ON THE HYPOTHESIS OF "NO INTERACTION" IN THREE-DIMENSIONAL CONTINGENCY TABLES

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The hypothesis of "no interaction" in three-dimensional contingency tables is considered from the point of view of the associated underlying model. A general and computationally simple class of test procedures based on Wald's criterion (1943) is discussed and illustrated.

The paper is expository in the sense that many of the results given appear elsewhere in the literature. New results are included, wherever necessary, for the sake of completion.

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1. Introduction

In recent years, much attention has been given to the hypothesis of "no interaction" in three-dimensional contingency tables. Often, however, there has been ambiguity in

(i) the specification of the underlying model,

(ii) the statement (in terms of this model) of the hypothesis to be tested,

(iii) the interpretation (with respect to the model) of the results of the test.

In this expository paper, the authors want to clarify the distinction between the different experimental situations which give rise to data in the form of a three-dimensional contingency table and to discuss the "no interaction" hypothesis appropriate to each.

In any experiment, we collect data of two particular types from each experimental unit (or subject). These are

(i) a description of the sub-population of units to which he belongs (or of the experimental conditions which he undergoes),

(ii) a description of what happens to him during and/or after the experiment.

In the case of contingency tables, we shall use the term "factor" to denote a "way of classification" describing some aspect of the sub-population of units to which a subject belongs (eg., the treatment combination assigned to him; the block to which he belongs); and the term "response", to denote a "way of classification" describing some aspect of what happens to him during and/or after the experiment (eg., lives or dies; passes or fails;

is of low grade, average grade, high grade). Hence, the data obtained from each experimental unit is simply a description of

(i) the sub-population to which he belongs in terms of a combination of factor categories,

(ii) what happens to him during and/or after the experiment in terms of a combination of response categories.

The data in any multi-dimensional contingency table then are simply the frequencies with which subjects belonging to the same combination of factor categories (i.e., the same sub-population) gave the same combination of response categories (i.e., did the same thing). Thus, one sees that the dimensions of such a table are precisely the factors and responses according to which each unit is classified (eg., treatment, block, lives or dies) with the levels being the associated sets of categories. There are three principal types of three-dimensional contingency tables of experimental interest

(i) the "three response, no factor" tables,

(ii) the "two response, one factor" tables,

(iii) the "one response, two factor" tables,

where the above phrasing specifies the nature of the dimensions of the table. At this point, we note that marginal frequencies depending on factor combinations only (i.e., all response categories have been summed over) are fixed numbers known prior to the actual performance of the experiment (eg., the number of subjects in each block; the number of subjects receiving any particular treatment combination); all other frequencies are random variables (except, of course, the total sample size). Let us now consider the "no interaction" hypothesis associated with each of the above three types of experimental situations.

With the "three response, no factor" tables, the experimenter is interested in the relationships among the different responses. The questions which concern him are analagous to the problems of independence and correlation in normal multivariate analysis. For example, are the three responses independent? Are two of the responses taken together independent of the third one? Is there any partial association between two of the responses within given levels of the third? Are the three responses pairwise independent? With this in mind, we see that the hypothesis of "no interaction" associated with this kind of experiment has nothing to do with the way in which factors combine to determine a response (because we are in a no factor situation) as is the case in analysis of variance; in actuality, it states that some particular measure of association (for which, there may be some choice) between any two of the responses is constant over the levels (categories) of the third (see Simpson (1951)). This statement about the association among the three responses is called the hypotheses of "no interaction" mainly because it is a "bridge" between certain weak hypotheses and certain strong hypotheses; for example, the conditions of "no interaction" among responses and of pairwise independence of responses are together equivalent to the condition of complete independence of the three responses; also the conditions of "no interaction" among responses and of marginal independence of two of the responses with the third are together equivalent to the condition of joint independence of those two responses with the third. For a discussion of the above, the reader is referred to Roy and Kastenbaum (1956) and Lewis (1962).

If the experiment is of the "two response, one factor" type, then we are interested not only in the association between the responses but also in the effect of the factor on them. The questions we pose here are analagous to the problems arising in the normal multivariate analysis. For example, does the factor affect the joint distribution of the responses? Does it affect the marginal distributions of the responses? Or, are the two responses independent at each level of the factor? Again the hypothesis of "no interaction" is expressed in terms of responses; it specifies that some particular measure of association (for which, there may be some choice) is constant over the levels (categories) of the factor. One particular approach to this type of formulation is given by Goodman (1964).

The "one response, two factor" tables are of the greatest interest to us in this paper. Here, the experimenter is interested in the way in which the factors combine to determine the response. The problems of the statistician here are analagous to those arising in univariate normal analysis of variance. For example, does one of the factors affect (the distribution of) the response? Does either factor affect (the distribution of) the response? Is there any interaction between the factors in the way they affect (the distribution of) the response? The "no interaction" hypothesis for this experimental situation has a number of forms depending on what function(s) of the distribution of the response is of interest to us (for example, cell probabilities, logits, probits, relative probabilities, or essentially any function formed only from the response data within the same factor combination and for each factor combination). With any of these functions, two hypotheses of interest would be

(i) the factors affect (the distribution of) the function of the response additively,

(ii) the factors affect (the distribution of) the function of the response multiplicatively.

These formulations are analagous to the hypothesis of no interaction in analysis of variance as they

(i) are concerned with the interaction of factors (as opposed to the association between responses),

(ii) specify simple descriptions of the way in which the factors combine to determine (the distribution of) some function of the response.

The above considerations will be discussed in greater detail in the next section of the paper.

Finally, one should note that in many experimental situations, it is more or less obvious whether a particular dimension is a "response" or a "factor", so that the problem belongs to one of the three situations discussed in a unique manner. On the other hand, in some instances a particular dimension may be viewed either as a "response" or a "factor" in which case the problem can be approached from different points of view.

2. Formulations of the Hypothesis of "No Interaction"

2.1. The "Three Response, No Factor" Model

Let p_{ijk} (where we assume $p_{ijk} > 0$) denote the probability that an experimental unit belongs to the i th category of the first response, the j th category of the second response, and the k th category of the third response in an r x s x t contingency table; thus, the p's are subject to the constraint $\sum_{i,j,k} p_{ijk} = 1$ only. Recall now that the term "no interaction" here means that a particular measure of association between any two of the responses is the same over all levels (categories) of the third. Bartlett (1935) suggested

$$\psi(p_{11k}, p_{12k}, p_{21k}, p_{22k}) = \frac{p_{11k}p_{22k}}{p_{12k}p_{21k}} \qquad k = 1,2$$

as a measure of association in a 2 x 2 table with the third response at level k. The hypothesis of "no interaction" for this case is then

$$\frac{p_{111}p_{221}}{p_{121}p_{211}} = \frac{p_{112}p_{222}}{p_{122}p_{212}} \quad .$$

