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A nonlinear Lax-Milgram lemma arising in the modeling of elastomers

1. Introduction

Elastomers have been used by engineers since the mid 1800s in a variety of roles including bearings, springs, and shock absorbers. For example, rubber composites filled with inactive particles such as carbon black and silica are routinely used as passive vibration suppression devices. The advent of smart materials technology has sparked great interest in the development of rubber composites filled with active elements (piezoelectric, magnetic or conductive particles) for use as active vibration suppression devices. The dynamic mechanical behavior of even the inactively filled rubbers is complex, including nonlinear constitutive laws, large deformations even under small loads, loss of kinetic energy (damping), loss of potential energy (hysteresis), dependence on fillers and environment (e.g., temperature).

Many current modeling efforts focus on phenomenological formulations involving strain energy function (SEF) theories (see [8], [10], [11]). The other predominant phenomenological approach is based on Rivlin's finite strain theories [9], [11]. Both classes of models rely on use of the principal extension ratios (deformed length of unit vectors parallel to the (principal) axes of zero shear strain), and are (as currently employed in the industrial community) static in nature. While these theories can be used in estimating stress–strain curves, they ignore hysteresis and damping. Moreover, if one attempts to extend either approach to dynamic models, the use of the principal axes system leads to additional conceptual difficulties due to a moving coordinate system.

In this note we outline the preliminary theoretical foundations for a computational methodology to treat estimation and control of elastomers. For a motivating example, we consider a slender rod under simple extension so that the zero shear axis (i.e., the principal axis) is in the direction of extension. Consider then the simple example of an isotropic, incompressible rubber–like rod with unstressed length l under simple elongation with a finite applied stress in the x direction. The finite stress theory and the SEF formulation for a neo–Hookean material (see [9], [10], [11]) lead to an engineering stress

$$\sigma_{\rm eng} = \frac{E}{3}\left(1 + \frac{\partial w}{\partial x} - \left(1 + \frac{\partial w}{\partial x}\right)^{-2}\right)$$

where w is the deformation in the x direction and E is a *generalized* modulus of elasticity. This can be used in the (force balance) Timoshenko theory for longitudinal vibrations of a bar to obtain the following nonlinear model for the dynamic longitudinal displacement of a neo–Hookean rod in extension:

$$\rho A_c \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{EA_c}{3}\tilde{g}\left(\frac{\partial w}{\partial x}\right)\right) = f \ , \tag{1.1}$$

where $\tilde{g}(\xi) = 1 + \xi - (1+\xi)^{-2}$, A_c is the cross-sectional area of the rod, and ρ is the mass density of the material.

It is convenient in arguments for well-posedness and approximation to break the stress-strain law into the sum of a linear term and a nonlinear term. We thus define $g(\xi) = \tilde{g}(\xi) - \xi$, which for a neo–Hookean material is given by $g(\xi) = 1 - (1+\xi)^{-2}$. Then the model (1.1) can be rewritten

$$\rho A_c \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{EA_c}{3}\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{EA_c}{3} g\left(\frac{\partial w}{\partial x}\right)\right) = f \ . \tag{1.2}$$

Because the neo–Hookean g is not adequate for modeling most elastomers, we will instead consider a more general nonlinearity g which must be estimated using experimental data.

In general, one does not expect (1.2) to have classical (smooth) solutions. If, for instance, one clamps the top of the rod and applies a periodic sinusoidal force to the bottom of the rod, then classical solutions do not exist due to incompatible boundary and initial conditions. It is thus useful to write (1.2) in a more generalized sense,

$$w_{tt} + Aw + \mathcal{N}^* g(\mathcal{N} w) = f \quad \text{in } V^* \tag{1.3}$$

where $\mathcal{A} = A + \mathcal{N}^* g(\mathcal{N}\cdot)$ is an operator (nonlinear) from a space V of test functions to its dual (or conjugate dual) V^*, $\mathcal{N} = \frac{\partial}{\partial x}$, and $w(t) \equiv w(t,\cdot)$. For the system (1.2) of our motivating example one can choose $V = H^1_L(0,l) \equiv \{\phi \in H^1(0,l) | \phi(0) = 0\}$ and treat the boundary condition $w(t,0) = 0$ as an essential boundary condition. Here $H^1(0,l)$ is the usual Sobolev space of $L_2(0,l)$ functions with derivatives in $L_2(0,l)$. If we take as pivot space $H = L_2(0,l)$ in the Gelfand triple setting $V \hookrightarrow H \hookrightarrow V^*$, then it is readily shown that the operator $\mathcal{A} \sim \frac{\partial}{\partial x}\left(\frac{EA_c}{3}\tilde{g}\left(\frac{\partial w}{\partial x}\right)\right)$ has the form

