

## FORCED VIBRATIONS OF A SHELL INSIDE OF A NARROW WATER ANNULUS

G. BOWERS and G. HORVAY

*School of Engineering, University of Massachusetts, Amherst, Massachusetts 01002, U.S.A.*

### SUMMARY

The response of a shell to pulsating pressure in a narrow water filled annulus is determined. The analysis is patterned after L.E. Penzes (SMiRT-2 Paper E 5/1\*) according to which the imposed pump pressure over a portion  $\Delta x, \Delta \theta$  of the outer periphery (at radius  $b$ ) is replaced by an equivalent radial body force of amplitude  $Q$  acting over an appropriate radial thickness  $\Delta$ . The governing equation for the pressure fluctuation  $q'$  is established

$$(\nabla^2 + \omega^2/c^2)q' = Q \langle \Delta x \rangle \langle \Delta \theta \rangle [r^{-1} \langle \Delta \rangle + \delta \{r - (b - \Delta)\}]$$

In the foregoing  $\langle \rangle$  denotes the difference of two appropriate Heaviside unit functions and  $\delta$  is the Dirac delta function. The right side gives the divergence of the body force. Penzes (2nd SMiRT Conference) incorrectly omits the second term in the brackets. For a narrow annulus a cartesian analysis suffices, in which case the first term in the brackets is eliminated. The amplitude  $Q$  of the body force is determined by comparing its Fourier expansion into the normal modes of the annulus with the Fourier expansion into the normal modes of the annulus with the Fourier expansion of the pump pressure. The thickness  $\Delta$  is rather arbitrary. It may be conveniently chosen, as by Penzes, as the width of the annulus. While in the corrected analysis the amplitude of the response is smaller than that obtained by Penzes by a factor of about 8, the mode shapes closely agree with his results. The study is aimed at incorporating, in the forced vibration analysis of reactor core support baskets, the effect of entrained water mass, in continuation of an earlier study presented at the 2nd SMiRT Conference (SMiRT-2 Paper E 5/8).

1. INTRODUCTION

In two earlier papers [1], [2] (to be referred to hereinafter as I and II) the natural modes of a shell, surrounded by a water annulus inside of a container of slightly larger radius, were determined for beam modes and breathing modes, Fig. 1. Simple support was assumed at the shell ends. In particular, the magnitude of the entrained water weight, many times that of the shell weight, was also taken into account. This weight substantially lowers the frequencies that would be expected in air.

In the present paper the forced mode is expanded into natural modes, each mode being endowed with its appropriate entrained mass. However, the earlier analysis is not immediately applicable to the shell response problem, inasmuch as the exciting force acting on the shell must, first, be determined. This is done in accordance with the ingenious idea of Penzes [3] of replacing the pump pressure by an equivalent body force. In retracing Penzes' steps opportunity is taken to correct his governing equation (our (16a)), the forced wave equation, where a very important term was omitted.

The results of the analysis are contained in Figures 4a,b which show the axial and peripheral mode shapes that result from the assumed pump pressure. The shapes are indistinguishable from those of Penzes, but the amplitudes are smaller by a factor of about 8. This may partly be attributed to entrained water mass, partly to the missing term in the original version of (16a).

2. DETERMINATION OF THE EQUIVALENT BODY FORCE DUE TO PUMP PRESSURE

Euler's equation of motion for departures

$$p' = p - p_{ref}, \quad \rho' = \rho - \rho_{ref}, \quad v'_M = v - V_M \tag{1}$$

from the main flow parameters  $p_{ref}, \rho_{ref}, V_M$  is

$$v'_M + \rho_{ref}^{-1} \nabla p' = \frac{P}{M} \tag{2}$$

Here we regard (in accordance with Penzes [3, (1)], and in contrast to II), the body force

$$(P_x, P_\theta, P_r) = \frac{P}{M} = Q \sin \omega_f t \tag{3}$$

an attribute of the perturbed flow rather than that of the main flow. Accordingly, the forced wave equation for the pressure fluctuation

$$p' = -\rho \dot{\phi} \tag{4a}$$

becomes

$$(\nabla^2 - \partial^2/c^2 - \partial^2/\partial t^2) p' = \text{div} \frac{P}{M} \tag{5}$$

In the foregoing  $\phi$  denotes the potential for the velocity fluctuations:

$$v'_M = \nabla \phi \tag{4b}$$

The natural modes  $\psi_{mn}$  (associated with frequencies  $\omega_{mn}$ ) of the water annulus  $a \leq r \leq b$ , Fig. 1, appropriate to the boundary conditions

$$r = b: \quad \partial \phi / \partial r = 0; \quad r = a: \quad \partial \phi / \partial r = \partial \eta / \partial t \tag{6}$$

