbf{x}$ dominates $\mathbf{y}$ through S, denoted by $\mathbf{x} \succ^S \mathbf{y}$, if $x_i > y_i$, for $i \in S$ and $\sum\limits_{i \in S} x_i \leq v(S)$ (the members of S can attain rewards given by $\mathbf{x}$). $\mathbf{x}$ is said to dominate $\mathbf{y}$ (denoted by $\mathbf{x} \succ \mathbf{y}$) if there exists a coalition S such that $\mathbf{x} \succ^S \mathbf{y}$.*

Given two coalitions S_1 and S_2, since it is possible to have $\mathbf{x} \succ^{S_1} \mathbf{y}$ and $\mathbf{y} \succ^{S_2} \mathbf{x}$, $\mathbf{x} \succ \mathbf{y}$ and $\mathbf{y} \succ \mathbf{x}$ can hold simultaneously. Intuitively, coalition S always prefers an imputation which cannot be dominated by other imputations through S.

Now we can discuss the solution concepts in the following section.

2.2.2 Solution concepts

Stable set (von Neumann-Morgenstern solution)

Definition 2.19 *A subset* $V(\Gamma)$ *of all imputations* $X(\Gamma)$ *is called a stable set if it satisfies the following two conditions:*

(i) $\mathbf{x} \not\succ \mathbf{y}$ *for any* $\mathbf{x}, \mathbf{y} \in V(\Gamma)$ *(internal stability),*

(ii) *for any imputation* $\mathbf{x} \notin V(\Gamma)$ *there exists a* $\mathbf{y} \in V(\Gamma)$ *for which* $\mathbf{y} \succ \mathbf{x}$ *(external stability).*

There is no existence or uniqueness proof for the stable set.

Core

Definition 2.20 *The core* $C(\Gamma)$ *is the set of all imputations* $\mathbf{x}$ *satisfying*

(i) $\sum\limits_{i \in S} x_i \geq v(S)$, *for all* $S \subset N$,

(ii) $\sum\limits_{i \in N} x_i = v(N)$.

From the above condition *(i)*, we know that the core represents a certain kind of stability. Once an imputation $\mathbf{x}$ in the core is agreed upon, then it is of no coalition's interest to change the imputation $\mathbf{x}$. As proved in [55], the core of a weakly superadditive game is the set of nondominated imputations. For the essential constant-sum game, $C(\Gamma) = \phi$. For superadditive game, the core is a subset of the stable set. In this case, it is the unique stable set [36]. Since the core does not always exist, the solution concept of strong $\varepsilon - core$, which relaxes the condition $\sum\limits_{i \in S} x_i \geq v(S)$, is introduced.

Strong $\varepsilon - core$

Let $\mathbf{x} \in X(\Gamma)$, and $x(S) = \sum\limits_{i \in S} x_i$, $e(S, \mathbf{x}) = v(S) - x(S)$ is called the excess of S with respect to imputation $\mathbf{x}$. It represents the gain that is realized by the coalition S if its members form a coalition. From the definition of core, $e(S, \mathbf{x}) \leq 0$ for $\mathbf{x} \in C(\Gamma)$.

Definition 2.21 *The strong* $\varepsilon - core$ *is defined as* $C_\varepsilon(\Gamma) = \{\mathbf{x} \mid \mathbf{x} \in X(\Gamma),\ e(S, \mathbf{x}) \leq \varepsilon$ *for any* $S \neq N$ *and* $\phi\}$.

$C_\varepsilon(\Gamma)$ has a nice property. For any n-person game, there exists a real number, ε_0, such that, for all $\varepsilon \geq \varepsilon_0$, $C_\varepsilon(\Gamma)$ is not empty.

Kernel

Let $s_{ij}(\mathbf{x})=\max_{S\in T_{ij}} e(S,\mathbf{x})$ where $\mathbf{x}\in X(\Gamma)$ and $T_{ij}=\{S \mid S\subset N,\, i\in S,\, j\notin S\}$. $s_{ij}(\mathbf{x})$ measures player i's "strength" against player j. We say player i outweighs player j with respect to $\mathbf{x}$ if $s_{ij}(\mathbf{x})>s_{ji}(\mathbf{x})$ and $x_j>v(\{j\})$; moreover, player i and player j are in equilibrium with respect to $\mathbf{x}$ if neither of them outweighs the other. The kernel is the set of imputations with respect to which every two players are in equilibrium.

Definition 2.22 *The kernel* $K(\Gamma)$ *is defined as* $\{\mathbf{x} \mid \mathbf{x}\in X(\Gamma), (s_{ij}(\mathbf{x})-s_{ji}(\mathbf{x}))(x_j-v(\{j\}))\leq 0,$ *for all* $i,\, j,\, i\neq j\}$.

Any rational n-person game Γ has a nonempty kernel. If Γ has a non-empty core, then the intersection of the core and the kernel is also nonempty. Let $\mathbf{x}\in C(\Gamma)$, then $\mathbf{x}\in K(\Gamma)$ if and only if $s_{ij}(\mathbf{x})=s_{ji}(\mathbf{x})$ for all $i,\, j,\, i\neq j$.

Nucleolus

Rearrange $\Theta(\mathbf{x})=(e(S_1,\mathbf{x}),\, e(S_2,\mathbf{x}),\ldots,\, e(S_{2^n},\mathbf{x}))$ such that $\Theta_i(\mathbf{x})\geq\Theta_j(\mathbf{x}), 1\leq i\leq j\leq 2^n$, where S_i is a subset of N. Let $\preceq$ denote "lexicographic order", i.e., $\Theta(\mathbf{x})\preceq\Theta(\mathbf{y})$ if and only if there exists an index $i_0\geq 0$ such that $\Theta_v(\mathbf{x})=\Theta_v(\mathbf{y})$, for $v=0,\ldots,\, i_0$, and $\Theta_{i_0+1}(\mathbf{x})<\Theta_{i_0+1}(\mathbf{y})$.

Definition 2.23 *The nucleolus* $N(\Gamma)$ *is defined as* $\{\mathbf{x} \mid \mathbf{x}\in X(\Gamma)$ *and* $\Theta(\mathbf{x})\preceq\Theta(\mathbf{y}),\ \forall\ \mathbf{y}\in X(\Gamma)\}$.

The nucleolus seeks an imputation in such a manner that the maximum complaint by any coalition against the reward division is minimized. Kohlberg proved that, under certain conditions, $N(\Gamma)$ is the unique solution of a specific linear programming problem [18]. However, both the number of constraints and the number of variables in this LP grow exponentially with the number of players.

Shapley value

Let $\mathbf{x}$ be a payoff vector for game $(N,\, v)$ and $\mathbf{y}$ be the payoff vector of game $(N,\, \bar{v})$. Define a game $(N,\, v+\bar{v})$ such that $(v+\bar{v})(S)=v(S)+\bar{v}(S)$ for every $S\subset N$. Let the payoff vector for

$(N,\, v+\bar{v})$ be $\mathbf{z}$. Shapley introduced the following three axioms:

Axiom 1: Relabeling of players interchanges the players' rewards.

Axiom 2: If $v(S-\{i\}) = v(S)$ for all coalitions S, then $x_i = 0$.

Axiom 3: $\mathbf{z}(v+\bar{v}) = \mathbf{x}(v) + \mathbf{y}(\bar{v})$.

There is a unique vector $\mathbf{x}$, known as Shapley value, satisfies the above three axioms. $\mathbf{x}$ is defined as:

$$x_i(v) = \sum_{\{i\in S\subseteq N\}} \tfrac{(t-1)!(n-t)!}{n!}[v(S) - v(S-\{i\})], \text{ for } i = 1, ..., n, \text{ where } t = |S|.$$

Bargaining set

A payoff configuration $(\mathbf{x}; B)$ consists a payoff vector $\mathbf{x}$ and a set of coalitions, B. A pair $(\mathbf{y}, S)$, where S is a coalition and $\mathbf{y}$ is the corresponded payoff vector for members of S, is an "objection of player i against j to $\mathbf{x}$", if S includes i but not j and $y_k > x_k$, for all $k \in S$.

A pair $(\mathbf{z}, T)$ is a "counter objection" to the objection $(\mathbf{y}, S)$ of player i against j, if T includes j but not i, $z_k > x_k$, for all $k \in T\backslash S$, and $z_k > y_k$, for all $k \in T \cap S$.

Definition 2.24 *A payoff configuration* $(\mathbf{x};\ B)$ *is called stable if for each objection* $(\mathbf{y}; S)$ *of any player i against player j with respect to* $\mathbf{x}$*, there exists a counter objection to* $(\mathbf{y};\ S)$ *by player j. The "bargaining set" is the set of all such stable payoff configurations* [2]*.*

Clearly, the bargaining set not only gives the payoff vector, but also provides the answer as to how to form coalitions to realize the payoff vector.

2.2.3 Example

Consider the game $\Gamma = (N, v)$, where $n = 3$ and $v(\cdot)$ defined as follows:

S	$v(S)$	S	$v(S)$
$\emptyset$	0	$\{1,2\}$	30
$\{1\}$	0	$\{1,3\}$	30
$\{2\}$	0	$\{2,3\}$	80
$\{3\}$	0	$\{1,2,3\}$	100

This game is a superadditive, monotonic, normalized, essential and rational game.

0. imputations $X(\Gamma)$

$$\begin{cases} x_1 + x_2 + x_3 = 100 \\ x_i \geq 0, \; i = 1,2,3 \end{cases}$$

1. stable set $V(\Gamma)$

   Same as $C(\Gamma)$ since the game is superadditive.

2. core $C(\Gamma)$

$$\begin{cases} x_1 + x_2 \geq 30 \\ x_1 + x_3 \geq 30 \\ x_2 + x_3 \geq 80 \\ x_1 + x_2 + x_3 = 100 \\ x_i \geq 0, \; i = 1,2,3 \end{cases} \Longrightarrow \begin{cases} x_1 \leq 20 \\ x_2 \leq 70 \\ x_3 \leq 70 \\ x_1 + x_2 + x_3 = 100 \\ x_i \geq 0, \; i = 1,2,3 \end{cases}$$

4. kernel $K(\Gamma)$

By solving $\begin{cases} \max\{30 - x_1 - x_2, -x_2\} = \max\{30 - x_1 - x_3, -x_3\} \\ \max\{80 - x_2 - x_3, -x_2\} = \max\{30 - x_1 - x_3, -x_1\} \\ \max\{80 - x_2 - x_3, -x_3\} = \max\{30 - x_1 - x_2, -x_1\} \\ x_1 + x_2 + x_3 = 100 \\ x_i \geq 0, \; i = 1,2,3 \end{cases}$, we have $K(\Gamma) = \{(10, 45, 45)\}$.

5. nucleolus $N(\Gamma)$

   $N(\Gamma) = (10, 45, 45)$

6. Shapley value

   $(16.66, 41.67, 41.67)$

7. bargaining set

   $(0, 0, 0; 1, 2, 3)$; $(0, 30, 0; 12, 3)$; $(30, 0, 0; 13, 2)$; $(0, 80 - x_3, \; 30 \leq x_3 \leq 50; 1, 23)$;

   $(0 \leq x_1 \leq 20, \; 0 \leq x_2 \leq 70, \; 0 \leq x_3 \leq 70; \; 123 \mid x_1 + x_2 + x_3 = 100)$.

In this example, kernel $\subset$ core, nucleolus $\in$ core, nucleolus $\in$ kernel, Shapley value $\in$ core and core $\subset$ bargaining set.

2.2.4 Cooperative games with interval-valued payoffs

There are lots of papers that deal with the solution concepts for conventional cooperative games. However, when coalition values $v(S)$ are interval-valued, finding the solution concepts becomes tough. Since the comparison of interval values is non-standard, we need to find a way to redefine some concepts, for instance, superadditivity, in the interval-valued environment. So far we have not found any related literature.

Chapter 3

Max-t-norm Fuzzy Relational Equations

3.1 Introduction

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we are interested in finding different "solution sets" for the equation $\mathbf{x} \vartriangle A^I = \mathbf{b}^I$, where $\vartriangle$ is the *max-t-norm* composition and the *t-norm* is continuous. Throughout this chapter, we will frequently refer to two index sets, $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$.

The *max-t-norm* composition has two important properties, expressed by the following two lemmas.

Lemma 3.1 (*Monotonicity of composition*) (i) *Given* $A \in [0,1]^{m\times n}$, *for* $\mathbf{x}^1$, $\mathbf{x}^2 \in [0,1]^m$, *if* $\mathbf{x}^1 \leq \mathbf{x}^2$ *then* $\mathbf{x}^1 \vartriangle A \leq \mathbf{x}^2 \vartriangle A$.
(ii) *Given* $\mathbf{x} \in [0,1]^m$, *for* A^1, $A^2 \in [0,1]^{m\times n}$, *if* $A^1 \leq A^2$ *then* $\mathbf{x} \vartriangle A^1 \leq \mathbf{x} \vartriangle A^2$.

Proof. (i) Let $\mathbf{b}^1 = \mathbf{x}^1 \vartriangle A$ and $\mathbf{b}^2 = \mathbf{x}^2 \vartriangle A$. Due to the monotonicity property of the triangular norms, we have $b_j^1 = \max_{i\in I} t(x_i^1, a_{ij}) \leq \max_{i\in I} t(x_i^2, a_{ij}) = b_j^2$, $\forall\ j \in J$.

(ii) Let $\mathbf{b}^1 = \mathbf{x} \vartriangle A^1$ and $\mathbf{b}^2 = \mathbf{x} \vartriangle A^2$. We have $b_j^1 = \max_{i\in I} t(x_i, a_{ij}^1) \leq \max_{i\in I} t(x_i, a_{ij}^2) = b_j^2$, $\forall\ j \in J$. ■

Lemma 3.2 (*Continuity of composition*) *For a continuous t-norm,* (i) *given* $\mathbf{x} \in [0,1]^m$ *and* A^1, $A^2 \in [0,1]^{m\times n}$, *if* $A^1 \leq A^2$ *then for any* $\mathbf{b} \in [\mathbf{x} \vartriangle A^1, \mathbf{x} \vartriangle A^2]$ *there exists* $A \in [A^1, A^2]$

such that $\mathbf{x} \vartriangle A = \mathbf{b}$.

(ii) *given* $A \in [0,1]^{m\times n}$ *and* $\mathbf{x}^1$, $\mathbf{x}^2 \in [0,1]^m$, *if* $\mathbf{x}^1 \le \mathbf{x}^2$ *then for any* $\mathbf{b} \in [\mathbf{x}^1 \vartriangle A,\ \mathbf{x}^2 \vartriangle A]$ *there exists* $\mathbf{x} \in [\mathbf{x}^1, \mathbf{x}^2]$ *such that* $\mathbf{x} \vartriangle A = \mathbf{b}$.

(iii) *given* $A \in [0,1]^{m\times n}$ *and* $\mathbf{b}^2 \in [0,1]^{\mathbf{n}}$, *if there exists* $\mathbf{x}^2 \in [0,1]^m$ *such that* $\mathbf{x}^2 \vartriangle A = \mathbf{b}^2$, *then for any* $\mathbf{b}^1 \le \mathbf{b}^2$ *with* $\mathbf{b}^1 \in [0,1]^{\mathbf{n}}$, *there exists* $\mathbf{x}^1 \in [0,1]^m$, *such that* $\mathbf{x}^1 \le \mathbf{x}^2$ *and* $\mathbf{x}^1 \vartriangle A = \mathbf{b}^1$.

Proof. (i) Let $\mathbf{b}^1 = \mathbf{x} \vartriangle A^1$ and $\mathbf{b}^2 = \mathbf{x} \vartriangle A^2$. For $j \in J$, since $\max_{i\in I} t(x_i, a^1_{ij}) = b^1_j$ and $\max_{i\in I} t(x_i, a^2_{ij}) = b^2_j$, there exist i_1, $i_2 \in I$, such that $t(x_{i_1}, a^1_{i_1 j}) = b^1_j$ and $t(x_{i_2}, a^2_{i_2 j}) = b^2_j$. Note that we have a continuous *t-norm.* Since $t(x_{i_2}, a^1_{i_2 j}) \le b^1_j \le t(x_{i_2}, a^2_{i_2 j}) = b^2_j$, there exists $a_{i_2 j} \in [a^1_{i_2 j}, a^2_{i_2 j}]$ such that $t(x_{i_2}, a_{i_2 j}) = b_j \in [b^1_j, b^2_j]$. Let $a_{ij} = a^1_{ij}$, for $i \ne i_2$. We have $\max_{i\in I} t(x_i, a_{ij}) = \max\{t(x_{i_2}, a_{i_2 j}),\ \max_{i\in I,\ i\ne i_2} t(x_i, a^1_{ij})\} = b_j$, since $\max_{i\in I,\ i\ne i_2} t(x_i, a^1_{ij}) \le b^1_j \le b_j$. Consequently, we have found a matrix $A \in [A^1, A^2]$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$.

(ii) Let $\mathbf{b}^1 = \mathbf{x}^1 \vartriangle A$ and $\mathbf{b}^2 = \mathbf{x}^2 \vartriangle A$. For $j \in J$, since $\max_{i\in I} t(x^1_i, a_{ij}) = b^1_j$ and $\max_{i\in I} t(x^2_i, a_{ij}) = b^2_j$, there exist i_1, $i_2 \in I$, such that $t(x^1_{i_1}, a_{i_1 j}) = b^1_j$ and $t(x^2_{i_2}, a_{i_2 j}) = b^2_j$. For a continuous *t-norm*, since $t(x^1_{i_2}, a_{i_2 j}) \le b^1_j \le t(x^2_{i_2}, a_{i_2 j}) = b^2_j$, there exists $x_{i_2} \in [x^1_{i_2}, x^2_{i_2}]$ such that $t(x_{i_2}, a_{i_2 j}) = b_j \in [b^1_j, b^2_j]$. Let $x_i = x^1_i$, for $i \ne i_2$. We have $\max_{i\in I} t(x_i, a_{ij}) = \max\{t(x_{i_2}, a_{i_2 j}),\ \max_{i\in I,\ i\ne i_2} t(x^1_i, a_{ij})\} = b_j$, since $\max_{i\in I,\ i\ne i_2} t(x^1_i, a_{ij}) \le b^1_j \le b_j$. Consequently, we have found a vector $\mathbf{x} \in [\mathbf{x}^1, \mathbf{x}^2]$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$.

(iii) Let $\mathbf{x}^0 \in [0,1]^m$ be the zero vector and $\mathbf{b}^0 = \mathbf{x}^0 \vartriangle A$. By the definition of *t-norm*, $\mathbf{b}^0$ is also a zero vector. Thus $\mathbf{b}^0 \le \mathbf{b}^2$. From (ii) we know that, for any $\mathbf{b}^1 \in [\mathbf{b}^0, \mathbf{b}^2]$, there exists $\mathbf{x}^1 \in [\mathbf{x}^0, \mathbf{x}^2]$ such that $\mathbf{x}^1 \vartriangle A = \mathbf{b}^1$. ■

3.2 Tolerable solution set

3.2.1 Solution structure

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, recall that the set

$$\{\ \mathbf{x} \in [0,1]^m \mid \text{for each } A \in A^I,\ \text{there exists } \mathbf{b} \in \mathbf{b}^I \text{ such that } \mathbf{x} \vartriangle A = \mathbf{b}\}, \qquad (3.1)$$

denoted by $\Sigma_T(A^I, \mathbf{b}^I)$, is called the "tolerable solution set" of the equation $\mathbf{x} \vartriangle A^I = \mathbf{b}^I$.

In order to reveal the structure of $\Sigma_T(A^I, \mathbf{b}^I)$, consider the following fuzzy relational inequality system,

$$\begin{cases} \mathbf{x} \vartriangle \underline{A} \geq \underline{\mathbf{b}} \\ \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}} \\ \mathbf{x} \in [0,1]^m \end{cases}, \tag{3.2}$$

when $[\underline{A}, \overline{A}] = A^I$ and $[\underline{\mathbf{b}}, \overline{\mathbf{b}}] = \mathbf{b}^I$ are given.

Theorem 3.1 *Assume a given t-norm is continuous. Then* $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$ *if and only if* $\mathbf{x}$ *is a solution of* (3.2)*.*

Proof. (i) Let $\mathbf{x}$ be a solution of (3.2). Then $\mathbf{x} \vartriangle \underline{A} \geq \underline{\mathbf{b}}$, $\mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$ and $\mathbf{x} \in [0,1]^m$. The monotonicity property of composition implies that $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle A \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$ for any $A \in A^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and, consequently, $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$.

(ii) If $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, then for each $A \in A^I$, there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$. Hence there are $\mathbf{b}^1$ and $\mathbf{b}^2$ such that $\mathbf{x} \vartriangle \underline{A} = \mathbf{b}^1 \geq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \overline{A} = \mathbf{b}^2 \leq \overline{\mathbf{b}}$. Furthermore, since $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, $\mathbf{x} \in [0,1]^m$. ■

We now focus on (3.2) to find $\Sigma_T(A^I, \mathbf{b}^I)$.

Theorem 3.2 *Given* $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, *if* $\underline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$ *and* $\overline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$, *then* $\mathbf{x}^I \subset \Sigma_T(A^I, \mathbf{b}^I)$.

Proof. Let $\mathbf{x} \in \mathbf{x}^I$. If $\underline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$, then $\underline{\mathbf{x}} \vartriangle \underline{A} \geq \underline{\mathbf{b}}$. Hence $\mathbf{x} \vartriangle \underline{A} \geq \underline{\mathbf{x}} \vartriangle \underline{A} \geq \underline{\mathbf{b}}$. Similarly, we have $\mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{x}} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. Moreover, for $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$, we know $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in [0,1]^m$. Consequently, $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$ and $\mathbf{x}^I \subset \Sigma_T(A^I, \mathbf{b}^I)$. ■

Theorem 3.2 suggests further investigation of the structure of $\Sigma_T(A^I, \mathbf{b}^I)$ by examining the so-called maximum and minimal solutions as defined in Section 2.1. For conventional (constant-valued) fuzzy relational equations with a *max-t* composition, the solution set can be completely determined by one maximum solution and a finite number of minimal solutions, when t is continuous and the solution set is not empty [8]. The same solution structure holds for the tolerable solution set of interval-valued fuzzy relational equations with *max-min* composition, when the solution set is not empty [58, 23]. These results suggest further study as to whether this solution structure holds for interval-valued fuzzy relational equations with a general *max-t* composition. In what follows, we will prove that it does.

From Chapter 2, given $\acute{a}, \acute{b} \in [0,1]$, the φ operator associated with a given *t-norm* is defined by

$$\acute{a} \ \varphi \ \acute{b} = \sup\{\acute{x} \in [0,1] \mid t(\acute{x}, \acute{a}) \leq \acute{b}\}.$$

Using the φ operator, for a given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we define

$$\hat{\mathbf{x}}^+ = \left(\min_{j\in J}(\overline{a}_{ij} \ \varphi \ \overline{b}_j) \right)_{i\in I}. \tag{3.3}$$

Theorem 3.3 *If $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, then $\hat{\mathbf{x}}^+$ defined by (3.3) is the maximum solution of $\Sigma_T(A^I, \mathbf{b}^I)$.*

Proof. (i) $\forall j \in J$, $\max_{i\in I} t(\hat{x}_i^+, \overline{a}_{ij}) = \max_{i\in I} t(\min_{j_0\in J}(\overline{a}_{ij_0} \ \varphi \ \overline{b}_{j_0}), \overline{a}_{ij}) \leq \max_{i\in I} t((\overline{a}_{ij} \ \varphi \ \overline{b}_j), \overline{a}_{ij}) \leq \overline{b}_j$. Hence $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \leq \overline{\mathbf{b}}$.

(ii) For any $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, since $\mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$ and $\max_{i\in I} t(x_i, \overline{a}_{ij}) \leq \overline{b}_j$, $\forall j \in J$, we have $t(x_i, \overline{a}_{ij}) \leq \overline{b}_j$ and $x_i \leq \overline{a}_{ij} \ \varphi \ \overline{b}_j$, for $i \in I$. Consequently, $x_i \leq \min_{j\in J}(\overline{a}_{ij} \ \varphi \ \overline{b}_j) = \hat{x}_i^+$ and $\mathbf{x} \leq \hat{\mathbf{x}}^+$.

(iii) When $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, there exists at least one $\mathbf{x}$ such that $\mathbf{x} \vartriangle \underline{A} \geq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. By (ii) we have $\mathbf{x} \leq \hat{\mathbf{x}}^+$. Hence, $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \geq \underline{\mathbf{b}}$.

Combining (i), (ii) and (iii), we see that $\hat{\mathbf{x}}^+$ is the maximum solution of $\Sigma_T(A^I, \mathbf{b}^I)$. ∎

Corollary 3.1 *$\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$ if and only if $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \geq \underline{\mathbf{b}}$.*

Proof. If $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \geq \underline{\mathbf{b}}$, from the proof of Theorem 3.3(i), we have $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. Consequently, $\hat{\mathbf{x}}^+ \in \Sigma_T(A^I, \mathbf{b}^I)$ and $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$.

On the other hand, if $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, Theorem 3.3 guarantees that $\hat{\mathbf{x}}^+ \in \Sigma_T(A^I, \mathbf{b}^I)$. Hence, $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \geq \underline{\mathbf{b}}$. ∎

We now focus on the minimal solutions.

Lemma 3.3 *If $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, then for each $j \in J$, $\max_{i\in I} t(x_i, \overline{a}_{ij}) \leq \overline{b}_j$, and there exists $i_0 \in I$ such that $t(x_{i_0}, \underline{a}_{i_0 j}) \geq \underline{b}_j$.*

Proof. Since $\mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$, we have $\max_{i\in I} t(x_i, \overline{a}_{ij}) \leq \overline{b}_j$, for $j \in J$. Similarly, $\mathbf{x} \vartriangle \underline{A} \geq \underline{\mathbf{b}}$ implies that $\max_{i\in I} t(x_i, \underline{a}_{ij}) \geq \underline{b}_j$, for $j \in J$. Hence for each $j \in J$, there exists at least one $i_0 \in I$ such that $t(x_{i_0}, \underline{a}_{i_0 j}) \geq \underline{b}_j$. ∎

When $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, $\hat{\mathbf{x}}^+$ can be found by using Equation (3.3). We define the following index set for each $j \in J$:

$$I_j = \{i \in I \mid t(\hat{x}_i^+, \underline{a}_{ij}) \geq \underline{b}_j\}. \tag{3.4}$$

The Cartesian product of the index sets is denoted by

$$\Lambda = I_1 \times I_2 \times \cdots \times I_n. \tag{3.5}$$

Notice that from Lemma 2.1, $t(\hat{x}_i^+, \underline{a}_{ij}) \geq \underline{b}_j$ implies that $\hat{x}_i^+ \geq \underline{a}_{ij} \ \gamma \ \underline{b}_j$. Also notice that $t(\hat{x}_i^+, \underline{a}_{ij}) \leq \min\{\hat{x}_i^+, \underline{a}_{ij}\}$, therefore, $\underline{a}_{ij} \geq \underline{b}$ and $\underline{a}_{ij} \ \gamma \ \underline{b}_j$ is well defined. Lemma 3.3 guarantees that $I_j \neq \emptyset$, $\forall\ j \in J$, when $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$. Moreover, given $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in I^n$, we say $\mathbf{f} \in \Lambda$ if and only if $f_j \in I_j$, $\forall\ j \in J$. Now we can study the solution structure of $\Sigma_T(A^I, \mathbf{b}^I)$ in terms of the elements in Λ.

Definition 3.1 *Given* $\mathbf{f} \in \Lambda$, *we define*

$$J_{\mathbf{f}}^i = \{j \in J \mid f_j = i\},\ \forall\ i \in I, \tag{3.6}$$

and $F : \Lambda \to R^m$ *such that* $F(\mathbf{f}) = (F_1(\mathbf{f}),\ F_2(\mathbf{f}),\ \ldots,\ F_m(\mathbf{f}))$ *with*

$$F_i(\boldsymbol{f}) = \begin{cases} \max_{j \in J_{\mathbf{f}}^i} (\underline{a}_{ij} \ \gamma \ \underline{b}_j) & , \text{ if } J_{\mathbf{f}}^i \neq \emptyset \\ 0 & , \text{ if } J_{\mathbf{f}}^i = \emptyset \end{cases} . \tag{3.7}$$

Remark 3.1 *If* $\mathbf{f} \in \Lambda$, *then* $t(\hat{x}_{f_j}^+, \underline{a}_{f_j j}) \geq \underline{b}_j$, $\forall\ j \in J$. *Hence* $j \in J_{\mathbf{f}}^i$ *implies that* $t(\hat{x}_i^+, \underline{a}_{ij}) \geq \underline{b}_j$.