The above formulation was generalized to the $r \times s \times t$ tables by Roy and Kastenbaum (1956) who considered the hypothesis specified by the constraints

$$\text{(2.1)} \qquad \frac{p_{ijk}p_{rsk}}{p_{rjk}p_{isk}} = \frac{p_{ijt}p_{rst}}{p_{rjt}p_{ist}} \qquad \begin{matrix} i = 1,2, \ldots , (r-1) \\ j = 1,2, \ldots , (s-1) \\ k = 1,2, \ldots , (t-1) \end{matrix}$$

which may be equivalently re-written in terms of the constraints

$$\text{(2.1.a)} \qquad \frac{p_{ijk}p_{rsk}}{p_{rjk}p_{isk}} - \frac{p_{ijt}p_{rst}}{p_{rjt}p_{ist}} = 0 \qquad \begin{matrix} i = 1,2, \ldots , (r-1) \\ j = 1,2, \ldots , (s-1) \\ k = 1,2, \ldots , (t-1) \end{matrix}$$

or the "freedom equations"

$$\text{(2.1.b)} \qquad \frac{p_{ijk}p_{rsk}}{p_{rjk}p_{isk}} = \psi_{ij} \qquad \begin{matrix} i = 1,2, \ldots , (r-1) \\ j = 1,2, \ldots , (s-1) \end{matrix} \text{ for } k = 1,2, \ldots , t .$$

They offered a test of (2.1) which was in terms of restricted maximum likelihood estimators of the parameters and which was identical with Bartlett's test when $r = s = t = 2$. However, the application of this test requires the solution of $(r-1)(s-1)(t-1)$ simultaneous fourth degree (or third degree when $\min(r,s,t) = 2$) equations in as many unknowns; methods of obtaining the solution to this system of equations have been given by Kastenbaum and Lamphiear (1959) and Darroch (1962).

A simpler class of test procedures is based upon Wald's criterion (1943). These involve the unrestricted maximum likelihood estimators of the p_{ijk} (i.e., the $q_{ijk} = (n_{ijk}/N)$ where n_{ijk} is the frequency of the response combination (i, j, k) and $N = \sum_{i,jk} n_{ijk}$ is the total sample size). One example of a Wald test is the test due to Plackett (1962), following the approach of Woolf (1955). He considered the natural logarithms of the "freedom equations" in (2.1.b) and constructed a test by the method discussed in Section 3. A modification of Plackett's method which is simpler to apply is given by Goodman (1963b). For a more complete discussion of the above, the reader is referred to Section 3 and the Appendix; numerical illustrations are given in Section 4.

Finally, tests of the hypothesis of "no interaction" for the "three response, no factor" situation can be obtained by adopting the method of conditional distributions (in which one or more of the responses are viewed as factors) and using tests appropriate to the "two response, one factor" situation (see Goodman (1964)) or the "one response, two factor" situation (see Goodman (1963a)). The latter of these is subject to some questioning on philosophical grounds in view of the interpretation of the term "interaction" given in the Introduction.

2.2 The "Two Response, One Factor" Model

In this situation, let p_{ijk} (where we assume $p_{ijk} > 0$) denote the probability that an experimental unit from the k th category of the factor belongs to the i th category of the first response and the j th category of the second response in an $r \times s \times t$ contingency table; thus, the p's are subject to the t constraints

$$\sum_{i,j} p_{ijk} = 1 \qquad k = 1,2, \ldots , t \quad .$$

One possible formulation of the "no interaction" hypothesis is given by (2.1) and its equivalent forms where now the p_{ijk} appearing there also satisfy the above t constraints. Goodman (1964) has offered a number of methods based upon Wald's criterion to test this hypothesis.

An alternative formulation of the "no interaction" hypothesis appropriate to this model is in terms of another measure of association between the responses. Consider a 2 x 2 x 2 table; if we let

$$\phi(p_{11k}, p_{12k}, p_{21k}, p_{22k}) = \frac{p_{11k}}{p_{10k}p_{01k}} \qquad k = 1,2,$$

where $p_{10k} = \sum_j p_{1jk}$, $p_{01k} = \sum_i p_{i1k}$, (or the natural logarithm of ϕ) be a measure of association in a 2 x 2 table with the factor at level k, then the hypothesis of "no interaction" with respect to this measure is

$$\frac{p_{111}}{p_{101}p_{011}} = \frac{p_{112}}{p_{102}p_{012}} .$$

Generalizing the above formulation to an r x s x t table, we are led to the hypothesis specified by the constraints

$$(2.2) \qquad \frac{p_{ijk}}{p_{i0k}p_{0jk}} = \frac{p_{ijt}}{p_{i0t}p_{0jt}} \qquad \begin{array}{l} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \\ k = 1,2, \ldots, (t-1) \end{array} ;$$

this may be equivalently re-written in terms of the constraints

$$(2.2.a) \qquad \frac{p_{ijk}}{p_{i0k}p_{0jk}} - \frac{p_{ijt}}{p_{i0t}p_{0jt}} = 0 \qquad \begin{array}{l} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \\ k = 1,2, \ldots, (t-1) \end{array}$$

or the "freedom equations"

$$(2.2.b) \qquad \frac{p_{ijk}}{p_{i0k}p_{0jk}} = \phi_{ij} \qquad \begin{array}{l} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \end{array} \text{for } k = 1,2, \ldots, t.$$

Here, one can note that if log ϕ is taken as the measure of association, the hypothesis of "no interaction" would be re-formulated in terms of the natural logarithms of the constraints in (2.2). Statistical tests of (2.2) and its equivalent forms (or the formulation in terms of logarithms) may be obtained by applying Wald's procedure which is described in Section 3 and the Appendix. Numerical illustrations are given in Section 4.

2.3 The "One Response, Two Factor" Model

We now let p_{ijk} (where we assume $p_{ijk} > 0$) denote the probability that an experimental unit from the j th category of the first factor and the k th category of the second factor belongs to the i th category of the response in an $r \times s \times t$ contingency table; thus, the p's are subject to the st constraints

$$\sum_i p_{ijk} = 1 \qquad \begin{array}{l} j = 1,2, \ldots, s \\ k = 1,2, \ldots, t \end{array} .$$

In univariate normal analysis of variance, we are interested in whether the mean response of subjects from the (j,k) th factor combination can be written as the sum of an overall mean effect, an effect due to the first factor, and an effect due to the second factor. If this simple description of the way in which the factors determine (the distribution of) the response is valid, we say that there is no interaction between the two factors. With this in mind, we would say that there is"no interaction" between the two factors of our contingency table model if (the distribution of) the response (namely, $p_{1jk}, p_{2jk}, \ldots, p_{rjk}$ or some function thereof) for subjects from the (j,k) th factor combination can be described solely in terms of an "effect" depending on neither factor, an "effect" depending on the first factor only, and an "effect" depending on the second factor only. For example, an "additive" formulation of this type of hypothesis would be

$$(2.3) \qquad p_{ijk} = t_{i**} + t_{ij*} + t_{i*k} \qquad i = 1,2, \ldots, (r-1)$$

$$\text{for} \quad \begin{array}{l} j = 1,2, \ldots, s \\ k = 1,2, \ldots, t \end{array}$$

where $\sum_j t_{ij*} = \sum_k t_{i*k} = 0$. In the above t_{ij*} is a quantity independent of the second factor while t_{i*k} is a quantity independent of the first factor. The hypothesis (2.3) may be equivalently re-written in terms of the constraints

(2.3.a) $$p_{ijk} - p_{isk} - p_{ijt} + p_{ist} = 0 \qquad \begin{matrix} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \\ k = 1,2, \ldots, (t-1) \end{matrix}$$

or the "freedom equations"

(2.3.b) $$p_{ijk} - p_{isk} = \Delta_{ij} \qquad \begin{matrix} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \end{matrix} \text{ for } k = 1,2, \ldots, t .$$

Statistical tests of (2.3) and its equivalent forms may be obtained by applying Wald's procedure; for example, the test offered by Goodman (1963a) for $2 \times 2 \times t$ tables is a Wald test based upon the "freedom equation" formulation (2.3.b).

