# Asymptotic Techniques in Elastic-Plastic Analysis of Structures

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Summary

Elastic-plastic structures can nowadays be analyzed with the powerful numerical procedures of the finite element method. Nevertheless, in many engineering applications, analytical expressions capable of predicting with sufficient accuracy the stress distributions, the extent of the plastic zones and the load displacement behaviour could be of great practical value. For simple structures and loading stages not too far from the elastic limit, such analytical expressions may be obtained by using perturbation methods and asymptotic expansions. A small dimensionless parameter $\varepsilon$ is defined as the ratio of a length characterizing the extent of the narrow plastic zone, to a conveniently chosen typical dimension of the structure. Stresses and displacements are formally expanded as asymptotic series in terms of powers of $\varepsilon$. For each order of magnitude, the exact basic relations lead to a separate set of simplified differential equations which can be integrated analytically or numerically by using standard procedures. The method is very general and can be applied to several classes of plastic behaviour and of structural problems.

Three examples of very simple structures are chosen in particular to illustrate the applicability of the perturbation method to engineering problems: (1) For a simply supported beam under a concentrated load, $\varepsilon$ is chosen as the ratio of the largest axial extension of the plastic zone to twice the beam length. (2) The case of a long thin cylindrical shell subject to a gradually increasing radial ring of load is algebraically more involved but can be treated similarly. (3) For a simply supported circular plate under a ring of load, $\varepsilon$ is defined as the thickness ratio of the central plastic zone which is assumed to be sufficiently thin in the first stage of loading beyond the elastic limit. Both Tresca's and v. Mises' yield criteria with the associated flow rule (rate theory) can be used.

The method can also be applied to more involved geometries (in conjunction with linear FE-procedures) and more complex material behaviour.

## 1. Introduction

Elastic-plastic deformations of structures such as beams, plates and shells can nowadays be theoretically studied and computed quite effectively by applying classical nonlinear finite-element procedures for each individual problem. Analytical solutions might lead to a better physical understanding of whole classes of problems, but, as in most nonlinear systems, they are hardly available. Usually, nonlinear systems are treated by introducing some sort of linearisation procedure such as the incremental iteration schemes used in numerical methods. Consider the first stage of loading beyond the elastic limit, where the plastic domain remains small with respect to the dimensions of the structure. An asymptotic perturbation scheme based on a small geometrical parameter characterizing the extent of the plastic domain should lead to a "natural" linearization of the problem. In the present paper, I will try to show that such an analytical asymptotic technique is well adapted to the physics of the first stage of loading beyond the elastic limit and that it leads to linear equations which can be solved quite straight-forwardly. Instead of describing the general method of solution in abstract terms, I will apply it directly to three typical examples and illustrate its main aspects. This will also allow a comparison with specific results obtained by applying other methods. The asymptotic technique illustrated here can also be used in conjunction with finite-element procedures. This particular extension of the method will be left out of the scope of the present paper, to be reported elsewhere.

This paper extends and generalizes a publication on the same topic which appeared recently [1].

Regular and singular perturbations have been used with success in different domains of mechanics. Systematic and detailed information about the mathematical method itself and its application possibilities can be found for example in [2],[3],[4].

## 2. Simply Supported Beam

As a first example, in order to introduce the main ideas, a particularly simple case is chosen for which an exact analytical solution exists (see for example [5]).

Consider a beam of rectangular cross-section, simply supported at both ends and loaded by a gradually increasing concentrated force P in the middle (fig. 1).

Coordinates x,y, bending displacement v, stress $s_x$, bending moment m and loading force p can be defined as dimensionless variables by choosing the appropriate reference quantities.

Let $p_c = 1$ be the value of the load corresponding to plastic collapse. The elastic limit will then be reached for $p_e = 2/3$. When the load exceeds the elastic limit, plastic zones appear in the vicinity of the loading force, near both the upper and lower surfaces of the beam (fig. 2). The curves separating the elastic core from the plastic zones will be characterized by the distance function

$$y = \pm\eta(x) \qquad (\eta(x) > 0) \tag{2.1}$$

which is one of the unknown functions to be determined.

For $|x| \geq x_e$ all cross sections are elastic, so that

$$\eta(x_e) = 1. \tag{2.2}$$

Plastic collapse occurs for $\eta(0) = 0$.

