On Load Carrying Capacity of Frames Taking Into Account Finite Displacements

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Abstract

An approximate method that takes into account the influence of finite displacements upon the load carrying capacity of planar unbraced frames is described. It is the iterative procedure where Quadratic Programming is applied for evaluation of subsequent configurations of the frame, whereas Linear Programming serves to find ultimate load factors for each configuration. Numerical tests show that the procedure gives practically acceptable results being at the same time much cheaper than the exact geometrically and physically non-linear incremental analysis.

1. Introduction

Most of the contemporary codes and design rules in Civil Engineering are based upon a semi-probabilistic concept of safety factors against the so-called ultimate states [1]. A plastic collapse is one of such states. Therefore evaluation of the load that corresponds to this event is an important task and should be accomplished with a proper accuracy.

It is commonly known that a conventional rigid-plastic-small-displacement approach [2, 3] gives over-estimated values of the collapse load for unbraced skeleton frames. The reason is that finite lateral displacements occuring prior to the development of the plastic collapse mechanism diminish the resistance of the structure. In order to find the exact ultimate load one has to solve a complex non-linear incremental problem. Although theoretical and numerical aspects of the solution of physically and geometricallly non-linear problems are intensively studied in many countries, the present state of art in this domain is still far from every-day applicability. Calculations are cumbersome and lengthy despite the high capacity of modern computers. Hence the exact incremental finite displacement analysis is performed (if at all) at the stage of the final check of the structure. For a preliminary evaluation of trial designs one needs a simplified method that accounts for geometric effects but requires much less computer time than the exact analysis.

An example of such an approximate procedure is the so-called P-Δ method [4]. It has a disadvantage that results depend too much upon rather arbitrary estimation of the sway. In the present paper we propose an alternative method named LQP-procedure. It is based upon a sequential application of Linear Programming (LP) and Quadratic Programming (QP). The first gives subsequent estimates of the collapse load, while the second provides the deformed shape of the frame in current iteration.

2. Matrix description of elastic-perfectly-plastic behaviour

Consider an arbitrary structure made from an elastic-perfectly-plastic material. Let the structure be discretized, i.e. described by means of a finite number of parameters arranged into the following four vectors: the generalized strain $\underline{q} \in R^m$, stress $\underline{s} \in R^m$, displacement $\underline{w} \in R^n$ and load $\underline{p} \in R^n$. Under the assumption of small displacements the strain-displacement relation

$$\underline{q} = \underline{C}\,\underline{w} \qquad (1)$$

is linear, $\underline{C}$ being the (m x n)-matrix of compatibility. The transpose of this matrix enters the equilibrium equation

$$\underline{p} = \underline{C}^T \underline{s} \qquad (2)$$

Assuming the constitutive law of a plastic flow type with piecewise-linear yield surface we split the total strain into elastic and plastic parts:

$$\underline{q} = \underline{q}_e + \underline{q}_p = \underline{E}^{-1}\,\underline{s} + \int_0^T \underline{N}\,\dot{\underline{\lambda}}\,dt \qquad (3)$$

Here $\underline{E}$ is the (m x m)-matrix of elasticity and $\underline{N}$ is the (m x k)-matrix that appears in the condition of the admissibility of stresses

$$\underline{N}^T \underline{s} \leq \underline{c} \,. \qquad (4)$$

The vector $\underline{c} \in R^k$ contains plastic moduli of the structure. The vector of plastic multipliers $\dot{\underline{\lambda}} \in R^k$ must satisfy additional complementarity constraints that reflect the non-holonomic nature of the plastic behaviour.

The integral on the right hand side of equation (3) depends upon the loading history. Hence the exact response of the elastic-plastic structure can be obtained only in the incremental way, which means integration along the given loading path. Following Maier [5] we neglect in the sequel the path-dependence of the plastic strain. Then the response of the structure to the given load $\underline{p}$ can be found in one step solving the following pair of the dual QP-problems:

$$\max \left\{ -\frac{1}{2} \underline{s}^T \underline{E}^{-1} \underline{s} \;\middle|\; \underline{C}^T \underline{s} = \underline{p} \,, \; \underline{N}^T \underline{s} \leq \underline{c} \right\} , \qquad (5')$$

$$\min \left\{ \frac{1}{2} \underline{q}^T \underline{E}\, \underline{q} - \underline{p}^T \underline{w} + \underline{c}^T \underline{\lambda} \;\middle|\; \underline{C}\, \underline{w} - \underline{N}\, \underline{\lambda} - \underline{q} = \underline{0} \,, \; \underline{\lambda} \geqslant \underline{0} \right\} \qquad (5'')$$

