A STUDY OF THE DISTRIBUTION AND MOMENTS OF BUNDLE STRENGTH IN SEQUENTIAL BREAKAGE OF PARALLEL FILAMENTS

by

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INTRODUCTION

The very nature of a group of any kind in the physical universe is that some of its quantitative properties remain indeterminate until the corresponding inherent properties of the elements in the group are properly redefined within the group. In particular, when a group functions as an assembly of elements the complexity of the group structure often demands more than the simple rule of additivity for expressing a group property.

Consider a group of straight and parallel filaments sampled from an infinite population formed into a bundle by fixing or clamping the filaments together at both ends. When the prime objective of forming a bundle is to have an increased strength, the natural question arises as to how much and in what fashion the increase is brought about by having a designated number of filaments in a bundle. The engineering significance of this problem has long been recognized in as many fields of material science as there exist problems of breakage, failure or fracture. The filaments which constitute the elements of a bundle can be, for example, textile fibers, yarns, electric wires, rubber cords or even polymer chains. The strength of a bundle can be defined, though it may seem arbitrary, as the minimum axial load above which every filament in the bundle breaks, or the maximum load attainable in the bundle throughout its load-extension history which extends to the final filament break.

Much of the analytical difficulties in theory of bundle strength are directly attributed to the variation of tensile properties among the filaments that constitute a bundle. In particular, the difficulties are multipled when such variation is of a statistical nature and has to be imbedded into the physical system of sequential filament breakage that is non-stochastic in its nature. Consequently, the system of bundle failure has to be converted to probabilistic events in order to facilitate a statistical approach.

The initial analytical approach to the problem of bundle strength was made by Peirce (6) in 1926. By assuming an underlying distribution of filament strength, he was able to provide a likelihood estimate of bundle strength if the number of filaments in the bundle were very large. Though his result was found to be correct in a later work (2), it was based on a crude approximation with little mathematical rigor. Moreover, Peirce's results were obtained in the absence of the distribution of bundle strength, and consequently he was unable to provide the expected strength of a small bundle.

A method of deriving the probability distribution of bundle strength is credited to Daniels (2). On considering successive filament breakages in a bundle with a given number of filaments, he was able to convert a set of conditions under which the bundle breaks into a multiple integral of the joint density of filament strengths in order to obtain the distribution function of bundle strength. From the general form of the distribution of bundle strength, he was successful in providing the asymptotic form for large bundles which follows a normal distribution with mean and variance that can be determined from the underlying distribution of the filament strengths. It was shown that the estimate of mean bundle strength obtained by Peirce coincided with the asymptotic mean of Daniels. Although the mathematical context in the work of

Daniels carries certain virtues in its rigor, it certainly cannot escape a criticism for being too complex and extremely lengthy in his proof of asymptotic normality. In addition, the frequent expansions and approximations employed throughout his analysis obscure the essential physical significance attached to each stage. Other remarks on Daniels' work will be made in the major parts of this study as necessities arise.

Other statistical works on bundle strength include one by Coleman (1). Under the premise that the strengths of filaments obey an Weibull distribution he showed, by utilizing the major result of Daniels, that the efficiency of a large bundle decreases as the variance of filament strength increases. Coleman's studies on time dependence of mechanical breakdown of bundles, which are referred to in the foregoing paper, are irrelevant to the interest of this study and hence are excluded from discussion.

The scope of this study embraces not only those aspects explored by the previous workers, but also several other important areas formerly untreated. First, the distribution function of bundle strength is derived based on a simple yet precise event description of bundle breakage. This process eliminates the complex multiple integration of a joint density presented in the work of Daniels. Secondly, the asymptotic properties of strength for large bundles are deduced directly from the definition of bundle strength without involving the distribution function. Utilizing several measure theoretic modes of convergence the analytical difficulties found in Daniels' work are avoided. Next, emphasis is placed on the efficiency of bundles with respect to the number of their constituent filaments. Major inequalities concerning the moments of the bundle strength distribution are proved in order to compare bundles of different sizes. Also, the lower and upper bounds of bundle efficiencies are established in accord with the inequalities. Another part of this study is devoted to the generalization of Peirce's 'weakest-link theory' (6) for filament bundles. Changes in bundle strength are examined with relation to the changes in filament length. Finally, a statistical study is made under a newly proposed breakage model. The load being functionally dependent on extension, this model is intended to relax one of the postulates imposed on the idealized model on which the first part of this study is based.

THEORY

Restricted Model for Breakage of Bundles

In view of the fact that the breakage of a bundle can occur only through the breakage of the filaments constituting the bundle, it is imperative to characterize the tensile properties of the filaments in order to properly define the strength of a bundle. Whatever the properties of the filaments are, however, it is conceivable that a sequential breakage of filaments results when a bundle is subjected to a monotonically increasing load. Also, for any given load the bundle either breaks or sustains the load. If it sustains the load, the total load is shared by the surviving filaments in a certain fashion. As the total load increases again gradually, there comes a point where one of the surviving filaments breaks. At this point, the load given up by that particular filament is distributed to the remaining filaments, thus increasing their shares. Consequently, further breakages may occur until another equilibrium is reached, or else successive breakages may result in the breakage of the entire bundle.

It is perhaps important to visualize the bundle breakage as a result of gradual extension of the constituent filaments. Observing that every filament is extended the same amount during the loading of a bundle, the load in a bundle at a given extension can be considered as the resultant load of the filaments at the extension. Obviously, the bundle load does not necessarily increase for an increase in bundle extension as soon as filaments start to break. Hence, the maximum bundle load attainable in the load-extension history of a bundle is a reasonable criteria for defining the strength of a bundle. On the other hand, if an extension is considered as a mere consequence of applying load, the minimum load above which every filament breaks constitutes another side of the dual definition of bundle strength. It is trivial to observe that the two foregoing definitions are identical.

The breakage model to be applied for most part of this study is founded on two postulates: 1) the load in every single filament monotonically increases along with the increase in its extension before it breaks, 2) for any given extension, the load in a bundle is equally shared by the filaments in the bundle surviving at the particular extension. These two postulates bring two-fold simplification to the algebraic expression of bundle strength. The first postulate implies that the bundle strength has to be found among the sums of filament loads just prior to the point of first, second, . . . , and the last filament break and at these points the sums achieve their local maxima, followed by the breaks. The second postulate further simplifies these sums: when the filament strengths are arranged in an increasing order of magnitude as Y_1, Y_2, ..., Y_n, the load in an n filament bundle at the first, second, ..., and the n^{th} filament break can now be expressed as nY_1, $(n-1)\,Y_2$, ..., and Y_n. Therefore, according to the given definition, B_n, the strength of a bundle with n filaments is defined as the following:

$$B_n = \max \left\{nY_1,\ (n-1)\,Y_2,\ \ldots,\ Y_n\right\}$$

$$\text{or,} = \max_{1 \le k \le n} \left\{(n-k+1)\,Y_k\right\},\quad 0 \le Y_1 < Y_2 < \ldots < Y_n \qquad (1)$$

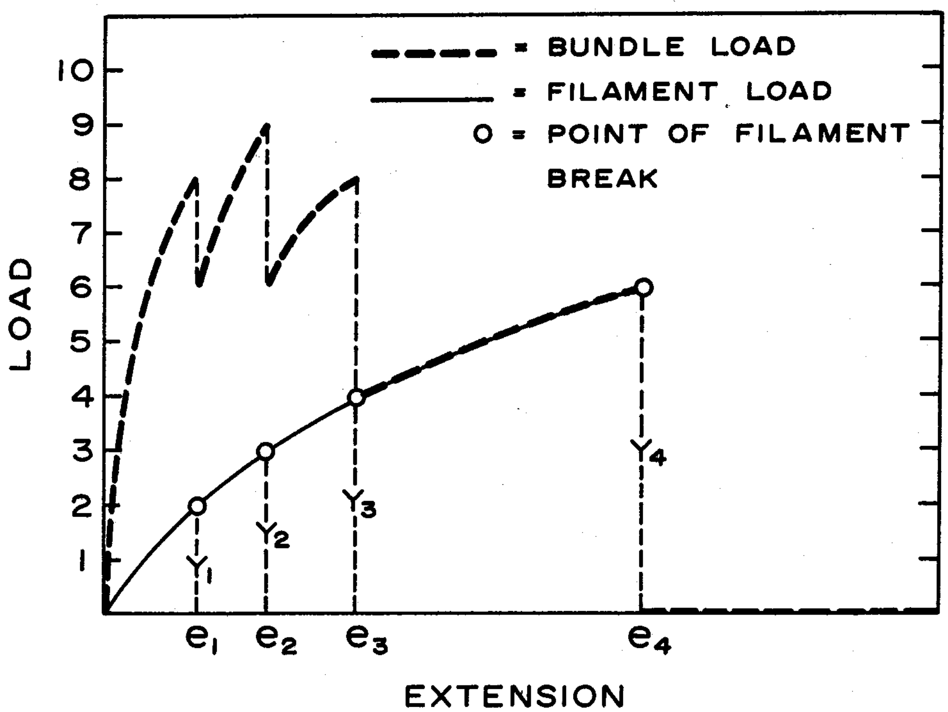

Figure 1. Load-extension curve of a four-filament bundle based on filament loads

The load-extension curve of a four-filament bundle is constructed in Figure 1 on the basis of its filament curves. The maximum bundle load is achieved at e_2, the extension at which the second breakage occurs, and is shown to be $3Y_2$ in this example.

The monotonicity assumption in the above restricted model is valid when a bundle is subjected to a dead load or to a constant rate of loading. The assumption, of course, is not compatible with the constant rate of extension experiments performed by the strain gauge type testers. The possible decrease in filament load due to stress-relaxation can be observed if the rate of extension is low. However, such possibility is remote even with a strain gauge tester since the rate of extension is quite high in ordinary strength tests so that no stress relaxation is observable. In the absence of the monotonicity assumption, the coefficients to be attached to $Y_1, Y_2, \ldots, Y_k$ are absolutely indeterminate and an analytic study of B_n hardly seems feasible. The second assumption of equal filament load is a fair approximation when the filaments are more or less uniform in their load-extension properties. Otherwise, this model may not be of value. For this reason a relaxed breakage model is examined in the last part of this study.

Distribution Function of Bundle Strength

Derivation

Suppose the filaments in a bundle are randomly sampled from a population of filaments for which the strength distribution is known. Let the strengths of the constituent filaments be $X_1, X_2, \ldots, X_n$ for a bundle of n filaments, and assume the underlying distribution function $F(x)$ to be absolutely continuous with the corresponding density $f(x)$. Instead of working with the definition of B_n given by Eq. (1), the following consideration of events leads to the distribution function of B_n in a simple way.

Let the event $E_i = (X_i \leq \frac{x}{n}, B_{n-1}$ (with all but $X_i) \leq x)$ i.e., the filament with strength X_i is breakable under load $\leq \frac{x}{n}$ and the bundle of (n-1) filaments without X_i is also breakable under load $\leq x$.

It is noticed, for the event E_i, that X_i does not have to be the weakest strength. Existence of some $X_j \leq X_i$ $(j \neq i)$ is allowed in the event E_i. Therefore, the event E_i simply provides a sufficient condition for a bundle to break rather than the actual breakage sequence. However, since any breakage can be described by one or more of E's, the union of E's exhausts all possible contingencies. Therefore,

$$S_n(x) = P_r(B_n \leq x)$$

$$= P_r(E_1 \cup E_2 \cup \dots \cup E_n)$$

by an elementary rule of inclusion and exclusion of events,

$$= \sum_{i=1}^{n} P_r(E_i) - \sum_{i \neq j}^{n} P_r(E_i E_j)$$

$$+ \sum_{i \neq j \neq k}^{n} P_r(E_i E_j E_k) - \dots + (-1)^{n+1} P_r(E_1 E_2 \dots E_n) .$$

In obtaining $P_r(E_i E_j)$ above, it is necessary to show that the following two events $E_i E_j$ and E_{ij} are equivalent.

$$E_i E_j = \left[X_i \leq \frac{x}{n},\ B_{n-1} \text{ (with all but } X_i) \leq x,\ X_j \leq \frac{x}{n}, \right.$$

$$\left. B_{n-1} \text{ (with all but } X_j) \leq x \right]$$

$$E_{ij} = \left[X_i \leq \frac{x}{n},\ X_j \leq \frac{x}{n},\ B_{n-2} \text{ (with all but } X_i \text{ and } X_j) \leq x \right]$$

To prove the identity $E_i E_j = E_{ij}$, it is sufficient to show that

$E_i E_j \Longleftrightarrow E_{ij}$.

It is trivial to see that $E_i E_j \Longrightarrow E_{ij}$ since the strength of a bundle cannot possibly be increased by deleting a filament. Therefore,

$$\left[B_{n-1} \text{ (with all but } X_i) \leq x \right] \Longrightarrow \left[B_{n-2} \text{ (with all but } X_i \text{ and } X_j) \leq x \right]$$

$$\left[B_{n-1} \text{ (with all but } X_j) \leq x\right] \Longrightarrow \left[B_{n-2} \text{ (with all but } X_i \text{ and } X_j) \leq x\right]$$

For completeness, the above is to be shown by using the definition of $B_n(x)$ given in Eq. (1).

$$\left[B_{n-1} \text{ (with all but } X_i) \leq x\right] = \max_{1 \leq k \leq n-1} \left\{(n-k)\, X_{(k)} \leq x\right\}$$

where, $X_{(k)}$ is the k^{th} order filament strength among the n-1 filaments excluding X_i.

$$= \left[(n-1)\, X_{(1)} \leq x,\ (n-2)\, X_{(2)} \leq x,\ \ldots,\ X_{(n-1)} \leq x\right]$$

Now suppose $X_j = X_{(\ell)}$ and consider the strength of the n-2 filament bundle with $X_{(1)}, X_{(2)}, \ldots, X_{(\ell-1)}, X_{(\ell+1)}, \ldots,$ and $X_{(n-1)}$ by deleting the filament corresponding to X_j, i.e., B_{n-2} (with all but X_i and X_j). Then

$$\left[B_{n-2} \text{ (with all but } X_i \text{ and } X_j) \leq x\right]$$

$$= \left[(n-2)\, X_{(1)} \leq x,\ \ldots,\ (n-\ell)\, X_{(\ell-1)} \leq x,\ (n-\ell-1)\, X_{(\ell+1)} \leq x,\ \ldots,\ X_{(n-1)} \leq x\right]$$

$$\supseteq \left[(n-1)\, X_{(1)} \leq x,\ \ldots,\ (n-\ell+1)\, X_{(\ell-1)} \leq x,\ (n-\ell)\, X_{(\ell)} \leq x,\ (n-\ell-1)\, X_{(\ell+1)} \leq x,\ \ldots,\ X_{(n-1)} \leq x\right]$$

$$= \left[B_{n-1} \text{ (with all but } X_i) \leq x\right].$$

Similarly,

$$\left[B_{n-2} \text{ (with all but } X_i \text{ and } X_j) \leq x\right] \supseteq \left[B_{n-1} \text{ (with all but } X_j) \leq x\right]$$

Therefore,

$$\left[B_{n-1}\ (\text{with all but } X_i) \leq x,\ B_{n-1}\ (\text{with all but } X_j) \leq x\right]$$

$$\subseteq \left[B_{n-2}\ (\text{with all but } X_i \text{ and } X_j) \leq x\right], \text{ and hence}$$

$$E_i E_j \subseteq E_{ij} \text{ or } E_i E_j \Longrightarrow E_{ij}\ .$$

Conversely, suppose E_{ij} is true, i.e.,

$$\left[X_i \leq \frac{x}{n}\ ,\ X_j \leq \frac{x}{n}\ ,\ B_{n-2}\ (\text{with all but } X_i \text{ and } X_j) \leq x\right]$$

and designate the strengths of the n-2 filaments excluding X_i and X_j by $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n-2)}$ in increasing order. Then, when $Z_{(\ell)} < X_j < Z_{(\ell+1)}$,

$$\left[X_j \leq \frac{x}{n}\ ,\ B_{n-2}\ (\text{with all but } X_i \text{ and } X_j) \leq x\right]$$

$$= \left[X_j \leq \frac{x}{n},\ (n-2)\ Z_{(1)} \leq x,\ \ldots,\ (n-\ell-1)\ Z_{(\ell)} \leq x,\ (n-\ell-2)\ Z_{(\ell+1)} \leq x,\ \ldots,\ Z_{(n-2)} \leq x\right]$$

$$= \left[X_j \leq \frac{x}{n},\ nZ_{(1)} \leq x,\ \ldots,\ nZ_{(\ell)} \leq x,\ (n-\ell-2)\ Z_{(\ell+1)} \leq x,\ \ldots,\ Z_{(n-2)} \leq x\right]$$

$$\subseteq \left[(n-1)\ Z_{(1)} \leq x,\ \ldots,\ (n-\ell)\ Z_{(\ell)} \leq x,\ (n-\ell-1)\ X_j \leq x,\ (n-\ell-2)\ Z_{(\ell+1)} \leq x,\ \ldots,\ Z_{(n-2)} \leq x\right]$$

$$= \left[B_{n-1}\ (\text{with all but } X_i) \leq x\right]$$

i.e., $\left[X_j \leq \frac{x}{n},\ B_{n-2}\ (\text{with all but } X_i \text{ and } X_j) \leq x\right]$

$$\subseteq \left[B_{n-1}\ (\text{with all but } X_i) \leq x\right].$$

Similarly,

$$\left[X_i \le \frac{x}{n},\ B_{n-2}\ (\text{with all but } X_i \text{ and } X_j) \le x\right]$$

$$\subseteq \left[B_{n-1}\ (\text{with all but } X_j) \le x\right].$$

Therefore, from above,

$$E_{ij} = \left[X_i \le \frac{x}{n},\ X_j \le \frac{x}{n},\ B_{n-2}\ (\text{with all but } X_i \text{ and } X_j) \le x\right]$$

$$\subseteq \left[X_i \le \frac{x}{n},\ B_{n-1}\ (\text{with all but } X_i) \le x\right] = E_i$$

and also

$$E_{ij} \subseteq \left[X_j \le \frac{x}{n},\ B_{n-2}\ (\text{with all but } X_j) \le x\right] = E_j$$

and hence $E_{ij} \subseteq E_i E_j$, or $E_{ij} \Longrightarrow E_i E_j$

It was shown, however, that

$$E_i E_j \subseteq E_{ij}, \text{ or } E_i E_j \Longrightarrow E_{ij}.$$

Therefore, $E_i E_j = E_{ij}$, or $E_i E_j \Longleftrightarrow E_{ij}$.