The following relations can be derived in the "elastic-plastic" interval $0 \leq x \leq x_e$ for $2/3 \leq p \leq 1$:

$$\begin{aligned} v'' &= -2\eta^{-1}, \\ m &= (1 - \tfrac{1}{3}\eta^2), \\ \eta^2 &= 3[1 - p(1 - 2x)], \\ x_e &= \tfrac{1}{2} - \tfrac{1}{3p}. \end{aligned} \tag{2.3}$$

The set (2.3) can be integrated with the help of easily formulated boundary and continuity conditions, taking $\eta$ as independent variable instead of x. From the well-known elasticity solution in $x_e \leq x \leq \frac{1}{2}$ one needs the value of v at $x = x_e$:

$$v(x_e) = \frac{1}{3p} v'(x_e) + \frac{2}{27p^2} \tag{2.4}$$

Instead of the corresponding exact solution, an asymptotic approximation will now be derived.

If the extent of the plastic zone with respect to the beam dimensions is small, the dimensionless value $x_e$ can be taken as a "vanishingly" small parameter

$$x_e =: \varepsilon \to 0, \tag{2.5}$$

$$\xi := x/\varepsilon, \quad \frac{d(\;)}{d\xi} =: (\;)^{\cdot} = (\;)'/\varepsilon; \tag{2.6}$$

one can expand the quantities in (2.3) in terms of $\varepsilon$ as follows

$$\begin{aligned} p &= \frac{2}{3}\cdot\frac{1}{1-2\varepsilon} = \frac{2}{3}(1 + 2\varepsilon + 4\varepsilon^2 + \ldots), \\ \eta &= 1 - 2(1-\xi)\varepsilon - [4(1-\xi) + 2(1-\xi)^2]\varepsilon^2 - \ldots, \\ v &= v_o + v_1\varepsilon + v_2\varepsilon^2 + v_3\varepsilon^3 + v_4\varepsilon^4 + \ldots \end{aligned} \tag{2.7}$$

Thus, the nonlinear differential equation $(2.3)_1$ can also be expanded as a system of easy linear differential equations for $v_o, v_1, v_2, \ldots$ . With the help of the expanded expressions for $\eta^{-1}$ on obtains

$$\ddot{v}_o = 0,$$
$$\ddot{v}_1 = 0,$$
$$\ddot{v}_2 = -2, \tag{2.8}$$
$$\ddot{v}_3 = -4(1 - \xi),$$
$$\ddot{v}_4 = -4[2(1 - \xi) + 3(1 - \xi)^2].$$

The set of linear equations (2.8) is of course much easier to integrate than the original expressions in (2.3). Using the boundary conditions and (2.4) one obtains eventually

$$v = v_P - \varepsilon^2\xi^2 - (2\xi^2 - \frac{2}{3}\xi^3)\varepsilon^3 - [10\xi^2 - \frac{16}{3}\xi^3 + \xi^4]\varepsilon^4 - \ldots, \tag{2.9}$$

where

$$v_P := v(0) = v_{max} = \frac{1}{6}(1 + 2\varepsilon + 4\varepsilon^2 + 20\varepsilon^3 + \ldots). \tag{2.10}$$

The expressions $(2.7)_1$ and (2.10) can be considered as the parametric equations of a load-displacement curve which has been compared with the corresponding function $v_P(p)$ of the exact solution in fig. 3. This exact expression is

$$v_P = \frac{1}{27p^2}[10 - 3(2 + p)\sqrt{3(1 - p)}]. \tag{2.11}$$

One sees that the approximation is quite acceptable (error less than 5 %) up to values $\varepsilon = 0{,}12$ ($\varepsilon_{max} = 0{,}167$ for $p = p_c$). Instead of (2.10) one can also use the asymptotically equivalent form free of the cubic term

$$v_P = \frac{1}{6} \cdot \frac{1-3\varepsilon-6\varepsilon^2}{1 - 5\varepsilon} + O(\varepsilon^4). \tag{2.12}$$

If one combines (2.12) and $(2.7)_1$, one obtains an excellent agreement with the exact solution up to values p reaching almost the collapse limit $p_c := 1$. Excellent approximations can be obtained similarly for $\eta(x)$.

## 3. Cylindrical Shell

Consider now the more difficult problem of a long and thin cylindrical shell subject to a gradually increasing radial ring load of specific value P per unit circumferential length (fig. 4).