( $\eta$  represents the outward displacement of the inner shell wall) are (see (4a), (II.12), (II.15))

$$p' / q = \left\{ \frac{\cos(\frac{m\pi x}{l})}{\sin(\frac{m\pi x}{l})} \right\} \cos n \theta [I_n(\kappa_m r) - \frac{I'_n(\kappa_m a)}{K'_n(\kappa_m a)} K_n(\kappa_m r)] X(x) \Theta(\theta) R(r) \sin \omega t \tag{7a}$$

$$\sim \left\{ \frac{\cos(\frac{m\pi x}{l})}{\sin(\frac{m\pi x}{l})} \right\} \cos n \theta [\cosh \lambda_{mn} \zeta - \tanh \lambda_{mn} L \sinh \lambda_{mn} \zeta] \sin \omega t$$

$$= X(x)\theta(\theta)Z(\zeta)\sin\omega t = \text{say, } \psi_{mn}(x, \theta, \zeta)\sin\omega t \quad (7b)$$

where  $\bar{q}$  is a pressure amplitude coefficient which serves to render the natural mode dimensionless, and (see (II.17))

$$r = a + \zeta, \quad \lambda_{mn}^2 = \kappa_m^2 + (n/a)^2 = -(\omega_{mn}/c)^2 + (m\pi/l)^2 + (n/a)^2 \quad (8)$$

The narrow annulus is regarded in (7b) as a repeating rectangle in the  $a\theta$  direction of length  $2\pi a$  each, Fig. 2.

The frequency equation in cylindrical coordinates may be written, on the assumption that the inner wall of the annulus is rigid as is the outer wall:

$$I_n'(\kappa a) K_n'(\kappa b) - K_n'(\kappa a) I_n'(\kappa b) = 0 \quad (9a)$$

or also (Penzes [3, (23)]) as

$$J_n'(\kappa a) Y_n'(\kappa b) - Y_n'(\kappa a) J_n'(\kappa b) = 0 \quad (9b)$$

For

$$|\kappa L|^2 \ll 1 \quad (10)$$

this simplifies to

$$J_n''(\kappa a)/J_n'(\kappa a) = Y_n''(\kappa a)/Y_n'(\kappa a) \quad (11)$$

The first 6 roots  $\kappa_j$ ,  $j=1, \dots, 6$  of (9b) have been determined for  $n=1, 2, 3$  over a range of  $b$  and  $a$  values by Dwight. Here  $j+1$  denotes the number of nodes of the radial function  $R(r)$ . (There are at least two nodes: at  $r=a$  and at  $r=b$ .)

For cartesian coordinates the equation corresponding to (9) is

$$\tanh \lambda L = 0 \quad (12a)$$

It is immediately noted that the root of smallest modulus (and the only real root) of this equation leads to the frequency condition

$$\lambda = 0: \quad (\omega_{mn}/c)^2 = (m\pi/l)^2 + (n/a)^2 \quad (12b)$$

In accordance with Penzes, we consider a forcing function  $Q_m$  with only a radial component

$$Q_r = Q(x-\xi_1) - (x-\xi_2) \delta(\theta + \mathcal{D}/2) - \frac{1}{2}(\theta - \mathcal{D}/2) \delta\{r-(b-\Delta)\} - \frac{1}{2}\{r-b^+\} \delta \quad (13a)$$

$$= Q\langle \xi \rangle \delta\mathcal{D} \delta\{\zeta-(L-\Delta)\} - \frac{1}{2}\{\zeta-L^+\} \delta \quad (13b)$$

of constant magnitude over a small volume element  $\Delta l[(b-\Delta/2)\mathcal{D}]$  adjacent to the outer wall, and extending infinitesimally,  $L^+-L$ , into the pump opening.  $\frac{1}{2}$  is the Heaviside unit function, and we denoted

$$d = \xi_2 - \xi_1, \quad x_1 = (\xi_1 + \xi_2)/2 \quad (14)$$

$d$  is the  $x$  extent,  $\mathcal{D}$  the  $\theta$  extent, and  $\Delta$  the  $r$  (or  $\zeta$ ) extent of the body force  $Q$ . In (13b) we introduced the abridged notations  $\langle \xi \rangle$ ,  $\langle \mathcal{D} \rangle$  for the corresponding expressions of (13a).