Remark 3.2 (3.6) *implies that if* $j \in J_{\mathbf{f}}^i$, *then* $t(\hat{x}_i^+, \underline{a}_{ij}) \geq \underline{b}_j$.

Now we examine the relationship between $\Sigma_T(A^I, \mathbf{b}^I)$ and $F(\Lambda) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Theorem 3.4 *Assume that* $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$.
(i) *If* $\mathbf{f} \in \Lambda$, *then* $F(\mathbf{f}) \in \Sigma_T(A^I, \mathbf{b}^I)$.
(ii) *For any* $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, *there exists* $\mathbf{f} \in \Lambda$ *such that* $F(\mathbf{f}) \leq \mathbf{x}$.

Proof. (i) From the definition of $J_{\mathbf{f}}^i$, for any $i \in I$, if $J_{\mathbf{f}}^i = \emptyset$, then $F_i(\mathbf{f}) = 0 \leq \hat{x}_i^+$. If $J_{\mathbf{f}}^i \neq \emptyset$, then $F_i(\mathbf{f}) = \max_{j \in J_{\mathbf{f}}^i}(\underline{a}_{ij} \ \gamma \ \underline{b}_j) = \max_{\{j \mid f_j = i\}}(\underline{a}_{ij} \ \gamma \ \underline{b}_j) \leq \hat{x}_i^+$, since $\hat{x}_i^+ \geq \underline{a}_{ij} \ \gamma \ \underline{b}_j$, for $j \in J$.

Therefore, $F_i(\mathbf{f}) \leq \hat{x}_i^+$, for any $i \in I$. Consequently, $\max_{i\in I} t(\overline{a}_{ij}, F_i(\mathbf{f})) \leq \max_{i\in I} t(\overline{a}_{ij}, \hat{x}_i^+) \leq \overline{b}_j$, $\forall\ j \in J$, i.e., $F(\mathbf{f}) \vartriangle \overline{A} \leq \overline{\mathbf{b}}$.

On the other hand, the definition of $J_{\mathbf{f}}^i$ implies that $j \in J_{\mathbf{f}}^{f_j}$, and $F_{f_j}(\mathbf{f}) = \max_{j_0 \in J_{\mathbf{f}}^{f_j}} (\underline{a}_{f_j j_0}\ \gamma\ \underline{b}_{j_0}) \geq \underline{a}_{f_j j}\ \gamma\ \underline{b}_j$, $\forall j \in J$. Therefore $t(\underline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \geq \underline{b}_j$ and consequently $\max_{i\in I} t(\underline{a}_{ij}, F_i(\mathbf{f})) \geq t(\underline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \geq \underline{b}_j$, $\forall\ j \in J$, i.e., $F(\mathbf{f}) \vartriangle \underline{A} \geq \underline{\mathbf{b}}$.

Combining $F(\mathbf{f}) \vartriangle \overline{A} \leq \overline{\mathbf{b}}$ and $F(\mathbf{f}) \vartriangle \underline{A} \geq \underline{\mathbf{b}}$, we know $F(\mathbf{f}) \in \Sigma_T(A^I, \mathbf{b}^I)$.

(ii) For each $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, according to Lemma 3.3, for each $j \in J$, there exists $i_j \in I$ such that $t(x_{i_j}, \underline{a}_{i_j j}) \geq \underline{b}_j$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = i_j, \forall\ j \in J$. In this way, $\mathbf{f} \in \Lambda$. Now, for any $i \in I$, if $J_{\mathbf{f}}^i = \emptyset$, then $F_i(\mathbf{f}) = 0 \leq x_i$. If $J_{\mathbf{f}}^i \neq \emptyset$, since $t(x_{i_j}, \underline{a}_{i_j j}) \geq \underline{b}_j$, for $j \in J$, $F_i(\mathbf{f}) = \max_{j \in J_{\mathbf{f}}^i}(\underline{a}_{ij}\ \gamma\ \underline{b}_j) = \max_{\{j|f_j=i\}}(\underline{a}_{ij}\ \gamma\ \underline{b}_j) = \max_{\{j|i_j=i\}}(\underline{a}_{ij}\ \gamma\ \underline{b}_j) \leq x_{i_j} = x_i$. This shows that $F_i(\mathbf{f}) \leq x_i$, $\forall\ i \in I$. Hence $F(\mathbf{f}) \leq \mathbf{x}$. ■

Theorem 3.4 implies that there exists at least one minimal solution of $\Sigma_T(A^I, \mathbf{b}^I)$, provided $\Sigma_T(A^I, \mathbf{b}^I)$ is not empty. Since the cardinality of Λ is finite, the number of minimal solutions must be finite. Denote the set of all minimal solutions by $N(\Sigma_T(A^I, \mathbf{b}^I))$. We have the following result.

Corollary 3.2 $N(\Sigma_T(A^I, \mathbf{b}^I)) \subset F(\Lambda) \subset \Sigma_T(A^I, \mathbf{b}^I)$.

This corollary reduces the search space for all the minimal solutions of $\Sigma_T(A^I, \mathbf{b}^I)$ to a relatively small set $F(\Lambda)$. Moreover, for $\mathbf{x}, \mathbf{x}' \in F(\Lambda)$, if $\mathbf{x} \leq \mathbf{x}'$, then $\mathbf{x}' \notin N(\Sigma_T(A^I, \mathbf{b}^I))$. Once we remove all such $\mathbf{x}'$ from $F(\Lambda)$, $N(\Sigma_T(A^I, \mathbf{b}^I))$ is identified.

3.2.2 The algorithm

Based on the above discussion, we propose an algorithm to solve *max-t-norm* interval-valued fuzzy relational equations.

Step 1. Specify the mathematical expression of the operators φ and γ associated with the *t-norm*.

Step 2. (Find the potential maximum solution) Compute $\hat{\mathbf{x}}^+$ according to (3.3).

Step 3. (Check feasibility) If $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \ngeq \underline{\mathbf{b}}$ then $\Sigma_T(A^I, \mathbf{b}^I) = \emptyset$, stop. Otherwise continue.

Step 4. Compute I_j for all $j \in J$ according to (3.4).

Step 5. Generate Λ according to (3.5).

Step 6. (Determine potential minimal solutions) For each $\mathbf{f} \in \Lambda$, calculate $F(\mathbf{f})$ according to (3.7). Set $N(\Sigma_T(A^I, \mathbf{b}^I)) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Step 7. (Exclude nonminimal solutions) For $\mathbf{x}, \mathbf{x}' \in N(\Sigma_T(A^I, \mathbf{b}^I))$ with $\mathbf{x} \leq \mathbf{x}'$, set $N(\Sigma_T(A^I, \mathbf{b}^I)) = N(\Sigma_T(A^I, \mathbf{b}^I)) \backslash \{\mathbf{x}'\}$.

Step 8. (Output) $\Sigma_T(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [\check{\mathbf{x}}^-, \hat{\mathbf{x}}^+] \mid \check{\mathbf{x}}^- \in N(\Sigma_T(A^I, \mathbf{b}^I))\}$.

Remark 3.3 $|I_j| \in [0, m]$ *for* $j \in J$. *Therefore* $|\Lambda| \leq m^n$. *Since we have to go through all elements of* Λ *to identify minimal solutions, the proposed algorithm is not polynomially bounded.*

3.2.3 Examples

Consider $(x_1\ x_2\ x_3) \vartriangle \begin{pmatrix} [0.3, 0.7] & [0.5, 0.6] \\ [0.4, 0.9] & [0.1, 0.7] \\ [0.2, 0.5] & [0.8, 1.0] \end{pmatrix} = \begin{pmatrix} [0.3, 0.8] & [0.5, 0.7] \end{pmatrix}$.

We have $\underline{A} = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$, $\overline{A} = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.7 \\ 0.5 & 1.0 \end{pmatrix}$, $\underline{\mathbf{b}} = (0.3\ 0.5)$, $\overline{\mathbf{b}} = (0.8\ 0.7)$.

This problem is adapted from [23], with $\underline{\mathbf{b}} = (0.3\ 0.5)$ instead of $\underline{\mathbf{b}} = (0.4\ 0.5)$. Otherwise there is no solution for Example 2.

Example 1: (*Max-min* composition). For the minimum operation $t(a, x) = \min(a, x)$, we have $a \varphi b = b$, for $a > b$. Hence $\hat{\mathbf{x}}^+ = (1\ 0.8\ 0.7)$. Since $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \geq \underline{\mathbf{b}}$, $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$. Then we have $I_1 = \{1, 2\}$, $I_2 = \{1, 3\}$, and $\Lambda = I_1 \times I_2 = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$.

From Table 3.1 we get $F(\Lambda) = \{(0.5\ 0\ 0),\ (0.3\ 0\ 0.5),\ (0.5\ 0.3\ 0),\ (0\ 0.3\ 0.5)\}$. Since $F(\Lambda)_1 \leq F(\Lambda)_3$, we can eliminate $F(\Lambda)_3$ from $N(\Sigma_T(A^I, \mathbf{b}^I))$. Finally we get $N(\Sigma_T(A^I, \mathbf{b}^I)) = \{(0.5\ 0\ 0), (0.3\ 0\ 0.5), (0\ 0.3\ 0.5)\}$ and $\Sigma_T(A^I, \mathbf{b}^I) = \{([0.5, 1]\ [0, 0.8]\ [0, 0.7]), ([0.3, 1]\ [0, 0.8]\ [0.5, 0.7]), ([0, 1]\ [0.3, 0.8]\ [0.5, 0.7])\}$.

Note: There are no tolerable solutions for the *max-product* composition in this example. In the next example we slightly modify $\mathbf{b}^I$ to illustrate a tolerable solution set.

$\mathbf{f} = (1,1)$	$J_{\mathbf{f}}^1 = \{1,2\}$ $J_{\mathbf{f}}^2 = \emptyset$ $J_{\mathbf{f}}^3 = \emptyset$	$F_1(\mathbf{f}) = 0.5$ $F_2(\mathbf{f}) = 0$ $F_3(\mathbf{f}) = 0$
$\mathbf{f} = (1,3)$	$J_{\mathbf{f}}^1 = \{1\}$ $J_{\mathbf{f}}^2 = \emptyset$ $J_{\mathbf{f}}^3 = \{2\}$	$F_1(\mathbf{f}) = 0.3$ $F_2(\mathbf{f}) = 0$ $F_3(\mathbf{f}) = 0.5$
$\mathbf{f} = (2,1)$	$J_{\mathbf{f}}^1 = \{2\}$ $J_{\mathbf{f}}^2 = \{1\}$ $J_{\mathbf{f}}^3 = \emptyset$	$F_1(\mathbf{f}) = 0.5$ $F_2(\mathbf{f}) = 0.3$ $F_3(\mathbf{f}) = 0$
$\mathbf{f} = (2,3)$	$J_{\mathbf{f}}^1 = \emptyset$ $J_{\mathbf{f}}^2 = \{1\}$ $J_{\mathbf{f}}^3 = \{2\}$	$F_1(\mathbf{f}) = 0$ $F_2(\mathbf{f}) = 0.3$ $F_3(\mathbf{f}) = 0.5$

Table 3.1: Calculation of minimal solutions for the max-min tolerable solutions

Example 2: (Max-product composition). For the algebraic product $t(a,x) = ax$, we have $a \ \varphi \ b = \frac{b}{a}$, for $a > b$. Hence $\hat{\mathbf{x}}^+ = (1 \ 8/9 \ 7/10)$. Since $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \geq \underline{\mathbf{b}}$, $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$. Then we have $I_1 = \{1,2\}$, $I_2 = \{1,3\}$, and $\Lambda = I_1 \times I_2 = \{(1,1),(1,3),(2,1),(2,3)\}$. Hence we have $F(\Lambda) = \{(1 \ 3/4 \ 0), \ (0 \ 3/4 \ 5/8), \ (1 \ 0 \ 0), \ (1 \ 0 \ 5/8)\}$. Since $F(\Lambda)_3 \leq F(\Lambda)_4$ and $F(\Lambda)_3 \leq F(\Lambda)_1$, we eliminate $F(\Lambda)_4$ and $F(\Lambda)_1$ from $N(\Sigma_T(A^I, \mathbf{b}^I))$. Finally we get $N(\Sigma_T(A^I, \mathbf{b}^I)) = \{(0 \ 3/4 \ 5/8), \ (1 \ 0 \ 0)\}$ and $\Sigma_T(A^I, \mathbf{b}^I) = \{([0,1] \ [3/4, 8/9] \ [5/8, 7/10]), \ (1 \ [0, 8/9] \ [0, 7/10])\}$.

3.3 United solution set

3.3.1 Solution structure

Given $A^I \in I([0,1]^{m \times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, recall that the set

$$\{ \ \mathbf{x} \in [0,1]^m \mid \text{there exist } A \in A^I \text{ and } \mathbf{b} \in \mathbf{b}^I \text{ such that } \mathbf{x} \vartriangle A = \mathbf{b}\}, \tag{3.8}$$

denoted by $\Sigma_U(A^I, \mathbf{b}^I)$, is called the "united solution set" of the equation $\mathbf{x} \vartriangle A^I = \mathbf{b}^I$.

Similar to the case of tolerable solution sets, let us consider the following fuzzy relational inequality system,

$$\begin{cases} \mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}} \\ \mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{b}} \\ \mathbf{x} \in [0,1]^m \end{cases}, \tag{3.9}$$

when $[\underline{A}, \overline{A}] = A^I$ and $[\underline{\mathbf{b}}, \overline{\mathbf{b}}] = \mathbf{b}^I$ are given.

Theorem 3.5 *Assume a given t-norm is continuous. Then* $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$ *if and only if* $\mathbf{x}$ *is a solution of* (3.9).

Proof. (i) Let $\mathbf{x}$ be a solution of (3.9). Then $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$. There are four possible cases:

Case 1 $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}} \leq \overline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A}$. From Lemma 3.2, there exists $A \in A^I$ such that $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 2 $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. In this case, $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$, for every $A \in A^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 3 $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. If we let $A = \overline{A}$, then $\mathbf{x} \vartriangle A \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 4 $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A}$. If we let $A = \underline{A}$, then $\mathbf{x} \vartriangle A \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

(ii) If $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, then there exist $A \in A^I$ and $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$. Hence $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$, $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \mathbf{x} \vartriangle \overline{A}$, and $\mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$. Furthermore, since $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, $\mathbf{x} \in [0, 1]^m$. ■

We now focus on (3.9) to find the structure of $\Sigma_U(A^I, \mathbf{b}^I)$.

Theorem 3.6 *Given* $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, *if* $\underline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$ *and* $\overline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$, *then* $\mathbf{x}^I \subset \Sigma_U(A^I, \mathbf{b}^I)$.

Proof. Let $\mathbf{x} \in \mathbf{x}^I$. If $\underline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$, then $\underline{\mathbf{x}} \vartriangle \overline{A} \geq \underline{\mathbf{b}}$. Hence $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{x}} \vartriangle \overline{A} \geq \underline{\mathbf{b}}$. Similarly, we have $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{x}} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$. For $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$, we know $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in [0, 1]^m$. Consequently, $\mathbf{x} \in [0, 1]^m$. Hence $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$ and $\mathbf{x}^I \subset \Sigma_U(A^I, \mathbf{b}^I)$. ■

Theorem 3.6 suggests further investigation of the structure of $\Sigma_U(A^I, \mathbf{b}^I)$. Moreover, $\Sigma_T(A^I, \mathbf{b}^I)$ and $\Sigma_U(A^I, \mathbf{b}^I)$ have something in common: each is a solution set of a fuzzy relational inequality system. This motivates further study as to whether $\Sigma_U(A^I, \mathbf{b}^I)$ has the same structure as $\Sigma_T(A^I, \mathbf{b}^I)$. In what follows, we will prove that, when the solution set is not empty, $\Sigma_U(A^I, \mathbf{b}^I)$ can be completely determined by one maximum solution and a finite number of minimal solutions.

From Chapter 2, given $\acute{a}, \acute{b} \in [0,1]$, the φ operator associated with a given *t-norm* is defined by

$$\acute{a} \ \varphi \ \acute{b} = \sup\{\acute{x} \in [0,1] \mid t(\acute{x}, \acute{a}) \leq \acute{b}\}.$$

Using the φ operator, for a given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we define

$$\hat{\mathbf{x}}^+ = \left(\min_{j\in J}(\underline{a}_{ij} \ \varphi \ \overline{b}_j)\right)_{i\in I}. \tag{3.10}$$

Theorem 3.7 *If* $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$*, then* $\hat{\mathbf{x}}^+$ *defined by* (3.10) *is the maximum solution of* $\Sigma_U(A^I, \mathbf{b}^I)$.

Proof. (i) $\forall j \in J$, $\max_{i\in I} t(\hat{x}_i^+, \underline{a}_{ij}) = \max_{i\in I} t(\min_{j_0\in J}(\underline{a}_{ij_0} \ \varphi \ \overline{b}_{j_0}), \underline{a}_{ij}) \leq \max_{i\in I} t((\underline{a}_{ij} \ \varphi \ \overline{b}_j), \underline{a}_{ij}) \leq \overline{b}_j$. Hence, we have $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \leq \overline{\mathbf{b}}$.

(ii) For any $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, we know that $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$ and $\max_{i\in I} t(x_i, \underline{a}_{ij}) \leq \overline{b}_j$, $\forall j \in J$. Therefore for $i \in I$, $t(x_i, \underline{a}_{ij}) \leq \overline{b}_j$ and $x_i \leq \underline{a}_{ij} \ \varphi \ \overline{b}_j$. Consequently, $x_i \leq \min_{j\in J}(\underline{a}_{ij} \ \varphi \ \overline{b}_j) = \hat{x}_i^+$ and $\mathbf{x} \leq \hat{\mathbf{x}}^+$.

(iii) When $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, there exists at least one $\mathbf{x}$ that satisfies $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$. By (ii) we have $\mathbf{x} \leq \hat{\mathbf{x}}^+$. Hence, $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \underline{\mathbf{b}}$.

Combining (i), (ii) and (iii), we see that $\hat{\mathbf{x}}^+$ is the maximum solution of $\Sigma_U(A^I, \mathbf{b}^I)$. ∎

Corollary 3.3 $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$ *if and only if* $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \underline{\mathbf{b}}$.

Proof. If $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \underline{\mathbf{b}}$, form the proof of Theorem 3.7(i), we have $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \leq \overline{\mathbf{b}}$, and hence, $\hat{\mathbf{x}}^+ \in \Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$.

On the other hand, if $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, Theorem 3.7 guarantees that $\hat{\mathbf{x}}^+ \in \Sigma_U(A^I, \mathbf{b}^I)$. Hence,
$\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \underline{\mathbf{b}}$. ∎

Now we focus on the minimal solutions.

Lemma 3.4 *If* $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$ *, then, for each* $j \in J$, $\max_{i\in I} t(x_i, \underline{a}_{ij}) \leq \overline{b}_j$*, and there exists* $i_0 \in I$ *such that* $t(x_{i_0}, \overline{a}_{i_0 j}) \geq \underline{b}_j$.

Proof. Since $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$, we have $\max_{i\in I} t(x_i, \underline{a}_{ij}) \leq \overline{b}_j$, for $j \in J$. Since $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{b}}$, we have $\max_{i\in I} t(x_i, \overline{a}_{ij}) \geq \underline{b}_j$, for $j \in J$. Hence for each $j \in J$, there exists at least one $i_0 \in I$ such that $t(x_{i_0}, \overline{a}_{i_0 j}) \geq \underline{b}_j$. ■

When $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, $\hat{\mathbf{x}}^+$ can be found by using Equation (3.10). Analogous to (3.4), we define the following index set for each $j \in J$:

$$I_j = \{i \in I \mid t(\hat{x}_i^+, \overline{a}_{ij}) \geq \underline{b}_j\}, \tag{3.11}$$

and the Cartesian product

$$\Lambda = I_1 \times I_2 \times \cdots \times I_n. \tag{3.12}$$

Notice that from Lemma 2.1, $t(\hat{x}_i^+, \overline{a}_{ij}) \geq \underline{b}_j$ implies that $\hat{x}_i^+ \geq \overline{a}_{ij}\ \gamma\ \underline{b}_j$. Also notice that $t(\hat{x}_i^+, \overline{a}_{ij}) \leq \min\{\hat{x}_i^+, \bar{a}_{ij}\}$, therefore, $\overline{a}_{ij} \geq \underline{b}_j$ and $\overline{a}_{ij}\ \gamma\ \underline{b}_j$ is well defined. Lemma 3.4 guarantees that $I_j \neq \emptyset$, $\forall\ j \in J$, when $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$. Moreover, given $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in I^n$, we say $\mathbf{f} \in \Lambda$ if and only if $f_j \in I_j, \forall\ j \in J$. Now we can study the solution structure of $\Sigma_U(A^I, \mathbf{b}^I)$ in terms of the elements in Λ.

Definition 3.2 *Given* $\mathbf{f} \in \Lambda$, *define*

$$J_{\mathbf{f}}^i = \{j \in J \mid f_j = i\},\ \forall\ i \in I, \tag{3.13}$$

and $F : \Lambda \to R^m$ *such that* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f}))$ *with*

$$F_i(\boldsymbol{f}) = \begin{cases} \max_{j \in J_{\mathbf{f}}^i} (\overline{a}_{ij}\ \gamma\ \underline{b}_j) & , \text{ if } J_{\mathbf{f}}^i \neq \emptyset \\ 0 & , \text{ if } J_{\mathbf{f}}^i = \emptyset \end{cases}. \tag{3.14}$$

Remark 3.4 *If* $\mathbf{f} \in \Lambda$, *then* $t(\hat{x}_{f_j}^+, \overline{a}_{f_j j}) \geq \underline{b}_j$, $\forall\ j \in J$.

Remark 3.5 *From* (3.13), $j \in J_{\mathbf{f}}^i$ *implies that* $t(\hat{x}_i^+, \overline{a}_{ij}) \geq \underline{b}_j$.

The relationship between $\Sigma_U(A^I, \mathbf{b}^I)$ and $F(\Lambda) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$ is given by the following theorem.

Theorem 3.8 *Assume that* $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$.
(i) *If* $\mathbf{f} \in \Lambda$, *then* $F(\mathbf{f}) \in \Sigma_U(A^I, \mathbf{b}^I)$.
(ii) *For any* $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, *there exists* $\mathbf{f} \in \Lambda$ *such that* $F(\mathbf{f}) \leq \mathbf{x}$.

Proof. (i) From the definition of $J^i_{\mathbf{f}}$, for any $i \in I$, if $J^i_{\mathbf{f}} = \emptyset$, then $F_i(\mathbf{f}) = 0 \leq \hat{x}^+_i$. If $J^i_{\mathbf{f}} \neq \emptyset$, then $F_i(\mathbf{f}) = \max_{j \in J^i_{\mathbf{f}}}(\overline{a}_{ij}\ \gamma\ \underline{b}_j) = \max_{\{j|f_j=i\}}(\overline{a}_{ij}\ \gamma\ \underline{b}_j) \leq \hat{x}^+_i$, since $\hat{x}^+_i \geq \overline{a}_{ij}\ \gamma\ \underline{b}_j$, for $j \in J$. Therefore, $F_i(\mathbf{f}) \leq \hat{x}^+_i$, for any $i \in I$. Consequently, $\max_{i\in I} t(\underline{a}_{ij}, F_i(\mathbf{f})) \leq \max_{i \in I} t(\underline{a}_{ij}, \hat{x}^+_i) \leq \overline{b}_j, \forall\ j \in J$, i.e., $F(\mathbf{f}) \vartriangle \underline{A} \leq \overline{\mathbf{b}}$.

On the other hand, the definition of $J^i_{\mathbf{f}}$ implies that $j \in J^{f_j}_{\mathbf{f}}$, and $F_{f_j}(\mathbf{f}) = \max_{j_0 \in J^{f_j}_{\mathbf{f}}}(\overline{a}_{f_j j_0}\ \gamma\ \underline{b}_{j_0}) \geq \overline{a}_{f_j j}\ \gamma\ \underline{b}_j$, $\forall j \in J$. Therefore $t(\overline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \geq \underline{b}_j$ and, consequently, $\max_{i \in I} t(\overline{a}_{ij}, F_i(\mathbf{f})) \geq t(\overline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \geq \underline{b}_j, \forall\ j \in J$, i.e., $F(\mathbf{f}) \vartriangle \overline{A} \geq \underline{\mathbf{b}}$.

Combining $F(\mathbf{f}) \vartriangle \underline{A} \leq \overline{\mathbf{b}}$ and $F(\mathbf{f}) \vartriangle \overline{A} \geq \underline{\mathbf{b}}$, we know $F(\mathbf{f}) \in \Sigma_U(A^I, \mathbf{b}^I)$.

(ii) For each $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, according to Lemma 3.4, for each $j \in J$, there exists $i_j \in I$ such that $t(x_{i_j}, \overline{a}_{i_j j}) \geq \underline{b}_j$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = i_j, \forall\ j \in J$. In this way, $\mathbf{f} \in \Lambda$. Now, for any $i \in I$, if $J^i_{\mathbf{f}} = \emptyset$, then $F_i(\mathbf{f}) = 0 \leq x_i$. If $J^i_f \neq \emptyset$, since $t(x_{i_j}, \overline{a}_{i_j j}) \geq \underline{b}_j$, for $j \in J$, $F_i(\mathbf{f}) = \max_{j \in J^i_{\mathbf{f}}}(\overline{a}_{ij}\ \gamma\ \underline{b}_j) = \max_{\{j|f_j=i\}}(\overline{a}_{ij}\ \gamma\ \underline{b}_j) = \max_{\{j|i_j=i\}}(\overline{a}_{ij}\ \gamma\ \underline{b}_j) \leq x_{i_j} = x_i$. This shows that $F_i(\mathbf{f}) \leq x_i$, $\forall\ i \in I$. Hence $F(\mathbf{f}) \leq \mathbf{x}$. ■

Theorem 3.8 implies that there exists at least one minimal solution of $\Sigma_U(A^I, \mathbf{b}^I)$, provided $\Sigma_U(A^I, \mathbf{b}^I)$ is not empty. Since the cardinality of Λ is finite, the number of minimal solutions must be finite. Denote the set of all minimal solutions by $N(\Sigma_U(A^I, \mathbf{b}^I))$. We have the following result.

Corollary 3.4 $N(\Sigma_U(A^I, \mathbf{b}^I)) \subset F(\Lambda) \subset \Sigma_U(A^I, \mathbf{b}^I)$.

This corollary reduces the search space for all the minimal solutions of $\Sigma_U(A^I, \mathbf{b}^I)$ to a relatively small set $F(\Lambda)$. Moreover, for $\mathbf{x}, \mathbf{x}' \in F(\Lambda)$, if $\mathbf{x} \leq \mathbf{x}'$, then $\mathbf{x}' \notin N(\Sigma_U(A^I, \mathbf{b}^I))$. Once we remove all such $\mathbf{x}'$ from $F(\Lambda)$, $N(\Sigma_U(A^I, \mathbf{b}^I))$ is identified.

3.3.2 The algorithm

Based on the above discussion, we propose an algorithm to solve *max-t-norm* interval-valued fuzzy relational equations.

Step 1. Specify the mathematical expression of the operators φ and γ associated with the *t-norm*.

Step 2. (Find the potential maximum solution) Compute $\hat{\mathbf{x}}^+$ according to (3.10).

Step 3. (Check feasibility and uniqueness) If $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \ngeq \underline{\mathbf{b}}$ then $\Sigma_U(A^I, \mathbf{b}^I) = \emptyset$, stop. Otherwise continue.

Step 4. Compute I_j for all $j \in J$ according to (3.11).

Step 5. Generate Λ according to (3.12).

Step 6. (Determine potential minimal solutions) For each $\mathbf{f} \in \Lambda$, calculate $F(\mathbf{f})$ according to (3.14). Set $N(\Sigma_U(A^I, \mathbf{b}^I)) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Step 7. (Exclude nonminimal solutions) For $\mathbf{x}, \mathbf{x}' \in N(\Sigma_U(A^I, \mathbf{b}^I))$ with $\mathbf{x} \leq \mathbf{x}'$, set $N(\Sigma_U(A^I, \mathbf{b}^I)) = N(\Sigma_U(A^I, \mathbf{b}^I)) \backslash \{\mathbf{x}'\}$.