The basic relations for the elastic-plastic phase of loading have been derived in [6] for a material obeying Tresca's yield criterion and, to reduce formalism, for elastic incompressibility. Explicit nonlinear expressions for the bending moment m and the azimutal normal force n can be given in terms of the radial displacement v and the curvature $k = -v''$ (dimensionless quantities) in the elastic-plastic interval $0 \le x \le x_e$. With a similar assumption as in (2.5) expansions for m,v,k, the function $\eta(x)$ for the limiting curve etc. can be considered. Again, one obtains eventually a set of very easily integrable linear differential equations. Of course, this case is algebraically more involved than the problem of section 2. Details can be found in [7]. The asymptotic solutions obtained are in good agreement with numerical results given in [6] up to values $\varepsilon \cong 0{,}35$ as illustrated in fig. 5 for the load-displacement curve. This is quite remarkable if one considers that for $\varepsilon = 0{,}487$ the elastic core vanishes at $x = 0$ and the curvature becomes here infinite (collapse is reached for much higher values of $\varepsilon$).

4. Circular Plate

As a third and last example, a simply supported thin circular plate subject to a gradually increasing ring load will be considered (fig. 6). Tresca's yield criterion leads with the associated incremental flow rule to integrable constitute relations, so that, as in section 3, the bending moments can be expressed explicitly as nonlinear functions of the corresponding curvatures. In [1] the asymptotic technique briefly outlined in section 2 has been successfully applied to that case to derive sets of easily integrable linear differential equations in the elastic-plastic domain. But the method remains powerful even if the material behaviour is more strongly nonlinear. For instance, in the case of v. Mises' yield criterion where the associated incremental flow rule leads to nonintegrable constitutive relations, the moments cannot be directly expressed as explicit functions of the curvatures by simple integration. Taking the thickness ratio of the plastic zone in the central part of the plate as a small parameter $\varepsilon$ characterizing the "plastic state", the strain increments can be formulated in terms of the increment $\delta\varepsilon$ and of the derivatives of the corresponding expansions with respect to the "plastic state variable" $\varepsilon$. Thus, the yield criterion and the associated flow rule lead to linear relations for the expansion coefficients of stresses and plastic strains, which can be used to find explicit (and linear) expressions for the expansion coefficients of the bending moments in terms of the corresponding bending curvatures for successive steps of approximation. Details are given in [7]. Eventually explicit analytic results for the bending displacement, for moments, stresses, for the limiting curve between the elastic and the plastic domains (fig. 7) can be found by straightforward analysis.
In the case of Tresca's yield criterion (see [1]), as soon as $\varepsilon$ reaches values comparable with $\nu^2$ ($\nu$ bending Poisson's ratio), the plastic zone extends radially faster and its radial with beyond the ring of load becomes of the order $\sqrt{\varepsilon}$. In the case of v. Mises' criterion no such effect appears and the radial width beyond the ring of load remains contained (fig. 7). Nevertheless, for both criteria the load-displacement relations are identical up to the second order approximation step including terms in $\varepsilon^2$.

## 5. Conclusion

The application of asymptotic techniques illustrated in the three examples chosen above leads to a physically meaningful linearization of the problem. As mentioned before, this linearization method can also be used for more difficult geometries in conjunction with finite element procedures to solve the linear sets of equations obtained. It can even be applied for more involved nonlinear constitutive behaviour if the extent of the inelastic domain remains small with respect to the dimensions of the structure. Further aspects and generalizations will be reported elsewhere.

## References


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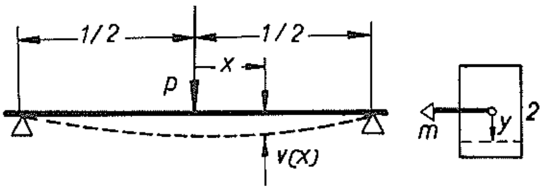