Although theoretically it is immaterial which one of the two above problems is actually solved, the kinematic formulation (5″) is advantageous from the numerical point of view. The reason is that it can be transformed into the problem:

$$\min \left\{ \frac{1}{2} \underline{\lambda}^T \underline{A}\, \underline{\lambda} + \underline{d}^T \underline{\lambda} \;\middle|\; \underline{\lambda} \geqslant \underline{0} \right\} , \qquad (6)$$

where

$$\underline{A} = \underline{N}^T \underline{E}^{-1} \underline{N} - \underline{N}^T \underline{E}^{-1} \underline{C} \, (\underline{C}^T \underline{E}^{-1} \underline{C})^{-1} \underline{C}^T \underline{E}^{-1} \underline{N} \,, \qquad (7)$$

$$\underline{d} = \underline{c} - \underline{N}^T \underline{E}^{-1} \underline{C} \, (\underline{C}^T \underline{E}^{-1} \underline{C})^{-1} \underline{p} \,. \qquad (8)$$

Numerical search of a minimum subject to the non-negativeness constraints only is much simpler than solution of a general constrained optimization problem.

Let $\hat{\underline{p}}$ be the given reference load, e.g. the service load. Assuming that overloading occurs in a proportional manner, i.e.

$$\underline{p} = \mu \, \hat{\underline{p}} \qquad (9)$$

one can find the ultimate load factor μ_{ult} which corresponds to the plastic collapse of the structure. This value μ_{ult} represents a safety factor and can be established by solving the following pair of dual LP-problems:

$$\max \left\{ \mu \;\middle|\; \mu \hat{\underline{p}} - \underline{C}^T \underline{s} = \underline{0}, \; \underline{N}^T \underline{s} \leq \underline{c} \right\} , \qquad (10')$$

$$\min \left\{ \underline{c}^T \dot{\underline{\lambda}} \;\middle|\; \underline{N}\, \dot{\underline{\lambda}} - \underline{C}\, \dot{\underline{w}} = \underline{0}, \; \hat{\underline{p}}^T \dot{\underline{w}} = 1 \right\} . \qquad (10'')$$

3. Description of LQP-method

A portal frame shown in Figure 1a will help us to explain the poposed method. The reference load is $\hat{\underline{p}} = \{P_1, P_2\} = \{11.76, 70.60\}$ (kN). We solve first the LP-problems (10) for the initial configuration of the frame. This solution yields the ultimate load factor $\mu_1 = 1.293$. Then we take $\underline{p}_1 = \mu_1 \hat{\underline{p}}$ as a given load and solve the QP-problem (6) obtaining $\underline{\lambda}_1$ and corresponding to it displacement $\underline{w}_1$. The value of sway, i.e. the top lateral displacement of the frame, is $\Delta_1 = 0.286$ (m).

Both μ_1 and Δ_1 overestimate the values $\mu^* = 1.210$ and $\Delta^* = 0.237$ that are obtained from the exact non-linear analysis. Adding the components of $\underline{w}_1$ to the initial co-ordinates of the nodes we find the deformed shape of the frame under the load $\underline{p}_1$ (Figure 1b). Now we take this configuration and compute the collapse load for it by means of (10). The result is $\mu_2 = 1.182$. Since the sway Δ_1 was excessive, the new ultimate load factor is smaller than μ^* . We feed the load $\underline{p}_2 = \mu_2 \hat{\underline{p}}$ into (6) and calculate $\underline{w}_2$. It gives the sway $\Delta_2 = 0.1415$ which is less than Δ^* . Allowing the nodes of the initial configuration to undergo displacements $\underline{w}_2$ we obtain the new shape of the frame (Figure 1c). Then we calculate μ_3 for this shape from (10), find $\underline{w}_3$ corresponding to $\underline{p}_3$ from (6), update the configuration and repeat the cycle.

