UNSTEADY THERMAL STRESSES IN A POLYGONAL PRISM WITH A CIRCULAR HOLE UNDER HEAT GENERATION

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SUMMARY

In designing some types of fuel elements, it is necessary to find the thermal stresses in a continuous medium perforated by cylindrical channels of circular cross section. In order to set up the problem, assume that the strength and heat transfer calculation are based on the phenomena that occur in a typical unit lattice cell, and that the heat generation is uniform over the whole section. Therefore, the problems reduce to solve the temperature and thermal stresses in a polygonal prism with a central circular hole under uniform heat generation and for insulated outer boundary.

In our previous papers we obtained the steady thermal stress distribution in such formed regions by use of the five elementary function's method:

— Y. Takeuti and T. Sekiya, Proceedings 8th Japan National Congress of Appl. Mech., 1958, p. 119.
— Y. Takeuti and T. Sekiya, *Z.A.M.M.*, 4, 1968, p. 237.

In starting or shutting down of the reactor, however, the problems become of the unsteady state. It is felt that no analysis has been given to the transient thermoelastic problem of the multiply-connected region. The problems considered here are concerned with unsteady thermal stresses in a polygonal prism with a circular hole under an uniform heat generation. The solving process is divided into two basic steps. First, the heat conduction equation is solved by the technique of Laplace transform. Second, the plane thermoelastic problem is solved by use of the five elementary function's method. Throughout the treatment of both problems, in order to satisfy the boundary conditions, we use the point-matching technique in conjunction with the least square method.

1. Introduction

The transient temperature and thermal stress distribution in a polygonal prism with a circular hole are investigated. The present writer has recently solved the steady plane thermoelastic problems in a multiply-connected region by the method of the five kinds of elementary stress functions[1],[2]. However, it seems that any attempt to give a theoretical solution of the transient thermoelastic problems in the multiply-connected region is likely to meet with considerable difficulty, and then no theoretical results have been obtained so far. As an extension of the five elementary function's method, this paper is concerned with the uncoupled quasi-static transient plane thermoelastic problems in a polygonal prism with a concentric circular hole under a constant heat generation. The analysis is developed by means of Laplace transforms and the point-matching method. In the first part of this paper a theoretical solution is given for the problem of transient heat conduction of a polygonal prism with a hole under a heat generation accompanied by the thermal insulation on the outer boundary. In the second part of this paper the transient two dimensional thermoelastic problem has been solved for the same region with the help of the five elementary functions method. Numerical work has been carried out for the problem of a hexagonal prism with a circular hole. This type of cell arrangement by the polygonal prism with a circular hole has often been used for a construction of the reactor. Therefore this kind of analyses is much applicable to the atomic power engineering problems. The solutions obtained from this analysis may also be applied to the case of a thin disk insulated on both flat faces.

2. Heat conduction problem

Consider a prism, as shown in Fig.1, bounded externally by a regular n-sided polygon and internally by a central circular hole. Let a be the inner radius of the circular hole and b be the outer boundary of the prism.

For transient heat conduction,

$$\frac{Q}{\lambda} + \Delta\tau = \frac{1}{\kappa}\cdot\frac{\partial\tau}{\partial t} \qquad (1)$$

where
- τ : temperature rise
- κ : thermal diffusivity
- λ : thermal conductivity
- Q : heat generation per unit area per unit time
- Δ : Laplacian = $\partial^2/\partial r^2 + r^{-1}\cdot\partial/\partial r + r^{-2}\cdot\partial^2/\partial\theta^2$

Boundary conditions for the heat conduction problem are

$$\tau = \tau_i \quad \text{at the inner boundary } r=a$$
$$\partial\tau/\partial x = 0 \quad \text{at the outer boundary } x=b \qquad (1')$$

Initial condition is

$$\tau = 0 \quad \text{at } t=0 \qquad (1'')$$

Applying the Laplace transform to eq.(1), we have

$$Q/\lambda p + \Delta\tau^* = (p/\kappa)\tau^* \tag{2}$$

where

$$\tau^* = \int_0^\infty e^{-pt}\ \tau\ dt \qquad (p : \text{parameter to transform})$$

Then the general solution of eq.(2) in plane coordinates can be represented as

$$\tau^* = Q\kappa/\lambda p^2 + (\bar{A}_o + \bar{B}_o\theta)\{\bar{C}_o I_o(qr) + \bar{D}_o K_o(qr)\} + \sum_{m=1}^{\infty} (\bar{C}_m I_m(qr) + \bar{D}_m K_m(qr))(\bar{A}_m \cos m\theta + \bar{B}_m \sin m\theta) \tag{3}$$

where

q^2 : p/κ

I_m, K_m : modified Bessel functions

$\bar{A}_m, \bar{B}_m, \bar{C}_m, \bar{D}_m$: coefficients of temperature function

By considering the symmetry of the problem, for a n-sided regular polygon, this solution reduces to

$$\tau^* = Q\kappa/\lambda p^2 + \bar{A}_o I_o(qr) + \bar{B}_o K_o(qr) + \sum_{m=1}^{\infty} \{\bar{A}_{nm} I_{nm}(qr) + \bar{B}_{nm} K_{nm}(qr)\} \cos nm\theta \tag{4}$$

By the use of boundary condition around the circular hole, we obtain the following relationships

$$\bar{A}_o = -Q\kappa/\lambda p^2 I_o(qa) - K_o(qa)\bar{B}_o/I_o(qa), \qquad \bar{A}_{nm} = -K_{nm}(qa)\bar{B}_{nm}/I_{nm}(qa) \tag{5}$$