Step 8. (Output) $\Sigma_U(A^I, \mathbf{b}^I) = \{\mathbf{x} \in [\check{\mathbf{x}}^-, \hat{\mathbf{x}}^+] \mid \check{\mathbf{x}}^- \in N(\Sigma_U(A^I, \mathbf{b}^I))\}$.

Remark 3.6 *$|I_j| \in [0, m]$ for $j \in J$. Therefore $|\Lambda| \leq m^n$. Since we have to go through all elements of Λ to identify minimal solutions, the proposed algorithm is not polynomially bounded.*

3.3.3 Example

$$\text{Consider } (x_1\ x_2\ x_3) \vartriangle \begin{pmatrix} [0.3, 0.7] & [0.5, 0.6] \\ [0.4, 0.9] & [0.1, 0.7] \\ [0.2, 0.5] & [0.8, 1.0] \end{pmatrix} = \begin{pmatrix} [0.4, 0.8] & [0.5, 0.7] \end{pmatrix}.$$

We have $\underline{A} = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$, $\overline{A} = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.7 \\ 0.5 & 1.0 \end{pmatrix}$, $\underline{\mathbf{b}} = (0.4\ 0.5)$, $\overline{\mathbf{b}} = (0.8\ 0.7)$.

This problem is from [23].

Example: (*Max-min* composition). For the minimum operation $t(a, x) = \min(a, x)$, we have $a\ \varphi\ b = b$, for $a > b$. Hence $\hat{\mathbf{x}}^+ = (1\ 1\ 0.7)$. Since $\hat{x}^+ \vartriangle \overline{A} \geq \underline{\mathbf{b}}$, $\Sigma_U(A^I, b^I) \neq \emptyset$. Then we have $I_1 = \{1, 2, 3\}$, $I_2 = \{1, 2, 3\}$, and $\Lambda = I_1 \times I_2 = \{(1, 1), (1, 3), (2, 1), (2, 3), (1, 2), (2, 2), (3, 1), (3, 2), (3, 3)\}$. Table 3.2 lists the first 4 cases.

We get $F(\Lambda) = \{(0.5\ 0\ 0), (0.4\ 0\ 0.5), (0.5\ 0.4\ 0), (0\ 0.4\ 0.5), (0.4\ 0.5\ 0), (0\ 0.5\ 0), (0.5\ 0\ 0.4), (0\ 0.5\ 0.4), (0\ 0\ 0.5)\}$. Since $F(\Lambda)_1 \leq F(\Lambda)_3$, $F(\Lambda)_1 \leq F(\Lambda)_7$, $F(\Lambda)_9 \leq F(\Lambda)_2$, $F(\Lambda)_9 \leq F(\Lambda)_4$, $F(\Lambda)_6 \leq F(\Lambda)_5$, $F(\Lambda)_6 \leq F(\Lambda)_8$, we can eliminate $F(\Lambda)_2$, $F(\Lambda)_3$, $F(\Lambda)_4$, $F(\Lambda)_5$, $F(\Lambda)_7$,

$\mathbf{f} = (1,1)$	$J^1_\mathbf{f} = \{1,2\}$ $J^2_\mathbf{f} = \emptyset$ $J^3_\mathbf{f} = \emptyset$	$F_1(\mathbf{f}) = 0.5$ $F_2(\mathbf{f}) = 0$ $F_3(\mathbf{f}) = 0$
$\mathbf{f} = (1,3)$	$J^1_\mathbf{f} = \{1\}$ $J^2_\mathbf{f} = \emptyset$ $J^3_\mathbf{f} = \{2\}$	$F_1(\mathbf{f}) = 0.4$ $F_2(\mathbf{f}) = 0$ $F_3(\mathbf{f}) = 0.5$
$\mathbf{f} = (2,1)$	$J^1_\mathbf{f} = \{2\}$ $J^2_\mathbf{f} = \{1\}$ $J^3_\mathbf{f} = \emptyset$	$F_1(\mathbf{f}) = 0.5$ $F_2(\mathbf{f}) = 0.4$ $F_3(\mathbf{f}) = 0$
$\mathbf{f} = (2,3)$	$J^1_\mathbf{f} = \emptyset$ $J^2_\mathbf{f} = \{1\}$ $J^3_\mathbf{f} = \{2\}$	$F_1(\mathbf{f}) = 0$ $F_2(\mathbf{f}) = 0.4$ $F_3(\mathbf{f}) = 0.5$

Table 3.2: Calculation of minimal solutions for the max-min united solutions

$F(\Lambda)_8$ from $N(\Sigma_U(A^I, b^I))$. Finally we get $N(\Sigma_U(A^I, b^I)) = \{(0.5\ 0\ 0), (0\ 0.5\ 0), (0\ 0\ 0.5)\}$ and $\Sigma_U(A^I, b^I) = \{([0.5, 1]\ [0, 1]\ [0, 0.7]),\ ([0, 1]\ [0.5, 1]\ [0, 0.7]),\ ([0, 1]\ [0, 1]\ [0.5, 0.7])\}$.

Note: there are no tolerable solutions for *max-product* composition in this example.

3.4 Controllable solution set

3.4.1 Solution structure

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, recall that the set

$$\{\ \mathbf{x} \in [0,1]^m \mid \text{for each } \mathbf{b} \in \mathbf{b}^I,\ \text{there exists } A \in A^I \text{ such that } \mathbf{x} \triangle A = \mathbf{b}\}, \tag{3.15}$$

denoted by $\Sigma_C(A^I, \mathbf{b}^I)$, is called the "controllable solution set" of the equation $\mathbf{x} \triangle A^I = \mathbf{b}^I$.

Similar to the previous two cases, consider the following fuzzy relational inequality system,

$$\begin{cases} \mathbf{x} \triangle \underline{A} \le \underline{\mathbf{b}} \\ \mathbf{x} \triangle \overline{A} \ge \overline{\mathbf{b}} \\ \mathbf{x} \in [0,1]^m \end{cases}, \tag{3.16}$$

where $[\underline{A}, \overline{A}] = A^I$ and $[\underline{\mathbf{b}}, \overline{\mathbf{b}}] = \mathbf{b}^I$ are given.

Theorem 3.9 *Assume a given t-norm is continuous. Then* $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$ *if and only if* $\mathbf{x}$ *is a solution of* (3.16).

Proof. (i) Let $\mathbf{x}$ be a solution of (3.16), then $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$, $\mathbf{x} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$ and $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}} \leq \overline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A}$. The continuity property of the composition implies that for any $\mathbf{b} \in \mathbf{b}^I$, there exists $A \in A^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$. Therefore $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$.

(ii) If $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, then for each $\mathbf{b} \in \mathbf{b}^I$, there exists $A \in A^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$. Hence $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$, $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \mathbf{x} \vartriangle \overline{A}$, and $\mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$, i.e., $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$. Furthermore, there exist $A^1, A^2 \in A^I$ such that $\mathbf{x} \vartriangle A^1 = \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle A^2 = \overline{\mathbf{b}}$, since $\underline{\mathbf{b}}, \overline{\mathbf{b}} \in \mathbf{b}^I$ and $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$. Therefore $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{b}} = \mathbf{x} \vartriangle A^1 \geq \mathbf{x} \vartriangle \underline{A}$ and $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}} = \mathbf{x} \vartriangle A^2 \leq \mathbf{x} \vartriangle \overline{A}$. That means $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. ■

We now focus on (3.16) to study the structure of $\Sigma_C(A^I, \mathbf{b}^I)$.

Theorem 3.10 *Given* $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, *if* $\underline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$ *and* $\overline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$, *then* $\mathbf{x}^I \subset \Sigma_C(A^I, \mathbf{b}^I)$.

Proof. Let $\mathbf{x} \in \mathbf{x}^I$. If $\underline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$, then $\underline{\mathbf{x}} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. Hence $\mathbf{x} \vartriangle \overline{A} \geq \underline{\mathbf{x}} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. Similarly, we have $\mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{x}} \vartriangle \underline{A} \leq \overline{\mathbf{b}}$. For $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$, we know $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in [0,1]^m$. Consequently, $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$ and $\mathbf{x}^I \subset \Sigma_C(A^I, \mathbf{b}^I)$. ■

Note that each of $\Sigma_T(A^I, \mathbf{b}^I)$ and $\Sigma_U(A^I, \mathbf{b}^I)$ can be obtained by solving a corresponding fuzzy relational inequality system. In what follows, we will prove that, when the solution set is not empty, $\Sigma_C(A^I, \mathbf{b}^I)$ can also be completely determined by one maximum solution and a finite number of minimal solutions.

From Chapter 2, given $\acute{a}, \acute{b} \in [0,1]$, the φ operator associated with a given *t-norm* is defined by

$$\acute{a} \ \varphi \ \acute{b} = \sup\{\acute{x} \in [0,1] \mid t(\acute{x}, \acute{a}) \leq \acute{b}\}.$$

Using the φ operator, for a given $A^I \in I([0,1]^{m \times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we define

$$\hat{\mathbf{x}}^+ = \left(\min_{j \in J} (\underline{a}_{ij} \ \varphi \ \underline{b}_j) \right)_{i \in I}. \tag{3.17}$$

Theorem 3.11 *If* $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, *then* $\hat{\mathbf{x}}^+$ *defined by* (3.17) *is the maximum solution of* $\Sigma_C(A^I, \mathbf{b}^I)$.

Proof. (i) $\forall j \in J$, $\max_{i \in I} t(\hat{x}_i^+, \underline{a}_{ij}) = \max_{i \in I} t(\min_{j_0 \in J}(\underline{a}_{ij_0} \ \varphi \ \underline{b}_{j_0}), \underline{a}_{ij}) \leq \max_{i \in I} t((\underline{a}_{ij} \varphi \underline{b}_j), \underline{a}_{ij}) \leq \underline{b}_j$, hence we have $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \leq \underline{\mathbf{b}}$.

(ii) For any $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, we know that $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$ and $\max_{i\in I} t(x_i, \underline{a}_{ij}) \leq \underline{b}_j$, $\forall j \in J$. Therefore, for $i \in I$, $t(x_i, \underline{a}_{ij}) \leq \underline{b}_j$ and $x_i \leq \underline{a}_{ij} \ \varphi \ \underline{b}_j$. Consequently, $x_i \leq \min_{j\in J}(\underline{a}_{ij} \ \varphi \ \underline{b}_j) = \hat{x}_i^+$ and $\mathbf{x} \leq \hat{\mathbf{x}}^+$.

(iii) When $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, there exists at least one $\mathbf{x}$ satisfies $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. By (ii) we have $\mathbf{x} \leq \hat{\mathbf{x}}^+$. Hence, $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \overline{\mathbf{b}}$.

Combining (i), (ii) and (iii), we see that $\hat{\mathbf{x}}^+$ is the maximum solution of $\Sigma_C(A^I, \mathbf{b}^I)$. ■

Corollary 3.5 $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$ *if and only if* $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \overline{\mathbf{b}}$.

Proof. If $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \overline{\mathbf{b}}$, from the proof of Theorem 3.11(i), we have $\hat{\mathbf{x}}^+ \vartriangle \underline{A} \leq \underline{\mathbf{b}}$, and, hence, $\hat{\mathbf{x}}^+ \in \Sigma_C(A^I, \mathbf{b}^I)$ and $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$.

On the other hand, if $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, then, from Theorem 3.11, $\hat{\mathbf{x}}^+ \in \Sigma_C(A^I, \mathbf{b}^I)$. Hence, $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. ■

We now focus on the minimal solutions.

Lemma 3.5 *If* $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, *then, for each* $j \in J$, $\max_{i\in I} t(x_i, \underline{a}_{ij}) \leq \underline{b}_j$, *and there exists* $i_0 \in I$ *such that* $t(x_{i_0}, \overline{a}_{i_0 j}) \geq \overline{b}_j$.

Proof. Since $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$, we have $\max_{i\in I} t(x_i, \underline{a}_{ij}) \leq \underline{b}_j$, for $j \in J$. Since $\mathbf{x} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$, we have $\max_{i\in I} t(x_i, \overline{a}_{ij}) \geq \overline{b}_j$, for $j \in J$. Hence for each $j \in J$, there exists at least one $i_0 \in I$ such that $t(x_{i_0}, \overline{a}_{i_0 j}) \geq \overline{b}_j$. ■

When $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, $\hat{\mathbf{x}}^+$ can be found by using Equation (3.17). Analogous to (3.4) and (3.11), we define the following index sets:

$$I_j = \{i \in I \mid t(\hat{x}_i^+, \overline{a}_{ij}) \geq \overline{b}_j\}, \ \forall\, j \in J, \tag{3.18}$$

with Cartesian product

$$\Lambda = I_1 \times I_2 \times \cdots \times I_n. \tag{3.19}$$

Notice that from Lemma 2.1, $t(\hat{x}_i^+, \overline{a}_{ij}) \geq \overline{b}_j$ implies that $\hat{x}_i^+ \geq \overline{a}_{ij} \ \gamma \ \overline{b}_j$. Also notice that $t(\hat{x}_i^+, \bar{a}_{ij}) \leq \min\{\hat{x}_i^+, \bar{a}_{ij}\}$, therefore, $\overline{a}_{ij} \geq \overline{b}_j$ and $\overline{a}_{ij} \ \gamma \ \overline{b}_j$ is well defined. Lemma 3.5 guarantees that $I_j \neq \emptyset$, $\forall\, j \in J$, when $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$. Moreover, given $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in I^n$, we say $\mathbf{f} \in \Lambda$ if and only if $f_j \in I_j$, $\forall\, j \in J$. We now study the structure of solution set $\Sigma_C(A^I, \mathbf{b}^I)$ in terms of the elements in Λ.

Definition 3.3 *Given* $\mathbf{f} \in \Lambda$, *we define*

$$J_{\mathbf{f}}^{i} = \{j \in J \mid f_j = i\}, \ \forall \ i \in I, \tag{3.20}$$

and $F : \Lambda \to R^m$ *with* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f}))$ *such that*

$$F_i(\boldsymbol{f}) = \begin{cases} \max_{j \in J_{\mathbf{f}}^{i}} (\overline{a}_{ij} \ \gamma \ \overline{b}_j) & , \ \textit{if} \ J_{\mathbf{f}}^{i} \neq \emptyset \\ 0 & , \ \textit{if} \ J_{\mathbf{f}}^{i} = \emptyset \end{cases} . \tag{3.21}$$

Remark 3.7 *If* $\mathbf{f} \in \Lambda$, *then* $t(\hat{x}_{f_j}^{+}, \overline{a}_{f_j j}) \geq \overline{b}_j$, $\forall \ j \in J$.

Remark 3.8 *From* (3.20), $j \in J_{\mathbf{f}}^{i}$ *implies that* $t(\hat{x}_i^{+}, \overline{a}_{ij}) \geq \overline{b}_j$.

The relationship between $\Sigma_C(A^I, \mathbf{b}^I)$ and $F(\Lambda) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$ is given below.

Theorem 3.12 *Assume that* $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$.
(i) *If* $\mathbf{f} \in \Lambda$, *then* $F(\mathbf{f}) \in \Sigma_C(A^I, \mathbf{b}^I)$.
(ii) *For any* $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, *there exists* $\mathbf{f} \in \Lambda$ *such that* $F(\mathbf{f}) \leq \mathbf{x}$.

Proof. (i) From the definition of $J_{\mathbf{f}}^{i}$, for any $i \in I$, if $J_{\mathbf{f}}^{i} = \emptyset$, then $F_i(\mathbf{f}) = 0 \leq \hat{x}_i^{+}$. If $J_{\mathbf{f}}^{i} \neq \emptyset$, then $F_i(\mathbf{f}) = \max_{j \in J_{\mathbf{f}}^{i}} (\overline{a}_{ij} \ \gamma \ \overline{b}_j) = \max_{\{j | f_j = i\}} (\overline{a}_{ij} \ \gamma \ \overline{b}_j) \leq \hat{x}_i^{+}$, since $\hat{x}_i^{+} \geq \overline{a}_{ij} \ \gamma \ \overline{b}_j$, for $j \in J$. Therefore, $F_i(\mathbf{f}) \leq \hat{x}_i^{+}$, for any $i \in I$. Consequently, $\max_{i \in I} t(\underline{a}_{ij}, F_i(\mathbf{f})) \leq \max_{i \in I} t(\underline{a}_{ij}, \hat{x}_i^{+}) \leq \underline{b}_j$, $\forall \ j \in J$, i.e., $F(\mathbf{f}) \vartriangle \underline{A} \leq \underline{\mathbf{b}}$.

On the other hand, the definition of $J_{\mathbf{f}}^{i}$ implies that $j \in J_{\mathbf{f}}^{f_j}$, and $F_{f_j}(\mathbf{f}) = \max_{j_0 \in J_{\mathbf{f}}^{f_j}} (\overline{a}_{f_j j_0} \ \gamma \ \overline{b}_{j_0}) \geq \overline{a}_{f_j j} \ \gamma \ \overline{b}_j$, $\forall \ j \in J$. Therefore $t(\overline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \geq \overline{b}_j$ and consequently $\max_{i \in I} t(\overline{a}_{ij}, F_i(\mathbf{f})) \geq t(\overline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \geq \overline{b}_j$, $\forall \ j \in J$, i.e., $F(\mathbf{f}) \vartriangle \overline{A} \geq \overline{\mathbf{b}}$.

Combining the facts that $F(\mathbf{f}) \vartriangle \underline{A} \leq \underline{\mathbf{b}}$ and $F(\mathbf{f}) \vartriangle \overline{A} \geq \overline{\mathbf{b}}$, we know $F(\mathbf{f}) \in \Sigma_C(A^I, \mathbf{b}^I)$.

(ii) For each $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, according to Lemma 3.5, for each $j \in J$, there exists $i_j \in I$ such that $t(x_{i_j}, \overline{a}_{i_j j}) \geq \overline{b}_j$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = i_j, \forall \ j \in J$. In this way, $\mathbf{f} \in \Lambda$. Now, for any $i \in I$, if $J_{\mathbf{f}}^{i} = \emptyset$, then $F_i(\mathbf{f}) = 0 \leq x_i$. If $J_{\mathbf{f}}^{i} \neq \emptyset$, since $t(x_{i_j}, \overline{a}_{i_j j}) \geq \overline{b}_j$, for $j \in J$, $F_i(\mathbf{f}) = \max_{j \in J_{\mathbf{f}}^{i}} (\overline{a}_{ij} \ \gamma \ \overline{b}_j) = \max_{\{j | f_j = i\}} (\overline{a}_{ij} \ \gamma \ \overline{b}_j) = \max_{\{j | i_j = i\}} (\overline{a}_{ij} \ \gamma \ \overline{b}_j) \leq x_{i_j} = x_i$. This shows that $F_i(\mathbf{f}) \leq x_i$, $\forall \ i \in I$. Hence $F(\mathbf{f}) \leq \mathbf{x}$. ■

Theorem 3.12 implies that there exists at least one minimal solution of $\Sigma_C(A^I, \mathbf{b}^I)$, provided $\Sigma_C(A^I, \mathbf{b}^I)$ is not empty. Since the cardinality of Λ is finite, the number of minimal solutions must be finite. Denote the set of all minimal solutions by $N(\Sigma_C(A^I, \mathbf{b}^I))$. We have the following result.

Corollary 3.6 $N(\Sigma_C(A^I, \mathbf{b}^I)) \subset F(\Lambda) \subset \Sigma_C(A^I, \mathbf{b}^I)$.

This corollary reduces the search space for all the minimal solutions of $\Sigma_C(A^I, \mathbf{b}^I)$ to a relatively small set $F(\Lambda)$. Moreover, for $\mathbf{x}$, $\mathbf{x}' \in F(\Lambda)$, if $\mathbf{x} \leq \mathbf{x}'$, then $\mathbf{x}' \notin N(\Sigma_C(A^I, \mathbf{b}^I))$. Once we remove all such $\mathbf{x}'$ from $F(\Lambda)$, $N(\Sigma_C(A^I, \mathbf{b}^I))$ is identified.

3.4.2 The algorithm

Based on the above discussion, we propose an algorithm to solve *max-t-norm* interval-valued fuzzy relational equations.

Step 1. Specify the mathematical expression of the operators φ and γ associated with the *t-norm*.

Step 2. (Find the potential maximum solution) Compute $\hat{\mathbf{x}}^+$ according to (3.17).

Step 3. (Check feasibility and uniqueness) If $\hat{\mathbf{x}}^+ \vartriangle \overline{A} \ngeq \overline{\mathbf{b}}$ then $\Sigma_C(A^I, \mathbf{b}^I) = \emptyset$, stop. Otherwise continue.

Step 4. Compute I_j for all $j \in J$ according to (3.18).

Step 5. Generate Λ according to (3.19).

Step 6. (Determine potential minimal solutions) For each $\mathbf{f} \in \Lambda$, calculate $F(\mathbf{f})$ according to (3.21). Set $N(\Sigma_C(A^I, \mathbf{b}^I)) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Step 7. (Exclude nonminimal solutions) For $\mathbf{x}, \mathbf{x}' \in N(\Sigma_C(A^I, \mathbf{b}^I))$ with $\mathbf{x} \leq \mathbf{x}'$, set $N(\Sigma_C(A^I, \mathbf{b}^I)) = N(\Sigma_C(A^I, \mathbf{b}^I)) \backslash \{\mathbf{x}'\}$.

Step 8. (Output) $\Sigma_C(A^I, \mathbf{b}^I) = \{\ \mathbf{x} \in [\check{\mathbf{x}}^-, \hat{\mathbf{x}}^+] \mid \check{\mathbf{x}}^- \in N(\Sigma_C(A^I, \mathbf{b}^I))\}$.

Remark 3.9 $|I_j| \in [0, m]$ *for* $j \in J$. *Therefore* $|\Lambda| \leq m^n$. *Since we have to go through all elements of* Λ *to identify minimal solutions, the proposed algorithm is not polynomially bounded.*

3.5 Relationship among different solution sets

By the definitions of $\Sigma_T(A^I, \mathbf{b}^I)$, $\Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_C(A^I, \mathbf{b}^I)$, it is easy to see that $\Sigma_T(A^I, \mathbf{b}^I) \subset \Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_C(A^I, \mathbf{b}^I) \subset \Sigma_U(A^I, \mathbf{b}^I)$. Furthermore, we have the following theorem.

Theorem 3.13 *Given* $A^I \in I([0,1]^{m\times n})$ *and* $\mathbf{b}^I \in I([0,1]^n)$.

(a) $\Sigma_U(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [0,1]^m \mid [\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \cap \mathbf{b}^I \neq \emptyset\}$;

(b) $\Sigma_T(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [0,1]^m \mid [\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \subset \mathbf{b}^I\}$;

(c) $\Sigma_C(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [0,1]^m \mid [\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \supset \mathbf{b}^I\}$.

Proof. (a) If $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, then $\mathbf{x} \vartriangle A \in \mathbf{b}^I$ for some $A \in A^I$, hence $[\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \cap \mathbf{b}^I \neq \emptyset$.

When $\mathbf{x} \in \{ \mathbf{x} \in [0,1]^m \mid [\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \cap \mathbf{b}^I \neq \emptyset\}$, there are four possible cases.

Case 1 $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}} \leq \overline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A}$. From Lemma 3.2, there exists $A \in A^I$ such that $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$. Consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 2 $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. Hence for all $A \in A^I$, $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle A \leq \overline{\mathbf{b}}$. Consequently, there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 3 $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. If we let $A = \overline{A}$, then $\mathbf{x} \vartriangle A \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 4 $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \overline{\mathbf{b}} \leq \mathbf{x} \vartriangle \overline{A}$. If we let $A = \underline{A}$, then $\mathbf{x} \vartriangle A \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x} \vartriangle A = \mathbf{b}$. Consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

(b) If $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, then $\mathbf{x} \vartriangle A \in \mathbf{b}^I$, for any $A \in A^I$. Hence, $\mathbf{x} \vartriangle \underline{A} \in \mathbf{b}^I$ and $\mathbf{x} \vartriangle \overline{A} \in \mathbf{b}^I$. This means that $[\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \subset \mathbf{b}^I$.

When $\mathbf{x} \in \{ \mathbf{x} \in [0,1]^m \mid [\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \subset \mathbf{b}^I\}$, $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. Hence, for any $A \in A^I$, we have $\underline{\mathbf{b}} \leq \mathbf{x} \vartriangle \underline{A} \leq \mathbf{x} \vartriangle A \leq \mathbf{x} \vartriangle \overline{A} \leq \overline{\mathbf{b}}$. Consequently, $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$.

(c) If $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, then $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. Hence $[\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \supset \mathbf{b}^I$.

When $\mathbf{x} \in \{ \mathbf{x} \in [0,1]^m \mid [\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}] \supset \mathbf{b}^I\}$, we have $\mathbf{x} \vartriangle \underline{A} \leq \underline{\mathbf{b}}$ and $\mathbf{x} \vartriangle \overline{A} \geq \overline{\mathbf{b}}$. Hence $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$. ■

If we denote $[\mathbf{x} \vartriangle \underline{A}, \mathbf{x} \vartriangle \overline{A}]$ by $\mathbf{x} \vartriangle A^I$, then Theorem 3.13 asserts that:

(1) $\Sigma_U(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [0,1]^m \mid (\mathbf{x} \vartriangle A^I) \cap \mathbf{b}^I \neq \emptyset\}$;

(2) $\Sigma_T(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [0,1]^m \mid \mathbf{x} \vartriangle A^I \subset \mathbf{b}^I\}$;

(3) $\Sigma_C(A^I, \mathbf{b}^I) = \{ \mathbf{x} \in [0,1]^m \mid \mathbf{x} \vartriangle A^I \supset \mathbf{b}^I\}$.

Figures 3-1 to 3-3 are used to depict the meaning of Theorem 3.13. On the right hand side of each figure, the light shaded rectangular represents $\mathbf{b}^I$, and other rectangles represent $\mathbf{x} \triangle A^I$ for a solution $\mathbf{x}$.

As shown in previous sections, the three different solution sets can be found by solving a corresponding fuzzy relational inequality system. We also proved that the solution set can be represented by a set of interval vectors. If the solution set is not empty, by finding the maximum solution and the (finite number of) minimal solutions, we can represent the solution set by a finite set of interval vectors with the maximum solution as the upper bound and one of the minimal solutions as the lower bound. This solution structure is what De Baets called the "root system" [4], with the stem being the maximum solution and offshoots being the minimal solutions. Since the number of minimal solutions is finite, this root system is finitely generated.

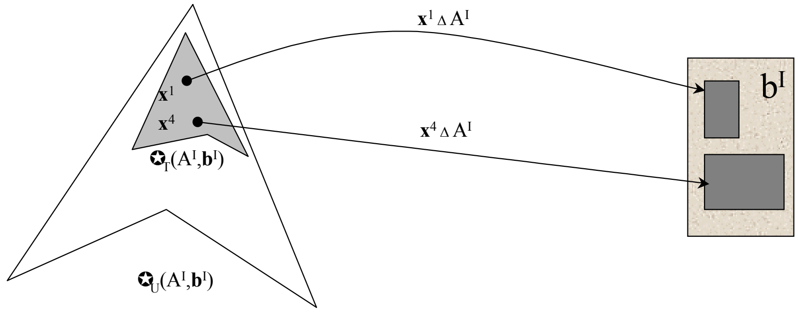

Figure 3-1: Max-t-norm tolerable solution set

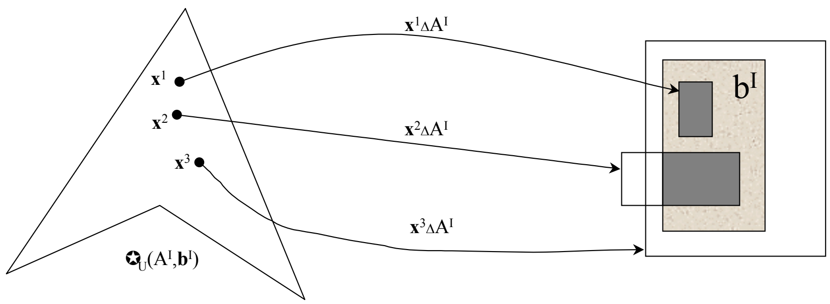

Figure 3-2: Max-t-norm united solution set

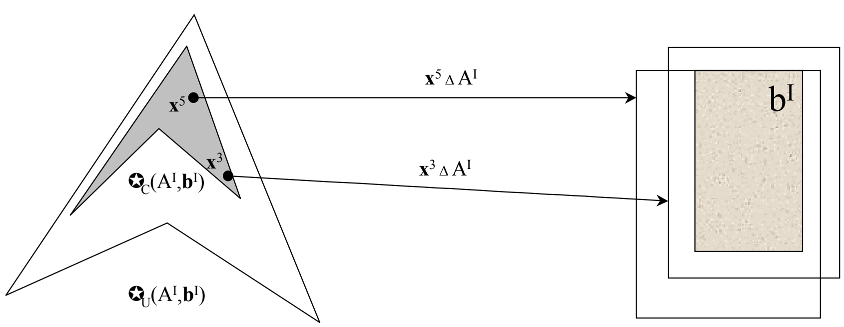

Figure 3-3: Max-t-norm controllable solution set

Chapter 4

Min-s-norm Fuzzy Relational Equations

In the previous chapters, the max-t-norm fuzzy relational equations was explored. Now the solutions to min-s-norm fuzzy relational equations will be developed.