$$\begin{aligned}(\mathcal{A}w)(\phi) &= -\langle \frac{EA_c}{3}\tilde{g}\left(\frac{\partial w}{\partial x}\right), \phi' \rangle_{L_2(0,l)} \\ &= -\langle \frac{EA_c}{3}\frac{\partial w}{\partial x}, \phi' \rangle_{L_2(0,l)} - \langle \frac{EA_c}{3} g\left(\frac{\partial w}{\partial x}\right), \phi' \rangle_{L_2(0,l)}\end{aligned}$$

as a mapping from V to V^* and reduces to the usual (and well-known) linear operator in $\mathcal{L}(V, V^*)$ in the case that $g \equiv 0$ (and thus $\tilde{g}(\xi) = \xi$).

Motivated by these considerations and specific examples, we discuss below some observations regarding theoretical foundations for such systems.

2. Well-posedness: a nonlinear Lax-Milgram lemma

In the previous section we were lead to abstract systems of the form

$$w_{tt} + \mathcal{A}w = f \quad \text{in } V^* \tag{2.1}$$

where $\mathcal{A}$ is a nonlinear mapping from V to V^* and $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ is the usual Gelfand triple. We shall denote the H norm by $|\cdot|$, the V norm by $|\cdot|_V$ and the usual duality product by $\langle \cdot, \cdot \rangle_{V^*,V}$ where H is the pivot space. As we indicated in the previous section, we are interested in the special case where the nonlinear mapping $\mathcal{A}$ has the form

$$\mathcal{A} = A + \mathcal{N}^* g(\mathcal{N}\cdot) \tag{2.2}$$

where $A \in \mathcal{L}(V, V^*)$, $\mathcal{N} \in \mathcal{L}(V, H)$, $g : H \to H$. In this section we address the *static* case, that is, we wish to solve

$$Aw + \mathcal{N}^* g(\mathcal{N}w) = f \tag{2.3}$$

for $w \in V$. Here A is associated in the usual way with a sesquilinear form $\sigma : V \times V \to \mathbb{C}$ in that

$$\sigma(\phi, \psi) = \langle A\phi, \psi \rangle_{V^*,V} \quad \phi, \psi \in V \ .$$

We assume that σ satisfies the following assumptions.

(A1) The form σ is Hermitian (symmetric) on $V \times V$.

(A2) The form σ is V–continuous, i.e., for some $c_1 > 0$

$$|\sigma(\phi, \psi)| \le c_1 |\phi|_V |\psi|_V$$

for all ϕ, $\psi \in V$.

(A3) The form σ is strictly V–elliptic, i.e., for some $k_1 > 0$

$$\sigma(\phi, \phi) \ge k_1 |\phi|_V^2$$

for all $\phi \in V$.

The linear operator $\mathcal{N}$ in the nonlinear term is assumed to satisfy

(N1) $\mathcal{N} \in \mathcal{L}(V, H)$ with $|\mathcal{N}\phi|_H \le \sqrt{k}|\phi|_V$ for $\phi \in V$.
The nonlinear mapping g satisfies

(N2) $g : H \to H$ is continuous and $|g(\phi)| \le C_1|\phi| + C_2$ for $\phi \in H$ where C_1, C_2 are nonnegative constants;

(N3) $g(0) = 0$ and for some $\epsilon < 1$ we have for all $\phi, \psi \in H$

$$\langle g(\phi) - g(\psi), \phi - \psi \rangle \ge -\epsilon k_1 k^{-1} |\phi - \psi|^2 \ .$$

The conditions (A1) – (A3) are quite standard in the theory of linear systems while the conditions (N1) – (N3) are readily shown to hold for classes of the elastomer models discussed in Section 1. Under these assumptions, one can prove existence of unique solutions to (2.3) which can be viewed as a nonlinear version of the generalized Lax-Milgram lemma ([12] and [5]).