From this,

$$P_r(E_i E_j) = P_r(E_{ij}) = F^2(\tfrac{x}{n})\ S_{n-2}(x).$$

In general, it can be shown for $k \le n$ that $E_1 E_2 \ldots E_k = E_{12\ldots k}$, where

$$E_{12\ldots k} = \left[X_1 \le \frac{x}{n},\ X_2 \le \frac{x}{n},\ \ldots,\ X_k \le \frac{x}{n},\right.$$

$$\left. B_{n-k}\ (\text{with } X_{k+1}, X_{k+2}, \ldots, X_n) \le x\right].$$

The proof for the above can be made simple by using mathematical induction as follows: Suppose $E_1 E_2 \ldots E_{k-1} = E_{12\ldots(k-1)}$ holds. Then,

$E_1 E_2 \ldots E_k = E_{12\ldots(k-1)} E_k$, and it is sufficient to prove

$E_{12\ldots(k-1)} E_k = E_{12\ldots k}$

i.e.,
$$\left[X_1 \le \frac{x}{n},\ X_2 \le \frac{x}{n},\ \ldots,\ X_{k-1} \le \frac{x}{n},\ B_{n-k+1} \text{ (with } X_k, \ldots, X_n) \le x,\right.$$
$$\left. X_k \le \frac{x}{n},\ B_{n-1} \text{ (with all but } X_k) \le x\right]$$
$$= \left[X_1 \le \frac{x}{n},\ X_2 \le \frac{x}{n},\ \ldots,\ X_{k-1} \le \frac{x}{n},\ X_k \le \frac{x}{n},\ B_{n-k} \text{ (with } X_{k+1}, \ldots, X_n) \le x\right].$$

By the same argument applied to the two event case,

$$\left[B_{n-k+1} \text{ (with } X_k, \ldots, X_n) \le x\right] \Longrightarrow \left[B_{n-k} \text{ (with } X_{k+1}, \ldots, X_n) \le x\right].$$

Hence, $E_{12\ldots(k-1)} E_k \Longrightarrow E_{12\ldots k}$ is immediate. Conversely if

$E_{12\ldots k}$ is true, then

$$E_{12\ldots k} \Longrightarrow \left[X_k \le \frac{x}{n},\ B_{n-k} \text{ (with } X_{k+1}, \ldots, X_n) \le x\right]$$
$$\Longrightarrow \left[B_{n-k+1} \text{ (with } X_k, \ldots, X_n) \le x\right]$$

also by the same argument applied to the two event case. On the other hand,

$$E_{12\ldots k} \Longrightarrow \left[X_1 \le \frac{x}{n},\ B_{n-k} \text{ (with } X_{k+1}, \ldots, X_n) \le x\right]$$
$$\Longrightarrow \left[B_{n-k+1} \text{ (with } X_1, X_{k+1}, \ldots, X_n) \le x\right].$$

From above, $\left[X_1 \le \frac{x}{n},\ X_2 \le \frac{x}{n},\ B_{n-k} \text{ (with } X_{k+1}, \ldots, X_n) \le x\right]$

$$\Longrightarrow \left[X_2 \le \frac{x}{n},\ B_{n-k+1} \text{ (with } X_1, X_{k+1}, \ldots, X_n) \le x\right]$$

applying the same logic,

$$\Longrightarrow \left[B_{n-k+2} \text{ (with } X_1, X_2, X_{k+1}, \ldots, X_n) \le x \right].$$

Continuing the above process, it can be shown that

$$E_{12\ldots k} \Longrightarrow \left[X_1 \le \frac{x}{n}, \ldots, X_{k-1} \le \frac{x}{n}, B_{n-k} \text{ (with } X_{k+1}, \ldots, X_n) \le x \right]$$

$$\Longrightarrow \left[X_{k-1} \le \frac{x}{n}, B_{n-2} \text{ (with } X_1, X_2, \ldots, X_{k-2}, X_{k+1}, \ldots, X_n) \le x \right]$$

$$\Longrightarrow \left[B_{n-1} \text{ (with all but } X_k) \le x \right].$$

Now, observing the terms in the event $E_{12\ldots(k-1)}E_k$, it is obvious, by combining the information above, that

$$E_{12\ldots k} \Longrightarrow E_{12\ldots(k-1)}E_k \,.$$

Therefore, $E_1E_2 \ldots E_k = E_{12\ldots k}$

and, $P_r\ (E_1E_2 \ldots E_k) = P_r\ (E_{12\ldots k})$

$$= F^k(\tfrac{x}{n})\ S_{n-k}(x) \,.$$

Finally, from the foregoing expression for $S_n(x)$,

$$S_n(x) = nF(\tfrac{x}{n})\ S_{n-1}(x) - \binom{n}{2} \left[F(\tfrac{x}{n})\right]^2 S_{n-2}(x)$$

$$+ \binom{n}{3} \left[F(\tfrac{x}{n})\right]^3 S_{n-3}(x) - \ldots + (-1)^{n+1} \left[F(\tfrac{x}{n})\right]^n$$

i.e., $$S_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left[F(\tfrac{x}{n})\right]^k S_{n-k}(x), \quad 0 \le x \tag{2}$$

where $S_o(x) = 1$ and $S_1(x) = F(x)$.

Daniels' (2) Eq. (9.2) can be reduced to the above expression for $S_n(x)$ by a substitution, but the above derivation is free of the complex multiple integration and Taylor expansion present in his study.

Geometrical Expression of $S_n(x)$

The breakage regions of two and three filament bundles are easily expressed as an area and a volume respectively by assigning each filament strength to an axis as shown in Figure 2. The relevance of this figure to $S_2(x)$ and $S_3(x)$ becomes obvious when the scale x is converted to F(x).

Properties of $S_n(x)$

The recurrence relation in Eq. (2) satisfies all the requirements for $S_n(x)$ to be a probability distribution function when $S_{n-1}(x)$, $S_{n-2}(x)$, ..., F(x) are assumed to be well defined probability distribution functions. Clearly,

$$S_n(0) = 0$$

$$S_n(\infty) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} = 1$$

and the right continuity of $S_n(x)$ is also preserved. By successive substitutions, $S_n(x)$ can be rewritten in terms of $F(\frac{x}{n})$, $F(\frac{x}{n-1})$, ..., F(x) for n = 1, 2, 3 and 4 as follows.[1]

$$S_1(x) = F(x)$$

$$S_2(x) = 2F(\tfrac{x}{2})\, F(x) - \left[F(\tfrac{x}{2})\right]^2$$

$$S_3(x) = 6F(\tfrac{x}{3})\, F(\tfrac{x}{2})\, F(x) - 3\left[F(\tfrac{x}{3})\right]^2 F(x) - 3F(\tfrac{x}{3})\left[F(\tfrac{x}{2})\right]^2 + \left[F(\tfrac{x}{3})\right]^3$$

[1]The formulae following the Eq. (10.3) in Daniels' (2) paper are entirely erroneous. His formulae for B_2, B_3 and B_4 which correspond to $S_2(x)$, $S_3(x)$ and $S_4(x)$ should read as above when his b_k is replaced by $F(\frac{x}{k})$ for k = 1, 2, 3 and 4.

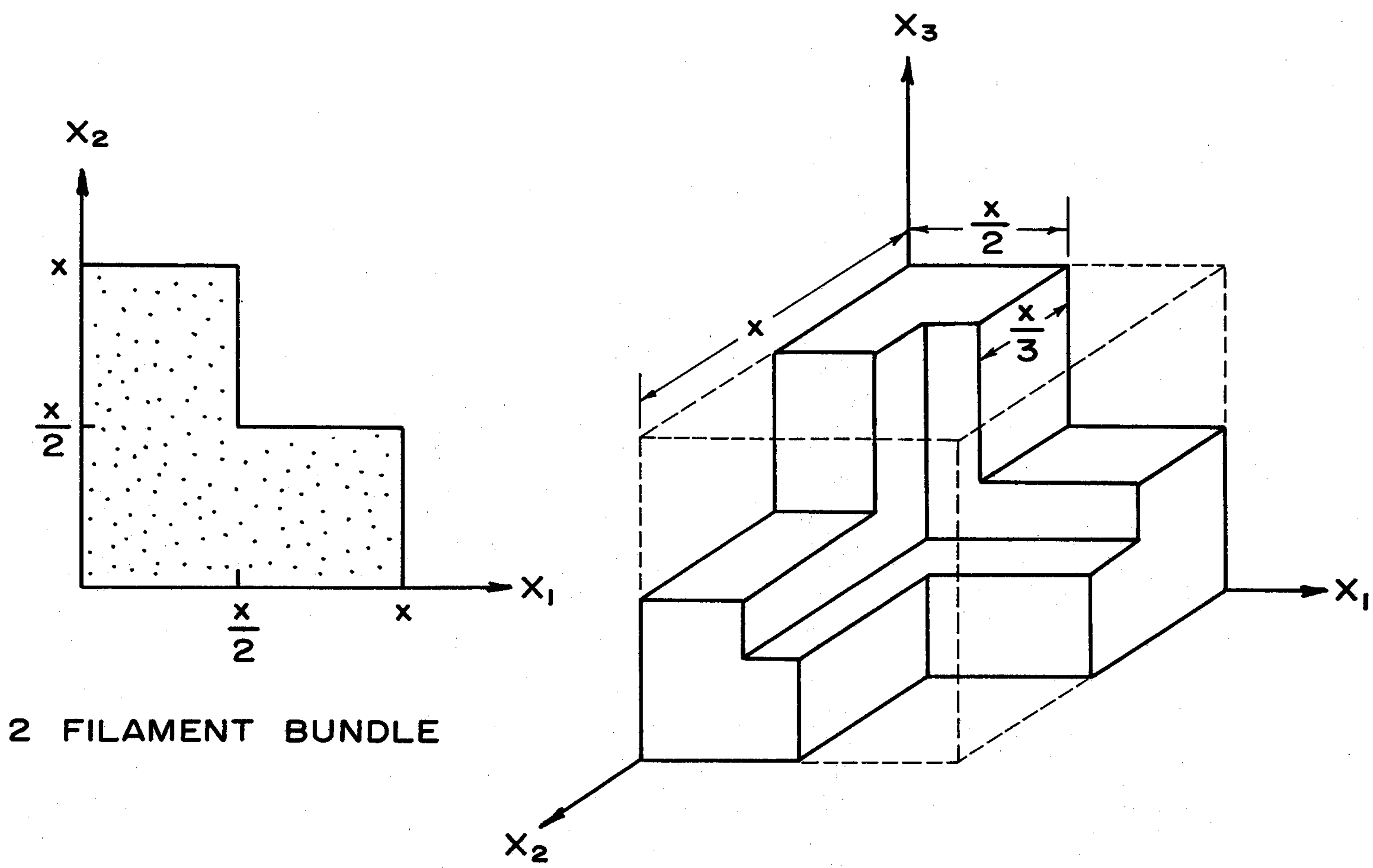

Figure 2. Geometrical expression of breakage region for bundles of two and three filaments

$$S_4(x) = 24F(\tfrac{x}{4})\ F(\tfrac{x}{3})\ F(\tfrac{x}{2})\ F(x) - 12\left[F(\tfrac{x}{4})\right]^2 F(\tfrac{x}{2})\ F(x)$$
$$- 12F(\tfrac{x}{4})\left[F(\tfrac{x}{3})\right]^2 F(x) - 12F(\tfrac{x}{4})\ F(\tfrac{x}{3})\left[F(\tfrac{x}{2})\right]^2$$
$$+ 6\left[F(\tfrac{x}{4})\right]^2\left[F(\tfrac{x}{2})\right]^2 + 4\left[F(\tfrac{x}{4})\right]^3 F(x) + 4F(\tfrac{x}{4})\left[F(\tfrac{x}{3})\right]^3$$
$$- \left[F(\tfrac{x}{4})\right]^4$$

As n increases, the expression for $S_n(x)$ in terms of the $F(\frac{x}{k})$, $k = 1, 2, \ldots, n$, rapidly inflates beyond a manageable limit. Consequently, for a large bundle, utilization of $S_n(x)$ for obtaining the expectation and variance of B_n is hardly feasible in practice. However, for relatively small n, the mean and variance can be readily found with the aid of a modern computing facility. Because of the analytic complexity of $S_n(x)$, the asymptotic properties of B_n are of especial value for approximating large bundle characteristics.

Examples with Small n

The effect of bundle size n on the properties of bundle strength B_n can be better visualized by examining $S_n(x)$ and its corresponding density $S_n'(x)$ for a known $F(x)$ and $f(x)$. For the case $n = 1, 2$ and 3, a normal distribution with mean 4 and unit variance is assumed for the filament strength, and the distribution and density function of B_2 and B_3 are obtained in Figures 3 and 4. The curves are based on values of $f(x)$ and $F(x)$ in normal probability table. It is shown in the figures that the expectation as well as the variance of B_n increases as n increases. Interestingly, $S_2'(x)$ and $S_3'(x)$ remain almost symmetric although their algebraic expressions hardly imply any such tendency. Of course, the skewness may rapidly develop as n increases.

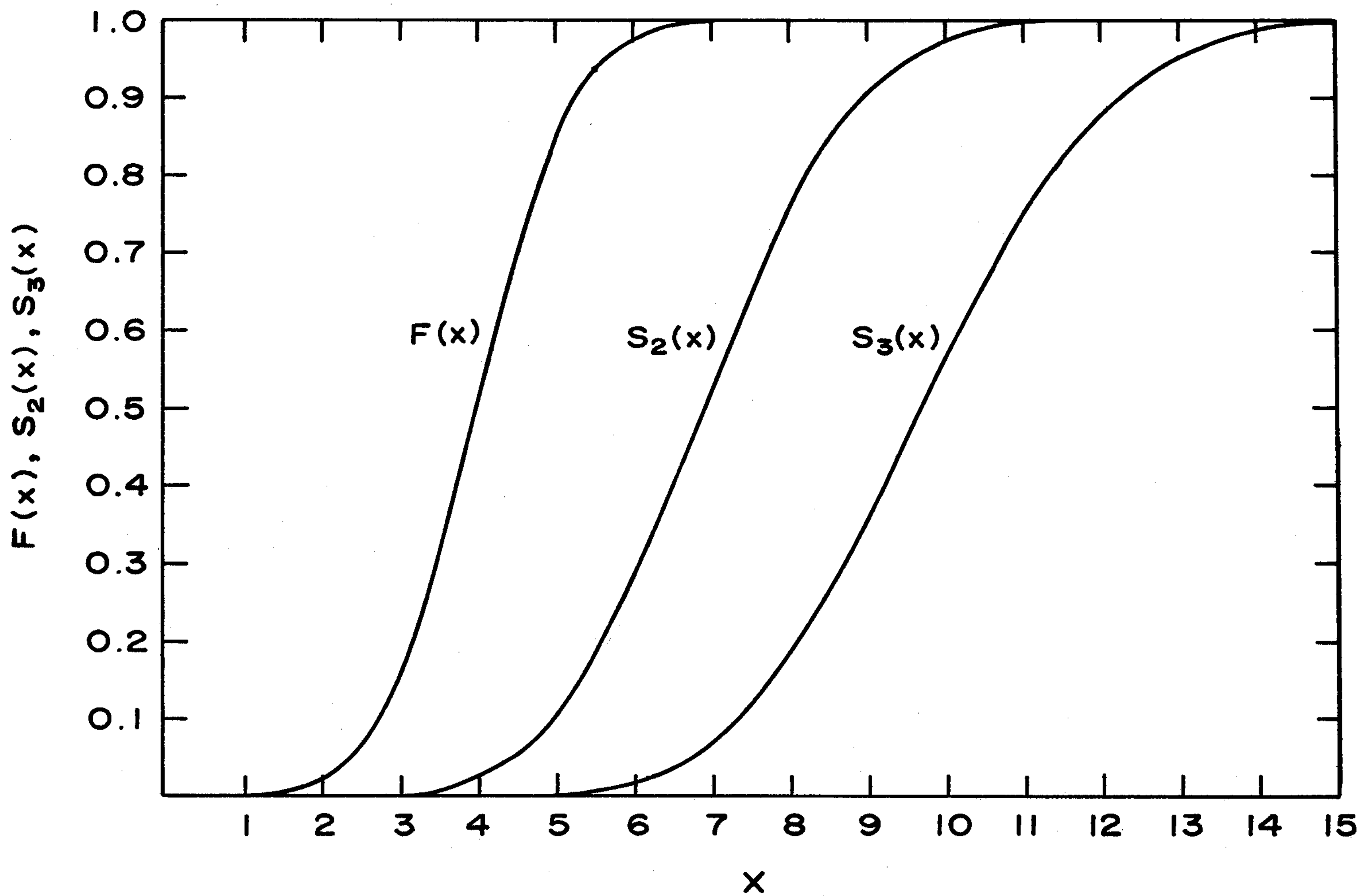

Figure 3. Distribution functions of bundle strength for n = 1, 2 and 3 based on filament strengths distributed as N(4,1)

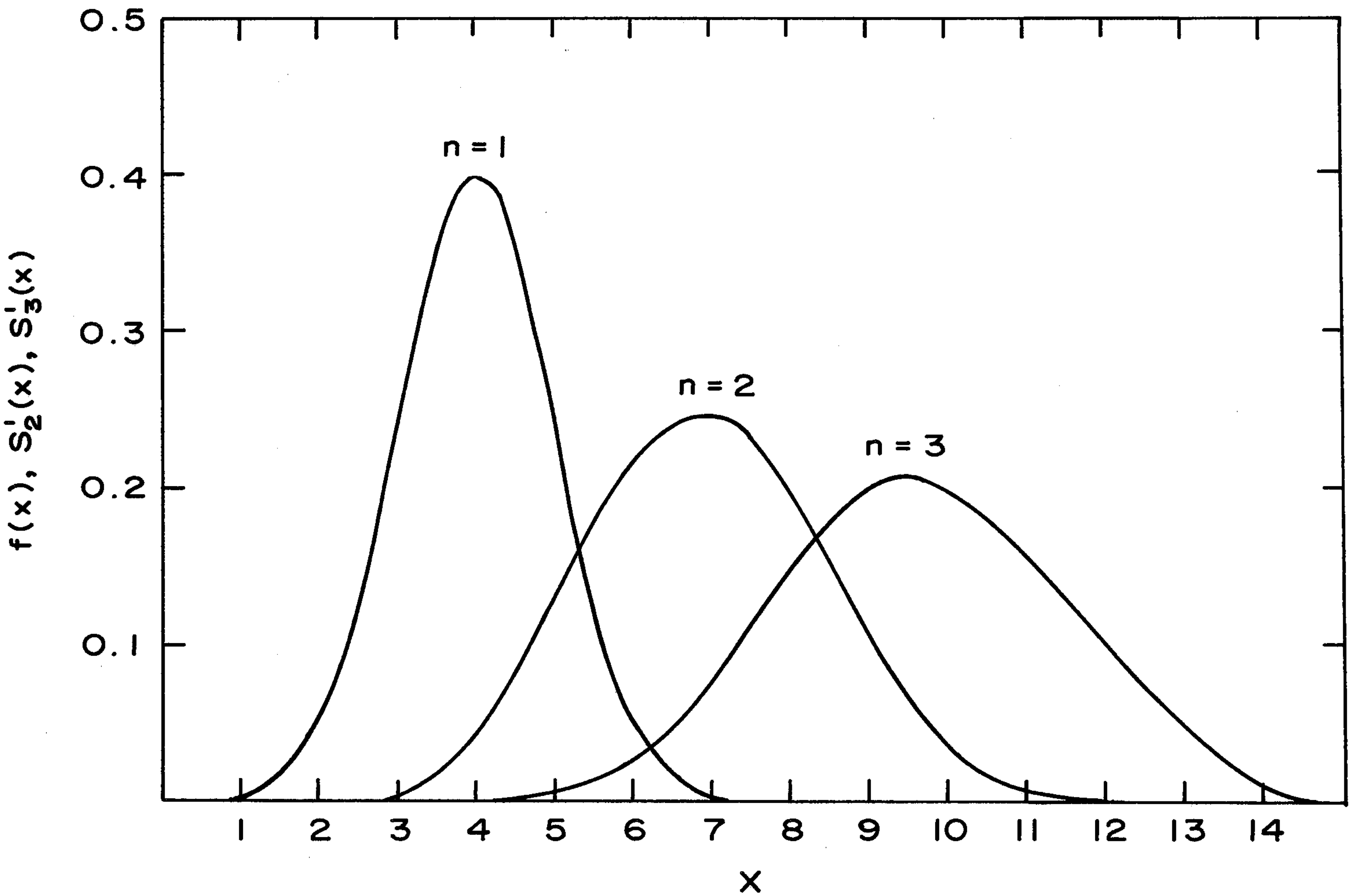

Figure 4. Density functions of bundle strength for n = 1, 2 and 3 based on filament strengths distributed as N(4,1)

For an exponential distribution, unlike the normal case, the expectation and variance of B_n can be obtained by simple integrations. For n = 1, 2 and 3, the major results are given in Table 1 based on $f(x) = \alpha e^{-\alpha x}$.