A "multiplicative" formulation of the hypothesis of "no interaction" for this model would be

(2.4) $$p_{ijk} = t_{ij*}t_{i*k} \qquad i = 1,2, \ldots, (r-1) \text{ for } \begin{matrix} j = 1,2, \ldots, s \\ k = 1,2, \ldots, t \end{matrix}$$

which may be re-written in terms of the natural logarithms of the p's as

(2.5) $$\log p_{ijk} = t'_{i**} + t'_{ij*} + t'_{i*k} \qquad i = 1,2, \ldots, (r-1)$$
$$\text{for } \begin{matrix} j = 1,2, \ldots, s \\ k = 1,2, \ldots, t \end{matrix}$$

where $\sum_j t'_{ij*} = \sum_k t'_{i*k} = 0$; now (2.4) is equivalent to the constraints

(2.4.a) $$\frac{p_{ijk}}{p_{isk}} - \frac{p_{ijt}}{p_{ist}} = 0 \qquad \begin{matrix} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \\ k = 1,2, \ldots, (t-1) \end{matrix}$$

or the "freedom equations"

(2.4.b) $$\frac{p_{ijk}}{p_{isk}} = \theta_{ij} \qquad \begin{matrix} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \end{matrix} \text{ for } k = 1,2, \ldots, t$$

while (2.5) is equivalent to the constraints

$$(2.5.a) \qquad \log p_{ijk} - \log p_{isk} - \log p_{ijt} + \log p_{ist} = 0 \qquad \begin{matrix} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \\ k = 1,2, \ldots, (t-1) \end{matrix}$$

or the "freedom equations"

$$(2.5.b) \qquad \log p_{ijk} - \log p_{isk} = \Gamma_{ij} \qquad \begin{matrix} i = 1,2, \ldots, (r-1) \\ j = 1,2, \ldots, (s-1) \end{matrix} \text{for } k = 1,2, \ldots, t.$$

Statistical tests of (2.4) and (2.5) and their equivalent forms may be obtained by applying Wald's procedure.

The formulation (2.4) is due to Roy and Bhapkar (1960); other possible formulations of the "no interaction" hypothesis appropriate to this type of experimental situation are considered by Goodman (1963a).

3. Test Criteria

3.1 Computational Procedures for Obtaining Wald's Criterion

There are a number of ways of obtaining Wald's test statistic for the hypotheses of "no interaction" discussed in the previous section. Some of these ways are equivalent in the sense that they lead to the same statistic; some others, though not equivalent in the above sense, do lead to asymptotically equivalent statistics.

There are essentially two approaches which lead to Wald's statistic. They differ in the sense that

(a) one involves constraints;

(b) the other involves "freedom equations".

With either approach, the computational procedure involves forming

(i) the unrestricted maximum likelihood estimators of the left-hand side of

(a) the constraints in terms of which the hypothesis is specified; for example (2.1.a), (2.2.a), (2.3.a), (2.4.a), (2.5.a);

(b) the "freedom equations" in terms of which the hypothesis is specified; for example, (2.1.b), (2.2.b), (2.3.b), (2.4.b), (2.5.b);

i.e., the corresponding expressions in terms of the q_{ijk} instead of the p_{ijk} where q_{ijk} is given by

1. n_{ijk}/N in the "three response, no factor" model
2. n_{ijk}/n_{00k} in the "two response, one factor" model
3. n_{ijk}/n_{0jk} in the "one response, two factor" model

with n_{ijk} being the number of experimental units in cell (i,j,k) and

$$n_{0jk} = \sum_i n_{ijk}, \quad n_{00k} = \sum_{i,j} n_{ijk}, \quad N = \sum_{i,j,k} n_{ijk};$$

(ii) the consistent estimate of the asymptotic variance-covariance matrix of the appropriate set of estimators (i.e., those of (a) or those of (b)) obtained by replacing p's by q's in the variance-covariance matrix of the linear parts of their corresponding Taylor expansions.

Now the estimates of the left hand side of

(a) the constraints in terms of which the hypothesis is specified

(b) the "freedom equations" in terms of which the hypothesis is specified

have asymptotically a multivariate normal distribution; hence the familiar chi-square statistic with (r-1)(s-1)(t-1) degrees of freedom to test for

(a) the vector of (r-1)(s-1)(t-1) components which are the estimators of the left hand sides of the constraints in terms of which the hypothesis is specified has zero mean vector;

(b) the t vectors of (r-1)(s-1) components which are the estimators of the left-hand sides of the "freedom equations" in terms of which the hypothesis is specified all have the same mean vector - namely, the corresponding right-hand sides of the "freedom equations".

Computationally, as has been observed by Plackett (1962) and Goodman (1963b),

(a) the constraint approach involves the inversion of one matrix of side $(r-1)(s-1)(t-1)$;

(b) the "freedom equation" approach involves the inversion of $(t+1)$ matrices of side $(r-1)(s-1)$.

The constraint approach and the "freedom equation" approach lead to the same test statistic if the constraints in terms of which the hypothesis is specified are contrasts; for example, (2.1.a), (2.2.a), (2.3.a), (2.4.a), (2.5.a) correspond in the above sense to (2.1.b), (2.2.b), (2.3.b), (2.4.b), (2.5.b) respectively. Before concluding this sub-section, we want to call to the reader's attention that if the hypothesis is specified in terms of natural logarithms (for example, Plackett's formulation of (2.1)), then Goodman (1963b) has offered a simpler computational procedure for the "freedom equation" approach.

3.2 Test Criteria for Some Special Cases

I. (2 x 2 x t) Tables from the "Three Response, No Factor" Situation:

1. Hypothesis: $\dfrac{p_{11k}p_{22k}}{p_{12k}p_{21k}} = \psi_{11}$ $\qquad k = 1,2, \ldots, t$.

Let $h_k = \dfrac{q_{11k}q_{22k}}{q_{12k}q_{21k}} = \dfrac{n_{11k}n_{22k}}{n_{12k}n_{21k}}$ $\qquad k = 1,2, \ldots, t$.

Let $v_{h_k} = h_k^2 \left(\dfrac{1}{n_{11k}} + \dfrac{1}{n_{12k}} + \dfrac{1}{n_{21k}} + \dfrac{1}{n_{22k}} \right)$ $\qquad k = 1,2, \ldots, t$.

The test statistic χ_h^2 is given by

$$\chi_h^2 = \sum_{k=1}^{t} (h_k - h)^2 \, v_{h_k}^{-1} = \sum_{k=1}^{t} h_k^2 \, v_{h_k}^{-1} - \Big(\sum_{k=1}^{t} h_k v_{h_k}^{-1} \Big)^2 \Big/ \Big(\sum_{k=1}^{t} v_{h_k}^{-1} \Big)$$

where $h = (\sum_{k=1}^{t} h_k v_{h_k}^{-1})/(\sum_{k=1}^{t} v_{h_k}^{-1})$; d.f. = t-1 .

If $t = 2$, $\chi_h^2 = (h_1-h_2)^2/(v_{h_1} + v_{h_2})$ and d.f. = 1 .