Accordingly, see Fig. 3,

$$\text{div } Q_m = Q\langle \xi \rangle \langle \mathcal{D} \rangle \delta\{r-(b-\Delta)\} = Q\langle \xi \rangle \langle \mathcal{D} \rangle \delta\{\zeta-(L-\Delta)\} \quad (15)$$

and the forced wave equation becomes

$$\text{Cylindrical coordinates: } (\Delta^2 + \omega^2/c^2)p' = \frac{Q}{r} + Q\langle \xi \rangle \langle \mathcal{D} \rangle \delta\{r-(b-\Delta)\} \quad (16a)$$

$$\text{Cartesian coordinates: } (\Delta^2 + \omega^2/c^2)p' = Q\langle \xi \rangle \langle \mathcal{D} \rangle \delta\{\zeta-(L-\Delta)\} \quad (16b)$$

The  $\delta$  term of (16a) is missing in Penzes' work.

Our present objective is to represent the pump pressure at  $r=b$ , extending over area  $db\mathcal{D}$  as a volume force, reaching into the liquid to a depth  $\Delta$ . Restricting ourselves henceforth to the cartesian representation, let accordingly

$$Q \delta\{\zeta-(L-\Delta)\} \sin \omega_f t = \sin \omega_f t \sum_{mn} Q_{mn} \psi_{mn} \quad (17a)$$

$$Q_{mn} = \frac{\int_{\xi_1}^{\xi_2} X_m(x) dx \int_{\theta_1 - \frac{\alpha}{2}}^{\theta_2 + \frac{\alpha}{2}} \theta_n(\theta) d\theta \int_{L-\Delta}^L Z_{mn}(\zeta) \{\zeta-(L-\Delta)\} d\zeta}{\int_0^{\xi} X_m^2 dx \int_{-\pi}^{\pi} \theta_n^2 d\theta \int_0^L Z_{mn}^2 d\zeta} \quad (17b)$$

be the expansion of the forcing function into normal modes. Then the expansion coefficients in the expression of the forced response pressure

$$p_f = q_f \sin \omega_f t = \left[ \sum_{m,n} C_{mn} \psi_{mn}(x, \theta, \zeta) \right] \sin \omega_f t \quad (18)$$

obey, by (16b), the relation

$$\left[ -(\pi n / \ell)^2 - n^2 / (a+L/2)^2 + \lambda_{mn}^2 + (\omega_f / c)^2 \right] C_{mn} = q_{mn} \quad (19)$$

(we followed Karman-Biot [5, p.348]), i.e.,

$$C_{mn} = \frac{c^2 Q_{mn}}{\omega_f^2 - \omega_{mn}^2} \cdot \frac{\frac{\ell}{\pi n} \left\{ \begin{array}{l} \sin \pi n \xi_2 / \ell - \sin \pi n \xi_1 / \ell \\ \cos \pi n \xi_1 / \ell - \cos \pi n \xi_2 / \ell \end{array} \right\} \frac{\sin n \alpha / 2}{n/2} \frac{\cosh \lambda \Delta}{\cosh \lambda L}}{(\pi L \ell / 2) \left[ \frac{1}{2} (1 - \tanh^2 \lambda L) + (\sinh 2 \lambda L) (4 \lambda L)^{-1} (1 + \tanh^2 \lambda L) - (\cosh 2 \lambda L - 1) (2 \lambda L)^{-1} \tanh \lambda L \right]} \quad (20a)$$

$$\approx \frac{8 Q c^2}{L \pi^2} \frac{\sin \frac{\pi n d}{\ell} \left\{ \frac{\cos \pi n x_1}{\sin \frac{\pi x_1}{\ell}} \right\} \sin n \alpha / 2}{mn (\omega_f^2 - \omega_{mn}^2)} \quad \text{for } |\lambda L|^2 \ll 1 \quad (20b)$$

We wrote the bracket in (19), by II.(17b), as  $(\frac{2}{\omega_f} - \omega_{mn}^2) / c^2$ ; the long fraction on the right of (20a) was arrived at as the result of the integrations (17b). The bracket in (20a) reduces to 1 for  $|\lambda L|^2 \ll 1$ . In the numerator of (17b) we used in the  $\zeta$  integration the relation

$$\int_0^L Z(\zeta) \delta\{\zeta-(L-\Delta)\} d\zeta = \frac{\cosh \lambda \Delta}{\cosh \lambda L} \quad (21a)$$

$$\approx 1 \text{ for } |\lambda L|^2 \ll 1 \quad (21b)$$

Observe that  $Q_{mn}$  has the dimension  $lb/in^4$  of  $Q\delta$ , whereas  $C_{mn}$  has the dimension  $lb/in^2$  of  $q_f$ .

