

## STRESS CONCENTRATIONS AROUND PLAIN AND REINFORCED HOLES IN CYLINDRICAL AND SPHERICAL SHELLS

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### SUMMARY

In general, cutouts are introduced in pressure containing structures for access or visibility. The resulting stress concentrations can be reduced by providing reinforcement around the opening. For an efficient design of cutouts and reinforcements, it is necessary to understand correctly the effect of elastic discontinuities. The analysis available in literature, especially on thin shells with non-circular holes, is valid only for small values of curvature parameter  $\beta$  ( $\beta \ll 1$ ) defining the size of the hole in relation to the dimensions of the shell.

Here a procedure is presented for the stress analysis of pressurised cylindrical and spherical shells containing curvilinear holes valid also for  $\beta > 1$ . The two shapes of holes considered are elliptical and square holes with rounded corners. Stresses and displacements at any point in the shell are obtained as a superposition of the stresses due to a circular hole and the perturbation stresses due to departure of the given contour from a circle. In this paper only the first order perturbation is considered.

#### *Cylindrical Shell with a hole*

Starting from Donnell's shallow shell equations in polar coordinates, the solution is obtained in terms of Hankel functions of the first kind. The problem of a circular hole is solved first. The solution for a curvilinear hole is then obtained as a first order perturbation over the circular hole solution using conformal mapping technique. Boundary conditions are satisfied by a collocation procedure to obtain the unknown constants in the series solution.

It is shown that Wittrick's optimum reinforcement for a  $\sqrt{2}:1$  ellipse in a 2:1 stress field does not lead to a nearly neutral hole in pressurised cylindrical shells which implies that flat plate solutions cannot be applied to shell problems. The effect of the mechanisms by which the pressure load over the hole region is communicated to the hole edge is also studied.

#### *Spherical Shell with a hole*

A similar procedure is used for spherical shells starting from Reissner's shallow shell equations. The effect of all the parameters involved in the design of a reinforcement around the hole is studied in detail. The area and the moment of inertia of the cross-section of the reinforcement are found to play a significant role in reducing the stress concentration. It is found that the maximum membrane and bending stresses increase with increasing departure of the given hole shape from a circle.

Graphs and tables are presented giving the stress concentration factors for various cases which are helpful in designing efficient reinforcements around holes in cylindrical and spherical shells.

## 1. INTRODUCTION

It is well known that holes in structures in general and pressure vessels in particular introduce severe stress concentrations. For an efficient design an engineer must have a clear picture of the behaviour of the structure especially in the regions around the hole. The ideal situation would be to have a reinforced hole of such a shape as to leave the stress everywhere in the structure undisturbed and such a hole is referred to, in literature, as a neutral hole. It is possible to find neutral hole shapes in the case of plates under in-plane loading. However, it has been shown both experimentally and theoretically that flat plate solutions are not applicable to the case of shell problems as the curvature effects bring in large bending stresses. In this paper, we present an analysis for the case of thin pressurised cylindrical and spherical shells having elliptical holes and square holes with rounded corners.

## 2. CYLINDRICAL SHELL WITH A HOLE

The problem of stresses around a circular hole in cylindrical shells has been extensively investigated [1,2,3,4]. However the analysis of shells with non-circular holes have received considerably less attention. Savin and Guz [5,6] have solved the problem of curvilinear holes in cylindrical shells only for small values of the curvature parameter  $\beta$  ( $\beta \ll 1$ ). They employ a boundary perturbation method to reduce the problem to a series of boundary value problems in polar coordinates. The first approximation corresponds to the problem of a circular hole.

In practice one meets with large cut-outs for which  $\beta$  is more than unity. We present here a theoretical analysis to estimate approximately the stresses around holes in a cylindrical shell which is valid for moderately large values of  $\beta$  but small values of ' $\epsilon$ ', which defines the departure of the given hole contour from a circle. It is assumed that the hole is covered by a diaphragm which transmits the pressure force acting on it to the hole edge as a uniform shear. The method of solution involves a first order perturbation on the circular boundary. The analysis is applied to the case of an elliptical hole and a square hole with rounded corners using a collocation procedure.