Thus,

$$\tau^* = \frac{Q}{\lambda}\frac{\kappa}{p^2}\cdot\frac{I_o(qa) - I_o(qr)}{I_o(qa)} - \frac{\bar{B}_o}{I_o(qa)}\{I_o(qr)K_o(qa) - K_o(qr)I_o(qa)\}$$
$$- \sum_{m=1}^{\infty} \frac{\bar{B}_{nm}}{I_{nm}(qa)}\{I_{nm}(qr)K_{nm}(qa) - K_{nm}(qr)I_{nm}(qa)\} \cos nm\theta \tag{6}$$

The remaining unknown coefficients are now determined by satisfying the boundary conditions at the outer edge of the polygon. Substitution of eq. (6) into eq.(1') results in an infinite number of equations for the infinite number of coefficients. However, since the boundary x=b is not coordinate line for the temperature function, an exact solution cannot be found. To obtain an approximate solution, we use the point-matching technique to satisy the conditions at the selected points, that is, $r_j \cos\theta_j = b$ $(j=0,1,2,\ldots\ldots,s)$ on the outer boundary, we can determine $\bar{B}_o$ and $\bar{B}_{nm}$. The solutions obtained satisfy the prescribed conditions in the interior and on the inner boundary of the body exactly, and those on the outer boundary approximately. Then, τ^* becomes

$$\tau^* = \frac{Q}{\lambda}\cdot\frac{\kappa}{p^2}\cdot\frac{1}{I_o(qa)}\Bigg[\{I_o(qa) - I_o(qr)\} + \frac{\bar{\bar{D}}_o}{\bar{\bar{D}}}\{I_o(qr)K_o(qa) - K_o(qr)I_o(qa)\}$$
$$+ \sum_{m=1}^{s} \frac{\bar{\bar{D}}_{nm}}{\bar{\bar{D}}}\{I_{nm}(qr)K_{nm}(qa) - K_{nm}(qr)I_{nm}(qa)\} \cos nm\theta\Bigg] \tag{7}$$

in which $\bar{\bar{D}}$ denotes a determinant of the order (s+1) and is expressed as follows

$$\bar{\bar{D}} = \left| A_{ji} \right| \qquad (j,i = 1,2,3,\cdots,s+1) \qquad (8)$$

where

$$A_{ji}=q\{I_{n(i-1)-1}(qr_{j-1})K_{n(i-1)}(qa)+I_{n(i-1)}(qa)K_{n(i-1)-1}(qr_{j-1})\}\cos n(i-1)\theta_{j-1}$$
$$\times\cos\theta_{j-1} - \frac{n(i-1)}{r_{j-1}}\{I_{n(i-1)}(qr_{j-1})K_{n(i-1)}(qa)$$
$$- I_{n(i-1)}(qa)K_{n(i-1)}(qr_{j-1})\}\cos\{n(i-1)+1\}\theta_{j-1} \qquad (9)$$

Furthermore new determinant of the order (s+1_ obtained by interchanging (m+1)th column in $\bar{\bar{D}}$ by $I_1(qr_{j-1})\cos\theta_{j-1}$ is denoted as $\bar{\bar{D}}_{nm}$.

From an inverse Laplace transform, we finally determine the solution for the temperature distribution

$$= \frac{Qa^2}{4\lambda}\Big[1 - \frac{r^2}{a^2} + 2F_o \ln\frac{r}{a} + \sum_{m=1}^{s}\frac{2}{nm}F_{nm}\{(\frac{r}{a})^{nm} - (\frac{r}{a})^{-nm}\}\cos nm\theta$$

$$+8\sum_{\alpha=1}^{\infty}\frac{e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^2 J_o(a\mu_\alpha)}\sum_{m=0}^{s}H_{\alpha m}\{J_{nm}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)-J_{nm}(a\mu_\alpha)Y_{nm}(r\mu_\alpha)\}\cos nm\theta\Big] \quad (10)$$

in which μ_α are the positive roots of

$$\begin{vmatrix} G_o(r_o,a,\mu) & \cdots & G_{nm}(r_o,a,\mu) & \cdots & G_{ns}(r_o,a,\mu) \\ \vdots & & \vdots & & \vdots \\ G_o(r_s,a,\mu) & \cdots & G_{nm}(r_s,a,\mu) & \cdots & G_{ns}(r_o,a,\mu) \end{vmatrix} = 0 \qquad (11)$$

where

$$G_{nm}(r_j,a,\mu) = \mu\{J_{nm}(a\mu)Y_{nm-1}(r_j\mu)-J_{nm-1}(r_j\mu)Y_{nm}(a\mu)\}\cos nm\theta_j \cos\theta_j$$
$$+ \frac{nm}{r_j}\{J_{nm}(r_j\mu)Y_{nm}(a\mu)-J_{nm}(a\mu)Y_{nm}(r_j\mu)\}\cos(nm+1)\theta_j \qquad (12)$$

and J_{nm} and Y_{nm} are the Bessel functions.