Had we written  $L$  instead of  $L^+$  in (13) we would have obtained in (16b) the factor

$$\left[ \delta\{\zeta-(L-\Delta)\} - \frac{1}{2} \delta\{\zeta-L\} \right] \quad (13^*)$$

instead of just its first term, visualizing half the  $\delta$  in the annulus, half of it in the pump opening. Correspondingly, we would have arrived at, in place of (21), to the result

$$\int = \frac{\cosh \lambda \Delta - 1/2}{\cosh \lambda L} \approx 1/2 \quad (21^*)$$

We are inclined to believe that (21) represents the correct picture where the  $p_f$  interface is completely included in the  $\Delta$  region, and not (21\*) where it is only half included. An experimental verification of this conclusion is urgently desirable. Note that if the second  $\delta$  function in (13b) had  $\zeta-L$  as argument, then (21) would be replaced by

$$\int = \frac{\cosh \lambda \Delta - 1}{\cosh L} \approx 0 \quad (21^{**})$$

This is clearly an incorrect result, because the equivalent body force is completely eliminated by the fact that the  $\Delta$  interval does not abut to the boundary  $\zeta=L$ .

We now proceed to the basic idea of Penzes. We wish to replace the prescribed pressure due to the pump, at the container radius  $b$ ,

$$p_b = q_b \sin \omega t = p_f(x_1, \theta_1, L) = \sin \omega_f t \sum_{mn} C_{mn} \left\{ \frac{\cos \frac{m\pi x_1}{\ell}}{\sin \frac{m\pi x_1}{\ell}} \right\} \frac{\cos n \theta_1}{\cosh \lambda \frac{L}{mn}} \quad (22)$$

(the pump is located at  $x_1, \theta_1, L$ ) by an equivalent body force  $P$  of radial extent  $\Delta$ . In this fashion we bypass the very difficult mixed boundary value problem where over the pump opening the pressure is prescribed as  $p_b = p_f$ , over the remainder of the container wall the radial velocity is specified as  $v_r = 0$ . On introducing (20b) into (22), the amplitude  $Q$  is determined from

$$\frac{q_b}{Q} = \frac{8c^2}{L\pi^2} \sum_{mn} \left\{ \frac{\sin m\pi d/\ell \cos^2 m\pi x_1/\ell}{\sin m\pi d/\ell \sin^2 m\pi x_1/\ell} \right\} \cdot \frac{\sin \frac{1}{2} n \mathcal{Q} \cos n \theta_1}{mn(\omega_f^2 - \omega_{mn}^2)} \quad (23a)$$

$$\equiv \text{say, } \sum_{mn} e_{mn} \quad (23b)$$

Unless we take rather high modes also into account in the summation, where the smallness of  $L$  cannot overcome the effect of a large  $\frac{\lambda}{mn}$ , the  $\cosh \lambda \Delta / \cosh \lambda L$  factor in (20a) is essentially unity. Thus the thickness  $\Delta$  of  $Q$  may be assigned any value  $0 < \Delta < L$ . We choose for simplicity, in accordance with Penzes

$$\Delta = L \quad (24)$$

Note also that the zeroth  $m$  mode causes no difficulty in the summation, inasmuch as  $[\frac{m^{-1} \sin m\pi x/\ell}]_{m=0} = \pi x/\ell$ . The same holds true also for the zeroth  $n$  mode. But actually, our physical conditions preclude the  $m=0, n=0$  cases.

Having determined  $Q$  from (23) as

$$Q = q / \sum_{m,n=1,1}^{M,N} e_{mn} \quad (25)$$

we find the pressure distribution in the annulus from (18), (20).

For the case considered by Penzes the parameters are:

$$\begin{aligned} a &= 76", & b &= q + L = 86", & \ell &= 328", & \xi_1 &= 54", & \xi_2 &= 94" \\ \mathcal{Q} &= 26.6^\circ, & c &= 3850 \text{ ft/sec}, & \rho g &= 62.5 \text{ lb/ft}^3, & \gamma_s / \rho g &= 7.74 \end{aligned} \quad (26a)$$

$$q_b = 10 \text{ psi}, \quad \omega_f = 200\pi \text{ rad/sec} \quad (26b)$$

[II(30) differs slightly (due to assignments  $b=85^\circ, \ell=320''$ ) but insignificantly from the values (26a).] By (25) the equivalent body force has, for  $(M,N)=(10,10)$  the value

$$Q = 0.0395 \text{ lb/in}^3 \quad (27a)$$

Table I tabulates  $Q$  as various  $M,N$  are tried. It is seen that  $(M,N)=(5,5)$ , which is well within condition (21b), is adequate. Note that Penzes' value for  $Q$  is

$$Q_{\text{penzes}} = 1.55 \text{ lb/in}^3 \quad (27b)$$

From (20b) one finds the Fourier coefficients  $C_{mn}$  as listed in Table II; the associated  $\omega_{mn}$  (as found from (12b)) are also tabulated. For comparison Penzes' results for  $\omega_{mn}$  and  $C_{mn}$ , based on cylindrical geometry, are also shown. The (small) disagreement in frequencies may be interpreted as a curvature effect (ignored in the present work), whereas the disagreement in the expansion coefficients (roughly by a factor of 8) may be interpreted as due (a) to the factor of 40 of  $Q_{\text{penzes}}/Q$ , and a factor of 1/5 that may be attributed to the absence of the  $\delta$  terms in Penzes' expression for (16a).