2. Hypothesis: $\log p_{11k} - \log p_{12k} - \log p_{21k} + \log p_{22k} = \log \psi_{11}$ $k = 1,2,\ldots,t$.

Let $g_k = \log q_{11k} - \log q_{12k} - \log q_{21k} + \log q_{22k}$ $k = 1,2, \ldots, t$.

$= \log n_{11k} - \log n_{12k} - \log n_{21k} + \log n_{22k}$

Let $v_{g_k} = (\frac{1}{n_{11k}} + \frac{1}{n_{12k}} + \frac{1}{n_{21k}} + \frac{1}{n_{22k}})$ $k = 1,2, \ldots, t$.

The test statistic χ_g^2 is given by

$$\chi_g^2 = \sum_{k=1}^{t} (g_k-g)^2 v_{g_k}^{-1} = \sum_{k=1}^{t} g_k^2 v_{g_k}^{-1} - (\sum_{k=1}^{t} g_k v_{g_k}^{-1})^2/(\sum_{k=1}^{t} v_{g_k}^{-1})$$

where $g = (\sum_{k=1}^{t} g_k v_{g_k}^{-1})/(\sum_{k=1}^{t} v_{g_k}^{-1})$; d.f. = t-1 .

If $t=2$, $\chi_g^2 = (g_1-g_2)^2/(v_{g_1} + v_{g_2})$

$$= [\log(\frac{n_{111}n_{122}n_{212}n_{221}}{n_{121}n_{211}n_{112}n_{222}})]^2/[\sum_{i,j,k} \frac{1}{n_{ijk}}]$$

and d.f. = 1 .

In the results given above, χ_h^2 is a statistic due to Goodman (1963a), (1963b), (1964); and χ_g^2 is the statistic of Woolf (1955) and Plackett (1962).

3. Assume $t = 2$.

Hypothesis: $\frac{p_{111}p_{221}p_{122}p_{212}}{p_{121}p_{211}p_{112}p_{222}} - 1 = 0$.

Let $b = \frac{q_{111}q_{221}q_{122}q_{212}}{q_{121}q_{211}q_{112}q_{222}} = \frac{n_{111}n_{221}n_{122}n_{212}}{n_{121}n_{211}n_{112}n_{222}}$.

Let $v_b = b^2(\frac{1}{n_{111}} + \frac{1}{n_{121}} + \frac{1}{n_{211}} + \frac{1}{n_{221}} + \frac{1}{n_{112}} + \frac{1}{n_{122}} + \frac{1}{n_{212}} + \frac{1}{n_{222}})$

The test statistic χ_b^2 is given by

$$\chi_b^2 = (b-1)^2/v_b \qquad \text{and d.f.} = 1.$$

$$= (b-1)^2/b^2 \left(\sum_{i,j,k} \frac{1}{n_{ijk}} \right) .$$

Note that although the hypothesis formulated here is equivalent to (2.1), the test statistic χ_b^2 associated with it is different from the test statistic χ_h^2 associated with (2.1.a), (2.1.b).

II. (2 x 2 x t) Tables from the "Two Response , One Factor" Situation:

1. Hypothesis: $\dfrac{p_{11k}p_{22k}}{p_{12k}p_{21k}} = \psi_{11} \qquad k = 1,2, \ldots, t$.

Test statistic is χ_h^2 as given in I.1.

2. Hypothesis: $\log \left(\dfrac{p_{11k}p_{22k}}{p_{12k}p_{21k}}\right) = \log \psi_{11} \qquad k = 1,2, \ldots, t$.

Test statistic is χ_g^2 as given in I.2.

3. Hypothesis: $\dfrac{p_{11k}}{p_{10k}p_{01k}} = \phi_{11} \qquad k = 1,2, \ldots, t$.

Let $f_k = \dfrac{q_{11k}}{q_{10k}q_{01k}} = \left(\dfrac{n_{00k}n_{11k}}{n_{10k}n_{01k}}\right) \qquad k = 1,2, \ldots, t$.

Let $v_{f_k} = f_k^2 \left(\dfrac{1}{n_{11k}} - \dfrac{1}{n_{10k}} - \dfrac{1}{n_{01k}} - \dfrac{1}{n_{00k}} + \dfrac{2n_{11k}}{n_{01k}n_{10k}} \right) \qquad k = 1,2, \ldots, t.$

The test statistic χ_f^2 is given by

$$\chi_f^2 = \sum_{k=1}^{t}(f_k-f)^2 v_{f_k}^{-1} = \sum_{k=1}^{t} f_k^2 v_{f_k}^{-1} - \left(\sum_{k=1}^{t} f_k v_{f_k}^{-1} \right)^2 / \left(\sum_{k=1}^{t} v_{f_k}^{-1} \right)$$

where $f = \left(\sum_{k=1}^{t} f_k v_{f_k}^{-1}\right) / \left(\sum_{k=1}^{t} v_{f_k}^{-1} \right)$; $\qquad$ d.f. = t-1 .

If $t = 2$, $\chi_f^2 = (f_1-f_2)^2/(v_{f_1}+v_{f_2})$ and d.f. = 1 .

4. Hypothesis: $\log p_{11k} - \log p_{10k} - \log p_{01k} = \log \phi_{11} \qquad k = 1,2, \ldots, t.$

$$\epsilon_k = \log q_{11k} - \log q_{10k} - \log q_{01k} \qquad k = 1,2, \ldots, t .$$

$$v_{\epsilon_k} = \left(\frac{1}{n_{11k}} - \frac{1}{n_{10k}} - \frac{1}{n_{01k}} - \frac{1}{n_{00k}} + \frac{2n_{11k}}{n_{01k}n_{10k}}\right) \qquad k = 1,2, \ldots, t .$$

The test statistic χ^2_ϵ is given by

$$\chi^2_\epsilon = \sum_{k=1}^{t} (\epsilon_k - \epsilon)^2 v^{-1}_{\epsilon_k} = \sum_{k=1}^{t} \epsilon^2_k v^{-1}_{\epsilon_k} - (\sum_{k=1}^{t} \epsilon_k v^{-1}_{\epsilon_k})^2 / (\sum_{k=1}^{t} v^{-1}_{\epsilon_k})$$

where $\epsilon = (\sum_{k=1}^{t} \epsilon_k v^{-1}_{\epsilon_k}) / (\sum_{k=1}^{t} v^{-1}_{\epsilon_k})$; d.f. = t-1 .

If $t = 2$, $\chi^2_\epsilon = (\epsilon_1 - \epsilon_2)^2/(v_{\epsilon_1} + v_{\epsilon_2})$ and d.f. = 1 .

III. (2 x 2 x t) Tables from the "One Response, Two Factor" Situation:

1. Hypothesis: $p_{11k} - p_{12k} = \Delta_{11}$ $k = 1,2, \ldots, t$.

Let $d_k = q_{11k} - q_{12k}$ $k = 1,2, \ldots, t$.

Let $v_{d_k} = \frac{q_{11k} q_{21k}}{n_{01k}} + \frac{q_{12k} q_{22k}}{n_{02k}}$ $k = 1,2, \ldots, t$.