Table 1. $E(B_n)$ and $Var(B_n)$ for n = 1, 2 and 3 when $f(x) = \alpha e^{-\alpha x}$, $x \geq 0$

n	$E(B_n)$	$Var(B_n)$
1	$(\frac{1}{\alpha})$	$(\frac{1}{\alpha})^2$
2	$1.667\ (\frac{1}{\alpha})$	$1.444\ (\frac{1}{\alpha})^2$
3	$2.223\ (\frac{1}{\alpha})$	$1.761\ (\frac{1}{\alpha})^2$

In obtaining $E(B_n)$ and $Var(B_n)$ the following identities are useful.

$$E(B_n) = \int_0^\infty x\, S_n'(x)\, dx = \int_0^\infty \left[1 - S_n(x)\right] dx$$

$$E(B_n^2) = \int_0^\infty x^2\, S_n'(x)\, dx = 2 \int_0^\infty x \left[1 - S_n(x)\right] dx$$

It is noted in the table that the increase in expected bundle strength is not proportional to the increase in bundle size. It seems that the efficiency of a bundle decreases as its size increases. The general assertion of this phenomenon will be made in a later section.

For the cases where X_i are positive, but bounded random variables, special care has to be placed in the evaluation of $S_n(x)$. For example, when

$$\begin{aligned} f(x) &= 1 \quad , 0 \leq x \leq 1 \\ &= 0 \quad , \text{otherwise} \end{aligned}$$

the maximum possible value of B_n is n. Therefore, for n = 2,

$$S_2(x) = 2F(\frac{x}{2})\, F(x) - \left[F(\frac{x}{2})\right]^2$$

$$= 2(\frac{x}{2}) \cdot x - (\frac{x}{2})^2 = \frac{3}{4}x^2 \qquad , 0 \le x \le 1$$

$$= 2(\frac{x}{2}) \cdot 1 - (\frac{x}{2})^2 = x - \frac{1}{4}x^2 \qquad , 1 < x \le 2$$

$$= 1 \qquad , 2 < x$$

and for n = 3,

$$S_3(x) = 3F(\frac{x}{3})\, S_2(x) - 3\left[F(\frac{x}{3})\right]^2 F(x) + \left[F(\frac{x}{3})\right]^3$$

$$= 3\,(\frac{x}{3})\,(\frac{3}{4}x^2) - 3\,(\frac{x}{3})^2 \cdot x + (\frac{x}{3})^3 \qquad , 0 \le x \le 1$$

$$= 3\,(\frac{x}{3})\,(x - \frac{1}{4}x^2) - 3\,(\frac{x}{3})^2 \cdot 1 + (\frac{x}{3})^3 \qquad , 1 < x \le 2$$

$$= 3\,(\frac{x}{3}) \cdot 1 - 3\,(\frac{x}{3})^2 \cdot 1 + (\frac{x}{3})^3 \qquad , 2 < x \le 3$$

$$= 1 \qquad , 3 < x$$

and so on.

In Table 2, $E(B_n)$ and $Var(B_n)$ are obtained, for n = 1, 2 and 3, using the above results.

Table 2. $E(B_n)$ and $Var(B_n)$ for n = 1, 2 and 3 when $f(x) = 1,\ 0 \le x \le 1$

n	$E(B_n)$	$Var(B_n)$
1	$\frac{1}{2}$	$\frac{1}{12}$
2	$\frac{5}{6}$	$\frac{5}{36}$
3	$\frac{41}{36}$	$\frac{263}{(36)^2}$

Asymptotic Properties of Strength for Large Bundles

Due to the complex form of the probability distribution function of bundle strength as expressed by Eq. (2), the function $S_n(x)$ provides no simple operational ground which may lead to the asymptotic properties of bundle strength. The information as to how a particular choice of $F(x)$, the distribution of filament strength affects the properties of a large bundle is indeed difficult to be extracted from $S_n(x)$, regardless whether or not $F(x)$ is specified.

Therefore, the mode of attack in this section is to base the general reasoning on the basic definition of bundle strength and gradually unveil its inherent statistical nature.

Asymptotic Distribution of $\frac{B_n}{n}$

Instead of working with B_n, the strength of an n filament bundle, the asymptotic properties are to be examined for $\frac{B_n}{n}$, that is the average contribution of each constituent filament to the strength of the bundle.

From Eq. (1),

$$\frac{B_n}{n} = \max_{1 \leq k \leq n} \left\{ \frac{n-k+1}{n} Y_k \right\} , \quad 0 \leq Y_1 < Y_2 < \dots < Y_n \tag{3}$$

Define
$$\begin{aligned} F_n(x) &= 0 && , \ 0 \leq x \leq Y_1 \\ &= \frac{k}{n} && , \ Y_k < x \leq Y_{k+1}, \ k = 1, 2, \dots, n-1 \\ &= 1 && , \ Y_n < x \end{aligned}$$

i.e., $F_n(x)$ is the empirical distribution function corresponding to $F(x)$, and it is defined with left continuity instead of the usual right continuity. Then, $1 - F_n(x)$ is the step function as shown in Figure 5.

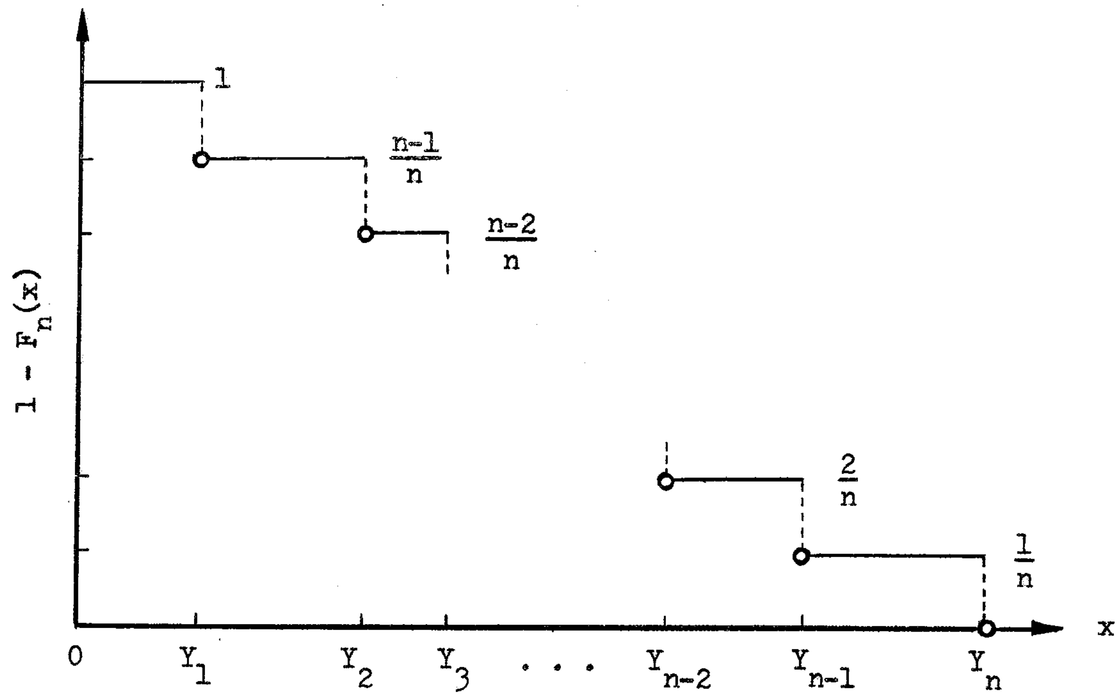

Figure 5. The step function $1 - F_n(x)$

According to the above definition, Eq. (3) is reduced to the following:

$$\frac{B_n}{n} = \max_{1 \leq k \leq n} \left\{ \left[1 - F_n(Y_k)\right] Y_k \right\}$$

$$= \max_{1 \leq k \leq n} \left[\max_{Y_{k-1} < x \leq Y_k} \left\{ \left[1 - F_n(x)\right] x \right\} \right] \quad \text{(where } Y_o = 0\text{)}$$

$$= \max_{0 < x \leq Y_n} \left\{ \left[1 - F_n(x)\right] x \right\}$$

$$= \max_{0 \leq x < \infty} \left\{ \left[1 - F_n(x)\right] x \right\} \qquad (4)$$

since $\left[1 - F_n(x)\right] x = 0$ for $x = 0$

and for $x > Y_n$.

The second equality above holds since $\left[1 - F_n(x)\right] x$ monotonically increases within each interval $Y_{k-1} < x \leq Y_k$ $(k = 1, 2, \ldots, n)$.

Thus, the original expression $\frac{B_n}{n}$, which was the maximum of $\left[1 - F_n(x)\right] x$ over the discrete points $Y_1, Y_2, \ldots, Y_n$, is now expressed in Eq. (4) as the maximum over the entire range of strengths for $x \geq 0$. Therefore, the asymptotic properties of $\frac{B_n}{n}$ is to be examined based on Eq. (4) from now on. Before considering the asymptotic behavior of $\frac{B_n}{n}$, the following theorem needs to be proved.

Theorem 1

The random variable $\left[1 - F_n(x)\right] x$ converges uniformly in probability to $\left[1 - F(x)\right] x$, i.e.,

$$P_r\left\{\sup_{0 \leq x < \infty} \left|\left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x\right| > \varepsilon\right\} \longrightarrow 0$$

as $n \longrightarrow \infty$ for an arbitrary $\varepsilon > 0$ if the second moment of filament strength $E(X^2)$ is finite, and $F(x)$ and $f(x)$ are absolutely continuous.

Proof:

$$P_r\left\{\sup_{0 \leq x < \infty} \left|\left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x\right| > \varepsilon\right\}$$

$$= P_r\left\{\max\left[\sup_{0 \leq x \leq Y_n} \left|\left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x\right| , \sup_{x > Y_n} \left[1 - F(x)\right] x\right] > \varepsilon\right\}$$

because, for $x > Y_n$, $1 - F_n(x) = 0$

$$\leq P_r\left\{\sup_{0 \leq x \leq Y_n} \left|\left[F_n(x) - F(x)\right] x\right| > \varepsilon\right\} + P_r\left\{\sup_{x > Y_n} \left[1 - F(x)\right] x > \varepsilon\right\} \quad (5)$$

The first term to the right of the above inequality gives:

$$P_r \left\{ \sup_{0 \leq x \leq Y_n} \left| \left[F_n(x) - F(x) \right] x \right| > \varepsilon \right\}$$

$$\leq P_r \left\{ \sup_{0 \leq x \leq Y_n} \left| F_n(x) - F(x) \right| Y_n > \varepsilon \right\}$$

$$= P_r \left\{ Y_n \cdot \sup_{0 \leq x \leq Y_n} \left| F_n(x) - F(x) \right| > \varepsilon \text{ , } Y_n \leq \lambda \sqrt{n} \right\}$$

$$+ P_r \left\{ Y_n \cdot \sup_{0 \leq x \leq Y_n} \left| F_n(x) - F(x) \right| > \varepsilon \text{ , } Y_n > \lambda \sqrt{n} \right\} \qquad (6)$$

where, λ is a positive constant.

In Eq. (6),

$$P_r \left\{ Y_n \cdot \sup_{0 \leq x < Y_n} \left| F_n(x) - F(x) \right| > \varepsilon \text{ , } Y_n \leq \lambda \sqrt{n} \right\}$$

$$\leq P_r \left\{ \lambda \sqrt{n} \cdot \sup_{0 \leq x \leq Y_n} \left| F_n(x) - F(x) \right| > \varepsilon \right\}$$

$$\leq P_r \left\{ \lambda \sqrt{n} \cdot \sup_{0 \leq x < \infty} \left| F_n(x) - F(x) \right| > \varepsilon \right\}$$

$$= P_r \left\{ \sqrt{n}\, D_n > \frac{\varepsilon}{\lambda} \right\}$$

where, $D_n = \sup_{0 \leq x < \infty} \left| F_n(x) - F(x) \right|$.

By Kolmogorov's Theorem (4) proved in 1933,

$$P_r \left\{ \sqrt{n}\, D_n < z \right\} \longrightarrow P_r \left\{ Z < z \right\}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 z^2} \qquad \text{for } z > 0$$

$$= 0 \qquad \text{for } z \leq 0$$

as $n \longrightarrow \infty$

i.e., $\sqrt{n}\, D_n \xrightarrow{d} Z$.

Therefore it is possible to write, for any z > 0

$$P_r\left\{\sqrt{n}\,D_n > z\right\} - P_r\left\{Z > z\right\} \le \frac{\delta}{8} \quad \text{for } n \ge n_1$$

and also z can be chosen to satisfy

$$P_r\left\{Z > z\right\} \le \frac{\delta}{8}$$

$$\therefore\ P_r\left\{\sqrt{n}\,D_n > z\right\} \le P_r\left\{Z > z\right\} + \frac{\delta}{8} \le \frac{\delta}{4} \quad \text{for } n \ge n_1 .$$

Letting $z = \frac{\varepsilon}{\lambda}$,

$$P_r\left\{\sqrt{n}\,D_n > \frac{\varepsilon}{\lambda}\right\} < \frac{\delta}{4} \text{ for } n \ge n_1 \tag{6a}$$

The last term in Eq. (6) is evaluated as follows:

$$P_r\left\{Y_n \cdot \sup_{0 \le x \le Y_n} \left|F_n(x) - F(x)\right| > \varepsilon\ ,\ Y_n > \lambda\sqrt{n}\right\}$$

$$\le P_r\left\{Y_n > \lambda\sqrt{n}\right\} .$$

To prove $P_r\left\{Y_n > \lambda\sqrt{n}\right\} \longrightarrow 0$ as $n \longrightarrow \infty$,

a theorem on p. 125 of Doob (3) is utilized here:

> <u>Theorem</u>: Let $Z_1, Z_2, \ldots$ be mutually independent random variables and let c be any positive constant. If
>
> $$\frac{1}{n}\sum_{j=1}^{n} Z_j \xrightarrow{p} 0$$
>
> then, $$\lim_{n\to\infty} \sum_{j=1}^{n} P_r\left\{\left|Z_j\right| > cn\right\} = 0$$

The condition, $E(X^2) < \infty$, given in Theorem 1 implies, by the weak law of large numbers, that

$$\frac{\sum_{j=1}^{n} X_j^2}{n} \xrightarrow{p} a \quad , \qquad 0 < a < \infty$$

or, $$\frac{\sum_{j=1}^{n} (X_j^2 - a)}{n} \xrightarrow{p} 0 \; .$$

Letting $X_j^2 - a = Z_j$, and applying the theorem by Doob,

$$\lim_{n \to \infty} \sum_{n=1}^{n} P_r \left\{ \left| X_j^2 - a \right| > cn \right\} = 0$$

Now, $P_r \left\{ Y_n > \lambda \sqrt{n} \right\} = P_r \left\{ Y_n^2 > \lambda^2 n \right\}$

$$= P_r \left\{ Y_n^2 - a > \lambda^2 n - a \right\} = P_r \left\{ \frac{Y_n^2 - a}{n} > \lambda^2 - \frac{a}{n} \right\}$$

$$\leq P_r \left\{ \frac{\left| Y_n^2 - a \right|}{n} > \lambda^2 - \frac{a}{n} \right\}$$

$$\leq \sum_{i=1}^{n} P_r \left\{ \frac{\left| X_i^2 - a \right|}{n} > \lambda^2 - \frac{a}{n} \right\} .$$

Let n_o be a number such that $\lambda^2 - \frac{a}{n} > 0$

for $n > n_o$ ($n_o = 1$ if $\lambda^2 > a$). Then, from above,

$$P_r \left\{ Y_n > \lambda \sqrt{n} \right\} \leq \sum_{i=1}^{n} P_r \left\{ \frac{\left| X_i^2 - a \right|}{n} > \lambda^2 - \frac{a}{n_o} \right\} \text{ for } n > n_o$$

letting $c = \lambda^2 - \frac{a}{n_o}$,

$$= \sum_{i=1}^{n} P_r \left\{ \left| X_i^2 - a \right| > cn \right\} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

or, $$\sum_{i=1}^{n} P_r \left\{ \left| X_i^2 - a \right| > cn \right\} < \frac{\delta}{4} \text{ for } n \geq n_2 \, .$$

Therefore, $P_r \left\{ Y_n \geq \lambda \sqrt{n} \right\} < \frac{\delta}{4}$ for $n \geq \max \left\{ n_o, n_2 \right\}$. (6b)

Combining the results given by Eqs. (6a) and (6b), Eq. (6) is now rewritten as

$$P_r \left\{ \sup_{0 \leq x \leq Y_n} \left| \left[F_n(x) - F(x) \right] x \right| > \varepsilon \right\} < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} \tag{7}$$

for $n \geq \max \left\{ n_o, n_1, n_2 \right\}$.