As a result we find the axial and the peripheral pressure values acting on the shell

surface  $r=0$ , Figs. 4a,b; each distribution is normalized in the figure to a maximum of 1. For comparison Penzes' corresponding graphs are also shown. The two sets of curves are essentially indistinguishable.

### 3. SHELL EQUILIBRIUM EQUATIONS

We adopt Kempner's right handed axis system (RHS), see [6]p.10, to the shell element  $dx \times Rd\theta \times h$ , Fig. 5a,b. ( $R=a+h/2$  is the mean radius; it is assumed that  $h/R \ll 1$ .) The membrane (N) and transverse shear force (Q) resultants, and moment and twist resultants (M), are illustrated in Figs. 5c,d. The z axis is drawn inward. The notation coincides with Timoshenko's ([7], p.82, 430, 508), and retains his unreasonable convention for twisting moments that

$$M_{xy} = -M_{yx} \quad (28a)$$

This notation is also used by Yu [8], Bleich and Baron [9]. Langhaar ([10], p.182) on the other hand uses LHS with z drawn outward, and the twisting moment  $M_{yx}$  chosen as ours, but  $M_{xy}$  chosen opposite to ours, so that

$$M_{xy} = M_{yx} \quad (28b)$$

The references [11],[12],[13] below adhere to this convention (28b).

Flügge ([11], p.40-16, [12], p.209) uses LHS with z drawn outward. Mnév [14] also uses LHS, but draws z inward. Lyons [13] uses a hybrid system in which y is drawn in RH sense, v the y-displacement in the LH sense, and w is drawn inward. The Donnell shell equilibrium equations are ([6], (32a); [7], (303)):

$$\begin{aligned} L_x\{u,v,w\} &\equiv u_{xx} + \frac{1-v}{2} u_{yy} + \frac{1+v}{2} v_{xy} - \frac{v}{R} w_x = -X/C \\ L_y\{u,v,w\} &\equiv v_{yy} + \frac{1-v}{2} v_{xx} + \frac{1+v}{2} u_{xy} - \frac{1}{R} w_y = -Y/C \\ L_z\{u,v,w\} &\equiv \frac{h^2}{12} \nabla^4 w + \frac{1}{R}(v u_x - v_y - w/R) = -Z/C \end{aligned} \quad (29)$$

where  $\mu_s$  denotes shell mass per unit area at radius a; X,Y,Z are the resultants of the surface forces acting on inner and outer surfaces (alternately, they are thickness times body forces acting inside of the shell wall); and

$$C = Eh/(1-\nu^2) = 12D/h^2 \quad (30)$$

$$-X = \mu_s \ddot{u} = -\omega^2 \mu_s u, \quad -Y = \mu_s \ddot{v} = -\omega^2 \mu_s v \quad (31a)$$

$$-Z = \mu_s \ddot{w} - p_h - f_r = -\omega^2 (\mu_s + \mu_h) w - f_r = -\omega^2 \mu w - f_r \quad (31b)$$

All the  $\mu_s$  terms in (31) represent inertia effects due to shell mass. The  $p_h$  (the  $\mu_h$  term) represents the inertia effect due to entrained water, appropriate to the frequency  $\omega$ . In addition we envisage the possibility that an externally applied radial forcing function

$$f_r = p_a \quad (32)$$

also may act at radius a.

The stress-resultant displacement relations are ([6], (15)):

$$\begin{aligned} N_x &= C[u_x + v(v_y - w/R)], & N_y &= C[v_y - w/R + \nu u_x] \\ N_{xy} &= N_{yx} = \frac{1-\nu}{2} C[u_y + v_x], & -M_{xy} &= M_{yx} = -D(1-\nu)w_{xy} \\ M_x &= -D[w_{xx} + \nu w_{yy}], & M_y &= -D[w_{yy} + \nu w_{xx}] \\ Q_x &= M_{x,x} + M_{yx,y} = -D\nu^2 w_x, & Q_y &= M_{y,y} - M_{xy,x} = -D\nu^2 w_y \end{aligned} \quad (33)$$

$$\begin{aligned} V_x &= Q_x - M_{xy,y} = -D[\nabla^2 w_x + (1-\nu)w_{xyy}] \\ V_y &= Q_y + M_{yx,x} = -D[\nabla^2 w_y + (1-\nu)w_{xxy}] \end{aligned} \quad (33) \text{ cont'd}$$

The last two expressions are the Kirchhoff effective edge shear force relations [6, (34)].