From a characteristic eq.(11), it seems to have complex roots. However, if we consider the physical situation on a characteristic equation in partial differential equation with parabolic type in which temperature must be finite and the vibrating temperature distribution is not allowed under our given initial and boundary conditions, it is sufficient to take the positive roots only. Moreover, the zero of $I_o(qa)$ becomes the removable singular points, and F_{nm} and $H_{\alpha m}$ are given by

$$F_{nm} = \frac{1}{\bar{\bar{F}}}\begin{vmatrix} \frac{a}{r_o}\cos\theta_o & \cdots & \frac{r_o}{a}\cos\theta_o & \cdots\{(\frac{r_o}{a})^{ns-1}\cos(ns-1)\theta_o+(\frac{r_o}{a})^{ns+1}\cos(ns+1)\theta_o\} \\ \vdots & & \vdots & \vdots \\ \frac{a}{r_s}\cos\theta_s & \cdots & \frac{r_s}{a}\cos\theta_s & \cdots\{(\frac{r_s}{a})^{ns-1}\cos(ns-1)\theta_s+(\frac{r_s}{a})^{ns+1}\cos(ns+1)\theta_s\} \end{vmatrix}$$

where

$$\bar{\bar{F}} = \begin{vmatrix} \frac{a}{r_0}\cos\theta_0 \cdots\cdots \{(\frac{r_0}{a})^{nm-1}\cos(nm-1)\theta_0 + (\frac{r_0}{a})^{-nm-1}\cos(nm+1)\theta_0\} \cdots\cdots \{(\frac{r_0}{a})^{ns-1}\cos(ns-1)\theta_0 + (\frac{r_0}{a})^{ns+1}\cos(ns+1)\theta_0\} \\ \vdots \\ \frac{a}{r_s}\cos\theta_s \cdots\cdots \{(\frac{r_s}{a})^{nm-1}\cos(nm-1)\theta_s + (\frac{r_s}{a})^{-nm-1}\cos(nm+1)\theta_s\} \cdots\cdots \{(\frac{r_s}{a})^{ns-1}\cos(ns-1)\theta_s + (\frac{r_s}{a})^{ns+1}\cos(ns+1)\theta_s\} \end{vmatrix} \tag{13}$$

$$H_{\alpha m} = \frac{\begin{vmatrix} G_o(r_o,a,\mu_\alpha) \cdots\cdots J_1(r_o\mu_\alpha)\cos\theta_o \cdots\cdots G_{ns}(r_o,a,\mu_\alpha) \\ \vdots \\ G_o(r_s,a,\mu_\alpha) \cdots\cdots J_1(r_s\mu_\alpha)\cos\theta_s \cdots\cdots G_{ns}(r_s,a,\mu_\alpha) \end{vmatrix}}{\begin{vmatrix} E_o(r_o,a,\mu_\alpha) \cdots G_{nm}(r_o,a,\mu_\alpha) \cdots G_{ns}(r_o,a,\mu_\alpha) \\ \vdots \\ E_o(r_s,a,\mu_\alpha) \cdots G_{nm}(r_s,a,\mu_\alpha) \cdots G_{ns}(r_s,a,\mu_\alpha) \end{vmatrix} + \cdots\cdots + \begin{vmatrix} G_o(r_o,a,\mu_\alpha) \cdots G_{nm}(r_o,a,\mu_\alpha) \cdots E_{ns}(r_o,a,\mu_\alpha) \\ \vdots \\ G_o(r_s,a,\mu_\alpha) \cdots G_{nm}(r_s,a,\mu_\alpha) \cdots E_{ns}(r_s,a,\mu_\alpha) \end{vmatrix}} \tag{14}$$

where

$$\begin{aligned} E_{nm}(r_j,a,\mu_\alpha) = \mu_\alpha \Big[& r_j\{J_{nm}(a\mu_\alpha)Y_{nm}(r_j\mu_\alpha) - J_{nm}(r_j\mu_\alpha)Y_{nm}(a\mu_\alpha)\} \\ & + a\{J_{nm-1}(r_j\mu_\alpha)Y_{nm-1}(a\mu_\alpha) - J_{nm-1}(a\mu_\alpha)Y_{nm-1}(r_j\mu_\alpha)\}\Big] \cos nm\theta_j \cos\theta_j \\ & + \frac{nm}{r_j}\Big[r_j\{J_{nm-1}(r_j\mu_\alpha)Y_{nm}(a\mu_\alpha) - J_{nm}(a\mu_\alpha)Y_{nm-1}(r_j\mu_\alpha)\} \\ & + a\{J_{nm}(r_j\mu_\alpha)Y_{nm-1}(r_j\mu_\alpha) - J_{nm-1}(a\mu_\alpha)Y_{nm}(r_j\mu_\alpha)\} \\ & - \frac{2nm}{\mu_\alpha}\{J_{nm}(r_j\mu_\alpha)Y_{nm}(a\mu_\alpha) - J_{nm}(a\mu_\alpha)Y_{nm}(r_j\mu_\alpha)\}\Big] \cos(nm+1)\theta_j \end{aligned} \tag{15}$$

Thus temperature distribution is obtained entirely.

3. Un-steady thermoelastic problem

Now the fundamental equation governing the two-dimensional thermal stress problems may be expressed in the known form,

$$\Delta\Delta\chi = -k\Delta\tau \tag{16}$$

where

χ : stress function

k : material constant, i.e., $k=\alpha_t E$ for plane stress problems and $k=\alpha_t E/(1-\sigma)$ for plane strain problems

α_t : coefficient of thermal expansion

E : Young's modulus

σ : Poisson's ratio

In this paper, a two-dimensional thermoelastic analysis is being developed by the five elementary function's method introduced firstly by one of the present authors [1],[2]. This method looks somewhat complex superficially but has many advantages to the problems of multiply-connected region. First merit of this method is to facilitate a physical meaning of the thermoelastic problems. Since, five elementary functions are decided to satisfy each alloted boundary conditions, the composite stress function χ perfectly satisfies all the boundary conditions as a whole. Referring to our previous work [2], it can be easily verified that what kind of temperature distribution is sufficient to ensure the presense of thermal stress for the given region and temperature conditions in a multiply-connected body. Second merit of this method is its mathematical convenience. We shall explain on this matter a little later.