Finally, the second term to the right of the inequality in Eq. (5) has to be evaluated. For this, the finiteness of $E(X^2)$ as an assumption has to be slightly modified. Due to the equalities

$$E(X^2) = \int_0^\infty x^2 f(x)\, dx$$

$$= 2 \int_0^\infty \left[1 - F(x) \right] x \, dx ,$$

the condition $E(X^2) < \infty$ immediately implies

$$\lim_{x \to \infty} \left[1 - F(x) \right] x = 0$$

i.e., there exists a point x' such that for $x > x'$, $\left[1 - F(x) \right] x < \varepsilon$ for an arbitrary $\varepsilon > 0$. Utilizing this,

$$P_r \left\{ \sup_{x > Y_n} \left[1 - F(x) \right] x > \varepsilon \right\}$$

$$= P_r \left\{ \sup_{x > Y_n} \left[1 - F(x) \right] x > \varepsilon \, , \, Y_n > x' \right\}$$

$$+ P_r \left\{ \sup_{x > Y_n} \left[1 - F(x) \right] x > \varepsilon \, , \, Y_n \leq x' \right\}$$

$$\leq 0 + P_r \left\{ Y_n \leq x' \right\} = \left[F(x') \right]^n < \frac{\delta}{2} \tag{8}$$

for $n \geq n_3$ say.

Now, Eq. (7) and Eq. (8) are substituted in Eq. (5) to yield the final inequality;

$$P_r\left\{ \sup_{0 \leq x < \infty} \left| \left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x \right| > \varepsilon \right\} < \frac{\delta}{2} + \frac{\delta}{2} = \delta \qquad (9)$$

$$\text{for } n \geq \max \left\{ n_o, n_1, n_2, n_3 \right\} .$$

Since ε and δ are arbitrary positive values, it follows that

$$P_r\left\{ \sup_{0 \leq x < \infty} \left| \left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x \right| > \varepsilon \right\} \longrightarrow 0$$

as $n \longrightarrow \infty$ and this completes the proof.

In the light of Theorem 1, the asymptotic properties of $\frac{B_n}{n}$ are deduced by noting the following. First, it is true from Theorem 1 that the modal value of $\left[1 - F_n(x)\right] x$ converges in probability to the modal value of $\left[1 - F(x)\right] x$ due to the well known inequality

$$\left| \sup_{0 \leq x < \infty} \left[1 - F_n(x)\right] x - \sup_{0 \leq x < \infty} \left[1 - F(x)\right] x \right|$$

$$\leq \sup_{0 \leq x < \infty} \left| \left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x \right|$$

which leads to

$$P_r\left\{ \left| \sup_{0 \leq x < \infty} \left[1 - F_n(x)\right] x - \sup_{0 \leq x < \infty} \left[1 - F(x)\right] x \right| > \varepsilon \right\}$$

$$\leq P_r\left\{ \sup_{0 \leq x < \infty} \left| \left[1 - F_n(x)\right] x - \left[1 - F(x)\right] x \right| > \varepsilon \right\} \longrightarrow 0$$

as $n \longrightarrow \infty$, i.e.,

$$\sup_{0 \leq x < \infty} \left[1 - F_n(x)\right] x \xrightarrow{p} \sup_{0 \leq x < \infty} \left[1 - F(x)\right] x$$

or, $$\frac{B_n}{n} \xrightarrow{p} \left[1 - F(x_o)\right] x_o \qquad (10)$$

where, x_o is the value of x at which $\left[1 - F(x)\right] x$ is maximized. Here, $\left[1 - F(x)\right]x$ is assumed to be a unimodal function. On the other hand, directly from Theorem 1,

$$\left[1 - F_n(x_o)\right] x_o \xrightarrow{p} \left[1 - F(x_o)\right] x_o \quad . \tag{11}$$

Therefore, combining Eqs. (10) and (11), it is concluded that

$$\frac{B_n}{n} - \left[1 - F_n(x_o)\right] x_o \xrightarrow{p} 0 \ . \tag{12}$$

The distribution of $\left[1 - F_n(x_o)\right]x_o$, however, depends on the basic properties of the sample distribution $F_n(x)$. Due to the fact that $nF_n(x)$ is same as the number of observations that fall below x among n total observations, $nF_n(x)$ follows a binomial distribution with F(x) as the occurrence probability. Moreover, it is well known that the distribution of $F_n(x)$ is asymptotically normal with mean F(x), and variance $\frac{1}{n} F(x)\left[1 - F(x)\right]$ for any $x > 0$. Therefore, the asymptotic distribution of $\left[1 - F_n(x_o)\right] x_o$ is normal with mean $\left[1 - F(x_o)\right] x_o$ and variance $\frac{1}{n} F(x_o) \left[1 - F(x_o)\right] x_o^2$ as x_o is a constant, i.e.,

$$\left[1 - F_n(x_o)\right] x_o \sim \text{A.N.} \left(\left[1 - F(x_o)\right] x_o, \frac{1}{n} F(x_o) \left[1 - F(x_o)\right] x_o^2 \right). \tag{13}$$

The major conclusion given in Daniels' (2) work that

$$\frac{B_n}{n} \sim \text{A.N.} \left(\left[1 - F(x_o)\right] x_o, \frac{1}{n} F(x_o)\left[1 - F(x_o)\right] x_o^2 \right) \tag{14}$$

is plausible when the asymptotic distribution of $\left[1 - F_n(x_o)\right] x_o$ is combined with the result given by Eq. (12). This is not to say that Eq. (12) is a sufficient condition for concluding that the asymptotic distribution of $\frac{B_n}{n}$ is exactly same as that of $\left[1 - F_n(x_o)\right] x_o$. However, it is the view of this author that Eq. (12) provides a better ground to approximate the asymptotic properties of $\frac{B_n}{n}$ by the above compared to the various approximation factors utilized in Daniels' (2) study.

Wasted Fraction of Filaments in a Bundle

Recalling the definition of B_n given by Eq. (1), the filaments with strength $Y_1, Y_2, \ldots, Y_{j-1}$ would contribute no part to the strength of a bundle if B_n is defined at the j^{th} filament breakage, and the maximum bundle load $B_n = (n-j+1)\, Y_j$ would be supported by the filaments with strength $Y_j, Y_{j+1}, \ldots, Y_n$. It is of interest, therefore, to estimate the quantity of wasted filaments in a bundle insofar as the strength of the bundle is concerned. For this, the order j of Y_j that corresponds to

$$\max_{1 \leq k \leq n} P_r \left\{ B_n = (n-k+1)\, Y_k \right\} \tag{15}$$

is a reasonable likelihood estimate of the number wasted. Noting that

$$P_r \left\{ B_n = (n-j+1)\, Y_j \right\} = P_r \left[\max_{1 \leq k \leq n} \left\{ (n-k+1)\, Y_k \right\} = (n-j+1)\, Y_j \right],$$

the event $B_n = (n-j+1)\, Y_j$ is converted to the following set of inequalities which leads to Eq. (16);

$$Y_j \geq \frac{n}{n-j+1} Y_1, \frac{n-1}{n-j+1} Y_2, \ldots, \frac{n-j+2}{n-j+1} Y_{j-1}, \frac{n-j}{n-j+1} Y_{j+1},$$

$$\frac{n-j-1}{n-j+1} Y_{j+2}, \ldots, \frac{1}{n-j+1} Y_n .$$

$$P_r \left\{B_n = (n-j+1)\, Y_j\right\}$$

$$= n! \int_0^{\infty} f(Y_j) \left[\int_0^{\frac{Z}{n}} \int_{Y_1}^{\frac{Z}{n-1}} \cdots \int_{Y_{j-2}}^{\frac{Z}{n-j+2}} f(Y_1)\, f(Y_2) \cdots f(Y_{j-1})\, dY_{j-1} \ldots dY_2 dY_1 \right]$$

$$\cdot \left[\int_{Y_j}^{\frac{Z}{n-j}} \int_{Y_{j+1}}^{\frac{Z}{n-j-1}} \cdots \int_{Y_{n-1}}^{Z} f(Y_{j+1})\, f(Y_{j+2}) \cdots f(Y_n)\, dY_n \ldots dY_{j+2} dY_{j+1} \right] dY_j \qquad (16)$$

where $Z = (n-j+1)\, Y_j$.

For a bundle with small n, Eq. (16) can be used to find the order j satisfying Eq. (15). When n becomes larger, however, the analytic difficulty of Eq. (15) prevents the practical use of the criteria given by Eq. (15).

An alternative way of estimating the wasted fraction is to utilize the value x_o at which $\left[1 - F(x)\right] x$ is maximized. The convergence of $\frac{B_n}{n}$ to $\left[1 - F(x_o)\right] x_o$ in probability, as shown by Eq. (10), suggests that $n \left[1 - F(x_o)\right]$ is close to the number of filaments surviving at the particular order of filament breakage where the bundle load is maximized to represent B_n. Therefore, the number wasted is estimated simply by $nF(x_o)$, and the wasted fraction by $F(x_o)$, when n is very large. For a small n, one of the following two methods can be used to estimate the wasted fraction:

a) If x_o is readily obtainable, find the order j for which $E(Y_j)$ is the nearest to x_o among all j = 1, 2, ..., n. Then, $\frac{j}{n}$ is an estimate of the wasted fraction.

b) If x_o is difficult to obtain, utilize the fact that x_o is determined from

$$\frac{d}{dx}\left[1 - F(x)\right] x = 0$$

$$\Rightarrow f(x_o)x_o = 1 - F(x_o) .$$

Setting $x_o = Y_j$ and taking expectations on both sides, it is shown that

$$E\left[f(Y_j)Y_j\right] = E\left[1 - F(Y_j)\right]$$

$$= \int_0^\infty \frac{n!}{(j-1)!(n-j)!}\left[F(Y_j)\right]^{j-1}\left[1 - F(Y_j)\right]^{n-j+1} dY_j$$

$$= \frac{n-j+1}{n+1}$$

i.e., $E\left[f(Y_j)Y_j\right] = \frac{n-j+1}{n+1}$. (17)

In solving Eq. (17) for j, the integer value j is the approximate one that satisfies Eq. (17) best. Of course, the left side of the equality is difficult to obtain for certain classes of distributions.

It can be shown, in the following example, that the methods a) and b) are consistent with the large n case where the wasted fraction is estimated by $F(x_o)$:

For $f(x) = \alpha e^{-\alpha x}$,

$$x_o = \frac{1}{\alpha} , \ F(x_o) = 1 - \frac{1}{e} \doteq 0.6321 .$$

Method (a) gives, by setting $x_o = E(Y_j)$,

$$\frac{1}{\alpha} = \frac{1}{\alpha}\left(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{n-j+1}\right)$$

$$\doteq \frac{1}{\alpha} \int_{n-j+1}^{n} \frac{1}{x}\, dx = \frac{1}{\alpha} \log \frac{n}{n-j+1} \qquad \text{for large } n$$

$$\Longrightarrow e = \frac{n}{n-j+1}$$

$$\Longrightarrow j \doteq 0.6321\, n + 1$$

$$\therefore \lim_{n \to \infty} \frac{j}{n} = 0.6321 .$$

Also, method (b) gives, by setting

$$E\left[f(Y_j)Y_j\right] = \frac{n-j+1}{n+1} ,$$

$$\Longrightarrow \left[\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{n-j+2}\right] = 1$$

$\Longrightarrow$ Similarly as above,

$$j \doteq 0.6321\, n + 1.6321$$

$$\therefore \lim_{n \to \infty} \frac{j}{n} = 0.6321 ,$$

and the results of (a) and (b) agree with $F(x_o)$.

Inequalities Concerning the Moments of the Bundle Strength Distribution

In order to answer the question as to how the size of a bundle governs its strength, the moments of B_n can be compared for different n. For this, the distribution function $S_n(x)$ has to be utilized and $F(x)$ has to be known exactly. Although this is the only way to obtain the moments of B_n, the results for limited n and a particular $F(x)$ hardly assert a general law applicable to other underlying distributions.

Comparisons of moments are made in this study on the basis of the definition of B_n given by Eq. (1). For a general unspecified $F(x)$, a few useful inequalities are obtained concerning the moments of B_n, and upper and lower bounds are established for the moments of $\frac{B_n}{n}$. The same inequalities are proved for a finite population case under the scheme of simple random sampling.

Monotonicity of $E(\frac{B_n}{n})^k$ with Unknown F(x)

The following lemma is essential in order to prove the theorem on monotonicity of $E(\frac{B_n}{n})^k$ in n.

Lemma 1

For any set of n+1 filaments ($n \geq 1$), let the filament strengths be $Y_1, Y_2, \ldots, Y_{n+1}$ in increasing order, and define

B_{n+1} = strength of the n+1 filament bundle with all Y's (filaments)

B_{ni} = strength of the n filament bundle with all Y's but Y_i,

$i = 1, 2, \ldots, n+1$

i.e., $B_{n1}, B_{n2}, \ldots, B_{n(n+1)}$ are the strengths of all possible n filament bundles that can be generated by the original n+1 filament bundle. Then, the following inequality holds between B_{n+1} and the B_{ni}.

$$n^k B_{n+1}^k \leq (n+1)^{k-1} \sum_{i=1}^{n+1} B_{ni}^k, \quad n, k \geq 1 \tag{18}$$

Proof:

a) k = 1 case

From the definition of B_n given by Eq. (1), the inequality

$$n\, B_{n+1} \leq \sum_{i=1}^{n+1} B_{ni}$$

can be rewritten as follows:

$$
\begin{aligned}
& n \cdot \max\left\{(n+1)Y_1,\ nY_2,\ (n-1)Y_3,\ \ldots,\ 3Y_{n-1},\ 2Y_n,\ Y_{n+1}\right\} \\
\leq\ & \max\left\{nY_2,\ (n-1)Y_3,\ (n-2)Y_4,\ \ldots,\ 3Y_{n-1},\ 2Y_n,\ Y_{n+1}\right\} \\
+\ & \max\left\{nY_1,\ (n-1)Y_3,\ (n-2)Y_4,\ \ldots,\ 3Y_{n-1},\ 2Y_n,\ Y_{n+1}\right\} \\
+\ & \max\left\{nY_1,\ (n-1)Y_2,\ (n-2)Y_4,\ \ldots,\ 3Y_{n-1},\ 2Y_n,\ Y_{n+1}\right\} \\
& \vdots \\
+\ & \max\left\{nY_1,\ (n-1)Y_2,\ (n-2)Y_3,\ \ldots,\ 3Y_{n-2},\ 2Y_n,\ Y_{n+1}\right\} \\
+\ & \max\left\{nY_1,\ (n-1)Y_2,\ (n-2)Y_3,\ \ldots,\ 3Y_{n-2},\ 2Y_{n-1},\ Y_{n+1}\right\} \\
+\ & \max\left\{nY_1,\ (n-1)Y_2,\ (n-2)Y_3,\ \ldots,\ 3Y_{n-2},\ 2Y_{n-1},\ Y_n\right\} . \qquad (19)
\end{aligned}
$$

If the inequality is proved for every possible choice of $B_{n+1} = (n-j+2)Y_j$, $j = 1, 2, \ldots, n+1$, then the proof will be complete. Suppose $B_{n+1} = (n-j+2)Y_j \quad (j = 1, 2, \ldots, n+1)$. Then, it is noted for the terms to the right of the inequality in Eq. (19) that

$$B_{n1},\ B_{n2},\ \ldots,\ B_{n(j-1)} \geq (n-j+2)Y_j$$

$$B_{nj} \geq (n-j+1)Y_{j+1} \geq (n-j+1)Y_j$$

$$B_{n(j+1)},\ B_{n(j+2)},\ \ldots,\ B_{n(n+1)} \geq (n-j+1)Y_j$$

$$
\begin{aligned}
\therefore \sum_{i=1}^{n+1} B_{ni} &\geq (j-1)\ (n-j+2)Y_j + (n-j+2)\ (n-j+1)Y_j \\
&= n(n-j+2)Y_j \\
&= n\ B_{n+1} \text{ by definition}
\end{aligned}
$$

i.e., $$n\ B_{n+1} \leq \sum_{i=1}^{n+1} B_{ni} .$$

b) $k \geq 2$ case

First note that $B^k_{n+1} = \max_{1 \leq j \leq n+1} \left\{(n-j+2)^k Y^k_j\right\}$ and similarly for every B^k_{ni} $(i = 1, 2, \ldots, n+1)$ the terms inside the brackets are raised to the k^{th} power. If the inequality is proved for every possible choice of B^k_{n+1}, then the proof will be complete. Suppose $B^k_{n+1} = (n-j+2)^k Y^k_j$ $(j = 1, 2, \ldots, n+1)$. Then similar to the case with $k = 1$, it is noted in Eq. (19) that

$$B^k_{n1}, B^k_{n2}, \ldots, B^k_{n(j-1)} \geq (n-j+2)^k Y^k_j$$

$$B^k_{nj} \geq (n-j+1)^k Y^k_{j+1} \geq (n-j+1)^k Y^k_j$$

$$B^k_{n(j+1)}, B^k_{n(j+2)}, \ldots, B^k_{n(n+1)} \geq (n-j+1)^k Y^k_j$$

$$\therefore \sum_{i=1}^{n+1} B^k_{ni} \geq \left[(j-1)(n-j+2)^k + (n-j+2)(n-j+1)^k\right] Y^k_j .$$

Therefore, to prove Eq. (18) one has to show

$$\left[(j-1)(n-j+2)^k + (n-j+2)(n-j+1)^k\right] Y^k_j \geq \frac{n^k}{(n+1)^{k-1}} (n-j+2)^k Y^k_j$$

$$= \frac{n^k}{(n+1)^{k-1}} B^k_{n+1}$$

or, $$\left[(j-1) + \frac{(n-j+1)^k}{(n-j+2)^{k-1}}\right] \geq \frac{n^k}{(n+1)^{k-1}} \quad (j = 1, 2, \ldots, n+1). \qquad (20)$$

For $j = n+1$, the above is true since $n \geq \frac{n^k}{(n+1)^{k-1}}$.