The Donnell equations, as stated in Flugge, are obtained from the Kempner and Timoshenko expressions by reversing the  $w$  sign. In order to obtain the Flugge stress resultants, leave  $M_{yx}$  unchanged, replace  $M_{xy}$  by  $-M_{xy}$  and, of course,  $w$  by  $-w$ .

4. FORCED VIBRATION RESPONSE OF SHELL

The axial, peripheral and (inward) radial responses

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sin \omega_f t \sum_{m,n} \begin{pmatrix} U_{mn} \left\{ \begin{matrix} \sin \frac{m\pi x}{\ell} \\ \cos \frac{n\pi x}{\ell} \end{matrix} \right\} \cos n\theta \\ V_{mn} \left\{ \begin{matrix} \cos \frac{m\pi x}{\ell} \\ \sin \frac{n\pi x}{\ell} \end{matrix} \right\} \sin n\theta \\ W_{mn} \left\{ \begin{matrix} \cos \frac{m\pi x}{\ell} \\ \sin \frac{n\pi x}{\ell} \end{matrix} \right\} \cos n\theta \end{pmatrix} \quad (34)$$

of the shell due to forcing function

$$q_f(x, \theta, 0) = q_a = \sum C_{mn} \left\{ \begin{matrix} \cos \frac{m\pi x}{\ell} \\ \sin \frac{n\pi x}{\ell} \end{matrix} \right\} \cos n\theta = \text{say, } \Sigma q_{mn} \quad (35)$$

are determined by (II.26) from

$$\begin{aligned} \Delta_{11} U \mp \Delta_{12} V \pm \Delta_{31} W &= 0 \\ \mp \Delta_{12} U + \Delta_{22} V + \Delta_{23} W &= 0 \\ \pm \Delta_{31} U + \Delta_{23} V + \Delta_{33} W &= -p_{mn}/C \end{aligned} \quad (36)$$

where

$$\begin{aligned} \Delta_{11} &= \frac{\mu s \omega_f^2}{C} - \left(\frac{m\pi}{\ell}\right)^2 - \frac{1-\nu}{2} \left(\frac{n}{a}\right)^2, & \Delta_{22} &= \frac{\mu s \omega_f^2}{C} - \frac{1-\nu}{2} \left(\frac{m\pi}{\ell}\right)^2 - \left(\frac{n}{a}\right)^2 \\ \Delta_{33} &= \frac{(\mu s + \mu_h) \omega_f^2}{C} - \frac{h^2}{12} \left[ \left(\frac{m\pi}{\ell}\right)^2 + \left(\frac{n}{a}\right)^2 \right] - \frac{1}{a^2} \\ \Delta_{12} &= \frac{(1+\nu)m n \pi}{2R\ell}, & \Delta_{23} &= \frac{n}{Ra}, & \Delta_{31} &= \frac{\nu m \pi}{R\ell} \end{aligned} \quad (37a)$$

$$\Delta_{mn}(\omega^2) \equiv \Delta_{11}\Delta_{22}\Delta_{33} - \Delta_{11}\Delta_{23}^2 - \Delta_{22}\Delta_{31}^2 - \Delta_{33}\Delta_{12}^2 + 2\Delta_{12}\Delta_{23}\Delta_{31} \quad (37b)$$

in the form

$$\begin{pmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{pmatrix} = \frac{C_{mn}}{C\Delta_{mn}} \begin{pmatrix} \pm(\Delta_{12}\Delta_{23} - \Delta_{22}\Delta_{31}) \\ \Delta_{11}\Delta_{23} - \Delta_{12}\Delta_{31} \\ \Delta_{11}\Delta_{22} - \Delta_{12}^2 \end{pmatrix} \quad (38)$$

We are dealing here, as elsewhere, with the lower  $x$  function, corresponding to the pinned-pinned condition, namely with the  $x$  variation (34) of  $(\cos, \sin, \sin)$ . Table III lists the amplitudes of the deflection components  $U_{mn}$ ,  $V_{mn}$ ,  $W_{mn}$ . The deflections  $u, v, w$  themselves are then given by (34). Since the  $x, \theta$  variations of  $u, v, w$  are prescribed, the derivatives of these displacements, hence also the stress resultants (33) are readily obtained, and the shell stresses may be calculated.

ACKNOWLEDGMENT

We are grateful to Combustion Engineering, Inc., Windsor, CT, for sponsoring this work, and for the financial support they provided for G. Bowers with a Research Assistantship at the University of Massachusetts. Thanks are due also to the University of Massachusetts Computing Center for use of their facilities.