The test statistic χ^2_d is given by

$$\chi^2_d = \sum_{k=1}^{t} (d_k - d)^2 v^{-1}_{d_k} = \sum_{k=1}^{t} d^2_k v^{-1}_{d_k} - (\sum_{k=1}^{t} d_k v^{-1}_{d_k})^2 / (\sum_{k=1}^{t} v^{-1}_{d_k})$$

where $d = (\sum_{k=1}^{t} d_k v^{-1}_{d_k}) / (\sum_{k=1}^{t} v^{-1}_{d_k})$; d.f. = t-1 .

If $t = 2$, $\chi^2_d = (d_1 - d_2)^2/(v_{d_1} + v_{d_2})$ and d.f. = 1 .

The statistic χ^2_d given here is due to Goodman (1963a).

2. Hypothesis: $p_{11k}/p_{12k} = \theta_{11}$ $k = 1,2, \ldots, t$.

Let $m_k = (q_{11k}/q_{12k})$ $k = 1,2, \ldots, t$.

Let $v_{m_k} = m^2_k \{ (\frac{1}{n_{11k}} + \frac{1}{n_{12k}}) - (\frac{1}{n_{01k}} + \frac{1}{n_{02k}}) \}$ $k = 1,2, \ldots, t$.

The test statistic χ^2_m is given by

$$\chi_m^2 = \sum_{k=1}^{t} (m_k - m)^2 v_{m_k}^{-1} = \sum_{k=1}^{t} m_k^2 \, v_{m_k}^{-1} - \left(\sum_{k=1}^{t} m_k v_{m_k}^{-1} \right)^2 / \left(\sum_{k=1}^{t} v_{m_k}^{-1} \right)$$

where $m = \left(\sum_{k=1}^{t} m_k v_{m_k}^{-1} \right) / \left(\sum_{k=1}^{t} v_{m_k}^{-1} \right)$; d.f. = t-1 .

If $t = 2$, $\chi_m^2 = (m_1 - m_2)^2 / (v_{m_1} + v_{m_2})$ and d.f. = 1 .

3. Hypothesis: $\log p_{11k} - \log p_{12k} = \Gamma_{11}$ $k = 1,2, \ldots, t$.

Let $\gamma_k = \log q_{11k} - \log q_{12k} = \log\left(\frac{n_{11k} n_{02k}}{n_{01k} n_{12k}}\right)$ $k = 1,2, \ldots, t$.

Let $v_{\gamma_k} = \left(\frac{1}{n_{11k}} + \frac{1}{n_{12k}}\right) - \left(\frac{1}{n_{01k}} + \frac{1}{n_{02k}}\right)$ $k = 1,2, \ldots, t$.

The test statistic χ_γ^2 is given by

$$\chi_\gamma^2 = \sum_{k=1}^{t} (\gamma_k - \gamma)^2 \, v_{\gamma_k}^{-1} = \sum_{k=1}^{t} \gamma_k^2 \, v_{\gamma_k}^{-1} - \left(\sum_{k=1}^{t} \gamma_k v_{\gamma_k}^{-1} \right)^2 / \left(\sum_{k=1}^{t} v_{\gamma_k}^{-1} \right)$$

where $\gamma = \left(\sum_{k=1}^{t} \gamma_k v_{\gamma_k}^{-1} \right) / \left(\sum_{k=1}^{t} v_{\gamma_k}^{-1} \right)$; d.f. = t-1 .

If $t = 2$, $\chi_\gamma^2 = (\gamma_1 - \gamma_2)^2 / (v_{\gamma_1} + v_{\gamma_2})$ and d.f. = 1 .

4. Numerical Illustrations

4.1 Bartlett's Data

The following data (see Table 1) of Hoblyn and Palmer are the results of an experiment designed to investigate the propagation of plum root stocks from root cuttings.

Table 1: Bartlett's Data

Time of Planting	At Once			In Spring		
Length of Cutting	Long	Short	Total	Long	Short	Total
Alive	156	107	263	84	31	115
Dead	84	133	217	156	209	365
Total	240	240	480	240	240	480

These data are of interest because they were used by Bartlett (1935) and Goodman (1964) to demonstrate their respective tests of a formulation of the hypothesis of "no interaction" which is appropriate to either the "three response, no factor" model or the "two response, one factor" model (namely, (2.1)). However, in actuality, "time of planting" and "length of cutting" are factors; and thus the experiment is really of the "one response, two factor" type. We feel that the data should be analyzed by methods appropriate to this model. In addition, we will demonstrate that the linear model (2.3) provides an excellent "fit" for the data with the chi-square value (d.f. = 1) to test (2.3) being 0.08 (as opposed to values of 1.85, 2.27 of Bartlett and 1.93, 2.26, 2.27 of Goodman to test (2.1)); and hence we conclude that the factors affect (the distribution of) the response additively in the sense of (2.3).

Analysis of the data:

(i) Notation: Let i denote the response: $i = \begin{cases} 1 & \text{if alive} \\ 2 & \text{if dead} \end{cases}$

Let j denote time of planting: $j = \begin{cases} 1 & \text{if at once} \\ 2 & \text{if in spring} \end{cases}$

Let k denote length of cutting: $k = \begin{cases} 1 & \text{if long} \\ 2 & \text{if short} \end{cases}$

(ii) Tests of Hypothesis:

$$H: \; p_{ijk} = t_{i**} + t_{ij*} + t_{i*k} \qquad i = 1 \;\text{ for }\; \begin{matrix} j = 1,2 \\ k = 1,2 \end{matrix}$$

where $\sum_{j=1}^{2} t_{ij*} = \sum_{k=1}^{2} t_{i*k} = 0$.

The hypothesis H may be expressed in terms of the constraint

$$H: (p_{111}-p_{121}) - (p_{112}-p_{122}) = 0 \; .$$

The Wald statistic to test H is $\chi_d^2 = (q_{111}-q_{121}-q_{112}+q_{122})^2/v$

where $q_{111} = n_{111}/n_{011} = \frac{156}{240} = .650$

$$q_{121} = n_{121}/n_{021} = \frac{84}{240} = .350$$

$$q_{112} = n_{112}/n_{012} = \frac{107}{240} = .446$$

$$q_{122} = n_{122}/n_{022} = \frac{31}{240} = .129$$

$$(q_{111} - q_{121} - q_{112} + q_{122}) = -0.017$$

$$v = \frac{q_{111}q_{211}}{n_{011}} + \frac{q_{121}q_{221}}{n_{021}} + \frac{q_{112}q_{212}}{n_{012}} + \frac{q_{122}q_{222}}{n_{022}}$$

$$= 0.0034$$

$$\chi_d^2 = \frac{0.0002890}{0.0033935} = 0.085 \qquad \text{d.f.} = 1 \quad .$$

Since χ_d^2 is small, we conclude that the data are consistent with the hypothesis H. In fact, we have the following estimates of the parameters appearing in H.

$$\hat{t}_{1**} = 0.393 \quad ,$$

$$\hat{t}_{11*} = 0.155 \quad ,$$

$$\hat{t}_{12*} = -0.155 \quad ,$$

$$\hat{t}_{1*1} = 0.107 \quad ,$$

$$t_{1*2} = -0.107 \quad .$$

and the following "fit" of the data to the hypothesis

$$q_{111} = 0.650 = (0.393 + 0.155 + 0.107) - 0.005 \ ,$$

$$q_{121} = 0.350 = (0.393 - 0.155 + 0.107) + 0.005 \ ,$$

$$q_{112} = 0.446 = (0.393 + 0.155 - 0.107) + 0.005 \ ,$$

$$q_{122} = 0.129 = (0.393 - 0.155 - 0.107) - 0.002 \ .$$

Hence, we conclude that the factors affect (the distribution of) the response additively in the sense of H.