Lemma 2.1. Under (A1) – (A3), (N1) – (N3), we have for each $f \in V^*$, equation (2.3) has a unique solution w in V.

Proof. We sketch the constructive arguments since they can also be used to guarantee convergence of certain classes of Galerkin approximations for (2.3). Let $\{\xi_j\}_{j=1}^{\infty}$ be a total set in V and let

$$V^M = \text{span}\{\xi_1, \xi_2, \ldots, \xi_M\} \quad M = 1, 2, \ldots \ .$$

Since (2.3) is equivalent to

$$\sigma(w, \phi) + \langle g(\mathcal{N}w), \mathcal{N}\phi \rangle = \langle f, \phi \rangle_{V^*,V} \tag{2.4}$$

for all $\phi \in V$, we may define a sequence $\{w^N\}$ of approximates by $w^N \in V^N$ and satisfies

$$\sigma(w^N, \phi) + \langle g(\mathcal{N} w^N), \mathcal{N}\phi\rangle = \langle f, \phi\rangle_{V^*,V} \tag{2.5}$$

for all $\phi \in V^N$.

Noting that $V^M \subset V^N$ for $N \geq M$, we observe that

$$\sigma(w^N, \phi) + \langle g(\mathcal{N} w^N), \mathcal{N}\phi\rangle = \langle f, \phi\rangle_{V^*,V} \tag{2.6}$$

for all $\phi \in V^M$ and for all $N \geq M$.

If we choose $\phi = w^N$ in (2.5) we obtain

$$\sigma(w^N, w^N) + \langle g(\mathcal{N} w^N), \mathcal{N} w^N\rangle = \langle f, w^N\rangle$$

and using (A3) with (N3) we find

$$k_1 |w^N|_V^2 - \epsilon k_1 k^{-1} |\mathcal{N} w^N|^2 \leq |f|_{V^*} |w^N|_V \ .$$

After an application of (N1) we have the uniform *a priori* estimates

$$(1-\epsilon) k_1 |w^N|_V \leq |f|_{V^*} \ . \tag{2.7}$$

From (N1) and (N2) we also find

$$|\mathcal{N} w^N|_H \leq \sqrt{k}|w^N|_V \leq |f|_{V^*}\sqrt{k}/(1-\epsilon)k_1 \equiv C|f|_{V^*} \tag{2.8}$$

and

$$|g(\mathcal{N} w^N)|_H \leq C_1 |\mathcal{N} w^N|_H + C_2 \leq C_1 C |f|_{V^*} + C_2 \ . \tag{2.9}$$

Using these estimates and taking the usual subsequences we find there exist a subsequence (again denoted by w^N), $w \in V$ and $h \in H$ such that

$$\begin{aligned} &w^N \to w \text{ weakly in } V \\ &g(\mathcal{N} w^N) \to h \text{ weakly in } H \\ &\mathcal{N} w^N \to \mathcal{N} w \text{ weakly in } H \ . \end{aligned}$$

Fixing M and taking the limit in (2.6) as $N \to \infty$ for the subsequences we obtain

$$\sigma(w, \phi) + \langle h, \mathcal{N}\phi\rangle = \langle f, \phi\rangle_{V^*,V} \tag{2.10}$$

for all $\phi \in V^M$. Since $\bigcup_{M=1}^{\infty} V^M$ is dense in V, we find that (2.10) actually holds for all $\phi \in V$. Thus for existence it remains only to argue that

$$\langle h, \mathcal{N}\phi\rangle = \langle g(\mathcal{N}w), \mathcal{N}\phi\rangle \quad \text{for all } \phi \in V \ .$$

Returning to (N3), we see that this along with the inequality in (N1) implies

$$\langle g(\mathcal{N}\phi) - g(\mathcal{N}\psi), \mathcal{N}\phi - \mathcal{N}\psi\rangle + k_1|\phi - \psi|_V^2 \geq 0$$

for all $\phi, \psi \in V$. In light of (A3), we thus find for all $\phi, \psi \in V$

$$\langle g(\mathcal{N}\phi) - g(\mathcal{N}\psi), \mathcal{N}\phi - \mathcal{N}\psi\rangle + \sigma(\phi - \psi, \phi - \psi) \geq 0 \ . \tag{2.11}$$