REFERENCES

1. G. Bowers and G. Horvay, "Beam modes of vibration of a thin cylindrical shell flexibly supported and immersed in water inside of a coaxial cylindrical container of slightly larger radius," *Nuclear Engineering and Design* 26 (1974), 291.
2. G. Horvay and G. Bowers, "Influence of entrained water mass on the vibration modes of a shell," *Journal of Fluids Engineering*, in press.
3. L.E. Penzes, "Theory of pump induced pulsating coolant pressure in pressurized water reactors," *Journal of Nuclear Engineering and Design* (1974)
4. H.B. Dwight, "Table of roots for natural frequencies in coaxial type cavities," *Journal Mathematics and Physics* 27 (1948), 84.
5. T. von Karman and M.A. Biot, Mathematical Methods in Engineering, McGraw Hill, 1940.
6. J. Kempner, "Energy expressions and differential equations for stress and displacement analysis of arbitrary cylindrical shell," *Journal of Ship Research*, June 1958, p. 8.
7. S. Timoshenko and S. Woinowski-Krieger, Theory of Plates and Shells, McGraw-Hill, 1959
8. Y.Y. Yu, "Free vibrations of thin cylindrical shells having finite lengths with freely supported and clamped edges," *Journal Applied Mechanics* 94 (1955) 547.
9. H.H. Bleich and M.L. Baron, "Free and forced vibrations of an infinitely long cylindrical shell in an infinite acoustic medium," *Journal Applied Mechanics* 93 (1954), 167.
10. H. Langhaar, Energy Methods in Applied Mechanics, Wiley, 1962.
11. W. Flugge, Handbook of Engineering Mechanics, McGraw Hill, 1962.
12. W. Flugge, Stresses in Shells, Springer, 1960.
13. W.C. Lyons, J.E. Russell, G. Herrmann, "Dynamics of submerged reinforced cylindrical shells," *Journal Engineering Mechanics Div., Proc. ASCE*, 1968, p. 397.
14. E.N. Mnév, "Small vibrations of a circular cylindrical shell immersed in a limited space filled with a compressible fluid," *Prikladnaya Mekhanika* 11 (1963), 133; translated from Ukrainian by Oak Ridge Consultants Bureau, 1972.



TABLE I. The equivalent body Force Q, formula (25).

M	N	$10^2 Q$
3	2	4.47
3	5	4.24
5	5	3.94
5	10	3.96
10	5	3.945
10	10	3.953
10	20	3.964

TABLE III. Shell deflections for  $\omega t = (2j+1)\pi/2$ , u at  $(x, \theta) = (0, 0)$ , v at  $(\ell/2, \pi/2)$ , w at  $(\ell/2, 0)$ .

m,n	$10^6 U_{mn}$	$10^6 V_{mn}$	$10^6 W_{mn}$
1 1	-7.97	-26.0	-25.9
1 2	-3.33	-11.8	-22.9
1 3	-1.44	- 6.61	-19.5
2 1	-3.45	-36.8	-60.3
2 2	-5.60	-20.0	-41.8
3 1	-2.53	30.4	85.8
3 2	-12.81	-69.6	-179.9
6 1	0.22	- 0.47	- 4.37
6 2	0.076	- 0.86	- 4.9

TABLE II. Annulus frequencies  $\omega_{mn}$ ; Fourier coefficients  $C_{mn}$ .

m,n	Bowers-Horvay		Penzes	
	$\omega$	C	$\omega$	C
1 1	120	1.255	115	9.85
1 2	206	0.171	195	1.405
1 3	299	0.067	280	0.557
2 1	173	0.416	168	2.67
2 2	241	0.169	230	1.363
3 1	237	0.145	229	0.902
3 2	290	0.089	278	0.692
6 1	443	-0.034	432	-0.199
6 2	474	-0.028	460	-0.2105

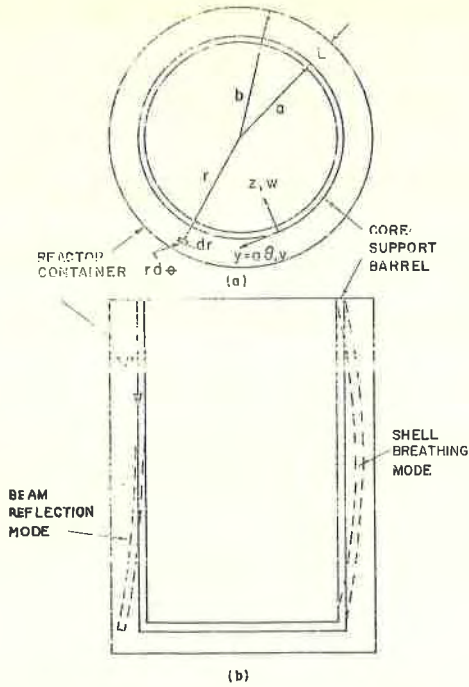


Fig. 1. Inner "shell" is separated from outer "container" by narrow water gap  $L$ .  
 (a) Horizontal cross-section. (b) Vertical cross-section illustrating beam mode ( $\cos n \theta$  variation,  $n = 1$ ), and breathing mode ( $n > 1$ ).