4.1 Introduction

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we are interested in finding different "solution sets" for the equation $\mathbf{x}ⓢA^I = \mathbf{b}^I$, where ⓢ is the *min-s-norm* composition and the *s-norm* is continuous. We will frequently refer to two index sets, $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$, throughout this chapter.

The *min-s-norm* composition has two important properties, expressed by the following two lemmas.

Lemma 4.1 (*Monotonicity of composition*) (i) *Given* $A \in [0,1]^{m\times n}$, *for* $\mathbf{x}^1$, $\mathbf{x}^2 \in [0,1]^m$, *if* $\mathbf{x}^1 \leq \mathbf{x}^2$ *then* $\mathbf{x}^1ⓢA \leq \mathbf{x}^2ⓢA$. (ii) *Given* $\mathbf{x} \in [0,1]^m$, *for* A^1, $A^2 \in [0,1]^{m\times n}$, *if* $A^1 \leq A^2$ *then* $\mathbf{x}ⓢA^1 \leq \mathbf{x}ⓢA^2$.

Proof. (i) Let $\mathbf{b}^1 = \mathbf{x}^1ⓢA$ and $\mathbf{b}^2 = \mathbf{x}^2ⓢA$. Due to the monotonicity property of the triangular conorms, we have $b^1_j = \min_{i\in I} s(x^1_i, a_{ij}) \leq \min_{i\in I} s(x^2_i, a_{ij}) = b^2_j$, $\forall\ j \in J$.

(ii) Let $\mathbf{b}^1 = \mathbf{x}ⓢA^1$ and $\mathbf{b}^2 = \mathbf{x}ⓢA^2$. We have $b_j^1 = \min_{i\in I} s(x_i, a_{ij}^1) \le \min_{i\in I} s(x_i, a_{ij}^2) = b_j^2$, $\forall\ j \in J$. ■

Lemma 4.2 (*Continuity of composition*) *For a continuous s-norm,* (i) *given* $\mathbf{x} \in [0,1]^m$ *and* A^1, $A^2 \in [0,1]^{m\times n}$, *if* $A^1 \le A^2$ *then for any* $\mathbf{b} \in [\mathbf{x}ⓢA^1, \mathbf{x}ⓢA^2]$ *there exists* $A \in [A^1, A^2]$ *such that* $\mathbf{x}ⓢA = \mathbf{b}$.
(ii) *given* $A \in [0,1]^{m\times n}$ *and* $\mathbf{x}^1$, $\mathbf{x}^2 \in [0,1]^m$, *if* $\mathbf{x}^1 \le \mathbf{x}^2$ *then for any* $\mathbf{b} \in [\mathbf{x}^1ⓢA, \mathbf{x}^2ⓢA]$ *there exists* $\mathbf{x} \in [\mathbf{x}^1, \mathbf{x}^2]$ *such that* $\mathbf{x}ⓢA = \mathbf{b}$.
(iii) *given* $A \in [0,1]^{m\times n}$ *and* $\mathbf{b}^1 \in [0,1]^n$, *if there exists* $\mathbf{x}^1 \in [0,1]^m$ *such that* $\mathbf{x}^1ⓢA = \mathbf{b}^1$, *then for any* $\mathbf{b}^2 \in [0,1]^n$ *with* $\mathbf{b}^2 \ge \mathbf{b}^1$, *there exists* $\mathbf{x}^2 \in [0,1]^m$, *such that* $\mathbf{x}^1 \le \mathbf{x}^2$ *and* $\mathbf{x}^2ⓢA = \mathbf{b}^2$.

Proof. (i) Let $\mathbf{b}^1 = \mathbf{x}ⓢA^1$ and $\mathbf{b}^2 = \mathbf{x}ⓢA^2$. Consider $j \in J$. We know that $\min_{i\in I} s(x_i, a_{ij}^1) = b_j^1$ and $\min_{i\in I} s(x_i, a_{ij}^2) = b_j^2$. Hence there exist i_1, $i_2 \in I$, such that $s(x_{i_1}, a_{i_1j}^1) = b_j^1$ and $s(x_{i_2}, a_{i_2j}^2) = b_j^2$. Remember that we are dealing with a continuous triangular conorm. Since $s(x_{i_1}, a_{i_1j}^2) \ge b_j^2 \ge s(x_{i_1}, a_{i_1j}^1) = b_j^1$, there exists $a_{i_1j} \in [a_{i_1j}^1, a_{i_1j}^2]$ such that $s(x_{i_1}, a_{i_1j}) = b_j \in [b_j^1, b_j^2]$. Now let $a_{ij} = a_{ij}^2$. For $i \ne i_1$, we have $\min_{i\in I} s(x_i, a_{ij}) = \min\{s(x_{i_1}, a_{i_1j}), \min_{i\in I,\ i\ne i_1} s(x_i, a_{ij}^2)\} = b_j$, since $\min_{i\in I,\ i\ne i_1} s(x_i, a_{ij}^2) \ge b_j^2 \ge b_j$. Consequently, we have found a matrix $A \in [A^1, A^2]$ such that $\mathbf{x}ⓢA = \mathbf{b}$.

(ii) Let $\mathbf{b}^1 = \mathbf{x}^1ⓢA$ and $\mathbf{b}^2 = \mathbf{x}^2ⓢA$. Consider $j \in J$. We know that $\min_{i\in I} s(x_i^1, a_{ij}) = b_j^1$ and $\min_{i\in I} s(x_i^2, a_{ij}) = b_j^2$. Hence there exist i_1, $i_2 \in I$, such that $s(x_{i_1}^1, a_{i_1j}) = b_j^1$ and $s(x_{i_2}^2, a_{i_2j}) = b_j^2$. Remember that we are dealing with a continuous triangular norm. Since $s(x_{i_1}^2, a_{i_1j}) \ge b_j^2 \ge s(x_{i_1}^1, a_{i_1j}) = b_j^1$, there exists $x_{i_1} \in [x_{i_1}^1, x_{i_1}^2]$ such that $s(x_{i_1}, a_{i_1j}) = b_j \in [b_j^1, b_j^2]$. Now let $x_i = x_i^2$. For $i \ne i_1$, we have $\min_{i\in I} s(x_i, a_{ij}) = \min\{s(x_{i_1}, a_{i_1j}), \min_{i\in I,\ i\ne i_1} s(x_i^2, a_{ij})\} = b_j$, since $\min_{i\in I,\ i\ne i_1} s(x_i^2, a_{ij}) \ge b_j^2 \ge b_j$. Consequently, we have found a vector $\mathbf{x} \in [\mathbf{x}^1, \mathbf{x}^2]$ such that $\mathbf{x}ⓢA = \mathbf{b}$.

(iii) Let $\mathbf{x}^3 \in [0,1]^m$ be the unit vector (all elements are 1) and $\mathbf{b}^3 = \mathbf{x}^3ⓢA$. By the definition of *s-norm*, $\mathbf{b}^3$ is also a unit vector. Thus $\mathbf{b}^3 \ge \mathbf{b}^1$. From (ii) we know that, for any $\mathbf{b}^2 \in [\mathbf{b}^1, \mathbf{b}^3]$, there exists $\mathbf{x}^2 \in [\mathbf{x}^1, \mathbf{x}^3]$ such that $\mathbf{x}^2ⓢA = \mathbf{b}^2$. ■

4.2 Tolerable solution set

4.2.1 Solution structure

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, recall that the set

$$\{\ \mathbf{x} \in [0,1]^m \mid \text{for each } A \in A^I,\ \text{there exists } \mathbf{b} \in \mathbf{b}^I \text{ such that } \mathbf{x}\circledS A = \mathbf{b}\}, \tag{4.1}$$

denoted by $\Sigma_T(A^I, \mathbf{b}^I)$, is called the "tolerable solution set" of the equation $\mathbf{x}\circledS A^I = \mathbf{b}^I$.

In order to reveal the structure of $\Sigma_T(A^I, \mathbf{b}^I)$, consider the following fuzzy relational inequality system,

$$\begin{cases} \mathbf{x}\circledS \underline{A} \geq \underline{\mathbf{b}} \\ \mathbf{x}\circledS \overline{A} \leq \overline{\mathbf{b}} \\ \mathbf{x} \in [0,1]^m \end{cases}, \tag{4.2}$$

where $[\underline{A}, \overline{A}] = A^I$ and $[\underline{\mathbf{b}}, \overline{\mathbf{b}}] = \mathbf{b}^I$ are given.

Theorem 4.1 *Assume a given s-norm is continuous. Then $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$ if and only if $\mathbf{x}$ is a solution of* (4.2) *.*

Proof. (i) Let $\mathbf{x}$ be a solution of (4.2). Then $\mathbf{x}\circledS \underline{A} \geq \underline{\mathbf{b}}$, $\mathbf{x}\circledS \overline{A} \leq \overline{\mathbf{b}}$ and $\mathbf{x} \in [0,1]^m$. The monotonicity property of composition implies that $\underline{\mathbf{b}} \leq \mathbf{x}\circledS \underline{A} \leq \mathbf{x}\circledS A \leq \mathbf{x}\circledS \overline{A} \leq \overline{\mathbf{b}}$ for any $A \in A^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}\circledS A = \mathbf{b}$, and, consequently, $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$.

(ii) If $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, then for each $A \in A^I$, there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}\circledS A = \mathbf{b}$. Hence there are $\mathbf{b}^1$ and $\mathbf{b}^2$ such that $\mathbf{x}\circledS \underline{A} = \mathbf{b}^1 \geq \underline{\mathbf{b}}$ and $\mathbf{x}\circledS \overline{A} = \mathbf{b}^2 \leq \overline{\mathbf{b}}$. Furthermore, since $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, $\mathbf{x} \in [0,1]^m$. ■

We now focus on (4.2) to find $\Sigma_T(A^I, \mathbf{b}^I)$.

Theorem 4.2 *Given $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, if $\underline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$ and $\overline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$, then $\mathbf{x}^I \subset \Sigma_T(A^I, \mathbf{b}^I)$.*

Proof. Let $\mathbf{x} \in \mathbf{x}^I$. If $\underline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$, then $\underline{\mathbf{x}}\circledS \underline{A} \geq \underline{\mathbf{b}}$. Hence $\mathbf{x}\circledS \underline{A} \geq \underline{\mathbf{x}}\circledS \underline{A} \geq \underline{\mathbf{b}}$. Similarly, we have $\mathbf{x}\circledS \overline{A} \leq \overline{\mathbf{x}}\circledS \overline{A} \leq \overline{\mathbf{b}}$. Moreover, for $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in \Sigma_T(A^I, \mathbf{b}^I)$, we know $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in [0,1]^m$. Consequently, $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$ and $\mathbf{x}^I \subset \Sigma_T(A^I, \mathbf{b}^I)$. ■

Theorem 4.2 suggests further investigation of the structure of $\Sigma_T(A^I, \mathbf{b}^I)$ by examining the so-called maximum and minimal solutions as defined in Section 2.1. For conventional (constant-valued) fuzzy relational equations with a *min-s* composition, the solution set can be completely determined by one minimum solution and a finite number of maximal solutions, provided s is continuous and the solution set is not empty [9]. This result suggests further study as to whether the same solution structure can apply to interval-valued fuzzy relational equations with a general *min-s* composition. In what follows, we will prove that it does.

From Chapter 2, given $\acute{a}, \acute{b} \in [0,1]$, the β operator associated with a given *s-norm* is defined by

$$\acute{a}\ \beta\ \acute{b} = \inf\{\acute{x} \in [0,1] \mid s(\acute{x}, \acute{a}) \geq \acute{b}\}.$$

Using the β operator, for a given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we define

$$\check{\mathbf{x}}^- = \left(\max_{j\in J}(\underline{a}_{ij}\ \beta\ \underline{b}_j)\right)_{i\in I}. \tag{4.3}$$

Theorem 4.3 *If $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, then $\check{\mathbf{x}}^-$ defined by (4.3) is the minimum solution of $\Sigma_T(A^I, \mathbf{b}^I)$.*

Proof. (i) $\forall j \in J$, $\min_{i\in I} s(\check{x}_i^-, \underline{a}_{ij}) = \min_{i\in I} s(\max_{j_0\in J}(\underline{a}_{ij_0}\ \beta\ \underline{b}_{j_0}), \underline{a}_{ij}) \geq \min_{i\in I} s((\underline{a}_{ij}\ \beta\ \underline{b}_j), \underline{a}_{ij}) \geq \underline{b}_j$. Therefore, $\check{\mathbf{x}}^- \circledS \underline{A} \geq \underline{\mathbf{b}}$.

(ii) For any $\mathbf{x} \in \sum_T(A^I, \mathbf{b}^I)$, we know that $\mathbf{x}\circledS\underline{A} \geq \underline{\mathbf{b}}$ and $\min_{i\in I} s(x_i, \underline{a}_{ij}) \geq \underline{b}_j$, $\forall j \in J$. Therefore, for $i \in I$, $s(x_i, \underline{a}_{ij}) \geq \underline{b}_j$ and $x_i \geq \underline{a}_{ij}\ \beta\ \underline{b}_j$. Consequently, $x_i \geq \max_{j\in J}(\underline{a}_{ij}\ \beta\ \underline{b}_j) = \check{x}_i^-$ and $\mathbf{x} \geq \check{\mathbf{x}}^-$.

(iii) When $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, there exists at least one $\mathbf{x}$ such that $\mathbf{x}\circledS\underline{A} \geq \underline{\mathbf{b}}$ and $\mathbf{x}\circledS\overline{A} \leq \overline{\mathbf{b}}$. By (ii) we have $\mathbf{x} \geq \check{\mathbf{x}}^-$. Hence, $\check{\mathbf{x}}^-\circledS\overline{A} \leq \overline{\mathbf{b}}$.

Combining (i), (ii) and (iii), we see that $\check{\mathbf{x}}^-$ is the minimum solution of $\Sigma_T(A^I, \mathbf{b}^I)$. ■

Corollary 4.1 *$\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$ if and only if $\check{\mathbf{x}}^-\circledS\ \overline{A} \leq \overline{\mathbf{b}}$.*

Proof. If $\check{\mathbf{x}}^-\circledS\overline{A} \leq \overline{\mathbf{b}}$, from the proof of Theorem 4.3, we have $\check{\mathbf{x}}^-\circledS\underline{A} \geq \underline{\mathbf{b}}$. Consequently $\check{\mathbf{x}}^- \in \Sigma_T(A^I, \mathbf{b}^I)$ and $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$.

On the other hand, if $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, Theorem 4.3 guarantees that $\check{\mathbf{x}}^- \in \Sigma_T(A^I, \mathbf{b}^I)$. Hence, $\check{\mathbf{x}}^-\circledS\overline{A} \leq \overline{\mathbf{b}}$. ■

We now focus on the maximal solutions.

Lemma 4.3 *If* $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$ *, then, for each* $j \in J$, $\min_{i\in I} s(x_i, \underline{a}_{ij}) \geq \underline{b}_j$*, and there exists* $i_0 \in I$ *such that* $s(x_{i_0}, \overline{a}_{i_0 j}) \leq \overline{b}_j$.

Proof. Since $\mathbf{x}Ⓢ\underline{A} \geq \underline{\mathbf{b}}$, we have $\min_{i\in I} s(x_i, \underline{a}_{ij}) \geq \underline{b}_j$, for $j \in J$. Similarly, $\mathbf{x}Ⓢ\overline{A} \leq \overline{\mathbf{b}}$ implies that $\min_{i\in I} s(x_i, \overline{a}_{ij}) \leq \overline{b}_j$, for $j \in J$. Hence for each $j \in J$, there exists at least one $i_0 \in I$ such that $s(x_{i_0}, \overline{a}_{i_0 j}) \leq \overline{b}_j$. ■

When $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$, $\check{\mathbf{x}}^-$ can be found by using Equation (4.3). We define the following index set for each $j \in J$:

$$I_j = \{i \in I \mid s(\check{x}_i^-, \overline{a}_{ij}) \leq \overline{b}_j\}, \tag{4.4}$$

and the Cartesian product of these index sets by

$$\Lambda = I_1 \times I_2 \times \cdots \times I_n. \tag{4.5}$$

Notice that from Lemma 2.2, $s(\check{x}_i^-, \overline{a}_{ij}) \leq \overline{b}_j$ implies that $\check{x}_i^- \leq \overline{a}_{ij} \ \omega \ \overline{b}_j$. Also notice that $s(\check{x}_i^-, \overline{a}_{ij}) \geq \max\{\check{x}_i^-, \overline{a}_{ij}\}$, therefore, $\overline{a}_{ij} \leq \overline{b}_j$ and $\overline{a}_{ij} \ \omega \ \overline{b}_j$ is well defined. Lemma 4.3 guarantees that $I_j \neq \emptyset$, $\forall \ j \in J$, when $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$. Moreover, given $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in I^n$, we say $\mathbf{f} \in \Lambda$ if and only if $f_j \in I_j, \forall \ j \in J$. Now we can study the structure of the solution set $\Sigma_T(A^I, \mathbf{b}^I)$ in terms of the elements in Λ.

Definition 4.1 *Given* $\mathbf{f} \in \Lambda$, *we define*

$$J_{\mathbf{f}}^i = \{j \in J \mid f_j = i\}, \ \forall \ i \in I \tag{4.6}$$

and $F : \Lambda \to R^m$ *such that* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f}))$ *with*

$$F_i(\boldsymbol{f}) = \begin{cases} \min_{j\in J_{\mathbf{f}}^i} (\overline{a}_{ij} \ \omega \ \overline{b}_j) & , \ \text{if} \ J_{\mathbf{f}}^i \neq \emptyset \\ 1 & , \ \text{if} \ J_{\mathbf{f}}^i = \emptyset \end{cases}. \tag{4.7}$$

Remark 4.1 *If* $\mathbf{f} \in \Lambda$, *then* $s(\check{x}_{f_j}^-, \overline{a}_{f_j j}) \leq \overline{b}_j$, $\forall \ j \in J$. *Hence* $j \in J_{\mathbf{f}}^i$, *and then* $s(\check{x}_i^-, \overline{a}_{ij}) \leq \overline{b}_j$.

Now we examine the relationship between $\Sigma_T(A^I, \mathbf{b}^I)$ and $F(\Lambda) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Theorem 4.4 *Assume that* $\Sigma_T(A^I, \mathbf{b}^I) \neq \emptyset$.

(i) *If* $\mathbf{f} \in \Lambda$, *then* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f})) \in \Sigma_T(A^I, \mathbf{b}^I)$.

(ii) *For any* $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, *there exists* $\mathbf{f} \in \Lambda$ *such that* $F(\mathbf{f}) \geq \mathbf{x}$.

Proof. (i) From the definition of $J^i_{\mathbf{f}}$, for any $i \in I$, if $J^i_{\mathbf{f}} = \emptyset$, then $F_i(\mathbf{f}) = 1 \geq \check{x}_i^-$. If $J^i_f \neq \emptyset$, then $F_i(\mathbf{f}) = \min_{j \in J^i_{\mathbf{f}}}(\overline{a}_{ij}\ \omega\ \overline{b}_j) = \min_{\{j|f_j=i\}}(\overline{a}_{ij}\ \omega\ \overline{b}_j) \geq \check{x}_i^-$, since $\check{x}_i^- \leq \overline{a}_{ij}\ \omega\ \overline{b}_j$ for $j \in J$. Therefore, $F_i(\mathbf{f}) \geq \check{x}_i^-$, for any $i \in I$. Consequently, $\min_{i \in I} s(\underline{a}_{ij}, F_i(\mathbf{f})) \geq \min_{i \in I} s(\underline{a}_{ij}, \check{x}_i^-) \geq \underline{b}_j, \forall\ j \in J$, i.e., $F(\mathbf{f})Ⓢ\underline{A} \geq \underline{\mathbf{b}}$.

On the other hand, the definition of J^i_f implies that $j \in J^{f_j}_f$, and $F_{f_j}(\mathbf{f}) = \min_{j_0 \in J^{f_j}_{\mathbf{f}}}(\overline{a}_{f_j j_0} \omega \overline{b}_{j_0}) \leq \overline{a}_{f_j j}\ \omega\ \overline{b}_j$, $\forall j \in J$. Therefore, $s(\overline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \leq \overline{b}_j$ and consequently $\min_{i \in I} s(\overline{a}_{ij}, F_i(\mathbf{f})) \leq s(\overline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \leq \overline{b}_j, \forall\ j \in J$, that is, $F(\mathbf{f})Ⓢ\overline{A} \leq \overline{\mathbf{b}}$.

Combining the facts that $F(\mathbf{f})Ⓢ\overline{A} \leq \overline{\mathbf{b}}$ and $F(\mathbf{f})Ⓢ\underline{A} \geq \underline{\mathbf{b}}$, we know $F(\mathbf{f}) \in \Sigma_T(A^I, \mathbf{b}^I)$.

(ii) For each $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, according to Lemma 4.3, for each $j \in J$, there exists $i_j \in I$ such that $s(x_{i_j}, \underline{a}_{i_j j}) \leq \underline{b}_j$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = i_j, \forall\ j \in J$. In this way, $\mathbf{f} \in \Lambda$. Now, for any $i \in I$, if $J^i_{\mathbf{f}} = \emptyset$, then $F_i(\mathbf{f}) = 1 \geq x_i$. If $J^i_{\mathbf{f}} \neq \emptyset$, then $F_i(\mathbf{f}) = \min_{j \in J^i_{\mathbf{f}}} (\overline{a}_{ij}\ \omega\ \overline{b}_j) = \min_{\{j|f_j=i\}}(\overline{a}_{ij}\ \omega\ \overline{b}_j) = \min_{\{j|i_j=i\}}(\overline{a}_{ij}\ \omega\ \overline{b}_j) \geq x_{i_j} = x_i$, since $s(x_{i_j}, \overline{a}_{i_j j}) \leq \overline{b}_j$, for $j \in J$. This shows that $F_i(\mathbf{f}) \geq x_i$, $\forall\ i \in I$. Hence $F(\mathbf{f}) \geq \mathbf{x}$. ■

Theorem 4.4 implies that there exists at least one maximal solution of $\Sigma_T(A^I, \mathbf{b}^I)$, provided $\Sigma_T(A^I, \mathbf{b}^I)$ is not empty. Since the cardinality of Λ is finite, the number of maximal solutions must be finite. Denote the set of all maximal solutions by $M(\Sigma_T(A^I, \mathbf{b}^I))$. We have the following result.

Corollary 4.2 $M(\Sigma_T(A^I, \mathbf{b}^I)) \subset F(\Lambda) \subset \Sigma_T(A^I, \mathbf{b}^I)$.

This corollary reduces the search space for all the maximal solutions of $\Sigma_T(A^I, \mathbf{b}^I)$ to a relatively small set $F(\Lambda)$. Moreover, for $\mathbf{x}$, $\mathbf{x}' \in F(\Lambda)$, if $\mathbf{x} \geq \mathbf{x}'$, then $\mathbf{x}' \notin M(\Sigma_T(A^I, \mathbf{b}^I))$. Once we remove all such $\mathbf{x}'$ from $F(\Lambda)$, $M(\Sigma_T(A^I, \mathbf{b}^I))$ is identified.

4.2.2 The algorithm

Based on the above discussion, we propose an algorithm to solve *min-s-norm* interval-valued fuzzy relational equations.

Step 1. Specify the mathematical expression of the operators β and ω associated with the *s-norm.*

Step 2. (Find the potential minimum solution) Compute $\check{\mathbf{x}}^-$ according to (4.3).

Step 3. (Check feasibility) If $\check{\mathbf{x}}^- \circledS\, \overline{A} \not\leq \overline{\mathbf{b}}$ then $\Sigma_T(A^I, \mathbf{b}^I) = \emptyset$, stop. Otherwise continue.

Step 4. Compute I_j for all $j \in J$ according to (4.4).

Step 5. Generate Λ according to (4.5).

Step 6. (Determine potential maximal solutions) For each $\mathbf{f} \in \Lambda$, calculate $F(\mathbf{f})$ according to (4.7). Set $M(\Sigma_T(A^I, \mathbf{b}^I)) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Step 7. (Exclude nonmaximal solutions) For $\mathbf{x}, \mathbf{x}' \in M(\Sigma_T(A^I, \mathbf{b}^I))$ with $\mathbf{x} \geq \mathbf{x}'$, set $M(\Sigma_T(A^I, \mathbf{b}^I)) = M(\Sigma_T(A^I, \mathbf{b}^I)) \backslash \{\mathbf{x}'\}$.

Step 8. (Output) $\Sigma_T(A^I, \mathbf{b}^I) = \{\mathbf{x} \in [\check{\mathbf{x}}^-, \hat{\mathbf{x}}^+] \mid \hat{\mathbf{x}}^+ \in M(\Sigma_T(A^I, \mathbf{b}^I))\}$.

Remark 4.2 *$|I_j| \in [0, m]$ for $j \in J$. Therefore $|\Lambda| \leq m^n$. Since we have to go through all elements of Λ to identify maximal solutions, the proposed algorithm is not polynomially bounded.*

4.2.3 Example

Consider $(x_1\ x_2\ x_3) \circledS \begin{pmatrix} [0.3, 0.7] & [0.5, 0.6] \\ [0.4, 0.9] & [0.1, 0.7] \\ [0.2, 0.5] & [0.8, 1.0] \end{pmatrix} = \begin{pmatrix} [0.3, 0.8] & [0.5, 0.7] \end{pmatrix}.$

We have $\underline{A} = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$, $\overline{A} = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.7 \\ 0.5 & 1.0 \end{pmatrix}$, $\underline{\mathbf{b}} = (0.3\ 0.5)$, $\overline{\mathbf{b}} = (0.8\ 0.7)$.

Example: (*Min-max* composition). For the maximum operation $s(a, x) = \max(a, x)$, we have $a\ \beta\ b = b$ for $a < b$. Hence $\check{\mathbf{x}}^- = (0\ 0.5\ 0.3)$. Since $\check{x}^- \circledS\, \overline{A} \leq \overline{\mathbf{b}}$, $\Sigma_T(A^I, b^I) \neq \emptyset$. Then we have $I_1 = \{1, 3\}$, $I_2 = \{1, 2\}$, and $\Lambda = I_1 \times I_2 = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$. From Table 4.1 we obtain $F(\Lambda) = \{(0.7\ 1\ 1), (0.8\ 1\ 1), (0.7\ 1\ 0.8), (1\ 0.7\ 0.8)\}$. Since $F(\Lambda)_2 > F(\Lambda)_1$ and $F(\Lambda)_2 > F(\Lambda)_3$, we can eliminate $F(\Lambda)_1$ and $F(\Lambda)_3$ from $M(\Sigma_T(A^I, \mathbf{b}^I))$. We get $M(\Sigma_T(A^I, \mathbf{b}^I)) = \{(0.8\ 1\ 1), (1\ 0.7\ 0.8)\}$ and $\Sigma_T(A^I, \mathbf{b}^I) = \{([0, 0.8]\ [0.5, 1]\ [0.3, 1]), ([0, 1]\ [0.5, 0.7]\ [0.3, 0.8])\}$.