Fig. 1 Simply supported beam

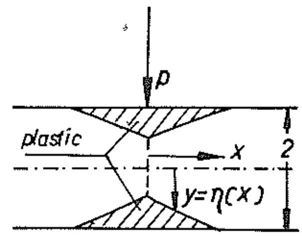


Fig. 2 Elastic and plastic domains in the beam

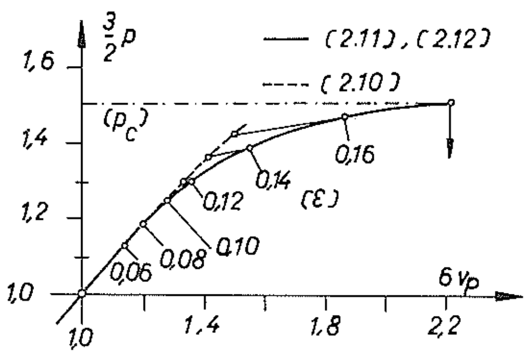


Fig. 3 Load-displacement curve for the beam

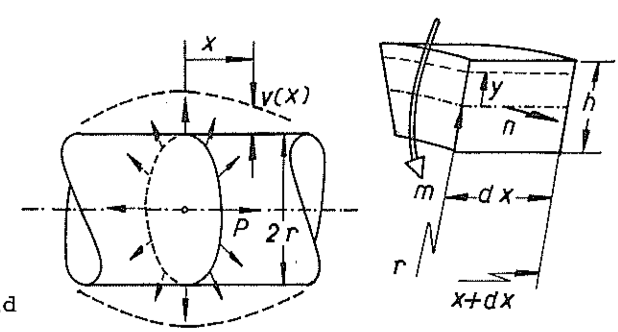


Fig. 4 Cylindrical shell under a ring of load

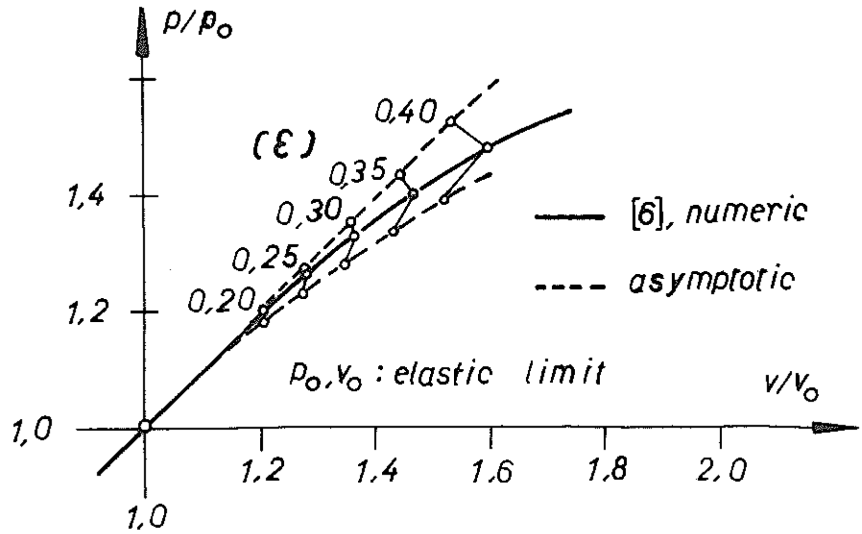


Fig. 5 Load-displacement curve for the cylindrical shell

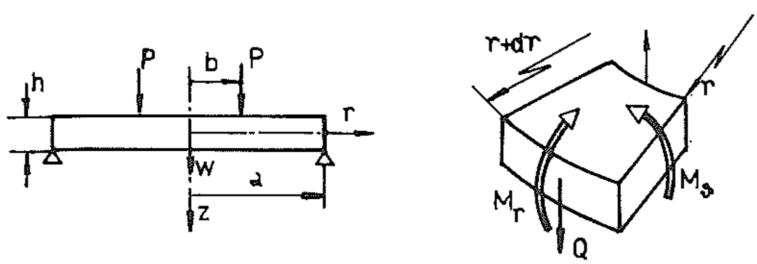


Fig. 6 Simply supported circular plate under a ring of load

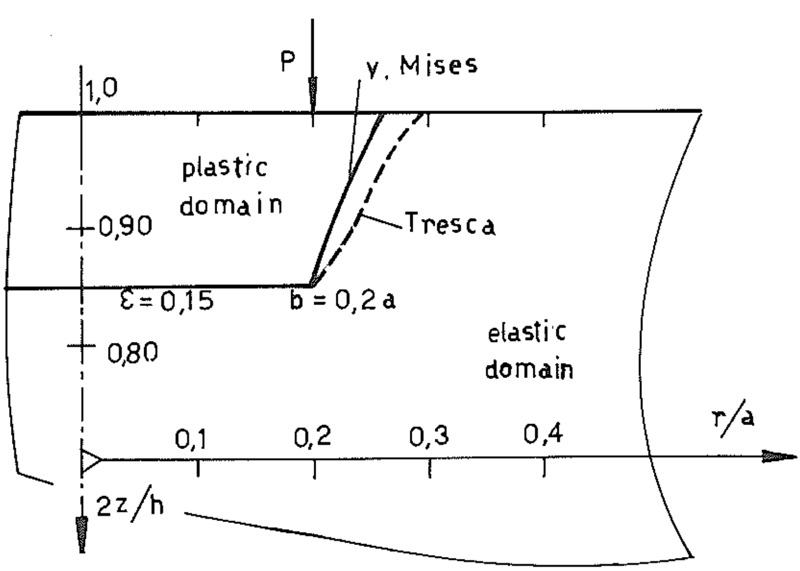


Fig. 7 Elastic and plastic domains in the plate