### 2.1 FORMULATION AND SOLUTION OF THE PROBLEM

The governing differential equation for a thin pressurised cylindrical shell (Figure 1) is [4]

$$\nabla^4 \phi + 8 \beta^2 \phi_{,\xi\xi} = 8 \beta^2, \quad \beta^2 = \frac{r_0^2}{8Rt} 12(1-\nu^2)^{1/2} \quad \dots(1)$$

where  $\phi = w - iF$ ;  $w = \bar{w} + w^*$ ; and  $F = \bar{F} + F^*$ .

$\bar{F}$  and  $\bar{w}$  are prescribed values far away from the hole, in which region  $F^*$  and  $w^*$  vanish. Eq.(1) reduces to

$$\nabla^4 \phi^* + 8 \beta^2 \phi^*_{,\xi\xi} = 0 \quad ..(2)$$

where  $\phi^* = w^* - iF^*$ .

Let the function  $Z = \omega(\zeta) = \zeta + \epsilon f(\zeta)$ ;  $\epsilon \ll 1$  ..(3)

$f(\zeta) = 1/\zeta$  for an ellipse,  $f(\zeta) = 1/\zeta^3$  for a square,

where  $Z = r e^{i\theta}$  and  $\zeta = \rho e^{i\gamma}$  map conformally the region outside the given hole in the  $(r, \theta)$  plane into the region outside a circular hole in the  $(\rho, \gamma)$  plane [5].

Eq.(3) implies

$$\begin{aligned} \phi^*(r, \theta) &= \sum_{j=0}^{\infty} \epsilon^j \phi_j^*(r, \theta) \\ \text{and } K &= \sum_{j=0}^{\infty} \epsilon^j K^{(j)} \end{aligned} \quad ..(4)$$

where

$$\phi_j^* = (E_1 - iE_2) Z \sum_{n=0,2,\dots}^{\infty} (A_{nj} + i B_{nj}) H_n^1(\beta r \sqrt{2i}) \cos n\theta$$

$$+ (E_3 - iE_4) Z \sum_{n=1,3,\dots}^{\infty} (A_{nj} + i B_{nj}) H_n^1(\beta r \sqrt{2i}) \cos n\theta$$

and 'K' represents any one of  $N_{nn}$ ,  $N_{ns}$ ,  $N_{ss}$ ,  $M_{nn}$ ,  $M_{ss}$ ,  $M_{ns}$ ,  $Q_n$  and  $Q_s$  (Figure 2).

### 2.2 UNREINFORCED HOLE

If we assume that the pressure load over the hole area is transferred to the hole edge as a uniform shear, the boundary conditions at the hole edge become,

$$\begin{aligned} N_{nn} &= N_{ns} = M_{nn} = 0 \\ Q_n' &= \sum_{j=0}^{\infty} \epsilon^j q_j \end{aligned} \quad ..(5)$$

Eq.(5) gives

$$\begin{aligned} N_{nn}^{(j)} &= N_{ns}^{(j)} = M_{nn}^{(j)} = 0 \\ Q_n^{(j)} &= q_j \quad (\text{for all } j) \end{aligned} \quad ..(6)$$

(See Appendix for details)

### 2.3 REINFORCED HOLE

Figure 3 shows the reinforcement forces and moments and shell forces and moments per unit length. Equations of equilibrium are

$$\begin{aligned} N_{nn} &= T \cdot \alpha_{,s} \quad ; \quad M_{nn} = M \cdot \alpha_{,s} + H_{,s} \\ N_{ns} &= -T_{,s} \quad ; \quad M_{ns} = -M_{,s} + H \cdot \alpha_{,s} - P \\ Q_n &= P_{,s} + T \cos \alpha \cdot \eta_{0,s} - q \end{aligned} \quad ..(7)$$

where  $d\eta_0$  is the angle made by the projection of the element  $ds$

perpendicular to the generator at the axis of the cylinder.