The sequence μ_i converges rapidly towards a certain value $\hat{\mu}$ (Figure 2). It happens for this particular example that $\hat{\mu} \approx \mu^*$ but in general we can expect $\hat{\mu}$ to be only a good estimate of μ^* . Numerical tests indicate that $\hat{\mu}$ usually exceeds μ^* slightly, being always much more accurate than μ_1 given by the conventional ultimate load analysis. The displacement $\underline{w}$ obtained from the LQP-method is of less practical significance. It underestimates exact deflections because $\underline{w}_i$ for subsequent iterations are obtained from the small displacement analysis of the udeformed frame.

The LQP-method was implemented in FORTRAN on the computer ODRA 1305. The package allows to calculate refined estimates of the ultimate load for arbitrary planar frames. The main difficulty in practical applications of the method arises from large dimensions of the matrices $\underline{C}$ and $\underline{N}$. It was necessary therefore to include an efficient generator of these matrices into the package, to exploit their sparsity and to organize carefully the flow of information between core and outer memories.

The principal flow chart of the program is shown in Figure 3. The LP-problem (10) is solved by means of the revised simplex method [7]. The QP-problem (6) is solved using Lemke's algorithm [8]. The intermediate results, i.e. the ultimate load factor μ_i, the stress at collapse $\underline{s}_i$, the collapse mechanism $\dot{\underline{q}}_i$ and $\dot{\underline{w}}_i$ as well as the current estimate of displacements $\underline{w}_i$, are printed after each iteration. When the difference between subsequent factors μ_{i-1}, μ_i becomes less than a prescribed treshold, then the iteration process is terminated. Otherwise the updated matrix $\underline{C}$ is computed and the procedure goes back to the generation of a new simplex matrix.

4. Numerical results

Since the proposed method has rather heuristic than strict mathematical justification it was necessary to check its accuracy, efficiency and stability by numerical experiments. Therefore a series of typical steel skeleton frames was calculated. For the sake of comparison each frame was analysed by means of five different methods. Two of them fall in the range of small dis placement theory, namely the incremental analysis by tangent stiffness method and the conventional ultimate load analysis by Linear Programming [9, 10]. The remaining three procedures take into account finite displacements: the P-Δ method [4], the incremental analysis in updated Lagrangian formulation [6] and the present LQP-method.

A detailed presentation of the numerical results will be given in a full version of the paper. In this compact we show only an example of a tall one-bay frame (Figure 4). Geometrically linear models yielded practically the same result: $\mu = 2.238$ from the incremental procedure and $\mu = 2.233$ from the LP-model (10). The value $\mu = 1.710$ obtained from the incremental updated Lagrangian procedure is to be considered numerically exact. The LQP-method converged after 6 iterations to $\mu = 1.810$ which is in error of about 5 %. This is in sharp contrast to the small displacement analysis where the ultimate load was overestimated by almost 25 %. The P-Δ method gave to low ultimate load ($\mu = 1.462$).

The LQP-method is much faster than the incremental finite displacement analysis. It took about 3 times less computer time for this particular frame. The gain increases together with the growing degree of redundancy.

5. Conclusion

The LQP-procedure seems to be a proper tool for estimation of the ultimate load for planar unbraced frames subjected to combined vertical and lateral loads. It takes into account the geometric softening with a sufficient accuracy preserving at the same time the low cost of an approximate method. Our numerical experiments confirmed the fact that slender steel frames should be calculated according to the finite displacement theory, because otherwise non-safe estimates of their load carrying capacity are obtained.