For $1 \leq j \leq n$, the monotonicity of $C(x)$, the left side of Eq. (20) with $j = x$, is examined by treating x as if it is continuous in $[1, n]$;

$$\frac{d}{dx} C(x) = \frac{d}{dx}\left[(x-1) + \frac{(n-x+1)^k}{(n-x+2)^{k-1}}\right]$$

$$= 1 - k(n-x+1)^{k-1}\,(n-x+2)^{-k+1} + (k-1)(n-x+1)^k(n-x+2)^{-k}$$

letting $y = n-x+1$,

$$= \frac{y^{k-1}}{(y+1)^k}\left[\frac{(y+1)^k}{y^{k-1}} - (k+1)\right] \qquad (y \neq 0)$$

$$= \frac{y^{k-1}}{(y+1)^k}\left[\frac{1}{y^{k-1}}\left\{y^k + \cancel{k}y^{k-1} + \binom{k}{2}y^{k-2} + \dots + ky+1\right\} - (\cancel{k}+1)\right]$$

$$= \frac{y^{k-1}}{(y+1)^k}\left[(y-1) + \binom{k}{2}\frac{1}{y} + \dots + \frac{k}{y^{k-2}} + \frac{1}{y^k}\right] > 0$$

for $y \geq 1$ or equivalently for $x \leq n$.

i.e., $C(x)$ is a monotone increasing function of x for $1 \leq x \leq n$.

Therefore, $\min_{1 \leq x \leq n} C(x) = C(1) = \dfrac{n^k}{(n+1)^{k-1}}$

and $C(x) > \dfrac{n^k}{(n+1)^{k-1}}$ for $1 < x \leq n$.

Combining the result for the case $j = n+1$, it is concluded that

$$(j-1) + \frac{(n-j+1)^k}{(n-j+2)^{k-1}} \geq \frac{n^k}{(n+1)^{k-1}}$$

for every $j = 1, 2, \dots, n+1$

i.e., Eq. (18) is true for $k \geq 2$, and this completes the proof of Lemma 1.

Note: Proof (b) above also applies to the case $k = 1$. However (a) is given for the simplicity of the proof available for that case.

Theorem 2

Under the restricted model for bundle breakage, $E(\frac{B_n}{n})^k$ $(k = 1, 2, 3, \ldots)$ decreases monotonically in n under any continuous distribution F(x) of filament strength provided the strengths of the constituent filaments $X_1, X_2, \ldots, X_n$ are independent and interchangeable random variables.

Proof:

Observe in Eq. (18) of Lemma 1 that $B_{n+1}, B_{n1}, B_{n2}, \ldots$ and $B_{n(n+1)}$ are functions dependent on all or all but one of $X_1, X_2, \ldots, X_{n+1}$. Let the joint density of $X_1, X_2, \ldots, X_{n+1}$ be $f(x_1, x_2, \ldots, x_{n+1})$ and the joint density of $X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots X_{n+1}$ be $f_j(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$ for $j = 1, 2, \ldots, n+1$. The inequality in Eq. (18) is preserved by multiplying both sides by $f(x_1, x_2, \ldots, x_{n+1})$ and integrating over $x_1, x_2, \ldots,$ and x_{n+1}. Therefore, utilizing the symmetry

$$\sum_{i=1}^{n+1} B_{ni} (y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1})$$

$$= \sum_{j=1}^{n+1} B_{nj} (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

Eq. (18) provides the inequality

$$n^k \int \underset{R}{\cdots} \int B_{n+1}^k(x_1, x_2, \ldots, x_{n+1}) f(x_1, x_2, \ldots, x_{n+1}) \prod_{i=1}^{n+1} dx_i$$

$$\leq (n+1)^{k-1} \sum_{j=1}^{n+1} \int \underset{R}{\cdots} \int B_{nj}^k (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

$$\cdot f (x_1, x_2, \ldots, x_{n+1}) \prod_{i=1}^{n+1} dx_i$$

$$= (n+1)^{k-1} \sum_{j=1}^{n+1} \int \underset{R_j}{\cdots} \int B_{nj}^k(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

$$\cdot f_j(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) \prod_{i \neq j}^{n+1} dx_i$$

where, R is the proper region of integration for $\left\{x_1, x_2, \ldots, x_{n+1}\right\}$, and R_j for $\left\{x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}\right\}$, $j = 1, 2, \ldots, n+1$.

Since the X_i are interchangeable, the above inequality is reduced to the following:

$$n^k E(B_{n+1}^k) \leq (n+1)^{k-1} \sum_{j=1}^{n+1} E(B_{nj}^k)$$

$$= (n+1)^k E(B_n^k)$$

i.e., $E\left(\frac{B_{n+1}}{n+1}\right)^k \leq E\left(\frac{B_n}{n}\right)^k$ for k, n $\geq$ 1

or, $E\left(\frac{B_n}{n}\right)^k \leq E\left(\frac{B_m}{m}\right)^k$ for $k = 1, 2, 3, \ldots$ and $1 \leq m < n$. (21)

As indicated by Theorem 2, the efficiency of a bundle, measured by $E(\frac{B_n}{n})$, is likely to decrease as n increases. This in fact is in accord with the same general trend often revealed by actual experiments. Unfortunately, the theorem does not provide any evidence as to whether $Var(\frac{B_n}{n})$ also will decrease monotonically in n.

Although the following theorem may be intuitively obvious, it is added here for completeness.

Theorem 3

Under the same assumptions and definitions given in Theorem 2, the inequality

$$E(B_n^k) \leq E(B_{n+1}^k) \tag{22}$$

holds for $k, n = 1, 2, 3, \ldots$.

<u>Proof</u>:

Observing the terms B_{n1}, B_{n2}, ..., $B_{n(n+1)}$ of the right side of Eq. (19), the coefficients of the Y_j's in a particular column are common whereas the orders of Y_j's can decrease from top to bottom. Therefore it is evident that

$$B_{n1} \geq B_{n2} \geq \cdots \geq B_{n(n+1)} .$$

But, $B_{n+1} = \max \left\{ (n+1)Y_1,\ B_{n1} \right\}$.

Therefore, $B_{n+1} \geq B_{n1} \geq B_{n2} \geq \cdots \geq B_{n(n+1)}$

and $\sum_{i=1}^{n+1} B_{ni}^k \leq (n+1)\, B_{n+1}^k$.

The same integration method used in the proof of Theorem 2 leads from the above inequality to

$$(n+1)\ E(B_n^k) \leq (n+1)\ E(B_{n+1}^k)$$

$$\text{i.e., } E(B_n^k) \leq E(B_{n+1}^k).$$

<u>Monotonicity of $E(\frac{B_n}{n})^k$ in Finite Populations</u>

Consider a finite population of N filaments and the all possible bundles of 1, 2, 3, ..., and N filaments that can be generated from the population. If the probability of selecting a particular n filament bundle ($n \leq N$) is determined under the scheme of simple random sampling, the k^{th} moment of bundle strength with n filaments is expressed by

$$E(B_n^k) = \frac{1}{\binom{N}{n}} \sum_{j=1}^{\binom{N}{n}} B_{nj}^k .$$

With this definition, the following theorem is given in parallel with Theorem 2.

Theorem 4

In a finite population of N filaments, $E(\frac{B_n}{n})^k$ decreases monotonically in n for $1 \leq n \leq N$ under the scheme of simple random sampling.

Proof:

Consider all possible bundles of n and n+1 filaments that can be generated from the population of N filaments for $1 \leq n \leq N-1$, and designate their strengths by

$$B_{nj} \quad , \quad j = 1, 2, \ldots, \binom{N}{n}$$

and $$B_{(n+1)\ell} \quad , \quad \ell = 1, 2, \ldots, \binom{N}{n+1} .$$

Note that each of the $\binom{N}{n+1}$ bundles of n+1 filaments can generate bundles of n filaments in exactly n+1 different ways, and that a particular bundle of n filaments can be generated by exactly N-n different bundles of n+1 filaments.

Define ${}_{\ell}B_{ni}$ = strength of a particular bundle (called i, say) of n filaments that is generated by a particular bundle (called ℓ, say) of n+1 filaments among the $\binom{N}{n+1}$ bundles of n+1 filaments, where

$$i = 1, 2, \ldots, n+1$$
$$j = 1, 2, \ldots, \binom{N}{n+1} .$$

Then, from the array of all possible ${}_{\ell}B_{ni}$'s for i = 1, 2, ..., n+1 and $\ell = 1, 2, \ldots, \binom{N}{n+1}$ a particular bundle of n filaments, say B_{nj}, will be found exactly N-n times for every $j = 1, 2, \ldots, \binom{N}{n}$. This is true from the preceding observation and the fact that ${}_{\ell}B_{ni}$ must belong to one of the B_{nj}'s, $j = 1, 2, \ldots, \binom{N}{n}$. Therefore, it follows that

$$\sum_{\ell=1}^{\binom{N}{n+1}} \sum_{i=1}^{n+1} {}_{\ell}B_{ni}^{k} = (N - n) \sum_{j=1}^{\binom{N}{n}} B_{nj}^{k}, \quad k = 1, 2, 3, \ldots \quad . \tag{23}$$

However, Eq. (18) of Lemma 1 gives for any $\ell = 1, 2, \ldots, \binom{N}{n+1}$ that

$$n^{k} B_{(n+1)\ell}^{k} \leq (n+1)^{k-1} \sum_{i=1}^{n+1} {}_{\ell}B_{ni}^{k} \quad .$$

Summing both sides on ℓ,

$$n^{k} \sum_{\ell=1}^{\binom{N}{n+1}} B_{(n+1)\ell}^{k} \leq (n+1)^{k-1} \sum_{\ell=1}^{\binom{N}{n+1}} \sum_{i=1}^{n+1} {}_{\ell}B_{ni}^{k}$$

$$= (n+1)^{k-1} (N-n) \sum_{j=1}^{\binom{N}{n}} B_{nj}^{k} \quad \text{by Eq. (23)}$$

or $$\sum_{\ell=1}^{\binom{N}{n+1}} \left[\frac{B_{(n+1)\ell}}{n+1}\right]^{k} \leq \frac{N-n}{n+1} \sum_{j=1}^{\binom{N}{n}} \left[\frac{B_{nj}}{n}\right]^{k}$$

$$\frac{1}{\binom{N}{n+1}} \sum_{\ell=1}^{\binom{N}{n+1}} \left[\frac{B_{(n+1)\ell}}{n+1}\right]^{k} \leq \frac{N-n}{n+1} \frac{1}{\binom{N}{n+1}} \sum_{j=1}^{\binom{N}{n}} \left[\frac{B_{nj}}{n}\right]^{k}$$

$$= \frac{1}{\binom{N}{n}} \sum_{j=1}^{\binom{N}{n}} \left[\frac{B_{nj}}{n}\right]^{k}$$

i.e., $E\left(\frac{B_{n+1}}{n+1}\right)^{k} \leq E\left(\frac{B_{n}}{n}\right)^{k}, \quad 1 \leq n \leq N-1, \qquad k = 1, 2, 3, \ldots$

and this proves the theorem.

Corollary

$$\left(\frac{B_n}{n}\right)^k \leq \frac{1}{n} \sum_{i=1}^{n} X_i^k , \quad k = 1, 2, 3, \ldots$$

and, in particular, for k = 1

$$\frac{B_n}{n} \leq \bar{X}_n \text{ or, } B_n = \sum_{i=1}^{n} X_i \tag{24}$$

i.e., the strength of an n filament bundle is less than or equal to the sum of its constituent filament strengths.

Proof:

Let N = n in Theorem 4. Then,

$$E\left(\frac{B_n}{n}\right)^k = \left(\frac{B_n}{n}\right)^k \leq E\left(\frac{B_1}{1}\right)^k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

i.e., $\left(\frac{B_n}{n}\right)^k \leq \frac{1}{n} \sum_{i=1}^{n} X_i^k$,

and for k = 1,

$$\frac{B_n}{n} \leq \bar{X}_n \quad .$$

The implication of this corollary is not startling at all. It is interesting, however, that such an assertion can be made based on the particular assumptions with which the breakage model was established.

Upper and Lower Bounds of $E(\frac{B_n}{n})^k$

Having established the monotonicity of $E(\frac{B_n}{n})^k$, it will be of practical value if there exists a lower bound of $E(\frac{B_n}{n})^k$ that is greater than the trivial lower bound zero. For this, the following lemma is given.

Lemma 2

If $Z_1, Z_2, Z_3, \ldots$ is a sequence of non-negative random variables that converges to a constant $c \geq 0$ in probability, i.e., if $Z_n \xrightarrow{p} c$ then, $\overline{\lim}\, E(Z_n^k) \geq c^k$, $c \geq 0$, $k = 1, 2, 3, \ldots$

Proof:

By the Markov inequality,

$$P_r\left\{Z_n \geq c - \varepsilon\right\} \leq \frac{E(Z_n^k)}{(c-\varepsilon)^k} \quad \text{for} \quad 0 \leq \varepsilon < c \text{ , or}$$

$$E(Z_n)^k \geq (c-\varepsilon)^k P_r\left\{Z_n \geq c - \varepsilon\right\} .$$

Taking $\overline{\lim}$ on both sides,

$$\begin{aligned} \overline{\lim}\, E(Z_n^k) &\geq (c-\varepsilon)^k\, \overline{\lim}\, P_r\left\{Z_n \geq c - \varepsilon\right\} \\ &\geq (c-\varepsilon)^k\, \overline{\lim}\, P_r\left\{c - \varepsilon \leq Z_n \leq c + \varepsilon\right\} \\ &= (c-\varepsilon)^k\, \overline{\lim}\, P_r\left\{\left|Z_n - c\right| \leq \varepsilon\right\} \\ &= (c-\varepsilon)^k \cdot 1 \quad \text{since } Z_n \xrightarrow{p} c \text{ .} \end{aligned}$$

By letting $\varepsilon \longrightarrow 0$ it follows that

$$\overline{\lim}\, E(Z_n^k) \geq c^k .$$

The lower bound of $E(\frac{B_n}{n})^k$ is easily obtainable by using the above lemma with $Z_n = \frac{B_n}{n}$ and $c = \left[1 - F(x_o)\right] x_o$. From Eq. (10), $\frac{B_n}{n} \xrightarrow{p} \left[1 - F(x_o)\right] x_o$. Therefore,

$$\overline{\lim}\, E(\frac{B_n}{n})^k \geq \left[1 - F(x_o)\right]^k x_o^k .$$

But, because $E(\frac{B_n}{n})^k$ is a monotone increasing function of n,

$\overline{\lim}\, E(\frac{B_n}{n})^k = \lim_{n\to\infty} E(\frac{B_n}{n})^k$. Combining this with the trivial upper bound $E(\frac{B_1}{1})^k = E(X^k)$, an upper and lower bound of $E(\frac{B_n}{n})^k$ can be given as the following:

$$\left[1 - F(x_o)\right]^k x_o^k \leq E(\frac{B_n}{n})^k \leq E(X^k) \quad \text{for } k,\ n = 1, 2, 3, \ldots \qquad (25)$$

Applying the bounds given above, an upper bound of $Var(\frac{B_n}{n})$ is now available;

$$0 \leq Var(\frac{B_n}{n}) = E(\frac{B_n}{n})^2 - E^2(\frac{B_n}{n})$$

$$\leq E(\frac{B_m}{m})^2 - \left[1 - F(x_o)\right]^2 x_o^2 , \quad 1 \leq m < n$$

$$\leq Var(\frac{B_m}{m}) + E^2(X) - \left[1 - F(x_o)\right]^2 x_o^2$$

or, with m = 1 it becomes simply

$$0 \leq Var(\frac{B_n}{n}) \leq E(X^2) - \left[1 - F(x_o)\right]^2 x_o^2$$

$$= Var(X) + E^2(X) - \left[1 - F(x_o)\right]^2 x_o^2 . \qquad (26)$$

The upper bound of $Var(\frac{B_n}{n})$ above can be improved by the inequality given in Eq. (24);

$$\frac{B_n}{n} \leq \bar{X}_n$$

which leads to $E(\frac{B_n}{n})^2 \leq E(\bar{X}_n^2) = \frac{1}{n} Var(X) + E^2(X)$.

Therefore,

$$0 \leq Var(\frac{B_n}{n}) \leq \frac{1}{n} Var(X) + E^2(X) - \left[1 - F(x_o)\right]^2 x_o^2 . \qquad (27)$$

It is of interest to know if the lower bound of $E(\frac{B_n}{n})^k$ given in Eq. (25) is the limit of $E(\frac{B_n}{n})^k$ itself. It can be shown to be so by utilizing a result on p. 163 of Loeve (5);

> If $|Y_n| \leq U_n$ with $U_n \xrightarrow{p} U$ and $\int U_n \longrightarrow \int U$ finite, then $Y_n \xrightarrow{p} Y$ implies that $\int Y_n \longrightarrow \int Y$.

By letting $Y_n = \frac{B_n}{n}$ and $U_n = \bar{X}_n$,

$Y_n \leq U_n$ from Eq. (24).

Further, $U_n \xrightarrow{p} \mu$ and $E(U_n) = E(X) = \mu$, finite (assume). Therefore, since $Y_n \xrightarrow{p} \left[1 - F(x_o)\right] x_o$,

$$E(Y_n) \longrightarrow \left[1 - F(x_o)\right] x_o$$

$$\underline{i}.\underline{e}., \lim_{n \to \infty} E(\frac{B_n}{n}) = \left[1 - F(x_o)\right] x_o .$$

For $k \geq 2$, let $Y_n = (\frac{B_n}{n})^k$ and $U_n = \bar{X}_n^k$.

Since $g(\bar{X}_n) = \bar{X}_n^k$ is a continuous function of $\bar{X}_n$, $\bar{X}_n \xrightarrow{p} \mu$ implies that $\bar{X}_n^k \xrightarrow{p} \mu^k$ $\underline{i}.\underline{e}.$, $U_n \xrightarrow{p} \mu^k$. Now it remains to show $E(U_n) \longrightarrow \mu^k$. When $k = 2$, $E(U_n) = E(\bar{X}_n^2) = \mu^2 + \frac{\sigma^2}{n} \longrightarrow \mu^2$ finite as $n \longrightarrow \infty$ if σ^2 (= Var (X)) is finite. Hence, $E(\frac{B_n}{n})^2 \longrightarrow \left[1 - F(x_o)\right]^2 x_o^2$.