In particular, since $V^M \subset V$ we may choose $\psi \in V^M$ for M fixed and $\phi = w^N$ for $N \geq M$ in (2.11), obtaining

$$\sigma(w^N - \psi, w^N - \psi) + \langle g(\mathcal{N}w^N) - g(\mathcal{N}\psi), \mathcal{N}w^N - \mathcal{N}\psi\rangle \geq 0 \ . \tag{2.12}$$

Returning to (2.5) we also have upon choosing $\phi = w^N - \psi$ with ψ arbitrary in V^M, $N \geq M$

$$\sigma(w^N, w^N - \psi) + \langle g(\mathcal{N}w^N), \mathcal{N}w^N - \mathcal{N}\psi\rangle = \langle f, w^N - \psi\rangle_{V^*,V} \ .$$

Subtracting this from (2.12) and letting $N \to \infty$, we obtain (for M fixed)

$$-\sigma(\psi, w - \psi) - \langle g(\mathcal{N}\psi), \mathcal{N}w - \mathcal{N}\psi\rangle + \langle f, w - \psi\rangle_{V^*,V} \geq 0 \tag{2.13}$$

for all $\psi \in V^M$. Again by density of $\bigcup_{M=1}^{\infty} V^M$, we obtain that this inequality actually holds for all $\psi \in V$.

If we choose $\phi = w$ in (2.10) we obtain

$$\sigma(w, w) + \langle h, \mathcal{N}w\rangle - \langle f, w\rangle_{V^*,V} = 0 \tag{2.14}$$

while a choice of $\phi = -\psi$, ψ arbitrary in V, in (2.10) yields

$$-\sigma(w, \psi) - \langle h, \mathcal{N}\psi\rangle + \langle f, \psi\rangle_{V^*,V} = 0 \ . \tag{2.15}$$

Adding (2.13), (2.14) and (2.15) we obtain

$$\sigma(w - \psi, w - \psi) + \langle h - g(\mathcal{N}\psi), \mathcal{N}w - \mathcal{N}\psi\rangle \geq 0 \tag{2.16}$$

for all $\psi \in V$.

For arbitrary $\phi \in V$ and $\lambda > 0$, choose ψ in (2.16) as $\psi = w - \lambda\phi$. We obtain

$$\lambda^2 \sigma(\phi, \phi) + \lambda \langle h - g(\mathcal{N}w - \lambda \mathcal{N}\phi), \mathcal{N}\phi \rangle \geq 0 \ . \tag{2.17}$$

Dividing by $\lambda > 0$ and then letting $\lambda \to 0$, we obtain (we use here the continuity of g from (N2))

$$\langle h - g(\mathcal{N}w), \mathcal{N}\phi \rangle \geq 0$$

for arbitrary $\phi \in V$; this, of course, implies that equality must hold for all $\phi \in V$. Thus (2.10) can be written

$$\sigma(w, \phi) + \langle g(\mathcal{N}w), \mathcal{N}\phi \rangle = \langle f, \phi \rangle_{V^*,V} \tag{2.18}$$

for arbitrary $\phi \in V$. Thus w is a solution to (2.3).

Turning to uniqueness, suppose w_1, w_2 are two solutions of (2.3) corresponding to a given $f \in V^*$. Then we have immediately that

$$\sigma(w_1 - w_2, \phi) + \langle g(\mathcal{N}w_1) - g(\mathcal{N}w_2), \mathcal{N}\phi \rangle = 0$$

for all $\phi \in V$. Choosing $\phi = w_1 - w_2$, we find

$$\sigma(w_1 - w_2, w_1 - w_2) + \langle g(\mathcal{N}w_1) - g(\mathcal{N}w_2), \mathcal{N}w_1 - \mathcal{N}w_2 \rangle = 0 \ .$$

Using (A3) and (N3), we obtain

$$k_1 |w_1 - w_2|_V^2 - \epsilon k_1 k^{-1} |\mathcal{N}w_1 - \mathcal{N}w_2|^2 \leq 0 \ ,$$

which with (N1) implies

$$k_1(1 - \epsilon)|w_1 - w_2|_V^2 \leq 0 \ .$$

Since $\epsilon < 1$, we must have $|w_1 - w_2|_V = 0$.