Now, when the region is double-connected, we see that the stress function χ may be expressed in terms of the system of five elementary functions χ_τ, χ_o, χ_{11}, χ_{21} and χ_{31} so that [2]

$$\chi = \chi_\tau + \chi_o + C_{h1}'\chi_{h1} \qquad (h=1,\ 2,\ 3) \tag{17}$$

However, pure thermal problems of zero traction on the boundary give $\chi_o=0$.

The functions on the right side of eq.(17) have to satisfy the following equations:

(i) Differential equations in the region,

$$\Delta\Delta\chi_\tau = -k\Delta\tau \tag{18}$$

$$\Delta\Delta\chi_{h1} = 0 \tag{19}$$

(ii) Boundary conditions on the m-th boundary $C_m(m=0,1)$,

$$(\chi_\tau)_{p_m} = (\partial_\nu\chi_\tau)_{p_m} = 0 \tag{20}$$

$$(\chi_{h1})_{p_m} = \left[(x_h)(\delta_{1h} + \delta_{2h}) + \delta_{3h}\right]\delta_{1m} \qquad \text{(h: not summed)} \tag{21}$$

$$(\partial_\nu\chi_{h1})_{p_m} = \left[(\nu_h)_{p_m}(\delta_{1h} + \delta_{2h})\right]\delta_{1m} \qquad \text{(h: not summed)} \tag{22}$$

where

ν_i = cos (x_i, ν)

p_m = an arbitrary point on the m-th boundary C_m

δ_{ij} = Kronecker's delta

∂_i = partial differentiation with respect to i

(iii) Michell's conditions,

$$\int_0^{2\pi} \partial_r\{\Delta(\chi_\tau+C_{hl}'\chi_{hl})+k\tau\}r\, d\theta = 0 \tag{23}$$

$$\int_0^{2\pi} (r\sin\theta \cdot \partial_r-\cos\theta \cdot \partial_\theta)\{\Delta(\chi_\tau+C_{hl}'\chi_{hl})+k\tau\}r\, d\theta = 0, \tag{24}$$

$$\int_0^{2\pi} (r\cos\theta \cdot \partial_r+\sin\theta \cdot \partial_\theta)\{\Delta(\chi_\tau+C_{hl}'\chi_{hl})+k\tau\}r\, d\theta = 0. \tag{25}$$

The remaining three constants C_{hl}' in eq.(17) are to be determined so as to satisfy the three integral relations of equations (23), (24) and (25). After simple considerations on the symmetry of the figure and temperature distribution of the body, and the Michell's conditions, we find that C_{11}' and C_{12}' in eq.(17) must be taken as zero. On the other hand, an example in the case of having a term C_{11}' was given in our previous work [3]. As the functions χ_{hl}'s themselves have no relation to temperature, χ_{hl} have the same values at a given region of same figures irrespective of the thermoelasticity or isothermal elasticity. Therefore, the mathematical procedure is not so troublesome, as one expects, because if we once calculate the exact values as much as possible on χ_{hm} and χ_τ, we can use them to any kinds of new problems such as unsteady or steady thermoelasticity and isothermal elasticity.

Then, in the present problem, we have

$$\chi = \chi_\tau + C_{31}'\chi_{31} \tag{26}$$

Now, it is convenient to split the solution of eq.(18) into a biharmonic function and a particular solution which will be denoted as $\chi_{\tau o}$ and $\chi_{\tau p}$, respectively. Then, each function satisfies

$$\Delta\Delta\chi_{\tau o} = 0 \tag{27}$$

$$\Delta\chi_{\tau p} = -k\tau \tag{28}$$

From eqs.(10), (27) and (28), we find that the acceptable solutions for $\chi_{\tau o}$ and $\chi_{\tau p}$ are given by

$$\chi_{\tau o} = \{A_o+B_or^2+C_o \ln r+ D_or^2 \ln r+ \sum_{m=1}^{\infty} (A_{nm}r^{nm}+B_{nm}r^{-nm}+C_{nm}r^{nm+2} +D_{nm}r^{-nm+2}) \cos nm\theta\} \tag{29}$$

$$\chi_{\tau p} = \frac{kQa^4}{4\lambda}\left[\frac{1}{16}\left(\frac{r}{a}\right)^4+8\sum_{\alpha=1}^{\infty}\frac{e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\sum_{m=0}^{\infty} H_{\alpha m}\{J_{nm}(r\mu_\alpha)Y_{nm}(a\mu_\alpha) -J_{nm}(a\mu_\alpha)Y_{nm}(r\mu_\alpha)\} \cos nm\theta\right] \tag{30}$$

Thus we may find χ_τ and two kinds of the unknown constants in eq.(29) are determined by the boundary conditions of eq.(20) at r=a. Then we have the following relations between unknown coefficients:

$$A_o=\frac{kQa^4}{\lambda}\left\{\frac{1}{64}-\sum_{\alpha=1}^{\infty}\frac{2H_{\alpha o}e^{-\kappa\mu_\alpha^2 t}}{\pi(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\right\}+\left(\frac{1}{2}-\ln a\right)C_o+\frac{1}{2}a^2D_o$$

$$B_o=\frac{kQa^4}{\lambda}\left\{-\frac{1}{32}\frac{1}{a^2}+\sum_{\alpha=1}^{\infty}\frac{2H_{\alpha o}e^{-\kappa\mu_\alpha^2 t}}{\pi(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\right\}-\frac{1}{2}a^{-2}C_o-\frac{1}{2}(2\ln a+1)D_o \tag{31}$$

$$A_{nm}=-\frac{nm+1}{nm}a^2C_{nm}-\frac{1}{nm}a^{-2(nm-1)}D_{nm}+\frac{2}{nm}\sum_{\alpha=1}^{\infty}\frac{H_{\alpha m}e^{-\kappa\mu_\alpha^2 t}}{\pi(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\cdot\frac{kQa^4}{\lambda}$$

$$B_{nm}=\frac{1}{nm}a^{2(nm+1)}C_{nm}-\frac{nm-1}{nm}a^2D_{nm}-\frac{2}{nm}\sum_{\alpha=1}^{\infty}\frac{H_{\alpha m}e^{-\kappa\mu_\alpha^2 t}}{\pi(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\cdot\frac{kQa^4}{\lambda}$$