Compatibility of displacements implies

$$\begin{aligned}
 T/AE_R &= \frac{1}{E_s t} [N_{ss} - \nu N_{nn}] \\
 M &= -E_R I W_{,ss} - E_R I W_{,n} \alpha',_s \\
 H &= \frac{E_R J W_{,ns}}{2(1+\nu)} \quad \dots(8)
 \end{aligned}$$

#### 2.4 METHOD OF SOLUTION

In this paper only the first order perturbation over the circular hole is considered. The boundary conditions are satisfied by a collocation procedure [4]. The constants  $A_{no}$ ,  $B_{no}$  can be obtained from the solution of a circular hole problem. The series for  $\phi_0^*$  is truncated at an odd value of 'n' leaving  $(2n+2)$  coefficients  $A_{no}$  and  $B_{no}$  to be determined. The set of algebraic equations required are generated by satisfying eqns.(6) for plain holes and eqns.(7) for reinforced holes at  $(n+1)/2$  equally spaced discrete points in the first quadrant. The series for  $\phi_0^*$  is truncated successively at higher values of 'n' until all the stresses calculated remain essentially the same ( $< 0.1$  per cent). A similar procedure is adopted for determining the constants  $A_{n1}$  and  $B_{n1}$  in  $\phi_1^*$ . The number of terms required for convergence in  $\phi_1^*$  was about the same as the number of terms chosen in  $\phi_0^*$  for any given  $\beta$ . In all numerical calculations  $\nu$  is taken as  $1/3$ . Knowing  $\phi_0^*$  and  $\phi_1^*$  and the transformation function  $f(\zeta)$ , one can obtain the stress resultants and moments at every point in the shell using the appendix.

#### 2.5 RESULTS

The stresses for the case of elliptical holes and square holes with rounded corners are presented in figures(4-9). For  $\sqrt{2}\beta = 10$ ; 39 terms were required for good convergence. The membrane and bending stresses for given values of  $\beta$  and  $\epsilon$  can be obtained from these graphs, e.g.

$$\frac{\sigma_{ss}}{\sigma_{\infty}} = \frac{\sigma_{ss}^{(0)}}{\sigma_{\infty}} + \epsilon \frac{\sigma_{ss}^{(1)}}{\sigma_{\infty}}$$

It is observed that the bending stresses are of the same order as the membrane stresses which are high for large size holes as well as for holes with large ' $\epsilon$ '.

Fig.10 shows the good agreement between the present solution and that of Savin [7], though the assumption made here regarding the transmission of pressure force on the cut-out region to the hole edge differs from that of Savin. This assumption primarily affects the bending stresses. In order to study the influence of reinforcement on an elliptical hole Wittrick's [8] optimum bead with  $\lambda = 0.8$ , was chosen with  $\mu = 6$ ;  $J/I = 2$  and  $E_R/E_s = 1$ . Figure.11 shows the beneficial role played by the

parameter  $\mu$ . The effect of choosing a non-uniform shear distribution (Case B) around the hole edge (shear proportional to  $\cos \theta \cos \alpha$ ) is to reduce the stress concentrations (figure 12) as compared to the case when the shear distribution is uniform (Case A). It is easily seen that the optimum reinforcement in the case of a flat plate is no more the optimum for a shell.

### 3. SPHERICAL SHELL WITH A HOLE

Starting from Reissner's [9] shallow pressurised spherical shell equations, the stress concentrations around plain and reinforced holes in spherical shells are obtained using a method similar to that described in section 2. The curvature parameter in this case is  $\beta^2 = (r_0^2/Rt)$   
 $12(1-\nu^2)^{1/2}$

The stress concentration factors for the principal stresses around unreinforced circular, elliptical and square holes in spherical shells are given in Table I. It is found that stress concentration increases with ' $\beta$ ' as well as  $\epsilon$ . A circular hole gives rise to a less severe stress concentration as compared to elliptical and square holes. A square hole results in a more severe stress concentration compared to an elliptical hole with the same  $\epsilon$ . Unlike the case of a circular hole, the contribution of the bending stresses is found to be of the same order as the membrane stresses for elliptical and square holes.