It is appropriate at this point to make several remarks about the above lemma and its proof. The inequality in (N3) can be recognized immediately as a type of monotonicity condition on the nonlinearity g. Conditions of this type and their use in arguments such as those involving (2.11)–(2.17) can be readily found in the literature on monotone operators and variational inequalities and are sometimes referred to as "Minty-Browder" type conditions and arguments (e.g., see [6], pp. 83–87). Indeed, the results above can be considered as a special case (or a corollary) of earlier results due to Minty [7] and Browder [4]. Specifically, if we consider Corollary 1.8, p. 87 of

[6], we see that if $\mathcal{A} : X \to X^*$ is a monotone, coercive nonlinear operator that is continuous on finite dimensional subspaces of X, then we are guaranteed existence of $w \in X$ such that

$$\langle \mathcal{A}w - f, v - w \rangle_{X^*,X} \geq 0$$

for all $v \in X$. Choosing $v = w + t\phi$ for t arbitrary, $\phi \in X$ arbitrary, we obtain

$$\langle \mathcal{A}w - f, t\phi \rangle_{X^*,X} \geq 0 \ .$$

This implies

$$\langle \mathcal{A}w - f, \phi \rangle_{X^*,X} = 0$$

for all $\phi \in X$ or $w \in X$ is a solution to $\mathcal{A}w = f$ in X^*. One could thus take an alternative approach to establish Lemma 2.1 by arguing that $\mathcal{A} = A + \mathcal{N}^*(g(\mathcal{N}\cdot))$ of (2.2), under certain conditions on A, $\mathcal{N}$, and g, satisfies the monotonicity, coercivity and continuity conditions of Corollary 1.8 of [6]. As we have already noted, we chose the constructive arguments outlined above since we can use them to obtain readily convergence of certain families of Galerkin approximations. This will be discussed in the next section.

3. Approximation and convergence

The arguments given in the previous section can be used almost immediately to establish convergence of certain types of Galerkin approximations that can readily be employed in computations. To see this, suppose we have a family H^N, $N = 1, 2, \ldots$, of finite dimensional approximation subspaces satisfying:

(C1) $H^N \subset V$ and for each $v \in V$, there exists $\{\tilde{v}_N\}$, $\tilde{v}_N \in H^N$ such that $|v - \tilde{v}_N|_V \to 0$ as $N \to \infty$;

(C2) $H^N \subset H^{N+1}$, $N = 1, 2, \ldots$.

The condition (C1) is a standard one from the theory of finite elements and is satisfied by many families for a given V (which, of course, is dictated by σ and H in specific examples of (2.3) or (2.4)). The condition (C2) is somewhat more unusual but is also satisfied by a number of schemes (piecewise linear splines, spectral families, etc.) as we shall see below. For the moment, assume (C2) holds. Then without loss of generality (i.e., by possibly reindexing the elements – see the example below) we may assume that each H^N can be written as

$$H^N = \text{span}\{\psi_1, \ldots, \psi_N\}$$

for some family of elements $\{\psi_i\}_{i=1}^{\infty}$. Because of (C1), we have that $\{\psi_i\}_{i=1}^{\infty}$ is total in V. We may now define the Galerkin approximations by $w^N = \sum_{j=1}^{N} w_j^N \psi_j$ where the w_j^N are defined by the nonlinear algebraic system (see (2.5))

$$\sigma(w^N, \phi) + \langle g(\mathcal{N} w^N), \mathcal{N}\phi\rangle = \langle f, \phi\rangle_{V^*,V} \tag{3.1}$$

for all $\phi \in H^N$. The arguments for existence and uniqueness of Section 2 guarantee that there is some subsequence $\{w^{N_k}\}$ of these Galerkin approximations such that $w^{N_k} \to w$ weakly in V, where w is the unique solution to (2.3) or equivalently (2.18). But solutions of (2.3) are unique and we see furthermore (from the arguments of Section 2) that *any* subsequence of the Galerkin sequence $\{w^N\}$ possesses, in turn, a subsequence which must converge to the unique solution w. Thus the original Galerkin sequence $\{w^N\}$ itself must converge weakly in V to the solution w of (2.3). If the embedding $V \hookrightarrow H$ is also compact (a condition that holds in many problems), we see that the Galerkin sequence converges strongly in H. Under reasonable assumptions, one can make additional arguments to obtain convergence of $w^N \to w$ strongly in V. But these results do not follow directly from the arguments in Section 2. Thus we have

Lemma 3.1. Under (C1), (C2) we have that the Galerkin approximations $\{w^N\}$ defined via (3.1) converge weakly in V to the unique solution of (2.3). If in addition $V \hookrightarrow H$ is compact, the convergence is strong in H.