4.2 Lessler's Data

In this sub-section, we consider three contingency tables from the unpublished data of Lessler. The experiment was conducted to study the nature of sexual symbolism with the basic observation being a subject's classification of an object: as being either masculine or feminine when that object is shown to him at the following exposure rates:

1/1000 second, 1/500 second, 1/100 second, 1/50 second, 1/5 second.

The subjects involved in the study were classified according to

(a) sex (males and females)

(b) those who were not told the purpose of the experiment and those who were told the purpose of the experiment (Group A and Group C respectively)

while the objects involved in it had been (by a previous study Lessler (1962)) assigned

(i) a cultural meaning (M or F) related to which sex used it

(ii) an anatomical meaning (M or F) related to which sex it inately evoked

(iii) an intensity (W or S) related to the strength of the degree to which the object was representative of its classification according to (i) and (ii).

Thus, as can be seen, the experiment is of the "five response, five factor type" with the factors being (a), (b), (i),(ii), (iii) above and the responses being the classifications at 1/1000, 1/500, 1/100, 1/50, 1/5 second.

The tables below are presented only to illustrate tests of the hypothesis of "no interaction"in various situations; no detailed analysis is demonstrated here.

I. A "Three Response, No Factor" Situation:

Table 2: Subject Type - Group C Males

Object Type - Culturally Masculine

Anatomically Feminine

Weak Intensity

Response at 1/5 sec.		M			F		
Response at 1/100 sec.		M	F	Total	M	F	Total
Response at 1/1000 sec.	M	184	10	194	7	20	27
	F	38	14	52	7	114	121
Total		222	24	246	14	134	148

$h_1 = 6.78$ $\quad v_{h_1} = 9.34$

$h_2 = 5.70$ $\quad v_{h_2} = 11.19$ $\quad \chi^2_h = 0.057$ $\quad$ d.f. = 1

$b = 1.19$ $\quad \chi^2_b = 0.047$ $\quad$ d.f. = 1

$v_b = 0.775$

$g_1 = 1.914$ $\quad v_{g_1} = 0.2032$ $\quad \chi^2_g = 0.055$ $\quad$ d.f. = 1

$g_2 = 1.740$ $\quad v_{g_2} = 0.3445$

Hence, we accept the hypothesis of "no interaction" and conclude that the association between (say) the response at 1/1000 second and the response at 1/100 second is the same within each of the categories of the response at 1/5 second.

II. A "Two Response, One Factor" Situation:

Table 3: Subject Type - Males

Object Type - Culturally Masculine

Anatomically Feminine

Weak Intensity

Subject Group		A			C		
Response at 1/5 second		M	F	Total	M	F	Total
Response at 1/1000 second	M	194	27	221	177	14	191
	F	52	121	173	30	63	93
Total		246	148	394	207	77	284

$f_1 = 1.41$ $v_{f_1} = 0.00230$ $\chi^2_f = 4.97$ d.f. = 1

$f_2 = 1.27$ $v_{f_2} = 0.00164$

$h_1 = 16.72$ $v_{h_1} = 19.48$ $\chi^2_h = 0.89$ d.f. = 1

$h_2 = 26.55$ $v_{h_2} = 89.02$

Hence, we reject the hypothesis that the measure of association in (2.2) is the same for both groups. However, we accept the hypothesis that the measure of association in (2.1) is the same for both groups.

III. A "One Response, Two Factor" Situation:

Table 4: Subject Type - Males

Object Type - Culturally Masculine

Weak Intensity

Subject Group		A			C		
Anatomical Meaning		M	F	Total	M	F	Total
Response at 1/1000 second	M	202	191	393	298	221	519
	F	82	93	175	96	173	269
Total		284	284	568	394	394	788

$d_1 = 0.0387$	$v_{d_1} = 0.001499$	$\chi^2_d = 9.47$	d.f. = 1
$d_2 = 0.1954$	$v_{d_2} = 0.001093$		
$m_1 = 1.0576$	$v_{m_1} = .003516$	$\chi^2_m = 9.82$	d.f. = 1
$m_2 = 1.3484$	$v_{m_2} = .005099$		
$\gamma_1 = .0583$	$v_{\gamma_1} = .003144$	$\chi^2_\gamma = 9.83$	d.f. = 1
$\gamma_2 = .3001$	$v_{\gamma_2} = .002804$		

Hence, we reject both the "additive" hypothesis (2.4) and "multiplicative" hypothesis (2.5) of no interaction between the effects of the two factors "group" and "Anatomical Meaning" on the probabilities p_{ijk}.

Appendix

Let us consider a situation in which N experimental units (n_{oj} of which are from the j th sub-population or factor category) are classified into categories according to some response. Let p_{ij} (where we assume all $p_{ij} > 0$) denote the probability of the i th response category by a subject from the j th sub-population; and let n_{ij} denote the corresponding frequency (where, of course, $n_{oj} = \sum_i n_{ij}$). We shall assume the product multinomial model

$$\text{(A.1)} \qquad \phi = \prod_{j=1}^{s} \{(n_{oj}!/\prod_{i=1}^{r} n_{ij}!)(\prod_{i=1}^{r} p_{ij}^{\,n_{ij}})\}$$

where $\sum_{i=1}^{r} p_{ij} = 1$, $\sum_{i=1}^{r} n_{ij} = n_{oj}$(fixed) $\quad j = 1,2, \ldots, s$. In (A.1), we allow i to be a multiple subscript

$$i = (i_1, i_2, \ldots, i_k) \text{ where } i_\alpha = 1,2, \ldots r_\alpha$$

and where $r = \prod_{\alpha=1}^{k} r_\alpha$; and we allow j to be a multiple subscript

$$j = (j_1, j_2, \ldots, j_\ell) \quad \text{where} \quad j_\beta = 1, 2, \ldots, s_\beta \quad .$$

This is a "k-response, ℓ-factor" situation. In certain incomplete designs, all sub-populations (or factor combinations) may not be included in the experiment; here, however, we shall assume that the design is complete, i.e.,

$$s = \prod_{\beta=1}^{\ell} s_\beta \quad .$$

One way of formulating hypotheses concerning the parameters p_{ij} in the model (A.1) is in terms of constraints

$$\text{(A.2)} \qquad F_m(\underset{\sim}{p}) = 0 \qquad m = 1, 2, \ldots, u < (r-1)s$$

where $\underset{\sim}{p}' = (p_{11}, \ldots, p_{r-1,1}, \ldots, p_{1s}, \ldots p_{r-1,s})$. In (A.2), the F_m's are given known functions. Let us denote the vector $(F_1(\underset{\sim}{p}), F_2(\underset{\sim}{p}), \ldots, F_u(\underset{\sim}{p}))$ by $\underset{\sim}{h}'(\underset{\sim}{p})$. Let us assume that the $F_m(\underset{\sim}{p})$

(i) possess continuous partial derivatives up to the second order with respect to the elements of $\underset{\sim}{p}$,