Tables II and III give the stress concentration factor for the principal stresses around symmetrically reinforced elliptical holes in spherical shells. It is seen that the area of reinforcement has a very predominant effect on the stress concentration and hence its choice requires utmost care. From Table II, it is evident that for  $\mu = 2$ , an increase in the value of  $\lambda$  from 0.2 to 1.0 results in 47 per cent reduction in stress concentration whereas increasing its value from 1 to 2 does not lead to any significant gain. It is seen that  $\mu$  also plays an important role in reducing the stress concentration although its influence is not as pronounced as of  $\lambda$ .

### ACKNOWLEDGEMENTS

A part of the work on holes in cylindrical shells presented above was done by the first author [10] at the Imperial College of Science and Technology, London under the supervision of Dr.G.A.O.Davies.

TABLE - I : Principal stress concentration factors for unreinforced circular, elliptical and square holes in spherical shells

$\sigma_p/\sigma_\infty$	$\beta$	0.1	0.5	1.0	2.0	4.0	6.0	10.0
Circular Hole	( $\epsilon=0$ )	2.07	2.75	3.70	5.38	8.40	11.30	17.02
Elliptical Hole	( $\epsilon=0.1$ )	2.47	3.19	4.17	5.74	11.17	19.89	43.62
	( $\epsilon=0.2$ )	2.87	3.64	4.64	6.61	16.91	31.56	73.38
Square Hole	( $\epsilon= -1/6$ )	4.10	5.10	6.46	8.45	15.94	27.96	64.82

TABLE - II : Stress concentrations ( $\sigma_p/\sigma_\infty$ ) around symmetrically reinforced elliptical holes in spherical shells  
 $\beta = 4$  ,  $J/I = 2.5$  ,  $\epsilon = (a-b)/(a+b) = 0.1$

$\mu$	$\lambda$	0.0	0.2	0.4	0.6	0.8	1.0	1.4	2.0
0		11.17	8.46	7.31	6.59	6.12	5.64	4.78	3.79
1		11.17	4.55	3.32	2.82	2.42	2.14	2.00	2.10
2		11.17	3.94	2.89	2.41	2.17	2.09	2.08	2.07
3		11.17	3.74	2.80	2.40	2.24	2.20	2.13	2.07
4		11.17	3.85	2.83	2.46	2.33	2.26	2.16	2.07
5		11.17	3.93	2.88	2.52	2.39	2.31	2.18	2.07
6		11.17	3.99	2.92	2.57	2.44	2.34	2.20	2.07

TABLE - III : Stress concentrations ( $\sigma_p/\sigma_\infty$ ) around symmetrically reinforced elliptical holes in spherical shells  
 $\beta = 4$  ,  $J/I = 2.5$  ,  $\epsilon = (a-b)/(a+b) = 0.2$

$\mu$	$\lambda$	0.0	0.2	0.4	0.6	0.8	1.0	1.4	2.0
0		16.91	14.73	13.26	11.92	10.64	9.39	7.12	4.56
1		16.91	7.04	4.83	3.73	3.04	2.55	2.37	2.82
2		16.91	5.88	3.92	2.98	2.56	2.54	2.77	3.00
3		16.91	5.23	3.43	2.82	2.74	2.83	2.97	3.09
4		16.91	4.78	3.22	2.96	2.96	3.02	3.09	3.14
5		16.91	4.47	3.32	3.10	3.13	3.15	3.16	3.17
6		16.91	4.41	3.43	3.25	3.24	3.24	3.22	3.20

APPENDIX

To obtain the stress transformation equations from (r,θ) coordinates to (ξ,γ) coordinates, we use

$$\begin{aligned} N_{nn} &= N_{rr} \cos^2 \psi + N_{\theta\theta} \sin^2 \psi + N_{r\theta} \sin 2\psi \\ N_{ss} &= N_{rr} \sin^2 \psi + N_{\theta\theta} \cos^2 \psi - N_{r\theta} \sin 2\psi \\ N_{ns} &= (N_{\theta\theta} - N_{rr}) \sin \psi \cos \psi + N_{r\theta} \cos 2\psi \\ Q'_n &= Q_r \cos \psi + Q_\theta \sin \psi + \partial M_{ns} / \partial s \end{aligned}$$