$\mathbf{f} = (1,1)$	$J^1_{\mathbf{f}} = \{1,2\}$ $J^2_{\mathbf{f}} = \emptyset$ $J^3_{\mathbf{f}} = \emptyset$	$F_1(\mathbf{f}) = 0.7$ $F_2(\mathbf{f}) = 1$ $F_3(\mathbf{f}) = 1$
$\mathbf{f} = (1,2)$	$J^1_{\mathbf{f}} = \{1\}$ $J^2_{\mathbf{f}} = \{2\}$ $J^3_{\mathbf{f}} = \emptyset$	$F_1(\mathbf{f}) = 0.8$ $F_2(\mathbf{f}) = 0.7$ $F_3(\mathbf{f}) = 1$
$\mathbf{f} = (3,1)$	$J^1_{\mathbf{f}} = \{2\}$ $J^2_{\mathbf{f}} = \emptyset$ $J^3_{\mathbf{f}} = \{1\}$	$F_1(\mathbf{f}) = 0.7$ $F_2(\mathbf{f}) = 1$ $F_3(\mathbf{f}) = 0.8$
$\mathbf{f} = (3,2)$	$J^1_{\mathbf{f}} = \emptyset$ $J^2_{\mathbf{f}} = \{2\}$ $J^3_{\mathbf{f}} = \{2\}$	$F_1(\mathbf{f}) = 1$ $F_2(\mathbf{f}) = 0.7$ $F_3(\mathbf{f}) = 0.8$

Table 4.1: Calculation of maximal solutions for the min-max tolerable solutions

4.3 United solution set

4.3.1 Solution structure

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, recall that the set

$$\{\, \mathbf{x} \in [0,1]^m \mid \text{there exist } A \in A^I \text{ and } \mathbf{b} \in \mathbf{b}^I \text{ such that } \mathbf{x}\circledS A = \mathbf{b}\}, \tag{4.8}$$

denoted by $\Sigma_U(A^I, \mathbf{b}^I)$, is called the "united solution set" of the equation $\mathbf{x}\circledS A^I = \mathbf{b}^I$.

Similar to the case of tolerable solution sets, let us consider the following fuzzy relational inequality system,

$$\begin{cases} \mathbf{x}\circledS \underline{A} \le \overline{\mathbf{b}} \\ \mathbf{x}\circledS \overline{A} \ge \underline{\mathbf{b}} \\ \mathbf{x} \in [0,1]^m \end{cases}, \tag{4.9}$$

where $[\underline{A}, \overline{A}] = A^I$ and $[\underline{\mathbf{b}}, \overline{\mathbf{b}}] = \mathbf{b}^I$ are given.

Theorem 4.5 *Assume a given s-norm is continuous. Then* $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$ *if and only if* $\mathbf{x}$ *is a solution of* (4.9).

Proof. (i) Let $\mathbf{x}$ be a solution of (4.9). Then $\mathbf{x}\circledS\overline{A} \ge \underline{\mathbf{b}}$ and $\mathbf{x}\circledS\underline{A} \le \overline{\mathbf{b}}$. There are four possible cases:

Case 1 $\mathbf{x}\circledS\underline{A} \le \underline{\mathbf{b}} \le \overline{\mathbf{b}} \le \mathbf{x}\circledS\overline{A}$. From Lemma 4.2, there exists $A \in A^I$ such that $\underline{\mathbf{b}} \le \mathbf{x}\circledS A \le \overline{\mathbf{b}}$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}\circledS A = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 2 $\underline{\mathbf{b}} \leq \mathbf{x}Ⓢ\underline{A} \leq \mathbf{x}Ⓢ\overline{A} \leq \overline{\mathbf{b}}$. In this case, $\underline{\mathbf{b}} \leq \mathbf{x}ⓈA \leq \overline{\mathbf{b}}$, for every $A \in A^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 3 $\mathbf{x}Ⓢ\underline{A} \leq \underline{\mathbf{b}} \leq \mathbf{x}Ⓢ\overline{A} \leq \overline{\mathbf{b}}$. If we let $A = \overline{A}$, then $\mathbf{x}ⓈA \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 4 $\underline{\mathbf{b}} \leq \mathbf{x}Ⓢ\underline{A} \leq \overline{\mathbf{b}} \leq \mathbf{x}Ⓢ\overline{A}$. If we let $A = \underline{A}$, then $\mathbf{x}ⓈA \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

(ii) If $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, then there exist $A \in A^I$ and $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$. Hence $\underline{\mathbf{b}} \leq \mathbf{x}ⓈA \leq \overline{\mathbf{b}}$. So we have $\underline{\mathbf{b}} \leq \mathbf{x}ⓈA \leq \mathbf{x}Ⓢ\overline{A}$ and $\mathbf{x}Ⓢ\underline{A} \leq \mathbf{x}ⓈA \leq \overline{\mathbf{b}}$. Furthermore, since $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, $\mathbf{x} \in [0,1]^m$. ■

We now focus on (4.9) to find $\Sigma_U(A^I, \mathbf{b}^I)$.

Theorem 4.6 *Given* $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, *if* $\underline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$ *and* $\overline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$, *then* $\mathbf{x}^I \subset \Sigma_U(A^I, \mathbf{b}^I)$.

Proof. Let $\mathbf{x} \in \mathbf{x}^I$. If $\underline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$, then $\underline{\mathbf{x}}Ⓢ\overline{A} \geq \underline{\mathbf{b}}$. Hence $\mathbf{x}Ⓢ\overline{A} \geq \underline{\mathbf{x}}Ⓢ\overline{A} \geq \underline{\mathbf{b}}$. Similarly, we have $\mathbf{x}Ⓢ\underline{A} \leq \overline{\mathbf{x}}Ⓢ\underline{A} \leq \overline{\mathbf{b}}$. For $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in \Sigma_U(A^I, \mathbf{b}^I)$, we know $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in [0,1]^m$. Consequently, $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$ and $\mathbf{x}^I \subset \Sigma_U(A^I, \mathbf{b}^I)$. ■

Theorem 4.6 suggests further investigation of the structure of $\Sigma_U(A^I, \mathbf{b}^I)$. Moreover, $\Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_T(A^I, \mathbf{b}^I)$ have something in common: each can be obtained by solving a fuzzy relational inequality system. This motivates further study as to whether $\Sigma_U(A^I, \mathbf{b}^I)$ has the same structure as $\Sigma_T(A^I, \mathbf{b}^I)$. In what follows, we will prove that, when the solution set is not empty, $\Sigma_U(A^I, \mathbf{b}^I)$ can be completely determined by one minimum solution and a finite number of maximal solutions.

From Chapter 2, given $\acute{a}$, $\acute{b} \in [0,1]$, the β operator associated with a given *s-norm* is defined by

$$\acute{a}\ \beta\ \acute{b} = \inf\{\acute{x} \in [0,1] \mid s(\acute{x}, \acute{a}) \geq \acute{b}\}.$$

Using the β operator, for a given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we define

$$\check{\mathbf{x}}^- = \left(\max_{j\in J}(\overline{a}_{ij}\ \beta\ \underline{b}_j)\right)_{i\in I}. \tag{4.10}$$

Theorem 4.7 *If* $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, *then* $\check{\mathbf{x}}^-$ *defined by* (4.10) *is the minimum solution of* $\Sigma_U(A^I, \mathbf{b}^I)$.

Proof. (i) $\forall j \in J$, $\min s(\check{x}_i^-, \overline{a}_{ij}) = \min_{i\in I} s(\max_{j_0\in J}(\overline{a}_{ij_0}\ \beta\ \underline{b}_{j_0}), \overline{a}_{ij}) \geq \min_{i\in I} s((\overline{a}_{ij}\ \beta\ \underline{b}_j), \overline{a}_{ij}) \geq \underline{b}_j$. Hence, we have $\check{\mathbf{x}}^- \circledS \overline{A} \geq \underline{\mathbf{b}}$.

(ii) For any $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, we know that $\mathbf{x} \circledS \overline{A} \geq \underline{\mathbf{b}}$ and $\min_{i\in I} s(x_i, \overline{a}_{ij}) \geq \underline{b}_j$, $\forall j \in J$. Therefore for $i \in I$, $s(x_i, \overline{a}_{ij}) \geq \underline{b}_j$ and $x_i \geq \overline{a}_{ij}\ \beta\ \underline{b}_j$. Consequently, $x_i \geq \max_{j\in J}(\overline{a}_{ij}\ \beta\ \underline{b}_j) = \check{x}_i^-$ and $\mathbf{x} \geq \check{\mathbf{x}}^-$.

(iii) When $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, there exists at least one $\mathbf{x}$ that satisfies $\mathbf{x} \circledS \underline{A} \leq \overline{\mathbf{b}}$ and $\mathbf{x} \circledS \overline{A} \geq \underline{\mathbf{b}}$. By (ii) we have $\mathbf{x} \geq \check{\mathbf{x}}^-$. Hence, $\check{\mathbf{x}}^- \circledS \underline{A} \leq \overline{\mathbf{b}}$.

Combining (i), (ii) and (iii), we see that $\check{\mathbf{x}}^-$ is the minimum solution of $\Sigma_U(A^I, \mathbf{b}^I)$. ■

Corollary 4.3 $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$ *if and only if* $\check{\mathbf{x}}^- \circledS\ \underline{A} \leq \overline{\mathbf{b}}$.

Proof. If $\check{\mathbf{x}}^- \circledS \underline{A} \leq \overline{\mathbf{b}}$, from the proof of Theorem 4.7(i), we have $\check{\mathbf{x}}^- \circledS \overline{A} \geq \underline{\mathbf{b}}$, and hence, $\check{\mathbf{x}}^- \in \Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$.

On the other hand, if $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, Theorem 4.7 guarantees that $\check{\mathbf{x}}^- \in \Sigma_U(A^I, \mathbf{b}^I)$. Hence, $\check{\mathbf{x}}^- \circledS \underline{A} \leq \overline{\mathbf{b}}$. ■

Now we can focus on the maximal solutions.

Lemma 4.4 *If* $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, *then, for each* $j \in J$, $\min_{i\in I} s(x_i, \overline{a}_{ij}) \geq \underline{b}_j$, *and there exists* $i_0 \in I$ *such that* $s(x_{i_0}, \underline{a}_{i_0 j}) \leq \overline{b}_j$.

Proof. Since $\mathbf{x} \circledS \overline{A} \geq \underline{\mathbf{b}}$, we have $\min_{i\in I} s(x_i, \overline{a}_{ij}) \geq \underline{b}_j$, for $j \in J$. Since $\mathbf{x} \circledS \underline{A} \leq \overline{\mathbf{b}}$, we have $\min_{i\in I} s(x_i, \underline{a}_{ij}) \leq \overline{b}_j$, for $j \in J$. Hence for each $j \in J$, there exists at least one $i_0 \in I$ such that $s(x_{i_0}, \underline{a}_{i_0 j}) \leq \overline{b}_j$. ■

When $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$, $\check{\mathbf{x}}^-$ can be found by using Equation (4.10). Analogous to (4.4), we define the following index set for each $j \in J$:

$$I_j = \{i \in I \mid s(\check{x}_i^-, \underline{a}_{ij}) \leq \overline{b}_j\}, \tag{4.11}$$

with the Cartesian product

$$\Lambda = I_1 \times I_2 \times \cdots \times I_n. \tag{4.12}$$

Notice that from Lemma 2.2, $s(\check{x}_i^-, \underline{a}_{ij}) \leq \overline{b}_j$ implies that $\check{x}_i^- \leq \underline{a}_{ij}\ \omega\ \overline{b}_j$. Also notice that $s(\check{x}_i^-, \underline{a}_{ij}) \geq \max\{\check{x}_i^-, \underline{a}_{ij}\}$, therefore, $\underline{a}_{ij} \leq \overline{b}_j$ and $\underline{a}_{ij}\ \omega\ \overline{b}_j$ is well defined. Lemma 4.4 guarantees that $I_j \neq \emptyset$, $\forall\ j \in J$, when $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$. Moreover, given $\mathbf{f} = (f_1, f_2, \cdots, f_n) \in I^n$, we say $\mathbf{f} \in \Lambda$ if and only if $f_j \in I_j, \forall\ j \in J$. Now we can study the structure of the solution set $\Sigma_U(A^I, \mathbf{b}^I)$ in terms of the elements in Λ.

Definition 4.2 *Given* $\mathbf{f} \in \Lambda$, *define*

$$J^i_{\mathbf{f}} = \{j \in J \mid f_j = i\},\ \forall\ i \in I \tag{4.13}$$

and $F : \Lambda \to R^m$ *such that* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f}))$ *with*

$$F_i(\boldsymbol{f}) = \begin{cases} \min_{j \in J^i_{\mathbf{f}}} (\underline{a}_{ij}\ \omega\ \overline{b}_j) & , \ if\ \ J^i_{\mathbf{f}} \neq \emptyset \\ 1 & , \ if\ \ J^i_{\mathbf{f}} = \emptyset \end{cases}. \tag{4.14}$$

Remark 4.3 *If* $\mathbf{f} \in \Lambda$ *then* $s(\check{x}^-_{f_j}, \underline{a}_{f_j j}) \leq \overline{b}_j$, $\forall\ j \in J$.

Remark 4.4 *From* (4.13), $j \in J^i_f$, *implies that* $s(\check{x}^-_i, \underline{a}_{ij}) \leq \overline{b}_j$.

The relationship between $\Sigma_U(A^I, \mathbf{b}^I)$ and $F(\Lambda) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$ is given by the following theorem.

Theorem 4.8 *Assume that* $\Sigma_U(A^I, \mathbf{b}^I) \neq \emptyset$.
(i) *If* $\mathbf{f} \in \Lambda$, *then* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f})) \in \Sigma_U(A^I, \mathbf{b}^I)$.
(ii)*For any* $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, *there exists* $\mathbf{f} \in \Lambda$ *such that* $F(\mathbf{f}) \geq \mathbf{x}$.

Proof. (i) From the definition of $J^i_{\mathbf{f}}$, for any $i \in I$, if $J^i_{\mathbf{f}} = \emptyset$, then $F_i(\mathbf{f}) = 1 \geq \check{x}^-_i$. If $J^i_f \neq \emptyset$, then $F_i(\mathbf{f}) = \min_{j \in J^i_{\mathbf{f}}}(\underline{a}_{ij}\ \omega\ \overline{b}_j) = \min_{\{j|f_j=i\}}(\underline{a}_{ij}\ \omega\ \overline{b}_j) \geq \check{x}^-_i$, since $\check{x}^-_i \leq \underline{a}_{ij}\ \omega\ \overline{b}_j$, for $j \in J$. Therefore, $F_i(\mathbf{f}) \geq \check{x}^-_i$, for any $i \in I$. Consequently, $\min_{i \in I} s(\overline{a}_{ij}, F_i(\mathbf{f})) \geq \min_{i \in I} s(\overline{a}_{ij}, \check{x}^-_i) \geq \underline{b}_j, \forall\ j \in J$, i.e., $F(\mathbf{f}) \circledS \overline{A} \geq \underline{\mathbf{b}}$.

On the other hand, the definition of $J^i_{\mathbf{f}}$ implies that $j \in J^{f_j}_{\mathbf{f}}$, and $F_{f_j}(\mathbf{f}) = \min_{j_0 \in J^{f_j}_{\mathbf{f}}} (\underline{a}_{f_j j_0} \omega \underline{b}_{j_0}) \leq \underline{a}_{f_j j}\ \omega\ \overline{b}_j$, $\forall j \in J$. Therefore $s(\underline{a}_{f_j j}, F_{f_j}(\mathbf{f}) \leq \overline{b}_j$ and, consequently $\min_{i \in I} s(\underline{a}_{ij}, F_i(\mathbf{f})) \leq s(\underline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \leq \overline{b}_j$, $\forall\ j \in J$, i.e., $F(\mathbf{f}) \circledS \underline{A} \leq \overline{\mathbf{b}}$.

Combining the facts that $F(\mathbf{f}) \circledS \underline{A} \leq \overline{\mathbf{b}}$ and $F(\mathbf{f}) \circledS \overline{A} \geq \underline{\mathbf{b}}$, we know $F(\mathbf{f}) \in \Sigma_U(A^I, \mathbf{b}^I)$.

(ii) For each $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, according to Lemma 4.4, for each $j \in J$, there exists $i_j \in I$ such that $s(x_{i_j}, \underline{a}_{i_j j}) \leq \overline{b}_j$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = i_j, \forall\ j \in J$. In this way, $f \in \Lambda$.

Now, for any $i \in I$, if $J_f^i = \emptyset$, then $F_i(\mathbf{f}) = 1 \geq x_i$. If $J_f^i \neq \emptyset$, since $s(x_{i_j}, \underline{a}_{i_j j}) \leq \overline{b}_j$ for $j \in J$, then $F_i(\mathbf{f}) = \min_{j \in J_{\mathbf{f}}^i}(\underline{a}_{ij} \ \omega \ \overline{b}_j) = \min_{\{j|f_j=i\}}(\underline{a}_{ij} \ \omega \ \overline{b}_j) = \min_{\{j|i_j=i\}}(\underline{a}_{ij} \ \omega \ \overline{b}_j) \geq x_{i_j} = x_i$. This shows that $F_i(\mathbf{f}) \geq x_i$, $\forall \ i \in I$. Hence $F(\mathbf{f}) \geq \mathbf{x}$. ■

Theorem 4.8 implies that there exists at least one maximal solution of $\Sigma_U(A^I, \mathbf{b}^I)$, provided $\Sigma_U(A^I, \mathbf{b}^I)$ is not empty. Since the cardinality of Λ is finite, the number of maximal solutions must be finite. Denote the set of all maximal solutions by $M(\Sigma_U(A^I, \mathbf{b}^I))$. We have the following result.

Corollary 4.4 $M(\Sigma_U(A^I, \mathbf{b}^I)) \subset F(\Lambda) \subset \Sigma_U(A^I, \mathbf{b}^I)$.

This corollary reduces the search space for all the maximal solutions of $\Sigma_U(A^I, \mathbf{b}^I)$ to a relatively small set $F(\Lambda)$. Moreover, for $\mathbf{x}$, $\mathbf{x}' \in F(\Lambda)$, if $\mathbf{x} \geq \mathbf{x}'$, then $\mathbf{x}' \notin M(\Sigma_U(A^I, \mathbf{b}^I))$. Once we remove all such $\mathbf{x}'$ from $F(\Lambda)$, $M(\Sigma_U(A^I, \mathbf{b}^I))$ is identified.

4.3.2 The algorithm

Based on the above discussion, we propose an algorithm to solve *min-s-norm* interval-valued fuzzy relational equations.

Step 1. Specify the mathematical expression of the operators β and ω associated with the *s-norm*.

Step 2. (Find the potential minimum solution) Compute $\check{\mathbf{x}}^-$ according to (4.10).

Step 3. (Check feasibility) If $\check{\mathbf{x}}^- \circledS \ \underline{A} \not\leq \overline{\mathbf{b}}$ then $\Sigma_U(A^I, \mathbf{b}^I) = \emptyset$, stop. Otherwise continue.

Step 4. Compute I_j for all $j \in J$ according to (4.11).

Step 5. Generate Λ according to (4.12).

Step 6. (Determine potential maximal solutions) For each $\mathbf{f} \in \Lambda$, calculate $F(\mathbf{f})$ according to (4.14). Set $M(\Sigma_U(A^I, \mathbf{b}^I)) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Step 7. (Exclude nonmaximal solutions) For $\mathbf{x}, \mathbf{x}' \in M(\Sigma_U(A^I, \mathbf{b}^I))$ with $\mathbf{x} \geq \mathbf{x}'$, set $M(\Sigma_U(A^I, \mathbf{b}^I)) = M(\Sigma_U(A^I, \mathbf{b}^I)) \backslash \{\mathbf{x}'\}$.

Step 8. (Output) $\Sigma_U(A^I, \mathbf{b}^I) = \{\mathbf{x} \in [\check{\mathbf{x}}^-, \hat{\mathbf{x}}^+] \mid \check{\mathbf{x}}^+ \in M(\Sigma_U(A^I, \mathbf{b}^I))\}$.

Remark 4.5 *$|I_j| \in [0, m]$ for $j \in J$. Therefore $|\Lambda| \leq m^n$. Since we have to go through all elements of Λ to identify maximal solutions, the proposed algorithm is not polynomially bounded.*

4.4 Controllable solution set

4.4.1 Solution structure

Given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, recall that the set

$$\{\ \mathbf{x} \in [0,1]^m \mid \text{for each } \mathbf{b} \in \mathbf{b}^I,\ \text{there exists } A \in A^I \text{ such that } \mathbf{x}\circledS A = \mathbf{b}\} \tag{4.15}$$

denoted by $\Sigma_C(A^I, \mathbf{b}^I)$, is called the "controllable solution set" of the equation $\mathbf{x}\circledS A^I = \mathbf{b}^I$.

Similar to the previous two cases, consider the following fuzzy relational inequality system,

$$\begin{cases} \mathbf{x}\circledS \underline{A} \leq \underline{\mathbf{b}} \\ \mathbf{x}\circledS \overline{A} \geq \overline{\mathbf{b}} \\ \mathbf{x} \in [0,1]^m \end{cases}, \tag{4.16}$$

where $[\underline{A}, \overline{A}] = A^I$ and $[\underline{\mathbf{b}}, \overline{\mathbf{b}}] = \mathbf{b}^I$ are given.

Theorem 4.9 *Assume a given s-norm is continuous. Then $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$ if and only if $\mathbf{x}$ is a solution of* (4.16)*.*

Proof. (i) Let $\mathbf{x}$ be a solution of (4.16). Then $\mathbf{x}\circledS \underline{A} \leq \underline{\mathbf{b}}$, $\mathbf{x}\circledS \overline{A} \geq \overline{\mathbf{b}}$ and $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x}\circledS \underline{A} \leq \underline{\mathbf{b}} \leq \overline{\mathbf{b}} \leq \mathbf{x}\circledS \overline{A}$. The continuity property of composition implies that for any $\mathbf{b} \in \mathbf{b}^I$, there exists $A \in A^I$ such that $\mathbf{x}\circledS A = \mathbf{b}$. Therefore $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$.

(ii) If $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, then for each $\mathbf{b} \in \mathbf{b}^I$, there exists $A \in A^I$ such that $\mathbf{x}\circledS A = \mathbf{b}$. Hence $\underline{\mathbf{b}} \leq \mathbf{x}\circledS A \leq \overline{\mathbf{b}}$, $\underline{\mathbf{b}} \leq \mathbf{x}\circledS A \leq \mathbf{x}\circledS \overline{A}$ and $\mathbf{x}\circledS \underline{A} \leq \mathbf{x}\circledS A \leq \overline{\mathbf{b}}$, i.e., $\mathbf{x}\circledS \overline{A} \geq \underline{\mathbf{b}}$ and $\mathbf{x}\circledS \underline{A} \leq \overline{\mathbf{b}}$. Furthermore, there exist A^1, $A^2 \in A^I$ such that $\mathbf{x}\circledS A^1 = \underline{\mathbf{b}}$ and $\mathbf{x}\circledS A^2 = \overline{\mathbf{b}}$, since $\underline{\mathbf{b}}$, $\overline{\mathbf{b}} \in \mathbf{b}^I$ and $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$. Therefore $\mathbf{x}\circledS \overline{A} \geq \underline{\mathbf{b}} = \mathbf{x}\circledS A^1 \geq \mathbf{x}\circledS \underline{A}$ and $\mathbf{x}\circledS \underline{A} \leq \overline{\mathbf{b}} = \mathbf{x}\circledS A^2 \leq \mathbf{x}\circledS \overline{A}$. That means $\mathbf{x}\circledS \underline{A} \leq \underline{\mathbf{b}}$ and $\mathbf{x}\circledS \overline{A} \geq \overline{\mathbf{b}}$. ■

We now focus on (4.16) to find $\Sigma_C(A^I, \mathbf{b}^I)$.

Theorem 4.10 *Given $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, if $\underline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$ and $\overline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$, then $\mathbf{x}^I \subset \Sigma_C(A^I, \mathbf{b}^I)$.*

Proof. Let $\mathbf{x} \in \mathbf{x}^I$. If $\underline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$, then $\underline{\mathbf{x}} ⓢ \overline{A} \geq \overline{\mathbf{b}}$. Hence $\mathbf{x} ⓢ \overline{A} \geq \underline{\mathbf{x}} ⓢ \overline{A} \geq \overline{\mathbf{b}}$. Similarly, we have $\mathbf{x} ⓢ \underline{A} \leq \overline{\mathbf{x}} ⓢ \underline{A} \leq \underline{\mathbf{b}}$. For $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in \Sigma_C(A^I, \mathbf{b}^I)$, we know $\underline{\mathbf{x}}$, $\overline{\mathbf{x}} \in [0,1]^m$. Consequently, $\mathbf{x} \in [0,1]^m$. Hence $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$ and $\mathbf{x}^I \subset \Sigma_C(A^I, \mathbf{b}^I)$. ■

Note that each of $\Sigma_T(A^I, \mathbf{b}^I)$ and $\Sigma_U(A^I, \mathbf{b}^I)$ can be obtained by solving a corresponding fuzzy relational inequality system. In what follows, we will prove that, when the solution set is not empty, $\Sigma_C(A^I, \mathbf{b}^I)$ can also be completely determined by one minimum solution and a finite number of maximal solutions.

From Chapter 2, given $\acute{a}, \acute{b} \in [0,1]$, the β operator associated with a given *s-norm* is defined by

$$\acute{a} \;\beta\; \acute{b} = \inf\{\acute{x} \in [0,1] \mid t(\acute{x}, \acute{a}) \geq \acute{b}\}.$$

Using the β operator, for a given $A^I \in I([0,1]^{m\times n})$ and $\mathbf{b}^I \in I([0,1]^n)$, we define

$$\check{\mathbf{x}}^- = \left(\max_{j\in J}(\overline{a}_{ij} \;\beta\; \overline{b}_j)\right)_{i\in I}. \tag{4.17}$$

Theorem 4.11 *If $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, then $\check{\mathbf{x}}^-$ defined by* (4.17) *is the minimum solution of $\Sigma_C(A^I, \mathbf{b}^I)$.*

Proof. (i) $\forall j \in J$, $\min_{i\in I} s(\check{x}_i^-, \overline{a}_{ij}) = \min_{i\in I} s(\max_{j_0\in J}(\overline{a}_{ij_0} \;\beta\; \overline{b}_{j_0}), \overline{a}_{ij}) \geq \min_{i\in I} s((\overline{a}_{ij} \;\beta\; \overline{b}_j), \overline{a}_{ij}) \geq \overline{b}_j$, hence we have $\check{\mathbf{x}}^- ⓢ \overline{A} \geq \overline{\mathbf{b}}$.

(ii) For any $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, we know that $\mathbf{x} ⓢ \overline{A} \geq \overline{\mathbf{b}}$ and $\min_{i\in I} s(x_i, \overline{a}_{ij}) \geq \overline{b}_j$, $\forall j \in J$. Therefore, for $i \in I$, $s(x_i, \overline{a}_{ij}) \geq \overline{b}_j$ and $x_i \geq \overline{a}_{ij} \;\beta\; \overline{b}_j$. Consequently, $x_i \geq \max_{j\in J}(\overline{a}_{ij} \;\beta\; \overline{b}_j) = \check{x}_i^-$ and $\mathbf{x} \geq \check{\mathbf{x}}^-$.

(iii) When $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, there exists at least one $\mathbf{x}$ satisfies $\mathbf{x} ⓢ \underline{A} \leq \underline{\mathbf{b}}$ and $\mathbf{x} ⓢ \overline{A} \geq \overline{\mathbf{b}}$. By (ii) we have $\mathbf{x} \geq \check{\mathbf{x}}^-$. Hence, $\check{\mathbf{x}}^- ⓢ \underline{A} \leq \underline{\mathbf{b}}$.

Combining (i), (ii) and (iii), we see that $\check{\mathbf{x}}^-$ is the minimum solution of $\Sigma_C(A^I, \mathbf{b}^I)$. ■

Corollary 4.5 $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$ *if and only if* $\check{\mathbf{x}}^- ⓢ \underline{A} \leq \underline{\mathbf{b}}$.

Proof. If $\check{\mathbf{x}}^- ⓢ \underline{A} \leq \underline{\mathbf{b}}$, from the proof of Theorem 4.11(i), we have $\check{\mathbf{x}}^- ⓢ \overline{A} \geq \overline{\mathbf{b}}$, and, hence, $\check{\mathbf{x}}^- \in \Sigma_C(A^I, \mathbf{b}^I)$ and $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$.

On the other hand, if $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, Theorem 4.11 guarantees that $\check{\mathbf{x}}^- \in \Sigma_C(A^I, \mathbf{b}^I)$. Hence, $\check{\mathbf{x}}^- ⓢ \underline{A} \leq \underline{\mathbf{b}}$. ■

Now we can focus on the maximal solutions.

Lemma 4.5 *If* $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$ *, then, for each* $j \in J$, $\min_{i\in I} s(x_i, \overline{a}_{ij}) \geq \overline{b}_j$*, and there exists* $i_0 \in I$ *such that* $s(x_{i_0}, \underline{a}_{i_0 j}) \leq \underline{b}_j$.