After substitution of these relations into eq.(29) and (30), the χ_τ can be rewritten in only four kinds of unknown coefficients.

$$\chi_\tau=\frac{kQa^4}{\lambda}\left[\frac{1}{64}\left\{\left(\frac{r}{a}\right)^4+1\right\}-\frac{1}{32}\left(\frac{r}{a}\right)^2\right]+\left[\frac{1}{2}\left\{1-\left(\frac{r}{a}\right)^2\right\}+\ln\frac{r}{a}\right]C_o+\left\{\frac{r^2}{a^2}\ln\frac{r}{a}+\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)\right\}a^2D_o$$
$$+\sum_{m=1}^{\infty}\left[\left\{-\frac{nm+1}{nm}\left(\frac{r}{a}\right)^{nm}+\frac{1}{nm}\left(\frac{r}{a}\right)^{-nm}+\left(\frac{r}{a}\right)^{nm+2}\right\}a^{nm+2}C_{nm}-\left\{\frac{1}{nm}\left(\frac{r}{a}\right)^{nm}+\frac{nm-1}{nm}\left(\frac{r}{a}\right)^{-nm}-\left(\frac{r}{a}\right)^{-nm+2}\right\}\right.$$
$$\left.\times a^{-nm+2}D_{nm}\right]\cos nm\theta+\frac{kQa^4}{\lambda}\left[\sum_{\alpha=1}^{\infty}\frac{2e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\left[H_{\alpha o}\left\{\frac{1}{\pi}\left(\frac{r^2}{a^2}-1\right)+J_o(r\mu_\alpha)Y_o(a\mu_\alpha)\right.\right.\right.$$
$$-J_o(a\mu_\alpha)Y_o(r\mu_\alpha)\Big\}+\sum_{m=1}^{\infty}\left[\frac{1}{nm\pi}\left\{\left(\frac{r}{a}\right)^{nm}-\left(\frac{r}{a}\right)^{-nm}\right\}+J_{nm}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)-Y_{nm}(r\mu_\alpha)J_{nm}(a\mu_\alpha)\right]$$
$$\left.\left.\times H_{\alpha m}\cos nm\theta\right]\right] \tag{32}$$

Similarly as before, for the determination of unknown constants C_o, D_o, C_{nm}, D_{nm} the point-matching technique is used again here in order to satisfy the outer boundary conditions at a selected finite set of points $(P_o)_i$ (i=0, 1, ..., s) at the outer boundary. Thus we have a set of following 2(s+1) simultaneous equations.

$$\left[\frac{1}{2}\left\{1-\left(\frac{r_i}{a}\right)^2\right\}+\ln\frac{r_i}{a}\right]C_o+\left\{\frac{r_i^2}{a^2}\ln\frac{r_i}{a}+\frac{1}{2}\left(1-\frac{r_i^2}{a^2}\right)\right\}a^2D_o$$
$$+\sum_{m=1}^{q}\left[\left\{-\frac{nm+1}{nm}\left(\frac{r_i}{a}\right)^{nm}+\frac{1}{nm}\left(\frac{r_i}{a}\right)^{-nm}+\left(\frac{r_i}{a}\right)^{nm+2}\right\}a^{nm+2}C_{nm}\right.$$
$$\left.-\left\{\frac{1}{nm}\left(\frac{r_i}{a}\right)^{nm}+\frac{nm-1}{nm}\left(\frac{r_i}{a}\right)^{-nm}-\left(\frac{r_i}{a}\right)^{-nm+2}\right\}a^{-nm+2}D_{nm}\right]\cos nm\theta_i$$

$$=-\frac{kQa^4}{\lambda}\left[\frac{1}{64}\{(\frac{r_i}{a})^4+1\}-\frac{1}{32}(\frac{r_i}{a})^2+\sum_{\alpha=1}^{\infty}\frac{2e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\quad H_{\alpha o}\{\frac{1}{\pi}(\frac{r_i^2}{a^2}-1)+J_o(r_i\mu_\alpha)Y_o(a\mu_\alpha)\right.$$

$$-J_o(a\mu_\alpha)Y_o(r_i\mu_\alpha)\}+\sum_{m=1}^{s}\left[\frac{1}{nm\pi}\{(\frac{r_i}{a})^{nm}-(\frac{r_i}{a})^{-nm}\}+J_{nm}(r_i\mu_\alpha)Y_{nm}(a\mu_\alpha)\right.$$

$$\left.\left.-Y_{nm}(r_i\mu_\alpha)J_{nm}(a\mu_\alpha)\right]\quad H_{\alpha m}\cos nm\theta_i\right] \tag{33}$$

$$(\frac{a}{r_i}-\frac{r_i}{a})a^{-1}\cos\theta_i\cdot C_o+2\frac{r_i}{a}\ln\frac{r_i}{a}\cdot a\cos\theta_i\cdot D_o$$

$$-\sum_{m=1}^{s}\left[\left[(nm+1)(\frac{r_i}{a})^{nm-1}\cos(nm-1)\theta_i+(\frac{r_i}{a})^{-nm-1}\cos(nm+1)\theta_i+(\frac{r_i}{a})^{nm+1}\{nm\right.\right.$$