In general,

$$E(U_n) = E(\bar{X}_n^k) = \frac{1}{n^k} E(X_1 + X_2 + \ldots + X_n)^k \longrightarrow \mu^k \text{ from the}$$

following reason:

Observe that $(X_1 + X_2 + .. + X_n)^k$ can be considered as the sum of n^k terms of order k, and there exist exactly $\frac{n!}{(n-k)!}$ terms each of which is the product of k different X's such as $X_1X_2 \ldots X_k$. The other $n^k - \frac{n!}{(n-k)!}$ terms contain less than k different X's such as $X_1^2X_2 \ldots X_{k-1}$, $X_1^2X_2^2X_3 \ldots X_{k-2}$, ..., X_1^k, so on. Therefore,

$$E(\bar{X}_n^k) = \frac{1}{n^k}\left[\frac{n!}{(n-k)!}\mu^k + \sum_i E(R_i)\right] \tag{28}$$

where R_i, $i = 1, 2, \ldots, n^k - \frac{n!}{(n-k)!}$, are the terms with less than k different X's in the expansion of $(X_1 + X_2 + .. + X_n)^k$. Under the assumption that $E(X^k)$ is finite, $E(R_i)$ is finite for every i. Letting $M = \max_i \left\{E(R_i)\right\}$, M is certainly finite, and

$$\frac{1}{n^k}\sum_i E(R_i) \leq \frac{1}{n^k}\left[n^k - \frac{n!}{(n-k)!}\right] M$$

$$= \left[1 - \frac{n(n-1)(n-2)\ \ldots\ (n-k+1)}{n^k}\right] M$$

$$= \left[1 - 1(1 - \frac{1}{n})\ (1 - \frac{2}{n})\ \ldots\ (1 - \frac{k-1}{n})\right] M \longrightarrow 0$$

$$\text{as } n \longrightarrow \infty$$

whereas, $\frac{1}{n^k}\frac{n!}{(n-k)!}\mu^k \longrightarrow \mu^k$ as $n \longrightarrow \infty$.

Therefore, using the results above, Eq. (28) yields

$$E(\bar{X}^k) \longrightarrow \mu^k \text{ as } n \longrightarrow \infty .$$

Hence, $$\lim_{n\to\infty} E\left(\frac{B_n}{n}\right)^k = \left[1 - F(x_o)\right]^k x_o^k \tag{29}$$

for every $k = 1, 2, \ldots$ if $E(X^k) < \infty$. Moreover, this result can be used to prove that

$$\mathrm{Var}\left(\frac{B_n}{n}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty .$$

Effect of Filament Length on Bundle Strength

Generalization of the Weakest-Link Theory in a Bundle

It is a known phenomenon that the strength of a filament is likely to decrease as its length is increased. The probabilistic expression for this, called "the weakest-link" theory by Peirce (6), is based on the assumption that a filament of length ℓ consists of a continuum of ℓ independent segments of unit length. Therefore, when the strengths of the ℓ segments are independently and identically distributed with $F^{(1)}(x)$ as their common distribution function, the distribution function of strength, $F^{(\ell)}(x)$, for the filament of length ℓ has the following expression:

$$F^{(\ell)}(x) = 1 - \left[1 - F^{(1)}(x)\right]^{\ell}, \quad \ell \geq 1 \tag{30}$$

Clearly, $F^{(\ell)}(x)$ is a monotonically increasing function of ℓ for $x \in (0, \infty)$ and corresponds to the distribution function of the first order statistic among ℓ samples.

A generalization of the weakest-link theory in a bundle of n filaments is immediate since the breakage of a bundle is realized by sequential breakages of the n weakest segments of the filaments. Hence, the distribution function of strength for a bundle of n filaments of length ℓ, $S_n^{(\ell)}(x)$, is given in the following with $F(x)$ replaced by $F^{(\ell)}(x)$ in Eq. (2).

$$S_n^{(\ell)}(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \left[1 - \left\{1 - F^{(1)}\left(\frac{x}{n}\right)\right\}^{\ell}\right]^{k} S_{n-k}^{(\ell)}(x), \ \ell \geq 1 \tag{31}$$

A slightly more general relation than Eq. (30),

$$F^{(\ell)}(x) = 1 - \left[1 - F^{(m)}(x)\right]^{\frac{\ell}{m}}, \tag{32}$$

can be used in order to obtain $S_n^{(\ell)}(x)$ in terms of $F^{(m)}(x)$ for the case $m \geq 2$.

Although it seems intuitively obvious that $S_n^{(\ell)}(x)$ is likely to be an increasing function of ℓ, unlike the case for $F^{(\ell)}(x)$, no indication is given in Eq. (31) to support such intuition. For this, the proof of the following theorem utilizes a different way of expressing $S_n(x)$ which is due to Daniels (2).

Theorem 5

$S_n^{(\ell)}(x)$, given by Eq. (31), is a monotonically increasing function of ℓ.

Proof:

In deriving $S_n(x)$, Daniels (2) starts with the multiple integral;

$$S_n(x) = n! \int_{0}^{\frac{x}{n}} \int_{Y_1}^{\frac{x}{n-1}} \cdots \int_{Y_{n-2}}^{\frac{x}{2}} \int_{Y_{n-1}}^{x} \prod_{i=1}^{n} f(Y_i)\, dY_n dY_{n-1} \cdots dY_2 dY_1$$

where $0 \leq Y_1 < Y_2 < \ldots < Y_n < \infty$

and by letting $Z_i = F(Y_i)$, $i = 1, 2, \ldots, n$,

$$S_n(x) = n! \int_0^{F(\frac{x}{n})} \int_{Z_1}^{F(\frac{x}{n-1})} \cdots \int_{Z_{n-2}}^{F(\frac{x}{2})} \int_{Z_{n-1}}^{F(x)} dZ_n\, dZ_{n-1} \cdots dZ_2\, dZ_1 \qquad (33)$$

where $0 \leq F(\frac{x}{n}) \leq F(\frac{x}{n-1}) \leq \ldots \leq F(x) \leq 1$.

Similarly, by letting $Z_i = F^{(\ell)}(Y_i)$, $i = 1, 2, \ldots, n$,

$$S_n^{(\ell)}(x) = n! \int_0^{F^{(\ell)}(\frac{x}{n})} \int_{Z_1}^{F^{(\ell)}(\frac{x}{n-1})} \cdots \int_{Z_{n-2}}^{F^{(\ell)}(\frac{x}{2})} \int_{Z_{n-1}}^{F^{(\ell)}(x)} dZ_n\, dZ_{n-1} \cdots dZ_2\, dZ_1$$

(34)

where, $0 \leq F^{(\ell)}(\frac{x}{n}) \leq F^{(\ell)}(\frac{x}{n-1}) \leq \ldots \leq F^{(\ell)}(x) \leq 1$.

In comparing Eqs. (33) and (34), the difference in the definitions of Z_i dissipates as Z_i become dummy variables. In Eq. (30) it is noted, by letting $F^{(1)}(x) = F(x)$, that $F(x) < F^{(\ell)}(x)$ for $x \in (0, \infty)$. Therefore,

$$F(\tfrac{x}{j}) < F^{(\ell)}(\tfrac{x}{j}), \; j = 1, 2, \ldots, n . \qquad (35)$$

Since the upper bound of the integration range monotonically decreases along with the order of integration, it is assured that

$Z_{n-j} < F(\frac{x}{j})$, $j = 1, 2, \ldots, n$ in Eq. (32)

and, $Z_{n-j} < F^{(\ell)}(\frac{x}{n})$, $j = 1, 2, \ldots, n$ in Eq. (33).

This implies the integrand is always positive at every stage of integration, and hence the higher the upper bound of the integration range, the greater the value after integration. Consequently Eq. (35) leads to

$$S_n(x) < S_n^{(\ell)}(x) \qquad \text{for } x \in (0, \infty) \text{ and } \ell \geq 2.$$

Also, from Eq. (32), $F^{(\ell)}(x) < F^{(m)}(x)$ for $1 \leq \ell < m$. Therefore, by a similar argument to the previous case, it can be shown that

$$S_n^{(\ell)}(x) < S_n^{(m)}(x) \quad \text{for } x \in (0, \infty) \text{ and } 1 \leq \ell < m,$$

i.e., $S_n^{(\ell)}(x)$ is a monotonically increasing function of ℓ.

Corollary

The expected bundle strength with n filaments of length ℓ, $E(B_n^{(\ell)})$ monotonically decreases in ℓ.

Proof:

$$E(B_n^{(\ell)}) = \int_0^\infty \left[1 - S_n^{(\ell)}(x)\right] dx > \int_0^\infty \left[1 - S_n^{(m)}(x)\right] dx = E(B_n^{(m)})$$

for $1 \leq \ell < m$ since $1 - S_n^{(\ell)}(x) > 1 - S_n^{(m)}(x)$ by Theorem 5. Therefore, $E(B_n^{(\ell)})$ monotonically decreases in ℓ.

Asymptotic Properties of $\frac{B_n^{(\ell)}}{n}$ for Large Bundles

The asymptotic properties previously derived for $\frac{B_n}{n}$ apply to $\frac{B_n^{(\ell)}}{n}$ by assuming the distribution of filament strength is $F^{(\ell)}(x)$ instead of F(x). Defining $x_o^{(\ell)}$ as the value of x at which $\left[1 - F^{(\ell)}(x)\right] x$ is maximized, it can be shown that $\frac{B_n^{(\ell)}}{n}$ tends to follow, asymptotically, a normal distribution with mean $\left[1 - F^{(\ell)}(x_o^{(\ell)})\right] x_o^{(\ell)}$

and variance $\frac{1}{n}\left[1 - F^{(\ell)}(x_o^{(\ell)})\right] F^{(\ell)}(x_o^{(\ell)}) \left[x_o^{(\ell)}\right]^2$, or mean $\left[1 - F(x_o^{(\ell)})\right]^{\ell} x_o^{(\ell)}$ and variance $\frac{1}{n}\left[1 - F(x_o^{(\ell)})\right]^{\ell} \left\{1 - \left[1 - F(x_o^{(\ell)})\right]^{\ell}\right\} \cdot \left[x_o^{(\ell)}\right]^2$ in terms of $F(x)$ which is assumed to be $F^{(1)}(x)$. The value of $x_o^{(\ell)}$ satisfies the relation;

$$\ell x_o^{(\ell)} = \frac{1-F(x_o^{(\ell)})}{f(x_o^{(\ell)})} \tag{36}$$

which reduces to

$$x_o = \frac{1-F(x_o)}{f(x_o)} \quad \text{when } \ell = 1 .$$

It is worth noting that the asymptotic mean of $\frac{B_n^{(1)}}{n}$ is greater than that of $\frac{B_n^{(\ell)}}{n}$. This is true since $\left[1 - F(x)\right] x > \left[1 - F(x)\right]^{\ell} x$ for every $x \in (0, \infty)$ and $\ell \geq 2$, i.e., one is uniformly dominated by the other, and therefore,

$$\max_{0 < x < \infty} \left[1 - F(x)\right] x > \max_{0 < x < \infty} \left[1 - F(x)\right]^{\ell} x$$

or, $\left[1 - F(x_o)\right] x_o > \left[1 - F(x_o^{(\ell)})\right]^{\ell} x_o^{(\ell)}$.

However, no general assertion can be made as to whether or not the asymptotic variance of $\frac{B_n^{(1)}}{n}$ is greater than that of $\frac{B_n^{(\ell)}}{n}$.

Using a normal distribution, an example is given in Figure 6 to show the change in $x_o^{(\ell)}$ as ℓ is increased. The points $x_o^{(1)}$, $x_o^{(2)}$, ... are determined in the Figure by the absissae corresponding to the intersecting points of $y = \ell x$ and $y = \frac{1-F(x)}{f(x)}$. The decrease in the

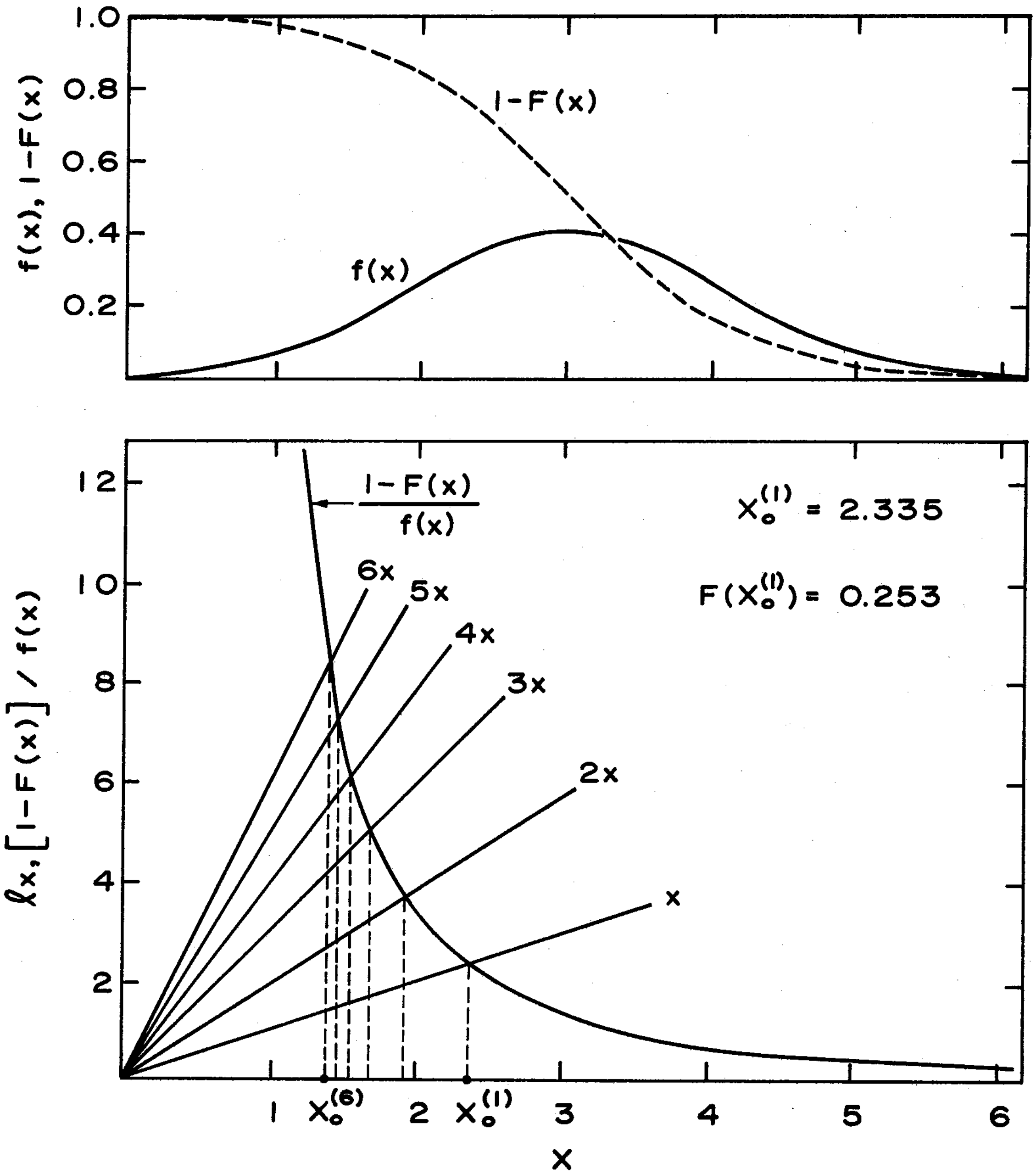

Figure 6. Graphical determination of $x_o^{(\ell)}$ based on filament strengths distributed as N(3,1)

value of $x_o^{(\ell)}$ in ℓ is obvious not only in the particular example but also with almost any continuous f(x). The rate of decrease becomes very slow as ℓ gets larger.

Effect of Length on the Wasted Fraction of Filaments in a Bundle

It was shown in a previous section that the wasted fraction can be estimated by $F(x_o)$ when the distribution of filament strength is given by F(x). Similarly, when the distribution is given by $F^{(\ell)}(x)$ for the filaments of length ℓ, the wasted fraction is estimable by $F^{(\ell)}(x_o^{(\ell)})$ or by $1 - \left[1 - F^{(1)}(x_o^{(\ell)})\right]^{\ell}$. Though one may suspect that the wasted fraction will increase along with the increase in ℓ, it can be false in general. The exact condition for the above to be true is given by

$$\left[1 - F^{(1)}(x_o^{(\ell)})\right]^{\ell} > \left[1 - F^{(1)}(x_o^{(m)})\right]^{m}, \quad 1 \le \ell < m.$$

In case $f(x) = \alpha e^{-\alpha x}$, $x > 0$, it is found that $x_o^{(\ell)} = \frac{1}{\ell\alpha}$ and $\left[1 - F^{(1)}(x_o^{(k)})\right]^k = \left[e^{-\frac{\alpha}{k\alpha}}\right]^k = e^{-1}$ for every k = 1, 2, ..., i.e., the wasted fraction is invariant in ℓ for large n. Incidentally, the asymptotic variance of $\frac{B_n^{(\ell)}}{n}$ for the above exponential case is given by $\frac{1}{n} e^{-1}(1 - e^{-1}) \left(\frac{1}{\ell\alpha}\right)^2$, and it decreases monotonically in ℓ.

Probability of Strength Retention in Bundles of Augmented Length

Consider a case where a single filament or a bundle has to be augmented by an extra length of filament or bundle to meet certain purposes. The length consumed in the augmenting process, such as knotting, is excluded from consideration. One practical problem in this situation is to maintain an adequate strength after augmentation, and it is of particular interest to examine the probability of retaining the original strength after the length is augmented. Furthermore, the changes in such probability can be examined relative to the length to be added as well as to the number of filaments in the bundle.