The stress resultants, can be expanded as a Taylor's series around the point (ξ,γ) and neglecting terms of order ε<sup>2</sup>, we get

$$K = K^{(0)} + \epsilon K^{(1)} + \Delta \xi \frac{\partial K^{(0)}}{\partial \xi} + \Delta \gamma \frac{\partial K^{(0)}}{\partial \gamma}$$

where K represents any one of N<sub>rr</sub>, N<sub>θθ</sub>, N<sub>rθ</sub>, Q<sub>r</sub>, etc., and

$$\begin{aligned} N_{rr}^{(j)} &= - \text{Imag} \left[ \frac{1}{\xi} \phi_{j,\xi} + \frac{1}{\xi^2} \phi_{j,\gamma\xi} \right] \\ N_{\theta\theta}^{(j)} &= - \text{Imag} \left[ \phi_{j,\xi\xi} \right] \\ N_{r\theta}^{(j)} &= - \text{Imag} \left[ \frac{1}{\xi^2} \phi_{j,\gamma} - \frac{1}{\xi} \phi_{j,\xi\gamma} \right] \\ Q_r^{(j)} &= - \text{Real} \left( \nabla^2 \phi_{j,\xi} \right) \end{aligned}$$

Defining

$$\begin{aligned} L_1 &= \frac{\bar{\zeta} f(\zeta) + \zeta \bar{f}(\bar{\zeta})}{2\xi} \frac{\partial}{\partial \xi} + \left[ \frac{f(\zeta) - \bar{f}(\bar{\zeta})}{2i\xi} \cos \gamma - \frac{f(\zeta) + \bar{f}(\bar{\zeta})}{2\xi} \sin \gamma \right] \frac{\partial}{\partial \gamma} \\ L_2 &= 2 \left[ \frac{\zeta \bar{f}(\bar{\zeta}) - \bar{\zeta} f(\zeta)}{2i\xi^2} + \frac{f'(\zeta) - \bar{f}'(\bar{\zeta})}{2i} \right] \end{aligned}$$

and restricting to terms of order ε only, one obtains

$$\begin{aligned} N_{nn} &= N_{nn}^{(0)} + \epsilon N_{nn}^{(1)} = N_{rr}^{(0)} + \epsilon \left[ N_{rr}^{(1)} + L_1 N_{rr}^{(0)} + L_2 N_{r\theta}^{(0)} \right] \\ N_{ss} &= N_{ss}^{(0)} + \epsilon N_{ss}^{(1)} = N_{\theta\theta}^{(0)} + \epsilon \left[ N_{\theta\theta}^{(1)} + L_1 N_{\theta\theta}^{(0)} - L_2 N_{r\theta}^{(0)} \right] \\ N_{ns} &= N_{ns}^{(0)} + \epsilon N_{ns}^{(1)} = N_{r\theta}^{(0)} + \epsilon \left[ N_{r\theta}^{(1)} + L_1 N_{r\theta}^{(0)} + (L_2/2) N_{\theta\theta}^{(0)} - N_{rr}^{(0)} \right] \\ Q_n &= Q_n^{(0)} + \epsilon Q_n^{(1)} = Q_r^{(0)} + \epsilon \left[ Q_r^{(1)} + L_1 Q_r^{(0)} + (L_2/2) Q^{(0)} \right] \\ w &= w^{(0)} + \epsilon \left[ w^{(1)} + L_1 w^{(0)} \right] \\ Q'_n &= Q_n + \partial M_{ns} / \partial s \end{aligned}$$

Similar expressions can be written for moments.