$$\left.\times\cos(nm-1)\theta_i+2\cos nm\theta_i\cos\theta_i\}\right]a^{nm+1}C_{nm}+\left[(\frac{r_i}{a})^{nm-1}\cos(nm-1)\theta_i\right.$$

$$-(nm-1)(\frac{r_i}{a})^{-nm-1}\cos(nm+1)\theta_i+(\frac{r_i}{a})^{-nm+1}\{nm\cos(nm+1)\theta_i$$

$$\left.\left.-2\cos nm\theta_i\cos\theta_i\}\right]a^{-nm+1}D_{nm}\right]$$

$$=-\frac{kQa^3}{\lambda}\left[\frac{1}{16}\{(\frac{r_i}{a})^3-\frac{r_i}{a}\}\cos\theta_i+\sum_{\alpha=1}^{\infty}\frac{2e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\left[H_{\alpha o}\left[\frac{2r_i}{\pi a}-a\mu_\alpha\{J_1(r\mu_\alpha)Y_o(a\mu_\alpha)\right.\right.\right.$$

$$\left.-J_o(a\mu_\alpha)Y_1(r\mu_\alpha)\}\right]\cos\theta_i+\sum_{m=1}^{s}H_{\alpha m}\left[\frac{1}{\pi}\{(\frac{r_i}{a})^{nm-1}\cos(nm-1)\theta_i+(\frac{r_i}{a})^{-nm-1}\cos(nm+1)\theta_i\}\right.$$

$$+a\mu_\alpha\{J_{nm-1}(r_i\mu_\alpha)Y_{nm}(a\mu_\alpha)-Y_{nm-1}(r_i\mu_\alpha)J_{nm}(a\mu_\alpha)\}\cos nm\theta_i\cos\theta_i$$

$$\left.\left.\left.-nm\frac{a}{r_i}\{J_{nm}(r_i\mu_\alpha)Y_{nm}(a\mu_\alpha)-Y_{nm}(r_i\mu_\alpha)J_{nm}(a\mu_\alpha)\}\cos(nm-1)\theta_i\right]\right]\right] \tag{34}$$

Appling these equations for the points $(P_o)_i$ ($i=0, 1, 2, \ldots, s, \ldots \delta$ and $\delta>s$) at the outer boundary, and solving these equations, we can determine the remaining coefficients. Thus χ_τ is now determined entirely. We now consider χ_{31} in the same way. The following function is applicable to this problem;

$$\chi_{31}=A_{30}+B_{30}(\frac{r}{a})^2+C_{30}\ln\frac{r}{a}+D_{30}(\frac{r}{a})^2\ln\frac{r}{a}+\sum_{m=1}^{\infty}(A_{3nm}(\frac{r}{a})^{nm}+B_{3nm}(\frac{r}{a})^{-nm}+C_{3nm}(\frac{r}{a})^{nm+2}$$

$$+D_{3nm}(\frac{r}{a})^{-nm+2})\cos nm\theta \tag{35}$$

For this function, the boundary conditions are

At $r = a$,

$$\chi_{31}=0\quad,\partial\chi_{31}/\partial r=0 \tag{36}$$

At $x = b$,

$$\chi_{31} = \partial\chi_{31}/\partial x = 0 \tag{36'}$$

And we also can determine the unknown coefficients A_{30}, B_{30}, ..., A_{3nm}, B_{3nm}, .. in the above equation by a point-matching technique. Therefore the last function χ_{31} may be calculated in a similar way.

Substituting eqs.(10), (29), (30) and (35) into eqs.(23), (24) and (25), we obtain

$$C_{31}' = -(D_o + Qa^4F_o/8\lambda)/D_{30} \tag{37}$$

Then we can determine the form of stress function from eq.(26).

For convenience, in accordance with the equivalent coefficients in χ_τ and χ_{31}, next notations are to be used

$$C_o^* = C_o + C_{30}C_{31}', \quad C_{nm}^* = C_{nm} + C_{3nm}C_{31}', \quad D_o^* = D_o + D_{30}C_{31}', \quad D_{nm}^* = D_{nm} + D_{3nm}C_{31}' \tag{38}$$

Thus the approximate solution of a stress function for this problem may be expressed entirely, and the thermal stress components corresponding to this stress function are given. Here we list the solutions by the final forms;

$$\sigma_{\theta\theta} = -(1+\frac{a^2}{r^2})a^{-2}C_o^* + 2(\ln\frac{r}{a}+1)D_o^* + \sum_{m=1}^{\infty}\Big[(nm+1)\{-(nm-1)(\frac{r}{a})^{nm-2} + (\frac{r}{a})^{-nm-2}$$

$$+(nm+2)(\frac{r}{a})^{nm}\}a^{nm}C_{nm}^* - (nm-1)\{(\frac{r}{a})^{nm-2} + (nm+1)(\frac{r}{a})^{-nm-2} - (nm-2)(\frac{r}{a})^{-nm}\}a^{-nm}D_{nm}^*\Big]$$

$$\times\cos nm\theta + \frac{kQa^2}{\lambda}\Big[\frac{1}{16}(3\frac{r^2}{a^2}-1) + \sum_{\alpha=1}^{\infty}\frac{2e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\Big(H_{\alpha o}\Big[\frac{2}{\pi} + (a\mu_\alpha)^2$$

$$\times\{J_o(a\mu_\alpha)Y_o(r\mu_\alpha) - J_o(r\mu_\alpha)Y_o(a\mu_\alpha)\} - \frac{a^2\mu_\alpha}{r}\{J_o(a\mu_\alpha)Y_1(r\mu_\alpha) - J_1(r\mu_\alpha)Y_o(a\mu_\alpha)\}\Big]$$