Proof. Since $\mathbf{x} \circledS \overline{A} \geq \overline{\mathbf{b}}$, we have $\min_{i\in I} s(x_i, \overline{a}_{ij}) \geq \overline{b}_j$, for $j \in J$. Since $\mathbf{x} \circledS \underline{A} \leq \underline{\mathbf{b}}$, we have $\min_{i\in I} s(x_i, \underline{a}_{ij}) \leq \underline{b}_j$, for $j \in J$. Hence for each $j \in J$, there exists at least one $i_0 \in I$ such that $s(x_{i_0}, \underline{a}_{i_0 j}) \leq \underline{b}_j$. ■

When $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$, $\check{\mathbf{x}}^-$ can be found by using Equation (4.17). Analogous to (4.4) and (4.11), we define the following index set for each $j \in J$:

$$I_j = \{i \in I \mid s(\check{x}_i^-, \underline{a}_{ij}) \leq \underline{b}_j\}, \tag{4.18}$$

with the Cartesian product

$$\Lambda = I_1 \times I_2 \times \cdots \times I_n. \tag{4.19}$$

Notice that from Lemma 2.2, $s(\check{x}_i^-, \underline{a}_{ij}) \leq \underline{b}_j$ implies that $\check{x}_i^- \leq \underline{a}_{ij} \ \omega \ \underline{b}_j$. Also notice that $s(\check{x}_i^-, \underline{a}_{ij}) \geq \max\{\check{x}_i^-, \underline{a}_{ij}\}$, therefore, $\underline{a}_{ij} \leq \underline{b}_j$ and $\underline{a}_{ij} \ \omega \ \underline{b}_j$ is well defined. Lemma 4.5 guarantees that $I_j \neq \emptyset$, $\forall\ j \in J$, when $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$. Moreover, given $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in I^n$, we say $\mathbf{f} \in \Lambda$ if and only if $f_j \in I_j$, $\forall\ j \in J$. Now we can study the structure of the solution set $\Sigma_C(A^I, \mathbf{b}^I)$ in terms of the elements in Λ.

Definition 4.3 *Given* $\mathbf{f} \in \Lambda$, *define*

$$J_{\mathbf{f}}^i = \{j \in J \mid f_j = i\}, \ \forall\ i \in I, \tag{4.20}$$

and $F : \Lambda \to R^m$ *with* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f}))$ *such that*

$$F_i(\boldsymbol{f}) = \begin{cases} \min_{j\in J_{\mathbf{f}}^i} (\underline{a}_{ij} \ \omega \ \underline{b}_j) & , \text{ if } J_{\mathbf{f}}^i \neq \emptyset \\ 1 & , \text{ if } J_{\mathbf{f}}^i = \emptyset \end{cases}. \tag{4.21}$$

Remark 4.6 *If* $\mathbf{f} \in \Lambda$, *then* $s(\check{x}_{f_j}^-, \underline{a}_{f_j j}) \leq \underline{b}_j$, $\forall\ j \in J$.

Remark 4.7 *From* (4.20), $j \in J_{\mathbf{f}}^i$ *implies that* $s(\check{x}_i^-, \underline{a}_{ij}) \leq \underline{b}_j$.

.The relationship between $\Sigma_C(A^I, \mathbf{b}^I)$ and $F(\Lambda) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$ is given below.

Theorem 4.12 *Assume that* $\Sigma_C(A^I, \mathbf{b}^I) \neq \emptyset$.
(i)*If* $\mathbf{f} \in \Lambda$, *then* $F(\mathbf{f}) = (F_1(\mathbf{f}), F_2(\mathbf{f}), \ldots, F_m(\mathbf{f})) \in \Sigma_C(A^I, \mathbf{b}^I)$.
(ii)*For any* $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, *there exists* $\mathbf{f} \in \Lambda$ *such that* $F(\mathbf{f}) \geq \mathbf{x}$.

Proof. (i) From the definition of $J_{\mathbf{f}}^i$, for any $i \in I$, if $J_{\mathbf{f}}^i = \emptyset$, then $F_i(\mathbf{f}) = 1 \geq \check{x}_i^-$. If $J_{\mathbf{f}}^i \neq \emptyset$, then $F_i(\mathbf{f}) = \min_{j \in J_{\mathbf{f}}^i}(\underline{a}_{ij} \ \omega \ \underline{b}_j) = \min_{\{j|f_j=i\}}(\underline{a}_{ij} \ \omega \ \underline{b}_j) \geq \check{x}_i^-$, since $\check{x}_i^- \leq \underline{a}_{ij} \ \omega \ \underline{b}_j$, for $j \in J$. Therefore, $F_i(\mathbf{f}) \geq \check{x}_i^-$, for any $i \in I$. Consequently, $\min_{i \in I} s(\overline{a}_{ij}, F_i(\mathbf{f})) \geq \min_{i \in I} s(\overline{a}_{ij}, \check{x}_i^-) \geq \overline{b}_j$, $\forall \ j \in J$, i.e., $F(\mathbf{f})Ⓢ\overline{A} \geq \overline{\mathbf{b}}$.

On the other hand, the definition of $J_{\mathbf{f}}^i$ implies that $j \in J_{\mathbf{f}}^{f_j}$, and $F_{f_j}(\mathbf{f}) = \min_{j_0 \in J_{\mathbf{f}}^{f_j}}(\underline{a}_{f_j j_0} \omega \underline{b}_{j_0}) \leq \underline{a}_{f_j j} \ \omega \ \underline{b}_j, \forall j \in J$. Therefore $s(\underline{a}_{f_j j}, F_{f_j}(\mathbf{f}) \leq \underline{b}_j$ and consequently $\min_{i \in I} s(\underline{a}_{ij}, F_i(\mathbf{f})) \leq s(\underline{a}_{f_j j}, F_{f_j}(\mathbf{f})) \leq \underline{b}_j$, $\forall \ j \in J$, i.e., $F(\mathbf{f})Ⓢ\underline{A} \leq \underline{\mathbf{b}}$.

Combining the facts that $F(\mathbf{f})Ⓢ\underline{A} \leq \underline{\mathbf{b}}$ and $F(\mathbf{f})Ⓢ\overline{A} \geq \overline{\mathbf{b}}$, we know $F(\mathbf{f}) \in \Sigma_C(A^I, \mathbf{b}^I)$.

(ii) For each $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, according to Lemma 4.5, for each $j \in J$, there exists $i_j \in I$ such that $s(x_{i_j}, \underline{a}_{i_j j}) \leq \underline{b}_j$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = i_j, \forall \ j \in J$. In this way, $\mathbf{f} \in \Lambda$. Now, for any $i \in I$, if $J_{\mathbf{f}}^i = \emptyset$, then $F_i(\mathbf{f}) = 1 \geq x_i$. If $J_{\mathbf{f}}^i \neq \emptyset$, since $s(x_{i_j}, \underline{a}_{i_j j}) \leq \underline{b}_j$, for $j \in J$, then $F_i(\mathbf{f}) = \min_{j \in J_{\mathbf{f}}^i} (\underline{a}_{ij} \ \omega \ \underline{b}_j) = \min_{\{j|f_j=i\}}(\underline{a}_{ij} \ \omega \ \underline{b}_j) = \min_{\{j|i_j=i\}}(\underline{a}_{ij} \ \omega \ \underline{b}_j) \geq x_{i_j} = x_i$. This shows that $F_i(\mathbf{f}) \geq x_i, \forall \ i \in I$. Hence $F(\mathbf{f}) \geq \mathbf{x}$. ■

Theorem 4.12 implies that there exists at least one maximal solution of $\Sigma_C(A^I, \mathbf{b}^I)$, provided $\Sigma_C(A^I, \mathbf{b}^I)$ is not empty. Since the cardinality of Λ is finite, the number of maximal solutions must be finite. Denote the set of all maximal solutions by $M(\Sigma_C(A^I, \mathbf{b}^I))$. We have the following result.

Corollary 4.6 $M(\Sigma_C(A^I, \mathbf{b}^I)) \subset F(\Lambda) \subset \Sigma_C(A^I, \mathbf{b}^I)$.

This corollary reduces the search space for all the maximal solutions of $\Sigma_C(A^I, \mathbf{b}^I)$ to a relatively small set $F(\Lambda)$. Moreover, for $\mathbf{x}$, $\mathbf{x}' \in F(\Lambda)$, if $\mathbf{x} \geq \mathbf{x}'$, then $\mathbf{x}' \notin M(\Sigma_C(A^I, \mathbf{b}^I))$. Once we remove all such $\mathbf{x}'$ from $F(\Lambda)$, $M(\Sigma_C(A^I, \mathbf{b}^I))$ is identified.

4.4.2 The algorithm

Based on the above discussion, we propose an algorithm to solve *min-s-norm* interval-valued fuzzy relational equations.

Step 1. Specify the mathematical expression of the operators β and ω associated with the *s-norm.*

Step 2. (Find the potential minimum solution) Compute $\check{\mathbf{x}}^-$ according to (4.17).

Step 3. (Check feasibility) If $\check{\mathbf{x}}^- Ⓢ \underline{A} \not\leq \underline{\mathbf{b}}$ then $\Sigma_C(A^I, \mathbf{b}^I) = \emptyset$, stop. Otherwise continue.

Step 4. Compute I_j for all $j \in J$ according to (4.18).

Step 5. Generate Λ according to (4.19).

Step 6. (Determine potential maximal solutions) For each $\mathbf{f} \in \Lambda$, calculate $F(\mathbf{f})$ according to (4.21). Set $M(\Sigma_C(A^I, \mathbf{b}^I)) = \{F(\mathbf{f}) \mid \mathbf{f} \in \Lambda\}$.

Step 7. (Exclude nonmaximal solutions) For $\mathbf{x}, \mathbf{x}' \in M(\Sigma_C(A^I, \mathbf{b}^I))$ with $\mathbf{x} \geq \mathbf{x}'$, set $M(\Sigma_C(A^I, \mathbf{b}^I)) = M(\Sigma_C(A^I, \mathbf{b}^I)) \backslash \{\mathbf{x}'\}$.

Step 8. (Output) $\Sigma_C(A^I, \mathbf{b}^I) = \{ \ \mathbf{x} \in [\check{\mathbf{x}}^-, \hat{\mathbf{x}}^+] \mid \check{\mathbf{x}}^- \in M(\Sigma_C(A^I, \mathbf{b}^I))\}$.

Remark 4.8 *$|I_j| \in [0, m]$ for $j \in J$. Therefore $|\Lambda| \leq m^n$. Since we have to go through all elements of Λ to identify maximal solutions, the proposed algorithm is not polynomially bounded.*

4.5 Relationship among different solution sets

By the definitions of $\Sigma_T(A^I, \mathbf{b}^I)$, $\Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_C(A^I, \mathbf{b}^I)$, it is easy to see that $\Sigma_T(A^I, \mathbf{b}^I) \subset \Sigma_U(A^I, \mathbf{b}^I)$ and $\Sigma_C(A^I, \mathbf{b}^I) \subset \Sigma_U(A^I, \mathbf{b}^I)$. Furthermore, we have the following theorem.

Theorem 4.13 *Given $A^I \in I([0,1]^{m \times n})$ and $\mathbf{b}^I \in I([0,1]^n)$.*
(a) $\Sigma_U(A^I, \mathbf{b}^I) = \{ \ \mathbf{x} \in [0,1]^m \mid [\mathbf{x}Ⓢ\underline{A}, \ \mathbf{x}Ⓢ\overline{A}] \cap \mathbf{b}^I \neq \emptyset\}$;
(b) $\Sigma_T(A^I, \mathbf{b}^I) = \{ \ \mathbf{x} \in [0,1]^m \mid [\mathbf{x}Ⓢ\underline{A}, \ \mathbf{x}Ⓢ\overline{A}] \subset \mathbf{b}^I\}$;
(c) $\Sigma_C(A^I, \mathbf{b}^I) = \{ \ \mathbf{x} \in [0,1]^m \mid [\mathbf{x}Ⓢ\underline{A}, \ \mathbf{x}Ⓢ\overline{A}] \supset \mathbf{b}^I\}$.

Proof. (a) If $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$, then $\mathbf{x}ⓈA \in \mathbf{b}^I$ for some $A \in A^I$, hence $[\mathbf{x}Ⓢ\underline{A}, \ \mathbf{x}Ⓢ\overline{A}] \cap \mathbf{b}^I \neq \emptyset$.

When $\mathbf{x} \in \{ \ \mathbf{x} \in [0,1]^m \mid [\mathbf{x}Ⓢ\underline{A}, \ \mathbf{x}Ⓢ\overline{A}] \cap \mathbf{b}^I \neq \emptyset\}$, there are four possible cases.

Case 1 $\mathbf{x}Ⓢ\underline{A} \leq \underline{\mathbf{b}} \leq \overline{\mathbf{b}} \leq \mathbf{x}Ⓢ\overline{A}$. From Lemma 4.2, there exists $A \in A^I$ such that $\underline{\mathbf{b}} \leq \mathbf{x}ⓈA \leq \overline{\mathbf{b}}$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 2 $\underline{\mathbf{b}} \le \mathbf{x}Ⓢ\underline{A} \le \mathbf{x}Ⓢ\overline{A} \le \overline{\mathbf{b}}$. So for all $A \in A^I$, $\underline{\mathbf{b}} \le \mathbf{x}ⓈA \le \overline{\mathbf{b}}$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 3 $\mathbf{x}Ⓢ\underline{A} \le \underline{\mathbf{b}} \le \mathbf{x}Ⓢ\overline{A} \le \overline{\mathbf{b}}$. If we let $A = \overline{A}$, then $\mathbf{x}ⓈA \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

Case 4 $\underline{\mathbf{b}} \le \mathbf{x}Ⓢ\underline{A} \le \overline{\mathbf{b}} \le \mathbf{x}Ⓢ\overline{A}$. If we let $A = \underline{A}$, then $\mathbf{x}ⓈA \subset \mathbf{b}^I$. Hence there exists $\mathbf{b} \in \mathbf{b}^I$ such that $\mathbf{x}ⓈA = \mathbf{b}$, and consequently, $\mathbf{x} \in \Sigma_U(A^I, \mathbf{b}^I)$.

(b) If $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$, then $\mathbf{x}ⓈA \in \mathbf{b}^I$, for any $A \in A^I$. So $\mathbf{x}Ⓢ\underline{A} \in \mathbf{b}^I$ and $\mathbf{x}Ⓢ\overline{A} \in \mathbf{b}^I$. Hence $[\mathbf{x}Ⓢ\underline{A},\ \mathbf{x}Ⓢ\overline{A}] \subset \mathbf{b}^I$.

If $\mathbf{x} \in \{\ \mathbf{x} \in [0,1]^m \mid [\mathbf{x}Ⓢ\underline{A},\ \mathbf{x}Ⓢ\overline{A}] \subset \mathbf{b}^I\}$, then $\underline{\mathbf{b}} \le \mathbf{x}Ⓢ\underline{A} \le \mathbf{x}Ⓢ\overline{A} \le \overline{\mathbf{b}}$. For any $A \in A^I$, we have $\underline{\mathbf{b}} \le \mathbf{x}Ⓢ\underline{A} \le \mathbf{x}ⓈA \le \mathbf{x}Ⓢ\overline{A} \le \overline{\mathbf{b}}$. Hence $\mathbf{x} \in \Sigma_T(A^I, \mathbf{b}^I)$.

(c) If $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$, then $\mathbf{x}Ⓢ\underline{A} \le \underline{\mathbf{b}}$ and $\mathbf{x}Ⓢ\overline{A} \ge \overline{\mathbf{b}}$. Hence $[\mathbf{x}Ⓢ\underline{A},\ \mathbf{x}Ⓢ\overline{A}] \supset \mathbf{b}^I$.

If $\mathbf{x} \in \{\ \mathbf{x} \in [0,1]^m \mid [\mathbf{x}Ⓢ\underline{A},\ \mathbf{x}Ⓢ\overline{A}] \supset \mathbf{b}^I\}$, then we have $\mathbf{x}Ⓢ\underline{A} \le \underline{\mathbf{b}}$ and $\mathbf{x}Ⓢ\overline{A} \ge \overline{\mathbf{b}}$. Hence $\mathbf{x} \in \Sigma_C(A^I, \mathbf{b}^I)$. ■

If we denote $[\mathbf{x}Ⓢ\underline{A},\ \mathbf{x}Ⓢ\overline{A}]$ by $\mathbf{x}ⓈA^I$, then we have:

(a) $\Sigma_U(A^I, \mathbf{b}^I) = \{\ \mathbf{x} \in [0,1]^m \mid (\mathbf{x}ⓈA^I) \cap \mathbf{b}^I \neq \emptyset\}$.

(b) $\Sigma_T(A^I, \mathbf{b}^I) = \{\ \mathbf{x} \in [0,1]^m \mid \mathbf{x}ⓈA^I \subset \mathbf{b}^I\}$.

(c) $\Sigma_C(A^I, \mathbf{b}^I) = \{\ \mathbf{x} \in [0,1]^m \mid \mathbf{x}ⓈA^I \supset \mathbf{b}^I\}$.

Figures 4-1 to 4-3 are used to depict the meaning of Theorem 4.13. On the right hand side of each figure, the light shaded rectangular represents $\mathbf{b}^I$, and other rectangles represent $\mathbf{x}ⓈA^I$ for a solution $\mathbf{x}$.

As shown in previous sections, the three different solution sets can be found by using similar procedures. We proved that each solution set is the solution set of a fuzzy relational inequality system. Then we proved that each solution set can be represented by a set of interval vectors. If a solution set is not empty, by finding the minimum solution first and then the (finite number of) maximal solutions, we can represent the solution set by a finite set of interval vectors with the minimum solution being the lower bound and one of the maximal solutions being the upper bound. This solution structure is what De Baets called the "crown system" [9], with the stem being the minimum solution and offshoots being the maximal solutions. Since the number of maximal solutions is finite, this crown system is finitely generated.

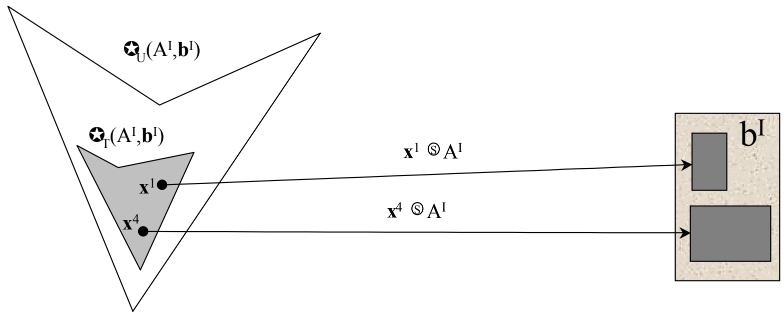

Figure 4-1: Min-s-norm tolerable solution set

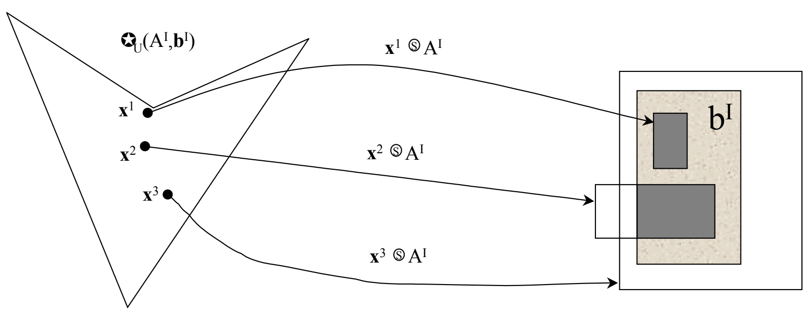

Figure 4-2: Min-s-norm united solution set

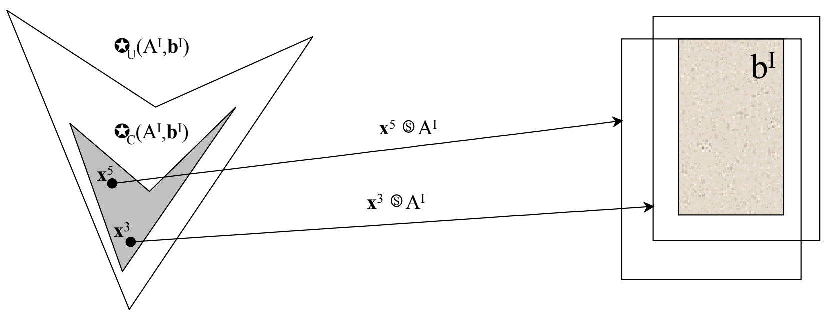

Figure 4-3: Min-s-norm controllable solution set

Chapter 5

Shapley Values of Interval-valued Cooperative Games

5.1 Introduction

The theory of cooperative games can be used to analyze interactive decision-making processes occurring in economics and other social sciences. In a cooperative game, players may form "coalitions" in order to increase their respective payoffs. The reward of each player is determined by the power of that player and with whom that player cooperates. Let $N = \{1, 2, \ldots, n\}$ be a set of players in an n-person game. A nonempty subset of N is called a coalition. Suppose S is a coalition. We denote the "value" of the coalition S by $v(S)$, which is the total payoff that the coalition earns (collectively). $v(\cdot)$ is defined on every subset of N. This function is also called the characteristic function, because N and $v(\cdot)$ uniquely determine a game, $\Gamma = (N, v)$. It is important to know what coalitions should be formed and how the payoffs should be distributed among players of each of these coalitions when a game is played. An answer to these questions is called a "*solution concept*" for cooperative games. Currently several solution concepts are available, for example, von Neumann-Morgenstern solution, core, kernal, nucleolus, and Shapley value; but none of them is perfect [55].

Shapley value [52] is a well known solution concept and has been used in many applications, since it is derived from several reasonable axioms and has a simple probabilistic interpretation. In real applications, since it may not be easy to get a crisp-valued characteristic function, we may wish to consider interval-valued characteristic functions. In such case, we will be interested in finding corresponding Shapley values for interval-valued cooperative games.

5.2 Shapley value

Recall that $N = \{1, 2, \ldots, n\}$ denotes a set of players in an n-person game, while $v(S)$ denotes the value (real numbers) of the coalition S, $S \subseteq N$. Define a superadditive game $\Gamma = (N, v)$ as a game such that

$$v(\emptyset) = 0,$$

$$v(S) \geq v(S \cap T) + v(S - T) \text{ for } S,\, T \subseteq N.$$

The definition implies that $v(S) \geq 0$ for $S \subseteq N$.
A *carrier* of (N, v) is any set $Q \subseteq N$ with

$$v(Q \cap S) = v(S) \text{ for } S \subseteq N.$$

Any set which contains a carrier of (N, v) is also a carrier of (N, v). The players outside of a carrier have no direct influence on the game because they contribute nothing to any coalition. The *sum* of two games $(N,\, v)$ and $(N,\, w)$ is defined as a game $(N,\, v+w)$ such that $(v+w)(S) = v(S) + w(S)$ for every $S \subseteq N$.

Let $\Pi(N)$ denote the set of permutations of N, i.e., the one to one mappings of N to itself. For $\pi \in \Pi(N)$, letting $\pi(S)$ denote the image of S under π, we may define the function πv by

$$\pi v(\pi(S)) = v(S), \text{ for } S \subseteq N.$$

$(N,\, \pi v)$ may be regarded as the "*abstract game*" [52] corresponding to $(N,\, v)$.

Let $x_i[v] \in \Re$ be the payoff (reward) for player i of game (N, v). Then $\mathbf{x}[v] = (x_1[v], x_2[v], ..., x_n[v])$ is called the "*payoff vector*" of game $(N,\, v)$.

Shapley introduced the following three axioms:

Axiom 1: (symmetry) For each π in $\Pi(N)$, $x_{\pi(i)}[\pi v] = x_i[v]$.

Axiom 2: (efficiency) For each carrier Q of (N, v), $\sum_{i \in Q} x_i[v] = v(Q)$.

Axiom 3: (law of aggregation) $\mathbf{x}[v + w] = \mathbf{x}[v] + \mathbf{x}[w]$.

Axiom 1 says that the relabeling of players interchanges the players' rewards. Axiom 2 says that the rewards represent a distribution of the total yield of the game. Axiom 3 states that when two independent games are combined, their rewards must be added player by player [52].

Shapley proved that, there is a unique vector $\mathbf{x}[v]$, later known as the *Shapley value*, that satisfies the above three axioms. The elements of this vector are

$$x_i[v] = \sum_{S \subseteq Q} \tfrac{(s-1)!(q-s)!}{q!}[v(S) - v(S - \{i\})], \text{ for } i = 1, ..., n,$$

where Q is any finite carrier of $(N,\, v)$, $s = |S|$, and $q = |Q|$.

The Shapley value has a simple interpretation [60]. Suppose that the q players arrive in a random order. That is, all possible orders of arrival are of the same probability $1/q!$. When player i arrives, the probability that all players in the set $S - \{i\}$ have already arrived is $(s-1)!(q-s)!/q!$. If player i joins the coalition $S - \{i\}$, player i contributes $v(S) - v(S - \{i\})$ to the coalition S. Therefore, the Shapley value implies that player i's reward should be the expected amount that player i adds to the coalition made up of the players who are present when that player i arrives.

5.3 Comparison (ranking) of interval numbers

Recall that given $\underline{a},\, \overline{a} \in \Re$ and $\underline{a} \leq \overline{a}$, the closed interval $[\underline{a},\, \overline{a}]$ defines an *interval number* $a^I = [\underline{a},\, \overline{a}] \triangleq \{a \in \Re \mid \underline{a} \leq a \leq \overline{a}\}$. Given $a^I = [\underline{a},\, \overline{a}]$ and $b^I = [\underline{b},\, \overline{b}]$, the four arithmetic operations $(+, -, \times, \div)$ are defined as [1]:

$$\begin{aligned}
a^I + b^I &= [\underline{a} + \underline{b},\, \overline{a} + \overline{b}], \\
a^I - b^I &= [\underline{a} - \overline{b},\, \overline{a} - \underline{b}], \\
a^I \times b^I &= [\min\{\underline{a}\underline{b},\, \underline{a}\overline{b},\, \overline{a}\underline{b},\, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b},\, \underline{a}\overline{b},\, \overline{a}\underline{b},\, \overline{a}\overline{b}\}], \\
a^I \div b^I &= [\underline{a},\, \overline{a}] \times [1/\overline{b}, 1/\underline{b}] \text{ given that } 0 \notin [\underline{b},\, \overline{b}].
\end{aligned}$$

For simplicity, $a^I b^I$ is used to represent $a^I \times b^I$, and a^I/b^I to represent $a^I \div b^I$. Any constant value $d \in \Re$ can be treated as $d = [d,\, d]$. Furthermore, $a^I = b^I$ means that $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$.

As in crisp computations, comparison of different numbers plays an important role in interval computations. We need to define $\approx$, $\succeq$ and $\succ$ for interval computations, corresponding to $=$,

$\geq$ and $>$, respectively. There are many different ranking methods available [51], but none of them is universally accepted. A ranking method should have some mechanism to tell "how much greater" when one interval is indicated to be greater than another. Also, intuitively, the ranking method should satisfy the following properties.

1. $a^I \succeq a^I$,
2. if $a^I \succeq b^I$ and $b^I \succeq a^I$, then $a^I \approx b^I$,
3. if $a^I \succeq b^I$ and $a^I \not\approx b^I$, then $a^I \succ b^I$,
4. if $a^I \succeq b^I$ and $b^I \succeq c^I$, then $a^I \succeq c^I$,
5. if $a^I \succeq b^I$ iff $a^I \approx b^I$ or $a^I \succ b^I$,
6. if $a^I \succeq b^I$, then $c^I + a^I \succeq c^I + b^I$ for any c^I,
7. if $a^I \succeq 0$ and $b^I \succeq 0$, then $a^I \times b^I \succeq 0$,
8. if $a^I \succeq 0$ and $b^I \not\succeq 0$, then $a^I \times b^I \not\succ 0$.

We introduce the following new ranking method.

Definition 5.1 *Given* $a^I = [\underline{a}, \overline{a}]$, $b^I = [\underline{b}, \overline{b}]$, *and measure*

$$P(a^I > b^I) = \frac{\max\{0, \bar{a} - \underline{b}\} + \max\{0, \underline{a} - \bar{b}\}}{\max\{0, \bar{a} - \underline{b}\} + \max\{0, \underline{a} - \bar{b}\} + \max\{0, \bar{b} - \underline{a}\} + \max\{0, \underline{b} - \bar{a}\}}, \tag{5.1}$$

we say $a^I \succeq b^I$ *if* $P(a^I > b^I) \geq 0.5$; $a^I \succ b^I$ *if* $P(a^I > b^I) > 0.5$; $a^I \approx b^I$ *if* $P(a^I > b^I) = 0.5$; *and* $a^I \prec b^I$ *if* $P(a^I > b^I) < 0.5$.