To indicate briefly an example which satisfies all of the conditions for the above lemma, consider the example for simple elongation of a rubber–like prismatic rod as outlined in Section 1. Then $H = L_2(0,l)$, $V = H_L^1(0,l)$ and for approximation elements we choose piecewise linear splines. That is, let ψ_j^K be the piecewise linear elements corresponding to discretization increments $\Delta x_j = l/K$. Hence for $j = 1, 2, \ldots, K$, ψ_j^K is piecewise linear, has value one at $x_j^K = jl/K$, and value zero outside (x_{j-1}^K, x_{j+1}^K). Define

$$Z^K = \text{ span } \{\psi_1^K, \ldots, \psi_K^K\}$$

and $H^N = Z^{2^N}$. Then conditions (C1) and (C2) above are readily established, as is the compact embedding of V into H.

The computations (as reported in [3], [2]) mentioned below were carried out using exactly the piecewise linear scheme described here.

We return briefly to consider strong V convergence of the Galerkin approximations without (C2) or the compact embedding requirement on V. We need an additional assumption on g for the arguments we give here.

(N4) For any $\phi \in H$ the Frechet derivative $g'(\phi)$ exists and satisfies $g'(\phi) \in \mathcal{L}(H)$ with $|g'(\phi)|_{\mathcal{L}(H)} \leq C_3$ for some constant C_3 independent of ϕ.

Lemma 3.2. Under (A1) – (A3), (N1) – (N4) and (C1), the Galerkin approximations $\{w^N\}$ defined via (3.1) converge strongly in V to the unique solution w of (2.3).

Proof. Let $\tilde{w}_N$ be chosen such that $\tilde{w}_N \to w$ in V as guaranteed by (C1). Since

$$|w^N - w|_V \leq |w^N - \tilde{w}_N|_V + |\tilde{w}_N - w|_V \ ,$$

it suffices to argue that $\Delta^N \equiv w^N - \tilde{w}_N \to 0$ in V. From (2.18) and (3.1) we have for any $\phi \in H^N$

$$\sigma(w^N - w, \phi) + \langle g(\mathcal{N} w^N) - g(\mathcal{N} w), \mathcal{N}\phi \rangle = 0 \ .$$

Choosing $\phi = \Delta^N = w^N - \tilde{w}_N$, we may write

$$\sigma(\Delta^N + \tilde{w}_N - w, \Delta^N) + \langle g(\mathcal{N} w^N) - g(\mathcal{N} w), \mathcal{N}\Delta^N \rangle = 0$$

or

$$\begin{aligned} \sigma(\Delta^N, \Delta^N) + \sigma(\tilde{w}_N - w, \Delta^N) \ &+\ \langle g(\mathcal{N} w^N) - g(\mathcal{N} \tilde{w}_N), \mathcal{N}\Delta^N \rangle \\ &+\ \langle g(\mathcal{N} \tilde{w}_N) - g(\mathcal{N} w), \mathcal{N}\Delta^N \rangle = 0 \ . \end{aligned}$$

From (A2), (A3) and (N3) we find

$$k_1 |\Delta^N|_V^2 \leq c_1 |\tilde{w}_N - w|_V |\Delta^N|_V + \epsilon k_1 k^{-1} |\mathcal{N}\Delta^N|_H^2 + |g(\mathcal{N}\tilde{w}_N) - g(\mathcal{N} w)|_H |\mathcal{N}\Delta^N|_H \ .$$

Thus, using (N1) we conclude

$$(1-\epsilon) k_1 |\Delta^N|_V \leq c_1 |\tilde{w}_N - w|_V + \sqrt{k} |g(\mathcal{N}\tilde{w}_N) - g(\mathcal{N} w)|_H \ . \tag{3.2}$$

But a standard calculation yields

$$\begin{aligned} |g(\mathcal{N}\tilde{w}_N) - g(\mathcal{N} w)|_H \ &= \ \left| \int_0^1 g'(\theta \mathcal{N}\tilde{w}_N + (1-\theta)\mathcal{N} w)[\mathcal{N}\tilde{w}_N - \mathcal{N} w] d\theta \right|_H \\ &\leq \ C_3 |\mathcal{N}\tilde{w}_N - \mathcal{N} w|_H \leq \sqrt{k} C_3 |\tilde{w}_N - w|_V \ . \end{aligned}$$

Thus from (3.2) we obtain

$$(1-\epsilon)k_1|\Delta^N|_V \le (c_1 + kC_3)|\tilde{w}^N - w|_V$$

which by (C1) gives the desired convergence.