$$+\sum_{m=1}^{\infty} H_{\alpha m}\Big[\frac{1}{\pi}\{(nm-1)(\frac{r}{a})^{nm-2} - (nm+1)(\frac{r}{a})^{-nm-2}\} + \{nm(nm+1)\frac{a^2}{r^2} - (a\mu_\alpha)^2\}$$

$$\times\{J_{nm}(r\mu_\alpha)Y_{nm}(a\mu_\alpha) - Y_{nm}(r\mu_\alpha)J_{nm}(a\mu_\alpha)\} - \frac{a^2\mu_\alpha}{r}\{J_{nm-1}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)$$

$$-Y_{nm-1}(r\mu_\alpha)J_{nm}(a\mu_\alpha)\}\Big]\cos nm\theta\Big)\Big] \tag{39}$$

$$\sigma_{rr} = (\frac{a^2}{r^2}-1)a^{-2}C_o^* + 2\ln\frac{r}{a}\cdot D_o^* + \sum_{m=1}^{\infty}\Big[-(nm+1)\{-(nm-1)(\frac{r}{a})^{nm-2} + (\frac{r}{a})^{-nm-2}$$

$$+(nm-2)(\frac{r}{a})^{nm}\}a^{nm}C_{nm}^* + (nm-1)\{(\frac{r}{a})^{nm-2} + (nm+1)(\frac{r}{a})^{-nm-2} - (nm+2)(\frac{r}{a})^{-nm}\}a^{-nm}D_{nm}^*\Big]$$

$$\times\cos nm\theta + \frac{kQa^2}{\lambda}\Big[\frac{1}{16}(\frac{r^2}{a^2}-1) + \sum_{\alpha=1}^{\infty}\frac{2e^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}\Big(H_{\alpha o}\Big[\frac{2}{\pi} + \frac{a^2\mu_\alpha}{r}\{J_o(a\mu_\alpha)Y_1(r\mu_\alpha)$$

$$-J_1(r\mu_\alpha)Y_o(a\mu_\alpha)\}\Big] + \sum_{m=1}^{A} H_{\alpha m}\Big[-\frac{1}{\pi}\{(nm-1)(\frac{r}{a})^{nm-2}-(nm+1)(\frac{r}{a})^{nm-2}\}$$

$$+\frac{a^2\mu_\alpha}{r}\{J_{nm-1}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)-Y_{nm-1}(r\mu_\alpha)J_{nm}(a\mu_\alpha)\}-nm(nm+1)\frac{a^2}{r^2}$$

$$\times\{J_{nm}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)-J_{nm}(a\mu_\alpha)Y_{nm}(r\mu_\alpha)\}\Big]\cos nm\theta\Big]\Big] \tag{40}$$

$$\sigma_{r\theta}=\sum_{m=1}^{A}\Big[(nm+1)\{(nm-1)(\frac{r}{a})^{nm-2}+(\frac{r}{a})^{-nm-2}-nm(\frac{r}{a})^{nm}\}a^{nm}C^*_{nm}+(nm-1)\{(\frac{r}{a})^{nm-2}$$

$$-(nm+1)(\frac{r}{a})^{-nm-2}+nm(\frac{r}{a})^{-nm}\}a^{-nm}D^*_{nm}\Big]\sin nm\theta+\frac{kQa^2}{\lambda}\sum_{\alpha=1}^{\infty}\sum_{m=1}^{A}\frac{2nme^{-\kappa\mu_\alpha^2 t}}{(a\mu_\alpha)^4 J_o(a\mu_\alpha)}$$

$$\times\Big[-\frac{1}{nm\pi}\{(nm-1)(\frac{r}{a})^{nm-2}+(nm+1)(\frac{r}{a})^{-nm-2}\}-\frac{a^2\mu_\alpha}{r}\{J_{nm-1}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)$$

$$-Y_{nm-1}(r\mu_\alpha)J_{nm}(a\mu_\alpha)\}+(nm+1)\frac{a^2}{r^2}\{J_{nm}(r\mu_\alpha)Y_{nm}(a\mu_\alpha)-J_{nm}(a\mu_\alpha)Y_{nm}(r\mu_\alpha)\}\Big]$$

$$\times\sin nm\theta \tag{41}$$

4. Numerical Results

The foregoing solution will be illustrated numerically by solving the problem of transient thermal stress distribution on the hexagonal region with a central circular hole under a constant heat generation specified by the following values:

$$n = 6 \quad , \quad b/a = 2$$

For simplicity and generality, the temperature and stress distribution in the cylinder or disk can now be non-dimensionalized by defining the following dimensionless variables:

$$T = \tau/\tau_i, \quad \rho = r/a, \quad t_d = \kappa t/a^2$$

and $$\widehat{\theta\theta} = \lambda\sigma_{\theta\theta}/kQa^2 \quad , \quad \widehat{\rho\rho} = \lambda\sigma_{rr}/kQa^2$$

Taking four points for the temperature problem and ten points for the stress problem on the outer boundary and the least square method for the stress problem is used. Variations of the temperature and hoop stress are shown in Figs.2-6 for various times. In the rest of the paper we will refer the accuracy of the point-matching method. We see that the values of σ_{11} and σ_{12} on the outer boundary x =b which are calculated from σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are nearly equal to zero. Therefore, the results were justified by the boundary conditions of zero tractions. Finally, according to our caluculations, the accuracy is not so dependent on the increase in the number of points on the boundary.

[Note]; Problem of no heat generation has already been accepted in ASME[4].