Figure 5-1 illustrates the value of $P(a^I > b^I)$ for several examples.

The relationship of a^I and b^I can be divided into three cases, as illustrated in Figure 5-2,

Case 1, $\bar{b} \leq \underline{a}$, and thus $P(a^I > b^I) = 1$.

Case 2, $\bar{a} \leq \underline{b}$, and thus $P(a^I > b^I) = 0$.

Case 3, $\bar{a} > \underline{b}$ and $\underline{a} < \bar{b}$, and thus $P(a^I > b^I) = \frac{\bar{a} - \underline{b}}{\bar{a} - \underline{a} + \bar{b} - \underline{b}}$.

Lemma 5.1 *Given* $a^I = [\underline{a}, \overline{a}]$ *and* $b^I = [\underline{b}, \overline{b}]$,

(i) $a^I \succeq b^I$ *iff* $\underline{a} + \bar{a} \geq \underline{b} + \bar{b}$;

(ii) $a^I \succ b^I$ *iff* $\underline{a} + \bar{a} > \underline{b} + \bar{b}$;

(iii) $a^I \approx b^I$ *iff* $\underline{a} + \bar{a} = \underline{b} + \bar{b}$;

(iv) $a^I \prec b^I$ *iff* $\underline{a} + \bar{a} < \underline{b} + \bar{b}$.

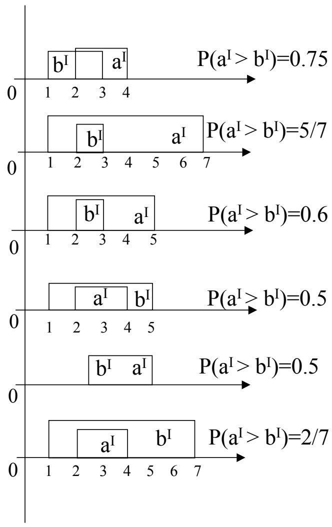

Figure 5-1: Examples of $P(a^I > b^I)$

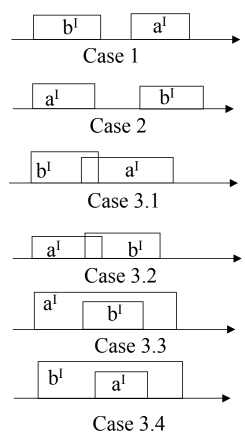

Figure 5-2: 3 cases of interval ranking

Proof. (i) If $a^I \succeq b^I$, then $P(a^I > b^I) \geq 0.5$. There are two possible cases.

Case 1, $\bar{b} \leq \underline{a}$, and thus $\underline{a} + \bar{a} \geq \underline{b} + \bar{b}$.

Case 2, $\bar{a} > \underline{b}$ and $\underline{a} < \bar{b}$, thus $\frac{\bar{a}-\underline{b}}{\bar{a}-\underline{a}+\bar{b}-\underline{b}} \geq 0.5$, and $\underline{a} + \bar{a} \geq \underline{b} + \bar{b}$.

If $\underline{a} + \bar{a} \geq \underline{b} + \bar{b}$, then $\bar{a} \not\leq \underline{b}$. There are two possible cases.

Case 1, $\bar{b} \leq \underline{a}$, and thus $P(a^I > b^I) = 1 \geq 0.5$ and $a^I \succeq b^I$.

Case 2, $\bar{a} > \underline{b}$ and $\underline{a} < \bar{b}$, thus $P(a^I > b^I) = \frac{\bar{a}-\underline{b}}{\bar{a}-\underline{a}+\bar{b}-\underline{b}} \geq 0.5$ and $a^I \succeq b^I$.

(ii) through (iv) can be proved in a similar way. ■

Lemma 5.1 indicates that the ranking method is based on the mid-points of the interval numbers.

Lemma 5.2 *The ranking method defined in 5.1 satisfies properties 1 to 8 listed at the beginning of section.*

Proof. Properties 1-6 can be easily verified by using Lemma 5.1.

For property 7, since $a^I \succeq 0$ and $b^I \succeq 0$, there are 3 cases.

Case 1, $\underline{a} \leq 0$, $\underline{b} \geq 0$. Therefore $a^I \times b^I = [\underline{a}\overline{b}, \overline{a}\overline{b}]$ and since $\underline{a} + \overline{a} \geq 0$, $\underline{a}\overline{b} + \overline{a}\overline{b} \geq 0$.

Case 2, $\underline{b} \leq 0$, $\underline{a} \geq 0$. Similar to case 1.

Case 3, $\underline{a} \leq 0$, $\underline{b} \leq 0$. Therefore $a^I \times b^I = [\min\{\underline{a}\overline{b}, \overline{a}\underline{b}\}, \overline{a}\overline{b}]$ and since $\underline{a} + \overline{a} \geq 0$ and $\underline{b} + \overline{b} \geq 0$, $\min\{\underline{a}\overline{b}, \overline{a}\underline{b}\} + \overline{a}\overline{b} \geq 0$.

For property 8, since $a^I \succeq 0$ and $b^I \not\succeq 0$, hence $\bar{a} > 0$ and $\underline{b} \leq 0$. There are 4 cases.

Case 1, $\underline{a} \geq 0$, $\bar{b} \geq 0$. Therefore $a^I \times b^I = [\overline{a}\underline{b}, \overline{a}\overline{b}]$ and since $\underline{b} + \bar{b} \leq 0$, $\overline{a}\underline{b} + \overline{a}\overline{b} \leq 0$.

Case 2, $\underline{a} \geq 0$, $\bar{b} \leq 0$. Therefore $a^I \times b^I = [\overline{a}\underline{b}, \underline{a}\overline{b}]$. $\overline{a}\underline{b} + \underline{a}\overline{b} \leq \underline{ab} + \underline{a}\overline{b} \leq 0$.

Case 3, $\underline{a} \leq 0$, $\bar{b} \geq 0$. Therefore $a^I \times b^I = [\min\{\overline{a}\underline{b}, \underline{a}\overline{b}\}, \underline{ab}]$ and $\min\{\overline{a}\underline{b}, \underline{a}\overline{b}\} + \underline{ab} \leq \underline{a}\overline{b} + \underline{ab} \leq 0$.

Case 4, $\underline{a} \leq 0$, $\bar{b} \leq 0$. Therefore $a^I \times b^I = [\overline{a}\underline{b}, \underline{ab}]$, and $\overline{a}\underline{b} + \underline{ab} = \underline{b}(\underline{a} + \overline{a}) \leq 0$. ■

Moreover, it is easy to verify the following assertions.

1. $a^I \approx b^I$ iff $a^I - b^I \approx 0$,
2. $a^I + b^I \approx b^I$ iff $a^I \approx 0$,
3. If $a^I \approx b^I \approx c^I$ then $a^I + b^I + c^I \approx 3a^I \approx 3b^I \approx 3c^I$,
4. $\alpha a^I - \beta a^I \approx (\alpha - \beta)a^I$, for α, $\beta \in \Re$,
5. If $a^I \approx b^I$, $b^I \approx c^I$ then $a^I \approx c^I$,
6. $a^I \approx b^I + c^I$ iff $a^I - b^I \approx c^I$,

7. If $a = b$, then $a \approx b$, for $a,\ b \in \Re$,
8. If $a^I = b^I + c^I$, then $b^I \approx a^I - c^I$,
9. If $a^I \succeq b^I + c^I$, then $a^I - b^I \succeq c^I$.

5.4 Shapley values for interval-valued cooperative games

5.4.1 Definitions

Let $N = \{1, 2, \ldots, n\}$ denote a set of players in an n-person game, while $v^I(S)$ denotes the value (interval number) of the coalition S, $S \subseteq N$. Define an interval-valued game $\Gamma = (N, v^I)$ such that

$$v^I(\emptyset) = 0,$$
$$v^I(S) \succeq 0 \text{ for } S \subseteq N.$$

A carrier of (N, v^I) is any set $Q \subseteq N$ with

$$v^I(S) = v^I(Q \cap S) \text{ for } S \subseteq N. \tag{5.2}$$

Any set which contains a carrier of (N, v^I) is also a carrier of (N, v^I). In contrary to Shapley's classic definition, the definition of games in this section extends to non-superadditive cases, but still requires that $v^I(S) \succeq 0$ for $S \subseteq N$. The reason will be explained later.

The sum of two games $(N,\ v^I)$ and $(N,\ w^I)$ is defined as a game $(N,\ v^I + w^I)$ such that $(v^I + w^I)(S) = v^I(S) + w^I(S)$ for every $S \subseteq N$.

Let $\Pi(N)$ denote the set of permutations of N, i.e., the one to one mappings of N to itself. For $\pi \in \Pi(N)$, letting $\pi(S)$ denote the image of S under π, we may define the function πv^I by

$$\pi v^I(\pi(S)) = v^I(S), \text{ for } S \subseteq N.$$

Let $\mathbf{x}^I[v^I]$ be an interval-valued payoff vector for game $(N,\ v^I)$. Similar to Shapley's axioms, we introduce the following three axioms:

Axiom 1: (symmetry) For each π in $\Pi(N)$, $x^I_{\pi(i)}[\pi v^I] = x^I_i[v^I]$.

Axiom 2: (efficiency) For each carrier Q of (N, v^I), $\sum_{i \in Q} x^I_i[v^I] = v^I(Q)$.

Axiom 3: (law of aggregation) $\mathbf{x}^I[v^I + w^I] = \mathbf{x}^I[v^I] + \mathbf{x}^I[w^I]$.

Axiom 1 says that relabeling of players interchanges the players' rewards. Axiom 2 says that the rewards represents a distribution of the total yield of the game. Axiom 3 states that when two independent games are combined, their rewards must be added player by player.

5.4.2 Determination of the interval Shapley value

Notice that interval numbers do not have inverses under addition and multiplication. For example,

$a^I + b^I = c^I$ does not mean that $a^I = c^I - b^I$. An example is $[1,2] - [1,2] = [-1\ 1] \neq 0$. For Axiom 3, this means that $\mathbf{x}^I[v^I + w^I] - \mathbf{x}^I[w^I] \neq \mathbf{x}^I[v^I]$. In order to deal with this problem, we need to define a new "*minus*" operator. Such operator $\ominus$ is defined as $a^I \ominus b^I = [\underline{a} - \underline{b},\, \overline{a} - \overline{b}]$. It can be seen that $a^I + b^I = c^I$ if and only if $a^I = c^I \ominus b^I$. Note that it is possible that

$\underline{a} - \underline{b} \geq \overline{a} - \overline{b}$.

Lemma 5.3 *If $(N,\, v^I)$ is a game, then, for any $c^I \succeq 0$, $(N,\, c^I v^I)$ is also a game.*

Proof. Since $(N,\, v^I)$ is a game, we have $v^I(\emptyset) = 0$ and $v^I(S) \succeq 0$.

From $v^I(\emptyset) = 0$, it follows that $c^I v^I(\emptyset) = 0$.

From $v^I(S) \succeq 0$ and $c^I \succeq 0$, by property 7 of Definition 5.1, $c^I v^I(S) \succeq 0$.

Hence, $(N,\, c^I v^I)$ is a game. ■

Lemma 5.4 *If Q is a finite carrier of $(N,\, v^I)$, then, for $j \notin Q$, $x_j[v^I] = 0$.*

Proof. Consider $j \notin Q$. Both Q and $Q \cup \{j\}$ are carriers of $(N,\, v^I)$ and $v^I(Q) = v^I(Q \cup \{j\})$. By Axiom 2, $\sum_{i \in Q} x_i^I[v^I] = v^I(Q) = v^I(Q \cup \{j\}) = \sum_{i \in Q} x_i^I[v^I] + x_j^I[v^I]$. Hence $x_i[v^I] = \sum_{i \in Q} x_i^I[v^I] \ominus \sum_{i \in Q} x_i^I[v^I] = 0$. ■

For any $R \subseteq N$, $R \neq 0$, define $v_R^I(\cdot)$ as

$$v_R^I(S) = \begin{cases} 1 & \text{if } S \supseteq R \\ 0 & \text{if } S \not\supseteq R \end{cases} . \tag{5.3}$$

Since $(N,\, v_R^I)$ is a game, from Lemma 5.3, for any $c^I \succeq 0$, $(N,\, c^I v_R^I)$ is also a game, and R is a carrier.

From now on, we shall use $r, s, n, \ldots$ to denote the cardinalities $|R|, |S|, |N|, \ldots$, respectively.

Lemma 5.5 *For $c^I \succeq 0$, $0 < r < \infty$, we have*

$$x_i^I[c^I v_R^I] = \begin{cases} c^I/r & \text{if } i \in R \\ 0 & \text{if } i \notin R \end{cases}.$$

Proof. Consider $i,\, j \in R$. Choose $\pi \in \Pi(N)$ so that $\pi(R) = R$ and $\pi(i) = j$. Then we have $\pi(v_R^I) = v_R^I$, $\pi(c^I v_R^I) = c^I v_R^I$, and hence, by Axiom 1, $x_j^I[c^I v_R^I] = x_{\pi(i)}^I[\pi(c^I v_R^I)] = x_i^I[c^I v_R^I]$. By Axiom 2, $c^I = c^I v_R^I(R) = \sum_{j \in R} x_j^I[c^I v_R^I] = r x_i^I[c^I v_R^I]$, for any $i \in R$. For $i \notin R$, from Lemma 5.4, $x_i^I[c^I v_R^I] = 0$. This complete the proof. ■

Definition 5.2 *For $i = 1, 2, ...,$ given $a_i^I = [\underline{a}_i,\, \bar{a}_i]$, define*

$$\sum_i^{\oplus} a_i^I = \sum_{a_i^I \succeq 0} a_i^I \ominus \sum_{a_i^I \not\succeq 0} (-a_i^I).$$

Lemma 5.6 *Given $a_i^I = [\underline{a}_i,\, \bar{a}_i]$, $N = \{1, 2, \ldots, n\}$, $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$, then*

$$\sum_{i \in N}^{\oplus} a_i^I = \sum_{i \in N_1}^{\oplus} a_i^I + \sum_{i \in N_2}^{\oplus} a_i^I.$$

Proof. $\sum_{i \in N}^{\oplus} a_i^I = \sum_{a_i^I \succeq 0, i \in N} a_i^I \ominus \sum_{a_i^I \not\succeq 0, i \in N} (-a_i^I)$

$$= \Big(\sum_{a_i^I \succeq 0, i \in N_1} a_i^I + \sum_{a_i^I \succeq 0, i \in N_2} a_i^I\Big) \ominus \Big(\sum_{a_i^I \not\succeq 0, i \in N_1} (-a_i^I) + \sum_{a_i^I \not\succeq 0, i \in N_1} (-a_i^I)\Big)$$

$$= \Big[\sum_{a_i^I \succeq 0, i \in N_1} \underline{a}_i + \sum_{a_i^I \succeq 0, i \in N_2} \underline{a}_i,\ \sum_{a_i^I \succeq 0, i \in N_1} \bar{a}_i + \sum_{a_i^I \succeq 0, i \in N_2} \bar{a}_i\Big]$$

$$\ominus \Big[\sum_{a_i^I \not\succeq 0, i \in N_1} (-\bar{a}_i) + \sum_{a_i^I \not\succeq 0, i \in N_2} (-\bar{a}_i),\ \sum_{a_i^I \not\succeq 0, i \in N_1} (-\underline{a}_i) + \sum_{a_i^I \not\succeq 0, i \in N_2} (-\underline{a}_i)\Big]$$

$$= \Big[\sum_{a_i^I \succeq 0, i \in N_1} \underline{a}_i + \sum_{a_i^I \succeq 0, i \in N_2} \underline{a}_i - \Big(\sum_{a_i^I \not\succeq 0, i \in N_1} (-\bar{a}_i) + \sum_{a_i^I \not\succeq 0, i \in N_2} (-\bar{a}_i)\Big),$$

$$\sum_{a_i^I \succeq 0, i \in N_1} \bar{a}_i + \sum_{a_i^I \succeq 0, i \in N_2} \bar{a}_i - \Big(\sum_{a_i^I \not\succeq 0, i \in N_1} (-\underline{a}_i) + \sum_{a_i^I \not\succeq 0, i \in N_2} (-\underline{a}_i)\Big)\Big]$$

$$= \Big[\sum_{a_i^I \succeq 0, i \in N_1} \underline{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_1} (-\bar{a}_i) + \sum_{a_i^I \succeq 0, i \in N_2} \underline{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_2} (-\bar{a}_i),$$

$$\sum_{a_i^I \succeq 0, i \in N_1} \bar{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_1} (-\underline{a}_i) + \sum_{a_i^I \succeq 0, i \in N_2} \bar{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_2} (-\underline{a}_i)\Big]$$

$$= \Big[\sum_{a_i^I \succeq 0, i \in N_1} \underline{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_1} (-\bar{a}_i),\ \sum_{a_i^I \succeq 0, i \in N_1} \bar{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_1} (-\underline{a}_i)\Big]$$

$$+ \Big[\sum_{a_i^I \succeq 0, i \in N_2} \underline{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_2} (-\bar{a}_i),\ \sum_{a_i^I \succeq 0, i \in N_2} \bar{a}_i - \sum_{a_i^I \not\succeq 0, i \in N_2} (-\underline{a}_i)\Big]$$

$= (\sum_{a_i^I \succeq 0, i \in N_1} a_i^I \ominus \sum_{a_i^I \not\succeq 0, i \in N_1} (-a_i^I)) + (\sum_{a_i^I \succeq 0, i \in N_2} a_i^I \ominus \sum_{a_i^I \not\succeq 0, i \in N_2} (-a_i^I))$

$= \overset{\oplus}{\sum_{i \in N_1}} a_i^I + \overset{\oplus}{\sum_{i \in N_2}} a_i^I$. ■

If $b_i^I = [\underline{b}_i, \overline{b}_i] \succeq 0$, $i \in N$, by definition and property 8 of Definition 5.1, $\overset{\oplus}{\sum_i} a_i^I b_i^I = \sum_{a_i^I \succeq 0} a_i^I b_i^I \ominus \sum_{a_i^I \not\succeq 0} (-a_i^I) b_i^I$. Also by definition, $a^I + \overset{\oplus}{\sum}(-a^I) = 0$.

Lemma 5.7 *Given* $a^I = [\underline{a}, \overline{a}] \succeq 0$, *for* $\alpha_i \in \Re$, $\overset{\oplus}{\sum_i}(\alpha_i a^I) = (\sum_i \alpha_i) a^I$.

Proof. $\overset{\oplus}{\sum_i}(\alpha_i a^I) = \overset{\oplus}{\sum_{\alpha_i \succeq 0}} \alpha_i a^I + \overset{\oplus}{\sum_{\alpha_i \not\succeq 0}} \alpha_i a^I$ (by Lemma 5.6)

$= \sum_{\alpha_i \succeq 0} \alpha_i a^I \ominus \sum_{\alpha_i \not\succeq 0} (-\alpha_i) a^I$

$= [(\sum_{\alpha_i \succeq 0} \alpha_i - \sum_{\alpha_i \not\succeq 0} (-\alpha_i))\underline{a}, (\sum_{\alpha_i \succeq 0} \alpha_i - \sum_{\alpha_i \not\succeq 0} (-\alpha_i))\bar{a}]$

$= [\sum \alpha_i \underline{a}, \sum \alpha_i \bar{a}]$

$= (\sum_i \alpha_i) a^I$. ■

Lemma 5.8 *If* (N, v^I) *has finite carriers, then* (N, v^I) *is a linear combination of games* (N, v_R^I)*:*

$$v^I(T) = \overset{\oplus}{\sum_{\substack{R \subseteq Q \\ R \neq \emptyset}}} c_R(v^I) v_R^I(T), \text{ for all } T \subseteq N, \tag{5.4}$$

where Q *is any finite carrier of* (N, v^I)*. The coefficients* $c_R(v^I)$ *are independent of* Q*, and are given by*

$$c_R(v^I) = \overset{\oplus}{\sum_{S \subseteq R}} (-1)^{r-s} v^I(S), 0 < r < \infty. \tag{5.5}$$

Proof. We need to verify that for all $T \subseteq N$ and any finite carrier Q of (N, v^I),

$$v^I(T) = \overset{\oplus}{\sum_{\substack{R \subseteq Q \\ R \neq \emptyset}}} c_R(v^I) v_R^I(T). \tag{5.6}$$

If $T \subseteq Q$, then

$\overset{\oplus}{\sum_{\substack{R \subseteq Q \\ R \neq \emptyset}}} c_R(v^I) v_R^I(T)$

$$= \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq Q\\ R\neq\emptyset}} \sideset{}{^{\oplus}}\sum_{S\subseteq R} (-1)^{r-s} v^I(S) v_R^I(T)$$

$$= \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq Q\\ R\neq\emptyset\\ T\supseteq R}} \sideset{}{^{\oplus}}\sum_{S\subseteq R} (-1)^{r-s} v^I(S) v_R^I(T) + \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq Q\\ R\neq\emptyset\\ T\not\supseteq R}} \sideset{}{^{\oplus}}\sum_{S\subseteq R} (-1)^{r-s} v^I(S) v_R^I(T) \quad \text{(by Lemma 5.6)}$$

$$= \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq Q\\ R\neq\emptyset\\ T\supseteq R}} \sideset{}{^{\oplus}}\sum_{S\subseteq R} (-1)^{r-s} v^I(S) + 0$$

$$= \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq T\\ R\neq\emptyset}} \sideset{}{^{\oplus}}\sum_{S\subseteq R} (-1)^{r-s} v^I(S) \quad \text{(since } T\subseteq Q)$$

$$= \sideset{}{^{\oplus}}\sum_{S\subseteq T} \Big(\sideset{}{^{\oplus}}\sum_{\substack{R\supseteq S\\ R\neq\emptyset}} (-1)^{r-s} v^I(S)\Big)$$

$$= \sideset{}{^{\oplus}}\sum_{S\subseteq T} \left[\sum_{r=s}^{t} (-1)^{r-s}\binom{t-s}{r-s}\right] v^I(S) \qquad \text{(by Lemma 5.7)}$$

$$= \sideset{}{^{\oplus}}\sum_{S\subseteq T} \left[\sum_{r=s}^{t} (-1)^{r-s}\binom{t-s}{r-s}\right] v^I(S) \quad \text{(denote } m = r-s)$$

$$= \sideset{}{^{\oplus}}\sum_{S\subseteq T} \left[\sum_{m=0}^{t-s} (-1)^{m}\binom{t-s}{m}\right] v^I(S)$$

$$= \sideset{}{^{\oplus}}\sum_{\substack{S\subseteq T\\ S=T}} \left[\sum_{m=0}^{t-s} (-1)^{m}\binom{t-s}{m}\right] v^I(S) + \sideset{}{^{\oplus}}\sum_{\substack{S\subseteq T\\ S\neq T}} \left[\sum_{m=0}^{t-s} (-1)^{m}\binom{t-s}{m}\right] v^I(S)$$

$$= \sideset{}{^{\oplus}}\sum_{\substack{S\subseteq T\\ S=T}} v^I(S) + \sideset{}{^{\oplus}}\sum_{\substack{S\subseteq T\\ S\neq T}} (1-1)^{t-s} v^I(S)$$

$$= v^I(T) \quad \text{(since } v^I(T) \succeq 0).$$

In general, by (5.2),

$$v^I(T) = v^I(Q\cap T) = \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq Q\\ R\neq\emptyset}} c_R(v^I) v_R^I(Q\cap T) = \sideset{}{^{\oplus}}\sum_{\substack{R\subseteq Q\\ R\neq\emptyset}} c_R(v^I) v_R^I(T). \ \blacksquare$$

Corollary 5.1 *If (N, w^I), (N, y^I) and $(N, w^I \ominus y^I)$ are all games and Axiom 3 is true, then* $\mathbf{x}^I[w^I \ominus y^I] = \mathbf{x}^I[w^I] \ominus \mathbf{x}^I[y^I]$.

Proof. Let $v^I = w^I \ominus y^I$. Then $w^I = v^I + y^I$. From Axiom 3, $\mathbf{x}^I[w^I] = \mathbf{x}^I[v^I] + \mathbf{x}^I[y^I]$. Therefore $\mathbf{x}^I[v^I] = \mathbf{x}^I[w^I] \ominus \mathbf{x}^I[y^I]$. ■

For simplicity, we denote "R contains i" by $R \ni i$. From Lemma 5.3, if $c_R(v^I) \succeq 0$ then $(N, c_R(v^I)v_R^I)$ is a game; otherwise, $(N, (-c_R(v^I))v_R^I)$ is a game. We can therefore apply Lemma 5.5 to Lemma 5.8 and obtain the formula:

$$
\begin{aligned}
x_i^I[v^I] &= x_i^I[\sum_{R\subseteq Q}^{\oplus} c_R(v^I)v_R^I] \quad (\text{for } i\in Q)\\
&= x_i^I[\sum_{\substack{R\subseteq Q\\ c_R(v^I)\succeq 0}} c_R(v^I)v_R^I \ominus \sum_{\substack{R\subseteq Q\\ c_R(v^I)\not\succeq 0}} (-c_R(v^I))v_R^I]\\
&= x_i^I[\sum_{\substack{R\subseteq Q\\ c_R(v^I)\succeq 0}} c_R(v^I)v_R^I] \ominus x_i^I[\sum_{\substack{R\subseteq Q\\ c_R(v^I)\not\succeq 0}} (-c_R(v^I))v_R^I] \quad (\text{by Corollary 5.1})\\
&= \sum_{\substack{R\subseteq Q\\ c_R(v^I)\succeq 0}} x_i^I[c_R(v^I)v_R^I] \ominus \sum_{\substack{R\subseteq Q\\ c_R(v^I)\not\succeq 0}} x_i^I[(-c_R(v^I))v_R^I] \quad (\text{by Axiom 3})\\
&= \sum_{\substack{R\subseteq Q,\ R\ni i\\ c_R(v^I)\succeq 0}} x_i^I[c_R(v^I)v_R^I] + \sum_{\substack{R\subseteq Q,\ R\not\ni i\\ c_R(v^I)\succeq 0}} x_i^I[c_R(v^I)v_R^I]\\
&\ominus \sum_{\substack{R\subseteq Q,\ R\ni i\\ c_R(v^I)\not\succeq 0}} x_i^I[(-c_R(v^I))v_R^I] \ominus \sum_{\substack{R\subseteq Q,\ R\not\ni i\\ c_R(v^I)\not\succeq 0}} x_i^I[(-c_R(v^I))v_R^I]\\
&= \sum_{\substack{R\subseteq Q,\ R\ni i\\ c_R(v^I)\succeq 0}} x_i^I[c_R(v^I)v_R^I] + 0 \ominus \sum_{\substack{R\subseteq Q,\ R\ni i\\ c_R(v^I)\not\succeq 0}} x_i^I[(-c_R(v^I))v_R^I] \ominus 0\\
&= \sum_{\substack{R\subseteq Q,\ R\ni i\\ c_R(v^I)\succeq 0}} c_R(v^I)/r \ominus \sum_{\substack{R\subseteq Q,\ R\ni i\\ c_R(v^I)\not\succeq 0}} (-c_R(v^I))/r\\
&= \sum_{R\subseteq Q,\ R\ni i}^{\oplus} c_R(v^I)/r.
\end{aligned}
$$

So we have

$$
x_i^I[v^I] \quad = \sum_{R\subseteq Q,\ R\ni i}^{\oplus} c_R(v^I)/r \quad (\text{for } i\in Q). \tag{5.7}
$$

The following Lemma is useful in simplifying equation (5.7).

Lemma 5.9 $\sum_{j=0}^{k}(-1)^j\frac{\binom{k}{j}}{a+j} = \frac{k!}{a(a+1)\ldots(a+k)}$.