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NOTATION

$A_{nj} ; B_{nj}$	constants occurring in $\phi_j^*$
$E_s, E_R$	elastic moduli of shell and reinforcement
$E_R^A, E_R^I,$ $E_R^J/2(1+\nu)$	extensional, bending and torsional rigidities of reinforcement
$(E_1 - iE_2)_z$	$\{ \exp [(1-i) \beta r \cos \theta] + \exp [-(1-i) \beta r \cos \theta] \} / 2$
$(E_3 - iE_4)_z$	$\{ \exp [(1-i) \beta r \cos \theta] - \exp [-(1-i) \beta r \cos \theta] \} (1+i)/4$
$F$	non-dimensional Airy's stress function for membrane stresses
$\bar{F}, \bar{w}$	asymptotic values of $F$ and $w$ away from the hole
$F^*$	$F - \bar{F}$
$\omega(\zeta)$	mapping function in Eq.(3)
$H_n^1$	Hankel function of the first kind, order $n$
$p$	pressure loading
$R$	radius of the shell middle surface
$r_0$	characteristic dimension of the problem
$r$	polar coordinates in $Z$ plane
$t$	shell thickness
$w$	non-dimensional normal displacement of shell
$w^*$	$w - \bar{w}$
$w_{,rr}$	$\partial^2 w / \partial r^2$
$\alpha$	angle between the normal to the hole contour at any point and the 'X' axis
$\beta$	curvature parameter
$\epsilon$	hole shape factor
$\lambda$	$A/r_0 t$
$\mu$	$I/\Delta t^2$
$\nu$	Poisson's ratio
$\xi$	$x/r_0 = r \cos \theta$
$\sigma_\infty$	$p R/t$
$\phi_j^*$	$w_j^* - iF_j^*$
$\psi$	angle between the normal at any point on the contour and the radial direction through that point
$\nabla^2$	Laplacian operator