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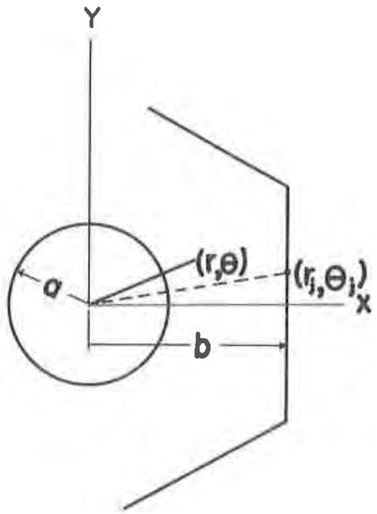

Fig.1 Regular n-sided polygon with a central circular hole.

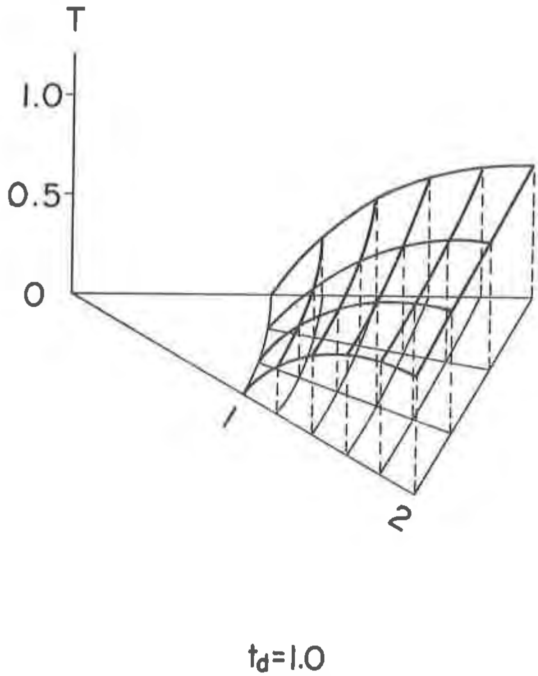

Fig.2 Temperature distribution for time t_d = 1.0.

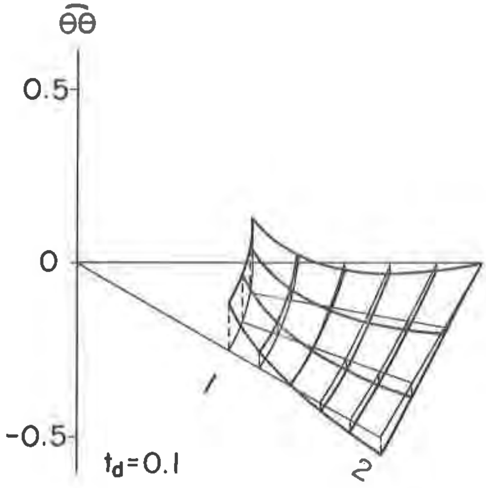

Fig.3 Stress distribution of $\widehat{\theta\theta}$ for time t_d = 0.1.

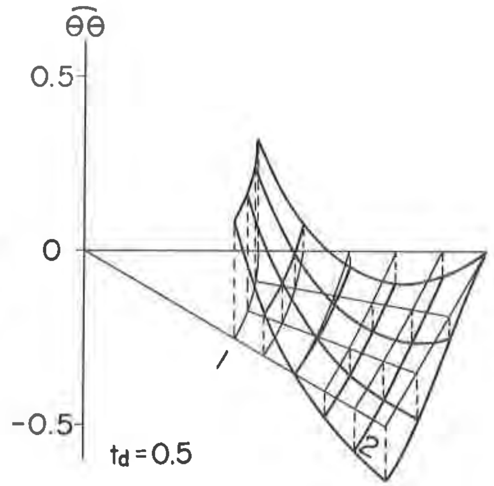

Fig.4 Stress distribution of $\widehat{\theta\theta}$ for time t_d = 0.5.

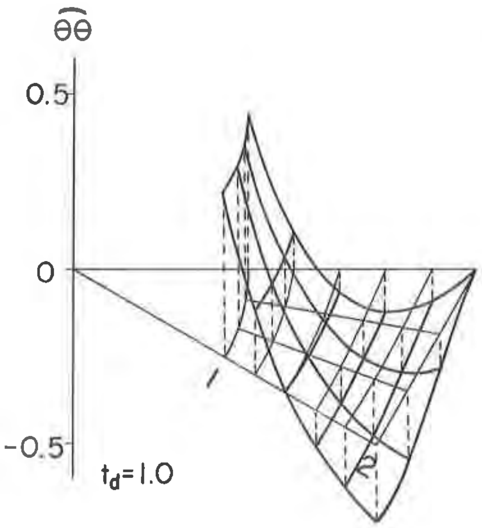

Fig.5 Stress distribution of $\widehat{\theta\theta}$ for time t_d = 1.0.

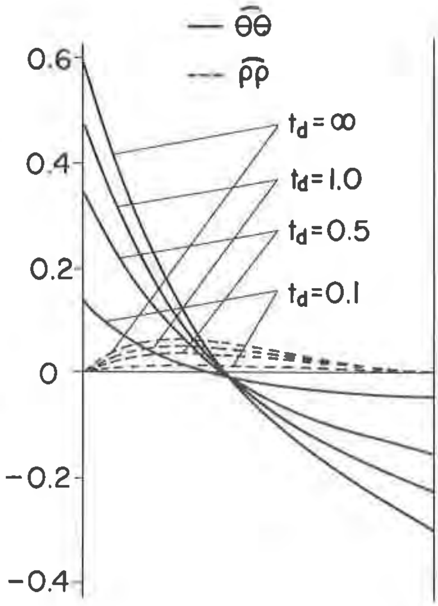

Fig.6 Variation in $\widehat{\theta\theta}$ and $\widehat{\rho\rho}$ along the θ = 0.