First, it is asserted that the strength of a filament or a bundle cannot be increased after the length is augmented. For a single filament, the weakest-link theory provides the trivial proof for the assertion. Precisely, when a filament of length m is added to a filament of length ℓ, the strength lowest among the $\ell + m$ segments is necessarily lower than or equal to that among the original ℓ segments. For a bundle of n filaments, the argument is as follows:

Let $\left\{Y_i^{(\ell)}\right\}$, i = 1, 2, ..., n be the order statistics corresponding to the strengths of the original n filaments of length ℓ, and let the strength of the particular filament with $Y_j^{(\ell)}$ be $X_j^{(\ell+m)}$ after augmentation. Then, from the single filament case, $Y_j^{(\ell)} \geq X_j^{(\ell+m)}$ for every j = 1, 2, ..., n. Therefore, when the order statistics corresponding to $\left\{X_i^{(\ell+m)}\right\}$ is called $\left\{Y_i^{(\ell+m)}\right\}$, it is trivial to show that $Y_j^{(\ell)} \geq Y_j^{(\ell+m)}$ for every j = 1, 2, ..., n. Therefore,

$$(n-j+1)Y_j^{(\ell)} \geq (n-j+1)Y_j^{(\ell+m)}, \quad j = 1, 2, \ldots, n \text{ and,}$$

$$\max\left\{nY_1^{(\ell)}, (n-1)Y_2^{(\ell)}, \ldots, Y_n^{(\ell)}\right\} \geq \max\left\{nY_1^{(\ell+m)}, (n-1)Y_2^{(\ell+m)}, \ldots, Y_n^{(\ell+m)}\right\}$$

or, $B_n^{(\ell)} \geq B_n^{(\ell+m)}$, where $B_n^{(\ell+m)}$ is understood as the strength of the bundle augmented to the bundle with $B_n^{(\ell)}$. Furthermore,

$$P_r\left[B_n^{(\ell)} = B_n^{(\ell+m)}\right] = 1 - P_r\left[B_n^{(\ell)} > B_n^{(\ell+m)}\right].$$

For a single filament case, the above is written as

$$P_r\left[X^{(\ell)} = X^{(\ell+m)}\right] = 1 - P_r\left[X^{(\ell)} > X^{(\ell+m)}\right]$$

$$= P_r\left[Y^{(\ell)} \leq Z^{(m)}\right]$$

where, $Y^{(\ell)}$ is the first order statistic among the original ℓ segments, and $Z^{(m)}$ is that of the m segments to be added.

Letting the distribution functions of $Y^{(\ell)}$ and $Z^{(m)}$ be $G(\xi)$ and $H(\eta)$ respectively, since $Y^{(\ell)}$ and $Z^{(m)}$ are independent,

$$P_r\left[X^{(\ell)} = X^{(\ell+m)}\right] = P_r\left[Y^{(\ell)} \leq Z^{(m)}\right]$$

$$= \int_0^\infty G(\eta)dH(\eta)$$

$$= \int_0^\infty \left\{1 - \left[1 - F(\eta)\right]^\ell\right\} \cdot m\left[1 - F(\eta)\right]^{m-1} f(\eta)d\eta$$

$$= m\int_0^\infty \left[1 - F(\eta)\right]^{m-1} f(\eta)d\eta - m\int_0^\infty \left[1 - F(\eta)\right]^{\ell+m-1} f(\eta)\,d\eta$$

$$= \frac{\ell}{\ell+m} \tag{37}$$

for any $F(x) = P_r\left[X^{(1)} \leq x\right]$ and $f(x) = F'(x)$ which are continuous.

For a bundle of n filaments, the event $B_n^{(\ell)} = B_n^{(\ell+m)}$ becomes more complex. With the foregoing definition of $Y_j^{(\ell)}$, let $Z_j^{(m)}$ be the strength of the particular filament of length m to be added to the filament with $Y_j^{(\ell)}$. Then, it is noted that $\left\{Y_i^{(\ell)}\right\}$, i = 1, 2, ..., n is an ordered set whereas $\left\{Z_i^{(m)}\right\}$, i = 1, 2, ..., n is not. Also, for every j = 1, 2, ..., n, $Y_j^{(\ell)}$ and $Z_j^{(m)}$ are understood as the first order statistics among the ℓ and m segments respectively. Then the following identities are easily understood;

$$P_r\left[B_n^{(\ell)} = B_n^{(\ell+m)}\right] = P_r\left\{\bigcup_{k=1}^{n}\left[B_n^{(\ell)} = B_n^{(\ell+m)},\ B_n^{(\ell)} = (n-k+1)Y_k^{(\ell)}\right]\right\}$$

$$= \sum_{k=1}^{n} P_r\left[Y_k^{(\ell)} \leq \min(Z_k^{(m)}, Z_{k+1}^{(m)}, \ldots, Z_n^{(m)})\right]$$

$$\cdot\ P_r\left[B_n^{(\ell)} = (n-k+1)Y_k^{(\ell)}\right]. \qquad (38)$$

In other words, the event $B_n^{(\ell)} = B_n^{(\ell+m)}$ requires the condition that if $B_n^{(\ell)}$ were represented by $(n-k+1)Y_k^{(\ell)}$, the filaments corresponding to $Y_k^{(\ell)}, Y_{k+1}^{(\ell)}, \ldots, Y_n^{(\ell)}$ should not decrease their strengths below $Y_k^{(\ell)}$ after the lengths are augmented, whereas no such restriction is necessary for the filaments corresponding to $Y_1^{(\ell)}, Y_2^{(\ell)}, \ldots, Y_{k-1}^{(\ell)}$. This is to say that the filament corresponding to $Y_k^{(\ell)}$ must retain its strength as well as the order of strength k after the length is augmented. The proof of this is elementary and excluded here.

Observe in Eq. (38) that $Y_k^{(\ell)}, Z_k^{(m)}, Z_{k+1}^{(m)}, \ldots, Z_n^{(m)}$ are all independent. Further, they are the first order statistics based on samples with a common strength distribution F(x) of a segment. Therefore, utilizing the notation for the single filament case,

$$P_r\left[Y_k^{(\ell)} \leq \min\ (Z_k^{(m)}, Z_{k+1}^{(m)}, \ldots, Z_n^{(m)})\right]$$

$$= P_r\left[Y^{(\ell)} \leq Z^{((n-k+1)m)}\right]$$

where, $Z^{((n-k+1)m)}$ represents the first order statistic among (n-k+1)m samples,

$$= \frac{\ell}{\ell+(n-k+1)m} \text{ by analogy to Eq. (37).}$$

Therefore, from Eq. (38),

$$P_r\left[B_n^{(\ell)} = B_n^{(\ell+m)}\right] = \sum_{k=1}^{n} \frac{\ell}{\ell+(n-k+1)m}\, p_{nk} \tag{39}$$

where, $p_{nk} = P_r\left[B_n^{(\ell)} = (n-k+1)Y_k^{(\ell)}\right]$

and $\sum_{k=1}^{n} p_{nk} = 1.$

But,

$$\sum_{k=1}^{n} \frac{\ell}{\ell+(n-k+1)m}\, p_{nk} \leq \frac{\ell}{\ell+m} \sum_{k=1}^{n} p_{nk} = \frac{\ell}{\ell+m}$$

i.e.,

$$P_r\left[B_n^{(\ell)} = B_n^{(\ell+m)}\right] \leq P_r\left[X^{(\ell)} = X^{(\ell+m)}\right] \tag{40}$$

for every $n = 2, 3, 4, \ldots$.

The inequality above indicates that a bundle is more prone to strength loss than a filament when their augmented lengths are same. The changes in probability of strength retention are well reflected in Eqs. (37) and (39) for the change in relative size of ℓ to m.

Relaxed Model for Breakage of Bundles

The breakage model introduced at the beginning of this study was called a 'restricted model' in the sense that the assumptions used in the model impose certain restrictions in its use. It was assumed that the bundle load at the k^{th} filament break is exactly $(n-k+1)Y_k$, or the load in any surviving filament at the k^{th} break is Y_k on the average.

This assumption is valid when the variation of load among the surviving filaments is low or the value Y_k is always close to the median of the filament loads at the moment of k^{th} filament break. Therefore, as soon as the load-extension characteristics of filaments deviate from the above ideal situation, the validity of the foregoing analysis becomes questionable.

Distribution of Bundle Load

The model to be given in the following is to overcome the deficiency of the previous one. In this model, the bundle load at any given extension point is expressed by the sum of the loads in the surviving filaments at that particular point, and the variation of load among filaments at a given extension is justified by considering the filament load as a random variable functionally dependent on extension.

Let $X_i(y)$ = load on a filament at extension y

β_i = a positive random variable representing the hypothetical asymptote of breaking load of a filament

Y_i = a random variable which defines the breaking extension of a filament

α = a positive constant inherent to the material

and

$$X_i(y) = \beta_i(1 - e^{-\alpha y}) \cdot 1_{[Y_i > y]} \quad , i = 1, 2, \ldots, n \qquad (41)$$

where, $1_{[Y_i > y]} = 0 \quad \text{if } Y_i \leq y$

$= 1 \quad \text{if } Y_i > y$.

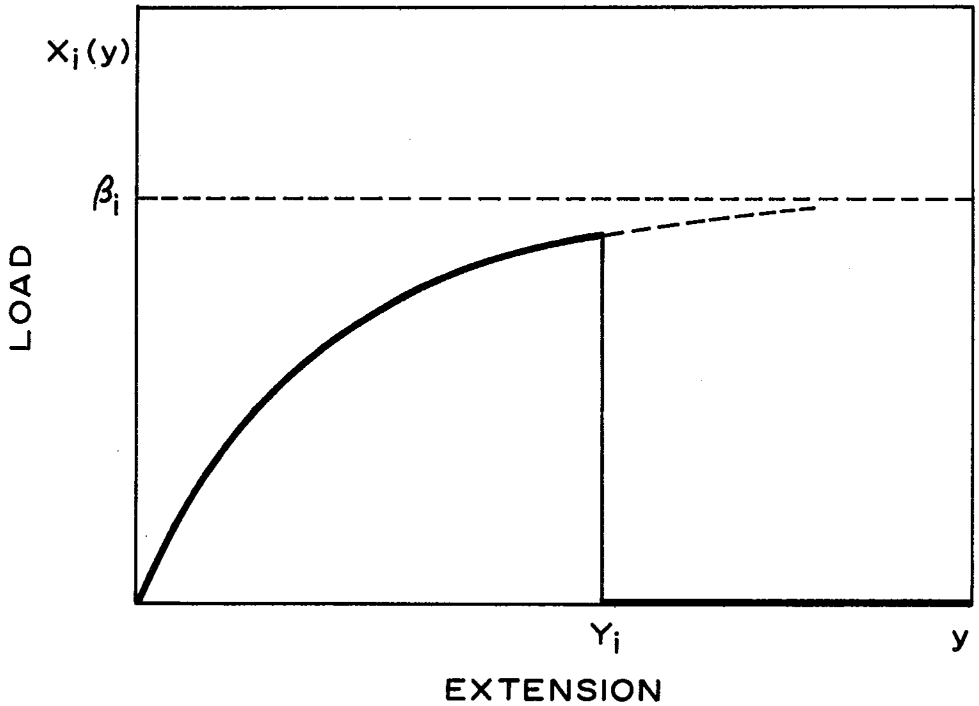

Figure 7. Load-extension curve of a filament with relaxed model

Then, the load-extension curve for a single filament is as given in Figure 7 based on Eq. (41). The shape of the load-extension curve shown in Figure 7 is not at all an arbitrarily chosen one. Experimental results with quasi-elastic filaments often indicate the validity of using an exponential curve as above with an asymptote. Although the shape of an experimental curve may differ at the portion prior to the so called 'yield point', the inaccuracy there can be ignored simply because the actual breakage occurs far beyond the yield point.

Now, the strength of a bundle can be defined as the maximum load attainable in a bundle as before. Since the bundle load at a given extension is the sum of the filament loads at that point, the bundle strength, B_n, is given by

$$B_n = \max_{0 \leq y < \infty} \sum_{i=1}^{n} X_i(y)$$

$$= \max_{0 \leq y < \infty} \sum_{i=1}^{n} \beta_i(1 - e^{-\alpha y}) \cdot 1_{[Y_i > y]} \,. \qquad (42)$$

The load-extension curve of a bundle based on $\sum_{i=1}^{n} X_i(y)$ resembles that given in Figure 1 except that there exists no regularity that governs the size of the jumps. Under the assumption that $X_i(y)$, $i = 1, 2, \ldots, n$ are independently and identically distributed and that β_i and Y_i are statistically independent, the distribution function of $\sum_{i=1}^{n} X_i(y)$ is the n-fold convolution of the distribution function of $X_i(y)$.

Define $\quad H(u) = P_r(\beta_i \leq u)$

$$F(v,y) = P_r\left[X_i(y) \leq v\right]$$

$$G(y) = P_r\,(Y_i \leq y)$$

$$\psi_\beta(m) = \int_0^\infty e^{imu}\, dH(u)$$

$$\theta_{X(y)}(s) = \int_0^\infty e^{isv}\, dF(v,y)$$

i.e., $\psi_\beta(m)$ and $\theta_{X(y)}(x)$ are the characteristic functions corresponding to the distribution of β_i and $X_i(y)$ respectively.

Then, $\quad F(v,y) = P_r\left[X_i(y) \leq v\right]$

$$= P_r\left[X_i(y) \leq v \,/\, Y_i > y\right] \cdot P_r(Y_i > y)$$

$$+ P_r\left[X_i(y) \leq v \,/\, Y_i \leq y\right] \cdot P_r(Y_i \leq y)$$

$$= P_r\left[\beta_i(1 - e^{-\alpha y}) \leq v\right]\left[1 - G(y)\right]$$

$$+ P_r\,(0 \leq v)\;\; G(y)$$

$$= P_r\,(\beta_i \leq \frac{v}{1 - e^{-\alpha y}})\left[1 - G(y)\right] + G(y)$$

$$= H\,(\frac{v}{1 - e^{-\alpha y}})\left[1 - G(y)\right] + G(y) \;. \qquad (43)$$

Note that $F(0, y) = G(y)$, *i.e.*, $F(v, y)$ has a jump at $v = 0$, but otherwise it has a corresponding continuous density of the following for $v > 0$;

$$f(v, y) = \frac{\partial}{\partial v} F(v, y)$$

$$= (\frac{1}{1-e^{-\alpha y}})\; h\,(\frac{v}{1-e^{-\alpha y}})\;\left[1 - G(y)\right] \;, \qquad v > 0$$

where, $h(v) = H'\,(v)$,

and therefore, $F(v, y)$ is a mixed-type distribution function.

Hence, $E\left[X_i(y)\right] = 0 \cdot F(0, y) + \int_0^\infty vf(v, y)\, dv$

$$= \int_0^\infty \frac{v}{1-e^{-\alpha y}}\;\; h(\frac{v}{1-e^{-\alpha y}})\left[1 - G(y)\right] dv$$

letting $u = \frac{v}{1-e^{-\alpha y}}$,

$$= \left[1 - G(y)\right]\,(1 - e^{-\alpha y}) \int_0^\infty u\, h(u)\, du$$

$$= \left[1 - G(y)\right]\,(1 - e^{-\alpha y})\; E(\beta) \qquad (44)$$

Similarly, $E\left[X_i^2(y)\right] = \left[1 - G(y)\right] \ (1 - e^{-\alpha y})^2 \ E(\beta^2)$

Therefore, $\mathrm{Var}\left[X_i(y)\right] = \left[1 - G(y)\right] \ (1 - e^{-\alpha y})^2 \ E(\beta^2)$

$$- \left[1 - G(y)\right]^2 \ (1 - e^{-\alpha y})^2 \ E^2(\beta)$$

$$= \left[1 - G(y)\right] \ (1 - e^{-\alpha y})^2 \left[E(\beta^2) - \left\{1 - G(y)\right\} E^2(\beta)\right]. \tag{45}$$

It is noted in Eq. (45) that

$$\mathrm{Var}\left[X_i(y)\right] \geq \left[1 - G(y)\right] \ (1 - e^{-\alpha y})^2 \ \mathrm{Var}(\beta).$$

The characteristic function $\theta_{X(y)}(s)$ corresponding to $F(v, y)$ can be obtained as follows:

$$\theta_{x(y)}(s) = \int_0^\infty e^{isv} \, dF(v, y)$$

$$= e^{is\cdot 0} F(0,y) + \int_0^\infty e^{isv} f(v,y) \, dv$$

$$= G(y) + \int_0^\infty e^{isv} \left(\frac{1}{1-e^{-\alpha y}}\right) \ h\left(\frac{v}{1-e^{-\alpha y}}\right) \left[1 - G(y)\right] \ dv$$

letting $u = \dfrac{v}{1-e^{-\alpha y}}$,

$$= G(y) + \left[1 - G(y)\right] \int_0^\infty e^{is(1-e^{-\alpha y})u} \, h(u) \, du$$

by the definition of $\psi_\beta(m)$,

$$= G(y) + \left[1 - G(y)\right] \ \psi_\beta\left[s(1 - e^{-\alpha y})\right] \tag{46}$$

By letting $L_n(y) = \sum_{i=1}^{n} X_i(y)$, i.e., the bundle load at extension y,

$$E\left[L_n(y)\right] = nE\left[X_i(y)\right] = n\left[1 - G(y)\right] \ (1-e^{-\alpha y}) \ E(\beta) \tag{47}$$

$$\mathrm{Var}\left[L_n(y)\right] = n\left[1-G(y)\right] \quad (1-e^{-\alpha y})^2 \left\{E(\beta^2) - \left[1-G(y)\right] E^2(\beta)\right\} \qquad (48)$$

and the characteristic function $\phi_{L_n(y)}(\omega)$ corresponding to the distribution of $L_n(y)$ is given in terms of $\theta_{X(y)}(\omega)$ as;

$$\phi_{L_n(y)}(\omega) = \left[\theta_{X(y)}(\omega)\right]^n$$

$$= \left\{G(y) + \left[1-G(y)\right] \psi_\beta \left[\omega(1-e^{-\alpha y})\right]\right\}^n \quad . \qquad (49)$$

Although the distribution of $L_n(y)$ is understood as the n-fold convolution of $F(v,y)$, it can be characterized by Eq. (49) as well.

The results given by Eqs. (47), (48) and (49) can be obtained by an alternative approach in which the physical meaning of the sum $\sum_{i=1}^{n} X_i(y)$ is more appealing than the previous method. The analysis given in the Appendix explicitly defines the bundle load with relation to the order of breakage extension.