4. Static inverse problems

Using static testing, one may estimate the nonlinearity g for a given sample. To eliminate the need to estimate E, consider instead $\hat{g} = \frac{E}{3}\tilde{g}$. Under a static tensile load f_i, with resulting end displacement Δ_i, the sample satisfies the steady state equation

$$\begin{aligned} &\frac{\partial}{\partial x}\left(A_c\hat{g}\left(\frac{\partial w_i}{\partial x}\right)\right) = 0 \qquad 0 < x < l \\ &A_c\hat{g}\left(\frac{\partial w_i}{\partial x}\right)(l) = f_i \\ &w_i(0) = 0 \\ &w_i(l) = \Delta_i \end{aligned}$$

where the nonlinearity $\hat{g}$ is unknown. We seek to find $\hat{g}$ minimizing

$$J(\hat{g}) = \sum_{i=1}^{k}\left|\Delta_i - w_i(l;\hat{g})\right|^2 \tag{4.1}$$

over some class of admissible functions $\hat{g} \in \mathcal{G}$, where $\{\Delta_i, f_i\}_{i=1}^k$ are data from a series of "static pull" experiments.

In general, problems such as those involving (4.1) are infinite dimensional in both state and parameter space and hence for computational purposes, finite dimensional approximations must be made. For state approximation, one typically uses Galerkin techniques such as those discussed above. For parameter space (i.e., approximation of $\hat{g}$), one may use a finite dimensional parameterization or representation. For example, one may approximate $\hat{g}$ using M approximating elements (e.g., linear splines)

$$\hat{g}_M(x) = \sum_{j=1}^{M} c_j\eta_j(x)\ .$$

The minimization problem is then to find $\vec{c} \in \mathbb{R}^M$ minimizing

$$J(\vec{c}) = \sum_{i=1}^{k}\left|\Delta_i - w_i^N(l;\vec{c})\right|^2\ . \tag{4.2}$$

We have used our methods (with a linear spline parameterization of $\hat{g}$) to fit the data from static tensile strain experiments performed at Lord Corporation. The results, which we only summarize here, can be found in detail in [2]. Briefly, one of the standard industrial techniques is to estimate a cubic Mooney–Rivlin SEF (see [10], [11]). It is not difficult to generate the stress–strain relationship which arises from the estimated SEF. Viewed in the stress–strain plane, the results from our method are nearly identical to the results from the SEF method. Mooney plots, which are in the reduced stress plane (see [10], pp. 95–99, [11], pp. 51–52), can also be generated from both methods. The reduced stress curve generated from our method approximates the curve generated from the data more closely than does the curve resulting from the SEF method. Thus the approach proposed in this note offers the possibility of improvement on existing industrial methods.

5. Concluding Remarks

The monotonicity arguments underlying the theory for static systems (2.3) given in Section 2 are also useful in establishing well–posedness for the dynamical analogues (1.3). While these analogous results are technically much more tedious to establish, the arguments are very much in the same spirit (e.g., *a priori* bounds and monotonicity conditions such as (N3)). Since damping is also an important issue in dynamical responses, this must be added to models such as (1.3). Under appropriate assumptions on the damping sesquilinear form and the additional assumption that g is of gradient type (i.e., there exists $G : H \to \mathbb{R}^1$ with the Frechet derivative of G given by $G'(\phi)\psi = \text{Re}\ \langle g(\phi), \psi\rangle$ for $\psi \in H$), one can use (A1)–(A3) and (N1)–(N4) to establish existence, uniqueness and certain regularity for solutions of (1.3). Details are given in [1]. Computational aspects of these problems for these dynamical systems are given in [3] and [2]. We are currently using the associated computational methods in design and analysis of dynamic tensile experiments.

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