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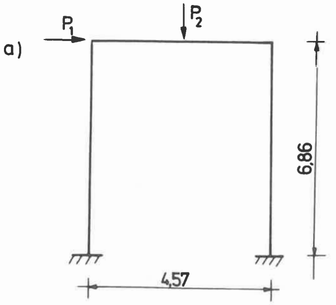

	Column	Beam
E (kN/m^2)	0.1975×10^9	$0,1975 \times 10^9$
A (m^2)	$0,385 \times 10^{-2}$	$0,385 \times 10^{-2}$
I (m^4)	$0,1921 \times 10^4$	$0,321 \times 10^4$
M_o (kNm)	51,1	60,6
N_o (kN)	892	892

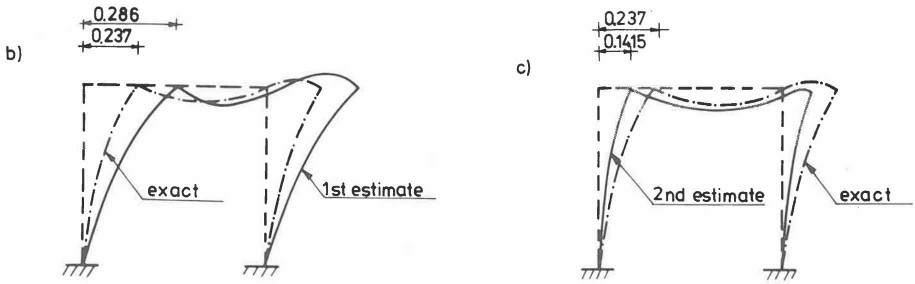

Figure 1. Calculation of a portal frame by LQP-method: a) initial data, b) results of the 1-st iteration, c) results of the 2-nd iteration.

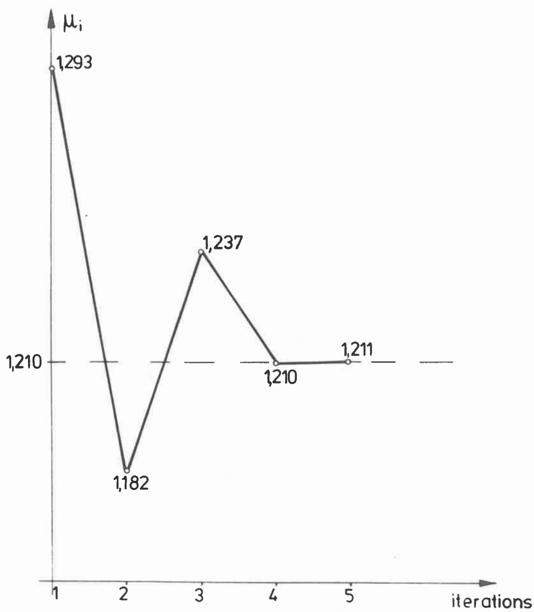

Figure 2. Convergence of the ultimate load factor fro the portal frame.

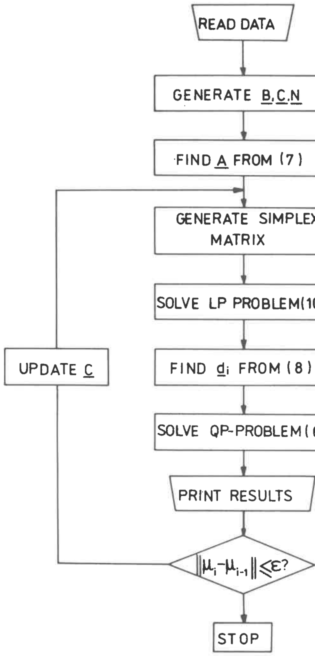

Figure 3. Principal flow chart of LQP-program.

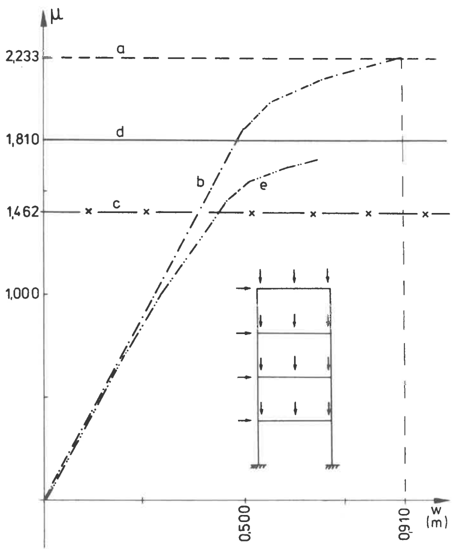

Figure 4. Comparison of results for tall frame: a) rigid-plastic solution, b) incremental small displacement solution, c) P-Δ solution, d) LQP solution, e) incremental finite displacement solution.