An Estimate of $E(\frac{B_n}{n})$

The distribution of bundle strength defined by Eq. (42) is in fact the distribution of $L_n(y^*) = \sum_{i=1}^{n} X_i(y^*)$, where y^* is the point which maximizes $L_n(y)$. Therefore, all major properties of $L_n(y^*)$ inherit that of $L_n(y)$ if y^* can be known. However, nothing is known about y^* other than the fact that y^* occurs at one of the breakage points Y_1, Y_2, ... and Y_n. This is so because $L_n(y)$ monotonically increases within the intervals $Y_{(k-1)} < y \leq Y_{(k)}$, k = 1, 2, ..., n with

$Y_{(0)} = 0$ and $0 \leq Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)} < \infty$, i.e. $\left\{Y_{(i)}\right\}$ are the ordered set of $\left\{Y_i\right\}$, $i = 1, 2, \ldots, n$, and hence $L_n(y)$ achieves its local maxima at $Y_{(1)}, Y_{(2)}, \ldots,$ and $Y_{(n)}$. Therefore,

$$B_n = \max_{0 \leq y < \infty} L_n(y) = \max_{1 \leq k \leq n} \sum_{i=1}^{n} X_i(Y_{(k)})$$

$$= \max_{1 \leq k \leq n} \sum_{i=1}^{n} X_i(Y_k) .$$

The distribution of the above random variable is hardly obtainable under the definition of $X_i(y)$. If the interest is in getting an approximate estimate of $E(\frac{B_n}{n})$, however, the following criteria is of value:

$$E(\frac{B_n}{n}) \doteq E\left[\frac{L_n(y_o)}{n}\right] = \left[1 - G(y_o)\right] \; (1 - e^{-\alpha y_o}) \; E(\beta) \tag{50}$$

where, y_o is determined from

$$\frac{dE\left[L_n(y)\right]}{dy} = 0$$ which leads, when $G(x)$ is continuous with $\frac{dG(y)}{dy} = g(y)$,

to
$$\frac{1 - G(y_o)}{g(y_o)} = \frac{1 - e^{-\alpha y_o}}{\alpha e^{-\alpha y_o}} . \tag{51}$$

In order to examine if the y_o that satisfies Eq. (51) provides the maximum for $E\left[L_n(y)\right]$, observe first that

$$E\left[L_n(0)\right] = E\left[L_n(\infty)\right] = 0 .$$

Therefore, when the solution y_o is unique it necessarily provides the maximum for $E[L_n(y)]$ when Eq. (50) is valid. If there exist two solutions y_{o1} and y_{o2} with $y_{o1} < y_{o2}$, then $E\left[L_n(y_{o1})\right]$ is the maximum since $\frac{d}{dy} E\left[L_n(y)\right]$ cannot be zero having started from $E\left[L_n(0)\right] = 0$. In case there exist more than two solutions that satisfy Eq. (51), the criteria of selection is reduced to those y_{oi}, $i = 1, 2, \ldots, k$, say, at which

$$\frac{d^2}{dy^2} E\left[L_n(y)\right] \Big|_{y=y_{oi}} < 0 \text{ holds, or, equivalently,}$$

$$-\frac{g'(y_{oi})}{g(y_{oi})} < \left(\frac{1+e^{-\alpha y_{oi}}}{1-e^{-\alpha y_{oi}}}\right) \alpha, \text{ where } g'(y) = \frac{d}{dy} g(y).$$

Then, y_o is selected among the y_{oi} to satisfy

$$E\left(\frac{B_n}{n}\right) \doteq E\left[\frac{L_n(y_o)}{n}\right] = \max_{1 \leq i \leq k} E\left[\frac{L_n(y_{oi})}{n}\right] .$$

Asymptotic Properties of $L_n(y)$ and B_n

It was shown in Eqs. (44) and (45) that

$$E\left[X_i(y)\right] = \left[1 - G(y)\right] \quad (1 - e^{-\alpha y})\, E(\beta) \quad \text{and}$$

$$\mathrm{Var}\left[X_i(y)\right] = \left[1 - G(y)\right] \; (1 - e^{-\alpha y})^2 \; \left\{E(\beta^2) - \left[1-G(y)\right] E^2(\beta)\right\} .$$

Therefore, since X_i are independently and identically distributed, by the central limit theorem,

$$\frac{L_n(y) - n\left[1 - G(y)\right] \quad (1-e^{-\alpha y})E(\beta)}{(1-e^{-\alpha y}) \quad \sqrt{n\left[1-G(y)\right] \left\{E(\beta^2) - \left[1-G(y)\right] \quad E^2(\beta)\right\}}} \xrightarrow{d} N(0,1) \tag{52}$$

as $n \longrightarrow \infty$ for every $y > 0$, and the above is also true with $y = y_o$ at which $E\left[L_n(y)\right]$ is maximized.

SUMMARY AND CONCLUSIONS

In defining the strength of a bundle of n parallel filaments, two breakage models were constructed based on the tensile properties of the constituent filaments. Under each model, studies were carried out in order to examine the statistical nature of the bundle strength.

The restricted model for bundle breakage was based on the postulates that the load in a filament monotonically increases along with the increase in its extension and that the total load in a bundle at any extension point is equally shared by the surviving filaments at the extension. The distribution function of bundle strength, $S_n(x)$, was derived under the restricted model by employing a probabilistic argument of events associated with the breakage of a bundle, and it was given in terms of $S_{n-1}(x)$, $S_{n-2}(x)$, ... and $F(x)$, the distribution function of the filament strength as well as in terms of $F(\frac{x}{n})$, $F(\frac{x}{n-1})$, ..., and $F(x)$. Although $S_n(x)$ facilitates computation of the moments of the random variable B_n, the strength of a bundle of n filaments, the complex feature of $S_n(x)$ limits its practical usefulness to relatively small n.

The asymptotic properties of $\frac{B_n}{n}$ were derived directly from the definition of B_n (Eq. (1)) without utilizing $S_n(x)$. By converting $\frac{B_n}{n}$ to a function of the empirical distribution of the filament strength, it was found that $\frac{B_n}{n}$ converges in probability to a constant determined from $F(x)$. For a large n, it was shown also that the distribution of $\frac{B_n}{n}$ tends to follow a normal distribution with mean and variance dependent on $F(x)$ (Eq. (14)). Further, the mean coincides the limiting constant and the variance approaches zero as n becomes large.

By defining the wasted fraction of filaments in a bundle as the ratio of the number of filaments broken before the load in the bundle reaches its maximum to the total number of filaments in the bundle, two different methods were proposed for its estimation. Theoretically, the strength of a bundle is not affected by eliminating the filaments that belong to the wasted fraction.

The average contribution of the individual filament to the strength of its bundle, $\frac{B_n}{n}$, tends to decrease as n increases. It was proved for any distribution function F(x) of filament strengths, where the constituent filament strengths are interchangeable random variables, that $E(\frac{B_n}{n})^k$ decreases monotonically in n for every k = 1, 2, 3, ... It was shown also that the lower bound of $E(\frac{B_n}{n})^k$ coincides its limit $x_o^k \left[1 - F(x_o)\right]^k$, where x_o is the mode of the function $x\left[1 - F(x)\right]$ assuming it is unique, whereas the upper bound can be given by either $E(X^k)$ or by $E(\bar{\bar{X}}_n^k)$. From above, the upper bounds of $Var(\frac{B_n}{n})$ were established (Eqs. (26) and (27)) and its limit was shown to approach zero as n becomes large.

In a finite population of filaments, $E(\frac{B_n}{n})^k$ was shown to decrease monotonically in n when the expectation was defined under the scheme of simple random sampling. A trivial, but useful case of the above was that $(\frac{B_n}{n})^k \leq \bar{\bar{X}}_n^k$ for any k > 0 if the n filaments were common in the definitions of B_n and $\bar{\bar{X}}_n$. In any event, the monotonicity of $E(\frac{B_n}{n})^k$ above implies the decrease of bundle efficiency along with the increase in its size n, i.e., the increase in bundle strength is not proportional to the increase in n, but always less on the average.

The effect of filament length on the strength of a bundle was examined by generalizing the weakest-link theory applicable to the strength of a filament. As expected, the probability of breakage in a bundle of n filaments was shown to increase along with the uniform increase of filament length within the bundle at any fixed load level, i.e., $S_n^{(\ell)}(x)$ was shown to increase monotonically in ℓ for every positive x. Hence, the expected bundle strength was shown to decrease for a uniform increase in filament length. The asymptotic mean and variance of $\frac{B_n^{(\ell)}}{n}$ were found for bundles of length ℓ by relating $F^{(\ell)}(x)$, the strength distribution for filaments of length ℓ, with $F^{(1)}(x)$, that for the filaments of unit length. The results were such that the asymptotic mean of $\frac{B_n}{n}$ was shown to decrease for an increase in filament length, however the same was not true in general for its asymptotic variance. No assertion could be made as to whether or not the wasted fraction is increased along with the increase in filament length.

In a bundle where the filaments of length ℓ are augmented by filaments of length m, the probability was obtained for the event that the strength of the augmented bundle with length ℓ+m is same as the strength of the original bundle with length ℓ (Eq. (39)). The results indicate that the probability of strength retention is decreased by either increasing m or decreasing ℓ, or by both. Also, such probability in a bundle was shown to be less than that in a single filament for fixed ℓ and m.

The relaxed model for bundle breakage was designed to allow the variation of loads among the surviving filaments at any particular extension of a bundle. Considering the filament loads as random variables functionally dependent on extension, the model defined the bundle load at a particular extension to be the sum of n independently, and identically distributed random variables representing the filament loads at the extension. The distribution function of bundle load was easily defined based on the distribution of filament load at any extension. The distribution function of bundle strength, however, was hardly definable in a closed form. Hence, the mean and variance of bundle strength were estimated by utilizing the mode of the expected bundle load. The asymptotic properties of bundle strength were deduced by using the central limit theorem. Although only a particular load function was given in the study, the modification will be immediate with other load functions which are similar to the one given in this study.

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APPENDIX

An Alternative Analysis of $L_n(y)$ with Relaxed Model

The bundle load $L_n(y)$ at extension y was previously given by the sum of the filament loads at that extension, and the definition of $X_i(y)$ given by Eq. (37) indicates that the contribution of $X_i(y)$ is nil for $y > Y_i$, *i.e.* after the filament is broken. Therefore, $L_n(y)$ is contributed by all n filaments at $0 \leq y \leq Y_{(1)}$, by (n-1) filaments at $Y_{(1)} < y \leq Y_{(2)}$, ..., by a single filament at $Y_{(n-1)} < y \leq Y_{(n)}$ and finally $L_n(y)$ is zero for $Y_{(n)} < y$ since no filament survives in that region. Here, $\left\{Y_{(i)}\right\}$ are the ordered set of $\left\{Y_i\right\}$, $i = 1, 2, \ldots, n$, the breaking extensions of the n filaments.

Without loss of generality, let the β's corresponding to $Y_{(1)}$, $Y_{(2)}, \ldots, Y_{(n)}$ be $\beta_1, \beta_2, \ldots, \beta_n$, respectively. Then from the argument above, $L_n(y)$ can be defined as follows:

$$
\begin{aligned}
L_n(y) = \sum_{i=1}^{n} X_i(y) &= \sum_{i=1}^{n} \beta_i(1-e^{-\alpha y}) &&, \ 0 \leq y \leq Y_{(1)} \\
&= \sum_{i=2}^{n} \beta_i(1-e^{-\alpha y}) &&, \ Y_{(1)} < y \leq Y_{(2)} \\
&\ \ \vdots \\
&= \beta_n(1-e^{-\alpha y}) &&, \ Y_{(n-1)} < y \leq Y_{(n)} \\
&= 0 &&, \ Y_{(n)} < y
\end{aligned}
$$

Therefore,

$$P_r\left[L_n(y) \le \xi\right] = P_r\left[L_n(y) \le \xi \;/\; 0 \le y \le Y_{(1)}\right] P_r(0 \le y \le Y_{(1)})$$

$$+ P_r\left[L_n(y) \le \xi \;/\; Y_{(1)} < y \le Y_{(2)}\right] P_r(Y_{(1)} < y \le Y_{(2)})$$

$$\vdots$$

$$+ P_r\left[L_n(y) \le \xi \;/\; Y_{(n-1)} < y \le Y_{(n)}\right] P_r(Y_{(n-1)} < y \le Y_{(n)})$$

$$+ P_r\left[L_n(y) \le \xi \;/\; Y_{(n)} < y\right] \; P_r \; (Y_{(n)} < y)$$

$$= P_r\left[\sum_{i=1}^{n} \beta_i(1-e^{-\alpha y}) \le \xi\right] \; \left[1 - G(y)\right]^{n}$$

$$+ P_r\left[\sum_{i=2}^{n} \beta_i(1-e^{-\alpha y}) \le \xi\right] \; \binom{n}{1} G(y) \; \left[1 - G(y)\right]^{n-1}$$

$$\vdots$$

$$+ P_r\left[\beta_n(1-e^{-\alpha y}) \le \xi\right] \; \binom{n}{n-1} \left[G(y)\right]^{n-1} \left[1-G(y)\right]$$

$$+ P_r\left[0 \le \xi\right] \; \left[G(y)\right]^{n}.$$

Let

$$X_i^{+}(y) = \beta_i(1-e^{-\alpha y})$$

$$T_k(y) = \sum_{i=1}^{k} X_i^{+}(y)$$

$$F_k(\xi, y) = P_r\left[T_k(y) \le \xi\right]$$

and $\qquad a_k = \binom{n}{k}\left[G(y)\right]^k \left[1-G(y)\right]^{n-k}.$

Note that $X_i^+(y)$ are independently and identically distributed since β_i are so, and that the distribution of $T_k(y)$ is the k-fold convolution of the distribution of $X_i^+(y)$. Hence,

$$P_r\left[L_n(y) \le \xi\right] = F_n(\xi, y)\, a_0 + F_{n-1}(\xi, y)\, a_1 + \ldots + F_1(\xi, y) a_{n-1} + a_n.$$

When the distribution of β_i has a continuous density, so does the distribution of $T_k(y)$. Therefore, $P_r\left[L_n(y) = 0\right] = a_n$, but otherwise $L_n(y)$ has a continuous density for $0 < \xi$, of the form;

$$dP_r\left[L_n(y) \le \xi\right] = a_0 dF_n(\xi, y) + a_1 dF_{n-1}(\xi, y) + \ldots + a_{n-1} dF_1(\xi, y)$$

and, $$E\left[L_n(\xi)\right] = \sum_{i=0}^{n-1} a_i \int_0^\infty \xi\, dF_{n-i}(\xi, y) + a_n \cdot 0$$

Because $\int_0^\infty \xi\, dF_k(\xi, y) = E\left[T_k(y)\right] = k(1-e^{-\alpha y})\, E(\beta)$,

$$E\left[L_n(\xi)\right] = \left[\sum_{i=0}^{n-1} a_i (n-i)\right] (1-e^{-\alpha y})\, E(\beta)$$

$$= n\left[1-G(y)\right] (1-e^{-\alpha y})\, E(\beta). \tag{53}$$

The last equality above is due to the identities

$$\sum_{i=0}^{n-1} (n-i)\, a_i = \sum_{i=0}^{n-1} \frac{n!}{i!(n-i-1)!} \left[G(y)\right]^i \left[1-G(y)\right]^{n-i}$$

$$= n\left[1-G(y)\right]$$

Similarly,

$$E\left[L_n^2(y)\right] = \sum_{i=0}^{n-1} a_i \int_0^\infty \xi^2 dF_{n-i}(\xi, y) + a_n \cdot 0$$

But, $\int_0^\infty \xi^2 dF_k(\xi) = E\left[T_k^2(y)\right] = E\left[\sum_{i=1}^{k} X_i^+(y)\right]^2$

$$= k\ E\left[X_i^+(y)\right]^2 + k(k-1)\ E^2\left[X_i^+(y)\right]$$

$$= k(1-e^{-\alpha y})^2\ E(\beta^2) + k(k-1)\ (1-e^{-\alpha y})^2 E^2(\beta)\ .$$

Therefore,

$$E\left[L_n^2(y)\right] = \left[\sum_{i=0}^{n-1} (n-i)a_i\right] (1-e^{-\alpha y})^2\ E(\beta^2)$$

$$+\left[\sum_{i=0}^{n-1} (n-i)(n-i-1)a_i\right] (1-e^{-\alpha y})^2\ E^2(\beta)$$

$$= n\left[1-G(y)\right]\ (1-e^{-\alpha y})^2 E(\beta^2) + n(n-1)\left[1-G(y)\right]^2 (1-e^{-\alpha y})^2 E^2(\beta)$$

since $\sum_{i=0}^{n-1} (n-i)a_i = n\left[1-G(y)\right]$ as before

and $\sum_{i=0}^{n-1} (n-i)(n-i-1)a_i = \sum_{i=0}^{n-2} \frac{n!}{i!\,(n-i-2)!}\left[G(y)\right]^i \left[1-G(y)\right]^{n-i}$

$$= n(n-1)\ \left[1-G(y)\right]^2\ .$$

Hence,

$$\mathrm{Var}\left[L_n(y)\right] = E\left[L_n^2(y)\right] - E^2\left[L_n(y)\right]$$

$$= n\left[1-G(y)\right]\ (1-e^{-\alpha y})^2\ \left[E(\beta^2) - \left\{1-G(y)\right\} E^2(\beta)\right] . \quad (54)$$

The characteristic function corresponding to the distribution of $L_n(y)$ is obtained as follows:

$$\phi_{L_n(y)}(\omega) = E\left[e^{i\omega L_n(y)}\right]$$

$$= \sum_{i=0}^{n-1} a_i \int_0^\infty e^{i\omega\xi}\ dF_{n-i}(\xi, y) + a_n e^o$$

$$\text{But,}\quad \int_0^\infty e^{i\omega\xi}\, dF_k(\xi, y) = \left[\int_0^\infty e^{i\omega\xi}\, dF_1(\xi, y)\right]^k$$

$$= E^k\left[e^{i\omega\beta(1-e^{-\alpha y})}\right]$$

$$= \left\{\psi_\beta\left[\omega(1-e^{-\alpha y})\right]\right\}^k$$

where $\psi_\beta(m) = \int_0^\infty e^{imu}\, dH(u)$

and $H(u) = P_r(\beta_i \leq u)$. Therefore,

$$\phi_{L_n(y)}(\omega) = \sum_{i=0}^{n-1} a_i \left\{\psi_\beta\left[\omega(1-e^{-\alpha y})\right]\right\}^{n-i} + a_n$$

$$= \sum_{i=0}^{n} \binom{n}{i} \left[G(y)\right]^i \left[1-G(y)\right]^{n-i} \left\{\psi_\beta\left[\omega(1-e^{-\alpha y})\right]\right\}^{n-i}$$

$$= \left\{G(y) + \left[1-G(y)\right] \psi_\beta\left[\omega(1-e^{-\alpha y})\right]\right\}^n \quad . \qquad (55)$$

The results given by Eqs. (53), (54) and (55) are same as that given by Eqs. (47), (48) and (49).