(ii) are independent in the sense that the $u \times (r-1)s$ matrix $\underset{\sim}{H}(\underset{\sim}{p})$ defined by $\underset{\sim}{H}(\underset{\sim}{p}) = \left[\frac{\partial F_m(\underset{\sim}{p})}{\partial p_{ij}} \right]$ is of rank u where $u < (r-1)s$. Let $\underset{\sim}{q}' = (q_{11}, \ldots, q_{r-1,1}, \ldots, q_{1s}, \ldots, q_{(r-1)s})$, where $q_{ij} = n_{ij}/n_{oj}$, denote the vector of unrestricted maximum likelihood estimators of the corresponding elements of $\underset{\sim}{p}'$; and let $\underset{\sim}{C}(\underset{\sim}{p})$ denote their variance-covariance matrix; i.e.,

$$\underset{\sim}{C}(\underset{\sim}{p}) = \begin{bmatrix} n_{01}^{-1}(\underset{\sim}{P}_1 - \underset{\sim}{p}_1\underset{\sim}{p}_1') & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & n_{02}^{-1}(\underset{\sim}{P}_2 - \underset{\sim}{p}_2\underset{\sim}{p}_2') & \cdots & \underset{\sim}{0} \\ \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & n_{0s}^{-1}(\underset{\sim}{P}_s - \underset{\sim}{p}_s\underset{\sim}{p}_s') \end{bmatrix}$$

where $\underset{\sim}{p}_j' = (p_{1j}, \ldots, p_{r-1,j})$ and $\underset{\sim}{P}_j = \text{diagonal}\,(p_{1j}, p_{2j}, \ldots, p_{r-1,j})$. Hence, $\underset{\sim}{H}(\underset{\sim}{p})\ \underset{\sim}{C}(\underset{\sim}{p})\ \underset{\sim}{H}'(\underset{\sim}{p})$ is the asymptotic variance-covariance matrix of

$\underset{\sim}{h}'(\underset{\sim}{q}) = (F_1(\underset{\sim}{q}), F_2(\underset{\sim}{q}), \ldots, F_u(\underset{\sim}{q}))$; i.e. of the unrestricted maximum likelihood estimators of the left-hand sides of the constraints in (A.2). By condition (ii) above, this matrix is non-singular and positive definite. If (A.2) is true and if $N \to \infty$ (in such a way that $Q_j = (n_{oj}/N)$ remains fixed), then the statistic

$$\text{(A.3)} \qquad \underset{\sim}{h}'(\underset{\sim}{q})[\underset{\sim}{H}(\underset{\sim}{q})\underset{\sim}{C}(\underset{\sim}{q})\underset{\sim}{H}'(\underset{\sim}{q})]^{-1}\underset{\sim}{h}(\underset{\sim}{q})$$

has a limiting χ^2-distribution with u degrees of freedom. We call (A.3) Wald's statistic.

We may also test (A.2) by applying Neyman's linearization technique and forming the minimum χ_1^2-statistic to test the linearized hypothesis (see Neyman 1949). The linearized hypothesis is

$$\text{(A.4)} \qquad F_m^*(\underset{\sim}{p}) = F_m(\underset{\sim}{q}) + \sum_{j=1}^{s}\sum_{i=1}^{r-1} \left(\frac{\partial F_m(\underset{\sim}{p})}{\partial p_{ij}}\bigg|_{\underset{\sim}{p}=\underset{\sim}{q}}\right)(p_{ij}-q_{ij}) = 0 \qquad m = 1, 2, \ldots, u$$

$$\text{or} \qquad \underset{\sim}{h}^*(\underset{\sim}{p}) = \underset{\sim}{h}(\underset{\sim}{q}) + \underset{\sim}{H}(\underset{\sim}{q})\,(\underset{\sim}{p} - \underset{\sim}{q}) = \underset{\sim}{0}$$

where $\underset{\sim}{q}$ here is regarded as fixed.

The minimum χ_1^2-test of (A.4) (and hence of (A.2)as proved by Neyman) is the minimum value of

$$\chi_1^2 = \sum_{j=1}^{s}\sum_{i=1}^{r} \frac{(n_{ij}-n_{oj}p_{ij})^2}{n_{ij}} \quad \text{where} \quad \sum_{i=1}^{r} p_{ij} = 1$$

subject to (A.4). Bhapkar (1961) showed that the minimum χ_1^2-test of (A.4) is given by

$$\text{(A.5)} \qquad \underset{\sim}{h}^{*\prime}(\underset{\sim}{q})\underset{\sim}{V}^{-1}\underset{\sim}{h}^*(\underset{\sim}{q})$$

where $\underset{\sim}{V}$ is the estimate of the covariance matrix of $\underset{\sim}{h}^*(\underset{\sim}{q})$ obtained by replacing p's by q's; i.e., $\underset{\sim}{V} = \underset{\sim}{H}(\underset{\sim}{q})\underset{\sim}{C}(\underset{\sim}{q})\underset{\sim}{H}'(\underset{\sim}{q})$. Since $\underset{\sim}{h}^*(\underset{\sim}{q}) = \underset{\sim}{h}(\underset{\sim}{q})$, we see that (A.4) and (A.5) are identical as was demonstrated by Bhapkar (1965).

<u>Theorem:</u> Let a hypothesis concerning the p_{ij} in (A.1) be formulated as follows:

$$g_\eta(\underset{\sim}{p}) = \sum_{\alpha=1}^{t} d_{\eta\alpha}\theta_\alpha \qquad \eta = 1,2, \ldots, \rho \tag{A.6}$$

where $t < \rho < (r-1)s$, where g_η are known functions satisfying the conditions (i) and (ii), and where the $d_{\eta\alpha}$'s are known constants such that the rank of the $(\rho \times t)$ matrix $\underset{\sim}{D}$ is ν. Then the Wald statistic to test (A.6) (i.e., the minimum χ_1^2-statistic for (A.6) linearized) is the same as the minimum sum of squares obtained by the generalized (weighted) least squares technique applied to the $g_\eta(\underset{\sim}{q})$ with the asymptotic variance-covariance matrix estimated by the "sample" variance-covariance matrix. This statistic has asymptotically a chi-square distribution with $(\rho - \nu)$ degrees of freedom under the hypothesis. For this paper, we need only the proof for the case where Rank $\underset{\sim}{D} = t$.

Proof: Linearizing the left-hand side of (A.6), we obtain

$$\underset{\sim}{g}(\underset{\sim}{q}) + \underset{\sim}{G}(\underset{\sim}{q})(\underset{\sim}{p} - \underset{\sim}{q}) = \underset{\sim}{D}\,\underset{\sim}{\theta} \tag{A.7}$$

where $\underset{\sim}{g}'(\underset{\sim}{q}) = (g_1(\underset{\sim}{q}), \ldots, g_\rho(\underset{\sim}{q}))$ and $\underset{\sim}{G}(\underset{\sim}{q}) = \left[\frac{\partial g_\eta(\underset{\sim}{p})}{\partial p_{ij}} \Big|_{\underset{\sim}{p} = \underset{\sim}{q}} \right]$ is a $\rho \times (r-1)s$ matrix of rank ρ. If we let

$$\underset{\sim}{f} = \underset{\sim}{g}(\underset{\sim}{q}) - \underset{\sim}{G}(\underset{\sim}{q})\underset{\sim}{q} \quad ,$$

$$\underset{\sim}{G} = \underset{\sim}{G}(\underset{\sim}{q}) \quad ,$$

$$\underset{\sim}{g} = \underset{\sim}{g}(\underset{\sim}{q}) \quad ,$$

then we may re-write (A.7) in the form

$$\underset{\sim}{G}\underset{\sim}{p} + \underset{\sim}{f} = \underset{\sim}{D}\,\underset{\sim}{\theta} \qquad t < \rho < (r-1)s \tag{A.8}$$

where Rank $\underset{\sim}{G} = \rho$ and Rank $\underset{\sim}{D} = t$. There exists a $(\rho-t) \times \rho$ matrix $\underset{\sim}{L}$ such that $\underset{\sim}{L}\,\underset{\sim}{D} = \underset{\sim}{0}$. Hence

$$\underset{\sim}{L}\,\underset{\sim}{G}\,\underset{\sim}{p} + \underset{\sim}{L}\,\underset{\sim}{f} = \underset{\sim}{L}\,\underset{\sim}{D}\,\underset{\sim}{\theta} = \underset{\sim}{0}\,.$$