Furthermore, we have

$$
\begin{aligned}
&\sum_{r=s}^{q}(-1)^{r-s}\binom{q-s}{r-s}/r \quad (\text{let } j=r-s)\\
&= \sum_{j=0}^{q-s}(-1)^j\frac{\binom{q-s}{j}}{s+j}\\
&= \frac{(q-s)!}{s(s+1)\ldots(s+q-s)}\\
&= \frac{(s-1)!(q-s)!}{q!}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{r=s+1}^{q}(-1)^{r-s}\binom{q-s-1}{r-s-1}/r\\
&= -\frac{s!(q-s-1)!}{q!}.
\end{aligned}
$$

Now we can substitute into (5.5) and simplify the result.

$$
\begin{aligned}
&\sum_{R\subseteq Q,\ R\ni i}^{\oplus} c_R(v^I)/r \\
&= \sum_{\substack{R\subseteq Q\\ R\ni i}}^{\oplus} \sum_{S\subseteq R}^{\oplus} (-1)^{r-s} v^I(S)/r \\
&= \sum_{S\subseteq Q}^{\oplus} \Big(\sum_{\substack{R\supseteq S\\ R\ni i}}^{\oplus} (-1)^{r-s}/r \Big) v^I(S) \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}}^{\oplus} \Big(\sum_{\substack{R\supseteq S\\ R\ni i}}^{\oplus} (-1)^{r-s}/r \Big) v^I(S) + \sum_{\substack{S\subseteq Q\\ S\not\ni i}}^{\oplus} \Big(\sum_{\substack{R\supseteq S\\ R\ni i}}^{\oplus} (-1)^{r-s}/r \Big) v^I(S) \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}}^{\oplus} \sum_{r=s}^{q} (-1)^{r-s} \binom{q-s}{r-s} v^I(S)/r + \sum_{\substack{S\subseteq Q\\ S\not\ni i}}^{\oplus} \sum_{r=s+1}^{q} (-1)^{r-s} \binom{q-s-1}{r-s-1} v^I(S)/r \quad \text{(see Figure 5-3)} \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}}^{\oplus} \frac{(s-1)!(q-s)!}{q!} v^I(S) + \sum_{\substack{S\subseteq Q\\ S\not\ni i}}^{\oplus} \frac{-s!(q-s-1)!}{q!} v^I(S) \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(S) \ominus \sum_{\substack{S\subseteq Q\\ S\not\ni i}} \frac{s!(q-s-1)!}{q!} v^I(S) \ \text{(Since } v^I(S)\succeq 0, \\
&\qquad\qquad \frac{(s-1)!(q-s)!}{q!} \geq 0,\ \frac{s!(q-s-1)!}{q!} \geq 0).
\end{aligned}
$$

That is,

$$
x_i^I[v^I] \quad = \sum_{\substack{S\subseteq Q\\ S\ni i}} v^I(S) \frac{(s-1)!(q-s)!}{q!} \ominus \sum_{\substack{S\subseteq Q\\ S\not\ni i}} v^I(S) \frac{s!(q-s-1)!}{q!} \tag{5.8}
$$

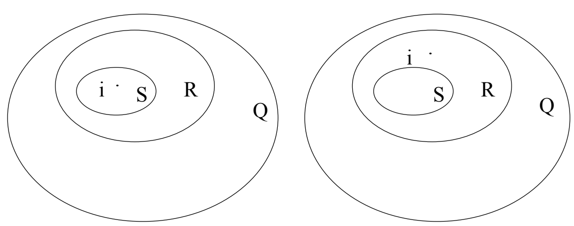

Figure 5-3: Set relation illustration

Introducing the quantities

$$p_q(s) = (s-1)!(q-s)!/q!, \tag{5.9}$$

we have the following theorem.

Theorem 5.1 *(Shapley Value) For games with finite carriers, there exists a unique payoff function* $\mathbf{x}^I[v]$ *satisfying Axioms 1-3, with elements*

$$x_i^I[v^I] = \sum_{S\subseteq Q} \frac{(s-1)!(q-s)!}{q!}[v^I(S) \ominus v^I(S-\{i\})], \text{ for } i = 1, ..., n, \tag{5.10}$$

where Q *is any finite carrier of* (N, v^I).

Proof. For $i \in Q$,

$$\begin{aligned}
&\sum_{S\subseteq Q} \tfrac{(s-1)!(q-s)!}{q!}[v^I(S) \ominus v^I(S-\{i\})] \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)[v^I(S) \ominus v^I(S-\{i\})] + \sum_{\substack{S\subseteq Q\\ S\not\ni i}} p_q(s)[v^I(S) \ominus v^I(S-\{i\})] \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)[v^I(S) \ominus v^I(S-\{i\})] + \sum_{\substack{S\subseteq Q\\ S\not\ni i}} p_q(s)[v^I(S) \ominus v^I(S)] \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)v^I(S) \ominus \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)v^I(S-\{i\}) + 0 \quad (\text{denote } S' = S-\{i\}) \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)v^I(S) \ominus \sum_{\substack{S'\cup\{i\}\subseteq Q\\ S'\not\ni i}} p_q(s'+1)v^I(S'\cup\{i\}-\{i\}) \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)v^I(S) \ominus \sum_{\substack{S'\subseteq Q\\ S'\not\ni i}} p_q(s'+1)v^I(S') \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} p_q(s)v^I(S) \ominus \sum_{\substack{S\subseteq Q\\ S\not\ni i}} p_q(s+1)v^I(S) \\
&= \sum_{\substack{S\subseteq Q\\ S\ni i}} v^I(S)\tfrac{(s-1)!(q-s)!}{q!} \ominus \sum_{\substack{S\subseteq Q\\ S\not\ni i}} v^I(S)\tfrac{s!(q-s-1)!}{q!}.
\end{aligned}$$

For $i \notin Q$, $x_i^I[v^I] = 0$, and $v^I(S) = v^I(S-\{i\})$, for $S \subseteq Q$. Therefore $\sum_{S\subseteq Q} \frac{(s-1)!(q-s)!}{q!}[v^I(S) \ominus v^I(S-\{i\})] = 0$, and (5.10) follows.

Since (5.10) does not depend on a particular finite carrier Q, $\mathbf{x}^I[v]$ is therefore well defined.

It is easy to verify that (5.10) satisfies Axiom 3. Let (N, v^I), (N, w^I), and $(N, v^I + w^I)$ be games. Then

$$x_i^I[v^I + w^I] = \sum_{S \subseteq Q} \frac{(s-1)!(q-s)!}{q!}[(v^I + w^I)(S) \ominus (v^I + w^I)(S - \{i\})]$$
$$= \sum_{S \subseteq Q} \frac{(s-1)!(q-s)!}{q!}[(v^I(S) + w^I(S)) \ominus (v^I(S - \{i\}) + w^I(S - \{i\}))]$$
$$= \sum_{S \subseteq Q} \frac{(s-1)!(q-s)!}{q!}[(v^I(S) \ominus v^I(S - \{i\})) + (w^I(S) \ominus w^I(S - \{i\}))]$$
$$= \sum_{S \subseteq Q} \frac{(s-1)!(q-s)!}{q!}[v^I(S) \ominus v^I(S - \{i\})] + \sum_{S \subseteq Q} \frac{(s-1)!(q-s)!}{q!}[w^I(S) \ominus w^I(S - \{i\})]$$
$$= x_i^I[v^I] + x_i^I[w^I], \text{ for } i = 1, ..., n.$$

Lemma 5.8 says that $(N,\ v^I)$ is a linear combination of games $(N, c_R(v^I)v_R^I)$, when $c_R(v^I) \succeq 0$, and $(N, -c_R(v^I)v_R^I)$, when $c_R(v^I) \not\succeq 0$. From Lemma 5.5, both $(N, c_R(v^I)v_R^I)$ and $(N, -c_R(v^I)v_R^I)$ have unique payoff functions which satisfy Axiom 1 and Axiom 2. Therefore, if there exists a payoff function of $(N,\ v^I)$ which satisfies Axiom 3, this function must be unique and also satisfy Axiom 1 and Axiom 2. As just proved, (5.10) satisfies Axiom 3, hence (5.10) is a unique payoff function satisfying Axioms 1-3. ■

5.4.3 Properties of the value

Corollary 5.2 *Given a superadditive game* (N, v^I), *i.e.,* $v^I(S) \succeq v^I(S \cap T) + v^I(S - T)$, *for* S, $T \subseteq N$, *for all* $i \in N$,

$$x_i^I[v^I] \succeq v^I(\{i\}),$$

with equality holding if and only if

$$v^I(S) = v^I(S - \{i\}) + v^I(\{i\}), \text{ for all } S \ni i.$$

Proof.

Case 1: For $i \in N$ and $i \notin Q$, from $v^I(S) \succeq v^I(S - \{i\}) + v^I(\{i\})$ and $v^I(S) = v^I(S - \{i\})$, we have $v^I(\{i\}) = 0$. Since $x_i^I[v^I] = 0$, we have $x_i^I[v^I] = v^I(\{i\})$.

Case 2: For $i \in N$ and $i \in Q$, from $v^I(S) \succeq v^I(S - \{i\}) + v^I(\{i\})$ we have $v^I(S) \ominus v^I(S - \{i\}) \succeq v^I(\{i\})$. Thus,

$$x_i^I[v^I] = \sum_{\substack{S \subseteq Q \\ i \in Q}} \frac{(s-1)!(q-s)!}{q!}[v^I(S) \ominus v^I(S - \{i\})]$$
$$= \sum_{\substack{S \subseteq Q \\ i \in S}} \frac{(s-1)!(q-s)!}{q!}[v^I(S) \ominus v^I(S - \{i\})]$$

$\succeq \sum\limits_{\substack{S \subseteq Q \\ S \ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(\{i\})$
$= \sum\limits_{s=1}^{q} \binom{q-1}{s-1} \frac{(s-1)!(q-s)!}{q!} v^I(\{i\})$
$= \sum\limits_{s=1}^{q} \frac{1}{q} v^I(\{i\})$
$= v^I(\{i\})$, with equality holding if and only if $v^I(S) = v^I(S - \{i\}) + v^I(\{i\})$, for all $S \ni i$.

This completes the proof. ■

Following Axiom 3, we have the following two results.

Corollary 5.3 *If (N, v^I) is decomposable, i.e., games (N, v_1^I), $(N, v_2^I), ..., (N, v_p^I)$ having pair-wise disjunct carriers $Q_1, Q_2, ..., Q_p$ satisfying $v^I = \sum\limits_{k=1}^{p} v_k^I$, then, for each $k = 1, 2, ..., p$ and $i \in Q_k$,*

$$x_i^I[v^I] = x_i[v_k^I].$$

Corollary 5.4 *If (N, v^I) and (N, w^I) are strategically equivalent games, i.e., there exist $a^I \succeq 0$ and $\beta_i^I \succeq 0$ for $i \in N$ such that*

$$v^I(S) = a^I w^I(S) + \sum_{i \in S} \beta_i^I, \text{ for all } S \subset N, \tag{5.11}$$

then

$$x_i^I[v^I] = a^I x_i^I[w^I] + \beta_i^I.$$

Furthermore, we have a formula for the payoff function.

Corollary 5.5 *If (N, v^I) is a constant-sum game, i.e., $v^I(S) + v^I(N - S) = v^I(N)$, for all $S \subseteq N$, then the payoff function is given by the formula*

$$x_i^I[v^I] = 2 \sum_{\substack{S \subseteq Q \\ S \ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(S) \ominus v^I(Q), \text{ for } i = 1, ..., q,$$

where Q is any finite carrier of (N, v^I).

Proof. For $i = 1, ..., q$,
$x_i^I[v^I] = \sum\limits_{\substack{S \subseteq Q \\ S \ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(S) \ominus \sum\limits_{\substack{T \subseteq Q \\ T \not\ni i}} \frac{t!(q-t-1)!}{q!} v^I(T)$ (let $T' = Q - T$)

$$= \sum_{\substack{S \subseteq Q \\ S \ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(S) \ominus \sum_{\substack{T' \subseteq Q \\ T' \ni i}} \frac{(q-t')!(t'-1)!}{q!} v^I(Q - T')$$
$$= \sum_{\substack{S \subseteq Q \\ S \ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(S) \ominus \sum_{\substack{S \subseteq Q \\ S \ni i}} \frac{(q-s)!(s-1)!}{q!} [v^I(Q) \ominus v^I(S)]$$
$$= 2 \sum_{\substack{S \subseteq Q \\ S \ni i}} \frac{(s-1)!(q-s)!}{q!} v^I(S) \ominus v^I(Q). \quad \blacksquare$$

The following two examples illustrate Shapley values for superadditive and non-superadditive games.

Example 1 *Consider a superadditive game* $\Gamma = (N, v^I)$ *with* $n = 3$ *and* $v^I(\cdot)$ *being defined by*

S	$v^I(S)$	S	$v^I(S)$
$\emptyset$	0	$\{1,2\}$	$[20, 40]$
$\{1\}$	0	$\{1,3\}$	$[20, 40]$
$\{2\}$	0	$\{2,3\}$	$[70, 90]$
$\{3\}$	0	$\{1,2,3\}$	$[90, 110]$

In this case, $x_1^I[v^I] = \frac{(2-1)!(3-2)!}{3!}(v^I(\{1,2\}) \ominus v^I(\{2\}))$
$$+ \frac{(2-1)!(3-2)!}{3!}(v^I(\{1,3\}) \ominus v^I(\{3\}))$$
$$+ \frac{(3-1)!(3-3)!}{3!}(v^I(\{1,2,3\}) \ominus v^I(\{2,3\}))$$
$$= \frac{1}{3!}[20, 40] + \frac{1}{3!}[20, 40] + \frac{1}{3}([90, 110] \ominus [70, 90])$$
$$= \frac{1}{3}[20, 40] + \frac{1}{3}([20, 20]$$
$$= \frac{1}{3}[40, 60] = [13.33, 20].$$

Similarly,
$$x_2^I[v^I] = \frac{1}{3!}[20, 40] + \frac{1}{3!}[70, 90] + \frac{1}{3}([90, 110] \ominus [20, 40])$$
$$= \frac{1}{6}[230, 270] = [38.33, 45]$$
$$x_3^I[v^I] = \frac{1}{3!}[20, 40] + \frac{1}{3!}[70, 90] + \frac{1}{3}([90, 110] \ominus [20, 40])$$
$$= \frac{1}{6}[230, 270] = [38.33, 45].$$

We can see that $x_1^I[v^I] + x_2^I[v^I] + x_3^I[v^I] = [90, 110] = v^I(\{1, 2, 3\})$.

Example 2 *Consider a non-superadditive game* $\Gamma = (N, v^I)$ *with* $n = 3$ *and* $v^I(\cdot)$ *being defined* *by*

S	$v^I(S)$	S	$v^I(S)$
$\emptyset$	0	$\{1,2\}$	$[20,40]$
$\{1\}$	0	$\{1,3\}$	$[20,40]$
$\{2\}$	0	$\{2,3\}$	$[70,90]$
$\{3\}$	0	$\{1,2,3\}$	$[50,60]$

In this case, $x_1^I[v^I] = \frac{(2-1)!(3-2)!}{3!}(v^I(\{1,2\}) \ominus v^I(\{2\}))$

$$+ \frac{(2-1)!(3-2)!}{3!}(v^I(\{1,3\}) \ominus v^I(\{3\}))$$

$$+\frac{(3-1)!(3-3)!}{3!}(v^I(\{1,2,3\}) \ominus v^I(\{2,3\}))$$

$$= \frac{1}{3!}[20,40] + \frac{1}{3!}[20,40] + \frac{1}{3}([50,60] \ominus [70,90])$$

$$= \frac{1}{3}[20,40] + \frac{1}{3}([-20,-30]$$

$$= \frac{1}{3}[0,10] = [0,3.33].$$

Similarly,

$$x_2^I[v^I] = x_3^I[v^I] = \frac{1}{6}[150,170] = [25,28.33].$$

We can see that $x_1^I[v^I] + x_2^I[v^I] + x_3^I[v^I] = [50,60] = v^I(\{1,2,3\})$.

The following example shows that equation (5.10) does not apply to games with negative characteristic functions.

Example 3 *Consider a "game"* $\Gamma = (N, v^I)$ *with* $n = 3$ *and* $v^I(\cdot)$ *being defined by*

S	$v^I(S)$	S	$v^I(S)$
$\emptyset$	0	$\{1,2\}$	$[20,40]$
$\{1\}$	0	$\{1,3\}$	$[-40,-20]$
$\{2\}$	0	$\{2,3\}$	$[70,90]$
$\{3\}$	0	$\{1,2,3\}$	$[90,110]$

In this case, $x_1^I[v^I] = \frac{(2-1)!(3-2)!}{3!}(v^I(\{1,2\}) \ominus v^I(\{2\}))$

$$+ \frac{(2-1)!(3-2)!}{3!}(v^I(\{1,3\}) \ominus v^I(\{3\}))$$

$$+\frac{(3-1)!(3-3)!}{3!}(v^I(\{1,2,3\}) \ominus v^I(\{2,3\}))$$

$$= \frac{1}{3!}[20,40] + \frac{1}{3!}[-40,-20] + \frac{1}{3}([90,110] \ominus [70,90])$$

$$= \frac{1}{3!}[-20,20] + \frac{1}{3}([20,20]$$

$$= \frac{1}{6}[20, 60].$$

$$x_2^I[v^I] = \frac{1}{3!}[20, 40] + \frac{1}{3!}[70, 90] + \frac{1}{3}([90, 110] \ominus [-40, -20])$$

$$= \frac{1}{3!}[90, 130] + \frac{1}{3}([130, 130]$$

$$= \frac{1}{6}[350, 390].$$

$$x_3^I[v^I] = \frac{1}{3!}[-40, -20] + \frac{1}{3!}[70, 90] + \frac{1}{3}([90, 110] \ominus [-40, -20])$$

$$= \frac{1}{3!}[30, 70] + \frac{1}{3}([130, 130]$$

$$= \frac{1}{6}[260, 330].$$

Obviously $x_1^I[v^I] + x_2^I[v^I] + x_3^I[v^I] = \frac{1}{6}[440, 560] \neq v^I(\{1, 2, 3\})$.

5.5 Conclusion

In this chapter, we have introduced a new method for ranking interval numbers. Based on this method, we defined interval-valued cooperative games. We adopted three axioms with desired properties for an interval-valued cooperative game. We proved that a unique payoff function, which is similar to the Shapley value function, exists and satisfies the devised axioms. Furthermore, this payoff function can be applied to non-superadditive games. However, it cannot applied to games with negative values.

Chapter 6

Summary and Future Research

6.1 Summary

In this dissertation, we have introduced three types of solution sets for interval-valued fuzzy relational equations. The three types are the united solution set, the tolerable solution set, and the controllable solution set. Using a constructive approach, we have proved that each solution set of a *max-t-norm* interval-valued fuzzy relational equation consists of one maximum solution and a finite number of minimal solutions, if the solution set is not empty. Some necessary and sufficient conditions for the existence of solutions are given. The solution set can be determined by first calculating the maximum solution and then the minimal ones. Similarly, we have proved that every solution set of a *min-s-norm* interval-valued fuzzy relational equation consists of one minimum solution and a finite number of maximal solutions, if the solution set is not empty. Again, necessary and sufficient conditions for the existence of solutions are given, and the solution set can be determined by first calculating the minimum solution and then the maximal ones. We have also designed algorithms to find the united solution set, tolerable solution set, and controllable solution set of interval-valued fuzzy relational equations.

For *max-t-norm* (*min-s-norm*) interval-valued fuzzy relational equations, we would like to find under which conditions there exists a unique minimal (maximal) solution. Moreover, we would like to find the conditions under which the solution set consists of a single solution vector. Unfortunately, the uniqueness cannot be determined without actually finding the whole solution sets.

In Chapter 5 we have studied interval-valued cooperative games. We introduced an interval-valued ranking method, which preserves several good properties of real number comparisons. Based on the ranking method, we defined "positive" interval-valued cooperative games. We adopted three axioms, namely symmetry, efficiency, and law of aggregation, for an interval-valued cooperative game. After introducing a new minus operator, we proved that a unique payoff function, which is similar to Shapley value function, exists and satisfies the devised axioms. Furthermore, this payoff function can be applied to non-superadditive games. However, it cannot be applied to games with negative values.

6.2 Future Research

Some potential directions for future research include:

1. Proving or disproving that the Shapley value function exists for games with negative values. The current Shapley function cannot be applied to interval-valued games with negative values. Even the classic (constant-valued) Shapley value cannot be applied to negative games. So one may have to extend the classic Shapley value first.

2. Finding Shapley values for games with incomplete coalitions. In reality, some coalitions may never be formed.

3. Finding an efficient method to calculate the nucleolus of some specific interval-valued cooperative games.

Since the nucleolus meets certain natural standards of equity and fairness among players in a cooperative game, it is one of the most popular solution concepts.

The definition of nucleolus entails comparisons between vectors of coalition-related excesses. Since there are 2^n possible coalitions in an n-person game, the length of any vector to be compared is 2^n. Therefore, computing the nucleolus by simply following its definition may take at least exponential time. In finding the nucleolus, we need to solve a series of linear programming problems, each having the following form:

Author	**LPs**	**Constraints**	**Variables**	**Coefficients**
Kohlberg[18]	1	$(2^n)!$	$O(n)$	
Owen[37]	2^n	$O(4^n)$	$O(2^n)$	
Maschler[28]	$O(4^n)$	$O(2^n)$	$O(2^n)$	$-1, 0, 1$
Sankaran[48]	$O(2^n)$	$O(2^n)$	$O(2^n)$	$-1, 0, 1$
Potters[46]	$O(n)$	$O(2^n)$	$O(2^n)$	$-1, 0, 1$
Fromen[15]	$O(n)$	$O(2^n)$	$O(n)$	

Table 6.1: Complexity of finding nucleolus

$$
\begin{aligned}
& z^* = min\ z \\
& s.t \quad z \succeq v^I(S) - x^I(S), \text{ for all } S \subset N \\
& \qquad \textstyle\sum_{i=1}^{n} x_i^I = v^I(N) \\
& \qquad x_i^I \succeq v^I(\{i\}),
\end{aligned}
$$

where the optimal objective value z^* represents the current largest excess. Following this approach, the best method found in literature uses n linear programming problems, and each problem has 2^n constraints and n variables. Table 6.1 lists some results from literature. Apparently the nucleolus cannot be found in polynomial time in this way.

However, it is well known that the nucleolus of some special classes of games can be computed in polynomial time, by using some mechanisms to reduce the number of coalitions. For example, Granot, Granot, and Zhu [15] introduced the concept of "characterization sets" to reduce the necessary coalitions to be considered. We hope to adopt the similar strategies to interval-valued games.

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Appendix A

Two Application Examples of Fuzzy Relational Equations

Example 1. For max-t application, we give a problem in optimal three-tiered multimedia streaming services (from Lee and Guu [21].)

Consider a stream media provider who uses a three-tiered multimedia architecture (See Figure A-1.) The provider is seeking an economic way to transmit a multimedia file to 6 clients.

x_i: capacity (bandwidth, i.e. bps) of server i,

a_{ij}: the bandwidth between server i and client j,

b_j: bandwidth (bps) required by client j,

c_i: cost/bps.

Suppose (after normalization)

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.8 & 0.8 & 0.9 & 1.0 \\ 0.5 & 0.6 & 0.8 & 0.9 & 1.0 & 0.8 \\ 0.6 & 0.8 & 0.5 & 0.9 & 0.4 & 0.4 \end{bmatrix}$$

$$\mathbf{b} = [0.6, 0.6, 0.8, 0.8, 0.9, 1.0]$$

$$\mathbf{c} = [0.7, 1, 2].$$

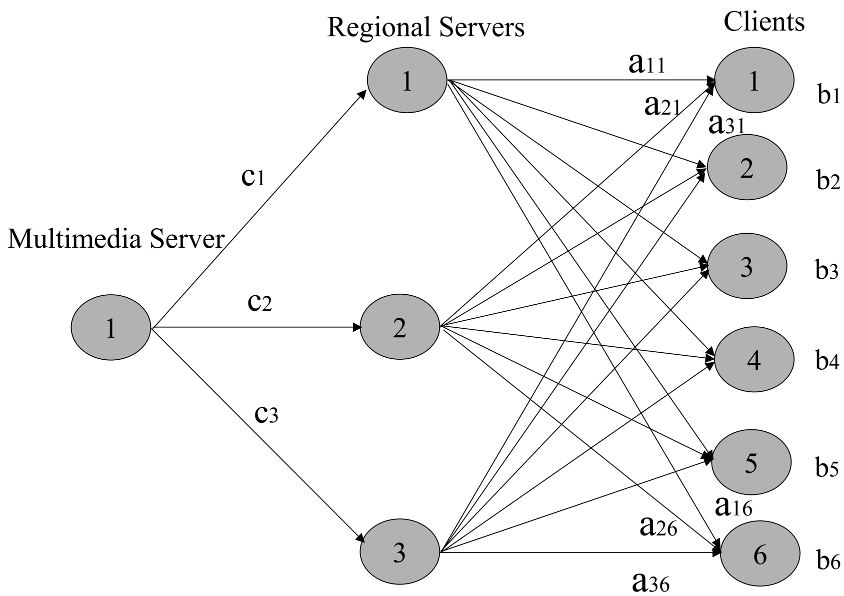

Figure A-1: Max-t application example

Then the constraints are

$$\begin{cases} \max(\min(x_1, 0.6), \min(x_2, 0.5), \min(x_3, 0.6)) = 0.6 \\ \vdots \\ \max(\min(x_1, 1.0), \min(x_2, 0.8), \min(x_3, 0.4)) = 1.0 \end{cases}.$$

Let $\triangle$ denotes *max-min,* the corresponding optimization problem is

$$\min \quad Z = 0.7x_1 + x_2 + 2x_3$$

$$s.t. \;\; \mathbf{x} \triangle A = \mathbf{b},$$

$$0 \le x_i \le 1, i = 1, 2, 3$$

After solving the problem, we have

$$\Sigma(A, \mathbf{b}) = [(1.0, 0.6, 0), (1.0, 0.8, 0.6)] \cup [(1.0, 0, 0.6), (1.0, 0.8, 0.6)],$$

and $x^* = (1.0, 0.8, 0.6)$.

Example 2. For min-s application, we give a problem involving long distance service provision..

A telecommunication service reseller has three long distance calling plans (See Figure A-2.) He is seeking maximum commission selling plans to clients.

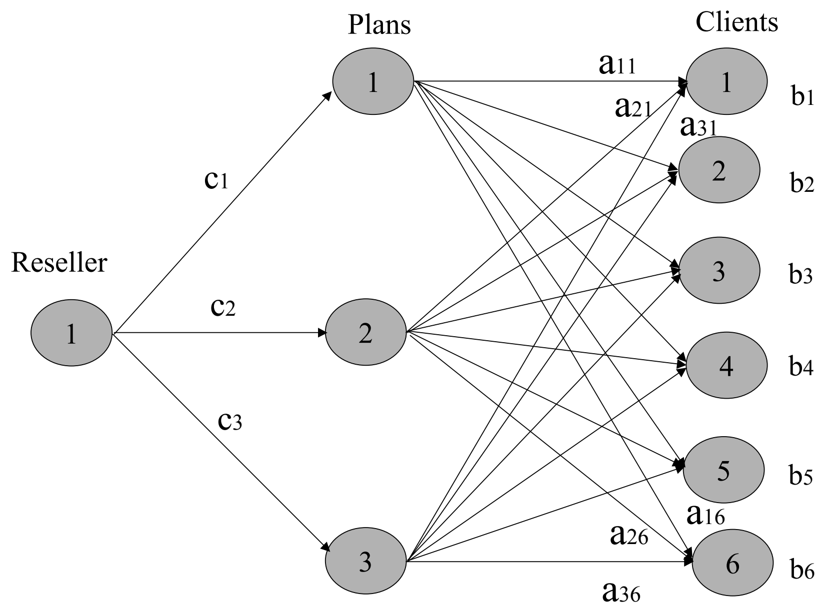

Figure A-2: Min-s application example

x_i: minimum monthly fee of plan i,

b_j: budget of customer j,

a_{ij}: estimated monthly usage if customer j uses plan i.

The constraints can be described as

$$\left\{\begin{array}{c} \min(\max(x_1, a_{11}), \max(x_2, a_{21}), \max(x_3, a_{31})) = b_1 \\ \vdots \\ \min(\max(x_1, a_{16}), \max(x_2, a_{26}), \max(x_3, a_{36})) = b_6 \end{array}\right. .$$

Let Ⓢ represents $\min - \max$ composition, the corresponding optimization problem (after normalization) is

$$
\begin{aligned}
\max \quad & Z = f(x_1, x_2, x_3) \\
s.t. \quad & \mathbf{x} \circledS A = \mathbf{b}, \\
& 0 \le x_i \le 1, i = 1, 2, 3
\end{aligned}
$$