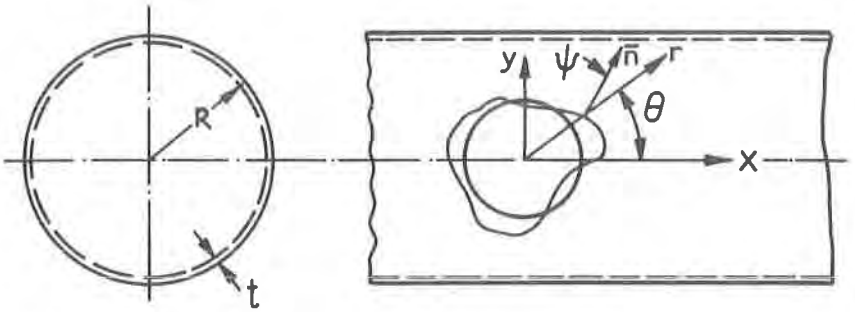


Figure 1 Cylindrical shell with an arbitrary shaped hole

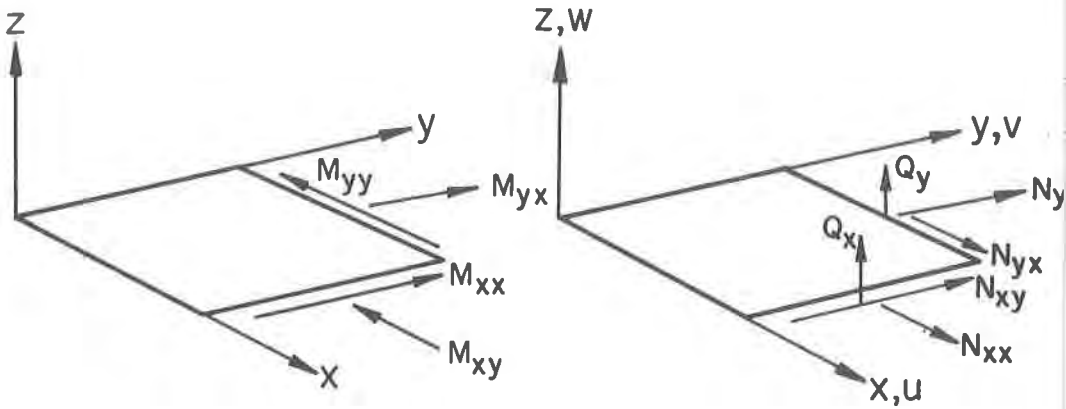


Figure 2 Stress resultants and displacements

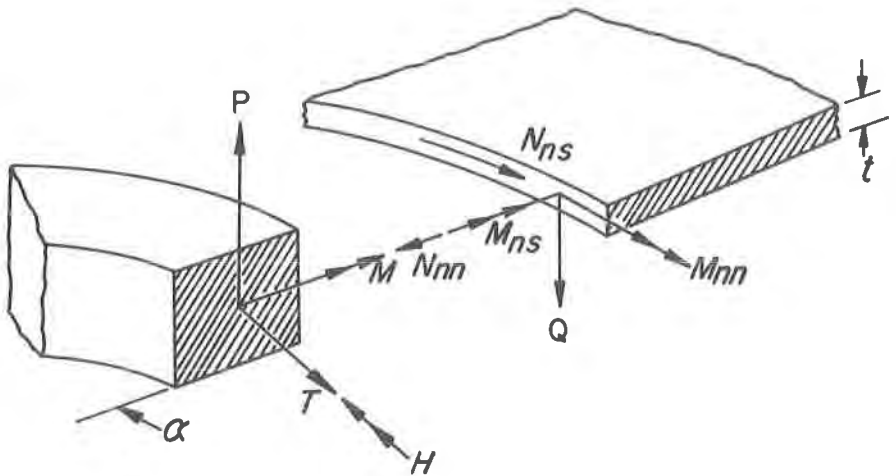


Figure 3 Forces at shell-reinforcement junction

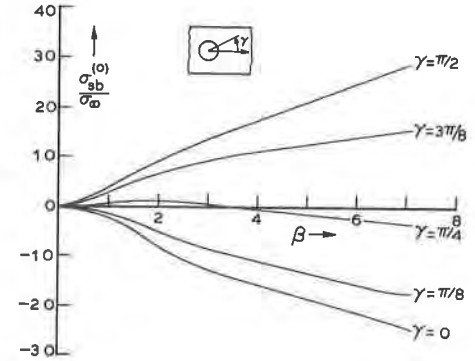
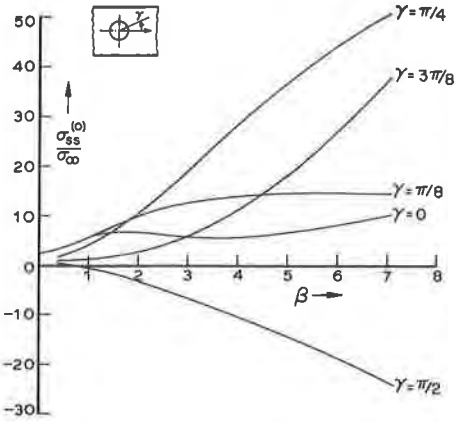


Figure 4 Direct stresses at hole edge ( $\epsilon = 0$ )

Figure 5 Bending stresses at hole edge (upper shell surface,  $\epsilon = 0$ )

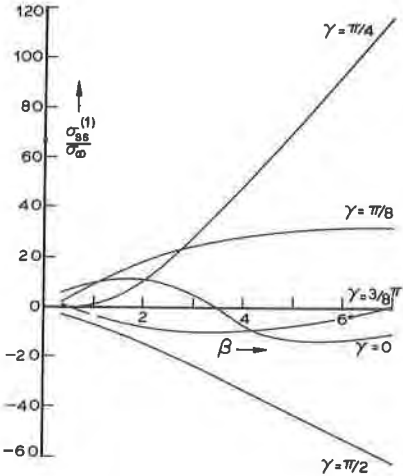


Figure 6 Membrane stress components (elliptical hole)

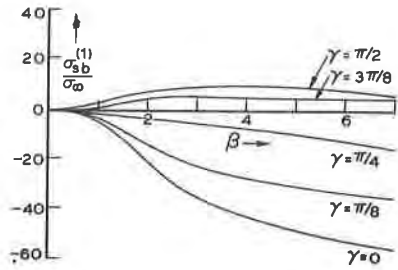


Figure 7 Bending stress components (elliptical hole - upper shell surface)