If we let $\underset{\sim}{\xi} = \underset{\sim}{G}\,\underset{\sim}{p}$, then (A.8) implies

$$\text{(A.9)} \qquad \underset{\sim}{L}\,\underset{\sim}{\xi} + \underset{\sim}{L}\,\underset{\sim}{f} = \underset{\sim}{0} \quad .$$

On the other hand (A.9) implies that $(\underset{\sim}{\xi} + \underset{\sim}{f})$ belongs to the vector space generated by the columns of $\underset{\sim}{D}$. so there exists $\underset{\sim}{\theta}$ such that $(\underset{\sim}{\xi} + \underset{\sim}{f}) = \underset{\sim}{D}\,\underset{\sim}{\theta}$; i.e., (A.9) implies (A.8). Hence (A.8) and (A.9) represent the same hypothesis. Let $\underset{\sim}{F} = \underset{\sim}{L}\,\underset{\sim}{G}$ and write (A.9) as

$$\text{(A.10)} \qquad \underset{\sim}{F}\,\underset{\sim}{p} + \underset{\sim}{L}\,\underset{\sim}{f} = \underset{\sim}{0} \quad .$$

The minimum χ_1^2-statistic to test (A.10) (and thus to test (A.8))is

$$\text{(A.11)} \qquad \chi_1^2 = (\underset{\sim}{q}'\underset{\sim}{F}' + \underset{\sim}{f}'\underset{\sim}{L}')(\underset{\sim}{F}\underset{\sim}{C}(\underset{\sim}{q})\underset{\sim}{F}')^{-1}(\underset{\sim}{F}\underset{\sim}{q} + \underset{\sim}{L}\,\underset{\sim}{f})$$

$$= (\underset{\sim}{q}'\underset{\sim}{G}'+\underset{\sim}{f}')\underset{\sim}{L}'(\underset{\sim}{L}\underset{\sim}{G}\underset{\sim}{C}(\underset{\sim}{q})\underset{\sim}{G}'\underset{\sim}{L}')^{-1}\underset{\sim}{L}(\underset{\sim}{G}\underset{\sim}{q} + \underset{\sim}{f})$$

$$= \underset{\sim}{g}'\underset{\sim}{L}'(\underset{\sim}{L}\,\underset{\sim}{S}\,\underset{\sim}{L}')^{-1}\underset{\sim}{L}\,\underset{\sim}{g}$$

where $\qquad \underset{\sim}{S} = \underset{\sim}{G}\,\underset{\sim}{C}\,(\underset{\sim}{q})\,\underset{\sim}{G}'$.

Consider now minimization of

$$S_e^2 = (\underset{\sim}{g} - \underset{\sim}{D}\,\underset{\sim}{\theta})'\,\underset{\sim}{S}^{-1}(\underset{\sim}{g} - \underset{\sim}{D}\,\underset{\sim}{\theta})$$

with respect to $\underset{\sim}{\theta}$. We have then

$$\hat{\underset{\sim}{\theta}} = (\underset{\sim}{D}'\underset{\sim}{S}^{-1}\underset{\sim}{D})^{-1}\underset{\sim}{D}'\underset{\sim}{S}^{-1}\underset{\sim}{g}$$

and

$$\min S_e^2 = \underset{\sim}{g}'[\underset{\sim}{S}^{-1} - \underset{\sim}{S}^{-1}\underset{\sim}{D}(\underset{\sim}{D}'\underset{\sim}{S}^{-1}\underset{\sim}{D})^{-1}\underset{\sim}{D}'\underset{\sim}{S}^{-1}]\underset{\sim}{g}\,.$$

If we let $\underset{\sim}{A} = \underset{\sim}{L}'(\underset{\sim}{L}\,\underset{\sim}{S}\,\underset{\sim}{L}')^{-1}\underset{\sim}{L}$ and $\underset{\sim}{B} = \underset{\sim}{S}^{-1} - \underset{\sim}{S}^{-1}\underset{\sim}{D}(\underset{\sim}{D}'\underset{\sim}{S}^{-1}\underset{\sim}{D})^{-1}\underset{\sim}{D}'\underset{\sim}{S}^{-1}$, then $\chi_1^2 = \min S_e^2$ if $\underset{\sim}{A} = \underset{\sim}{B}$. Observe that

(i) Rank $\underset{\sim}{A}$ = Rank $\underset{\sim}{B}$ = $\rho - t$,

(ii) $\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{B}\,\underset{\sim}{D} = \underset{\sim}{0}$,

(iii) $\underset{\sim}{D}'\underset{\sim}{A} = \underset{\sim}{D}'\underset{\sim}{B} = \underset{\sim}{O}$,

(iv) $\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{A} = \underset{\sim}{A}$, $\underset{\sim}{B}\,\underset{\sim}{S}\,\underset{\sim}{B} = \underset{\sim}{B}$.

From (i), (ii), (iii) we have that there exist nonsingular ($\rho \times \rho$) matrices $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ such that $\underset{\sim}{A} = \underset{\sim}{C}_1\underset{\sim}{B}$ and $\underset{\sim}{A}\,\underset{\sim}{C}_2 = \underset{\sim}{B}$. Hence ,

$$\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{B} = \underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{A}\,\underset{\sim}{C}_2 = \underset{\sim}{A}\,\underset{\sim}{C}_2 = \underset{\sim}{B}$$

and also

$$\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{B} = \underset{\sim}{C}_1\underset{\sim}{B}\,\underset{\sim}{S}\,\underset{\sim}{B} = \underset{\sim}{C}_1\underset{\sim}{B} = \underset{\sim}{A} \quad .$$

Thus $\underset{\sim}{A} = \underset{\sim}{B}$ and $\chi_1^2 = \min S_e^2$

Summary Table

Model	"Three Response, No Factor"	"Two Response, One Factor"	"One Response, Two Factor"
Hypothesis	(2.1)	(2.1) (2.2)	(2.3) (2.4) (2.5)
Test Statistic	χ_h^2 , χ_b^2 , χ_g^2	χ_h^2 , χ_g^2 χ_f^2, χ_e^2	χ_d^2 χ_m^2 χ_γ^2

In the above table, χ_h^2, χ_b^2, χ_g^2, χ_f^2, χ_e^2, χ_d^2, χ_m^2, χ_γ^2 are the generalizations for the $r \times s \times t$ tables of the statistics given in Sub-section 3.2 for the $2 \times 2 \times t$ tables.

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