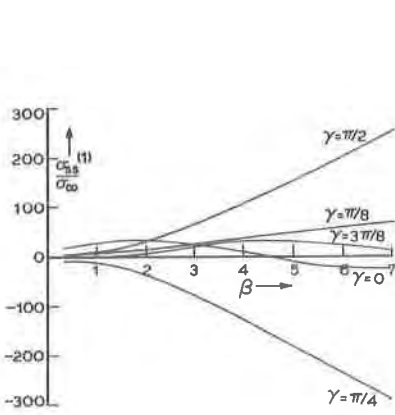


Figure 8 Membrane stress components (square hole)

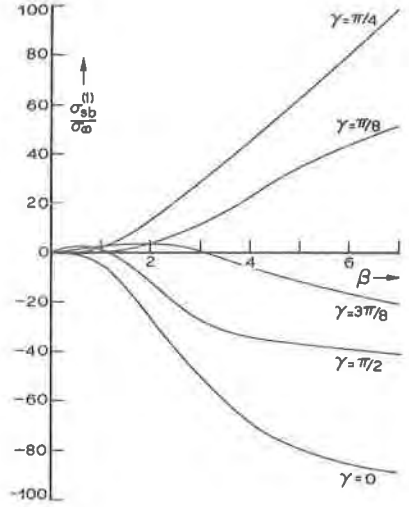


Figure 9 Bending stress components (square hole - upper shell surface)

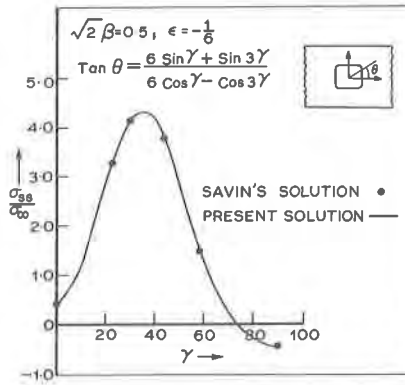


Figure 10 Stresses around a square hole

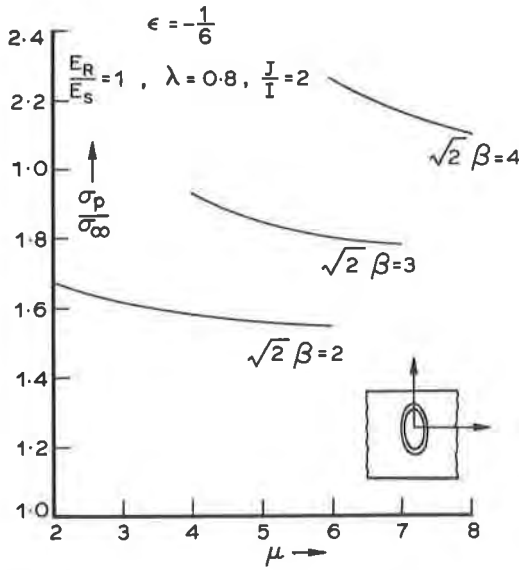


Figure 11 Reinforced elliptical hole (constant shear)

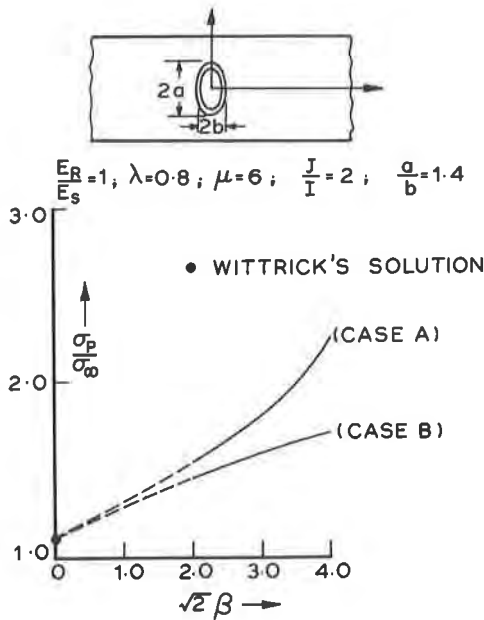


Figure 12 Reinforced elliptical hole (distributed shear)