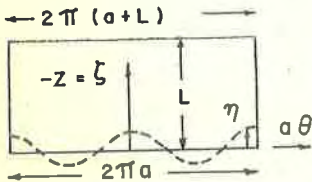


Fig. 2. Annulus represented as a rectangle.

DELTA FUNCTION

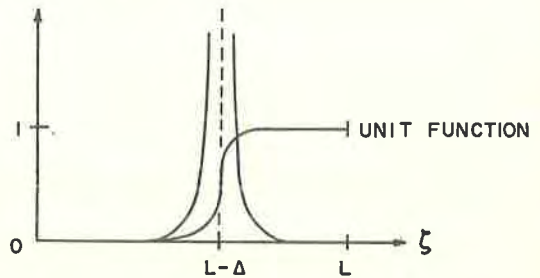


Fig. 3. Rise of the body force from 0 to 1 at  $\zeta = L - \Delta$  has the slope of a delta function.

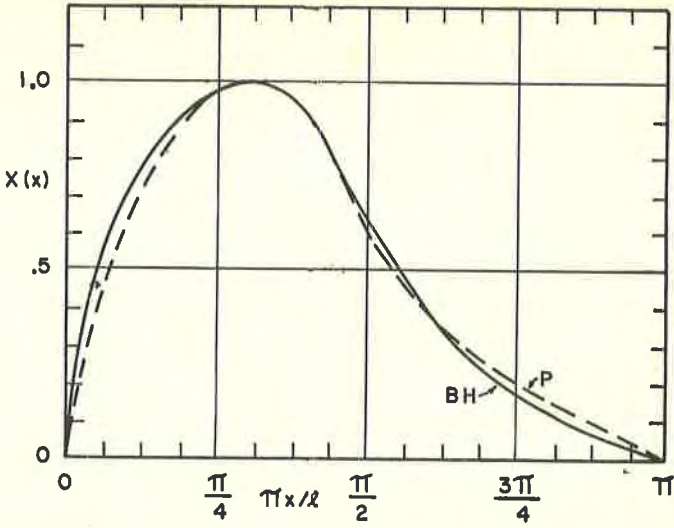
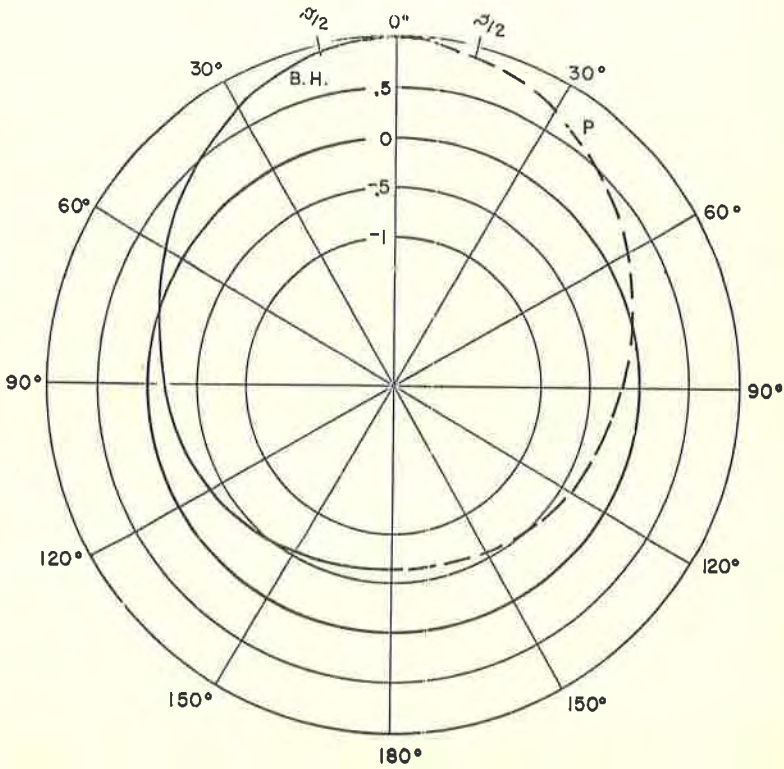


Fig. 4. Distribution of pressure on shell wall, (a) in axial direction, (b) in peripheral direction. Solid line: Bowers-Horvay; dashed line: Penzes.



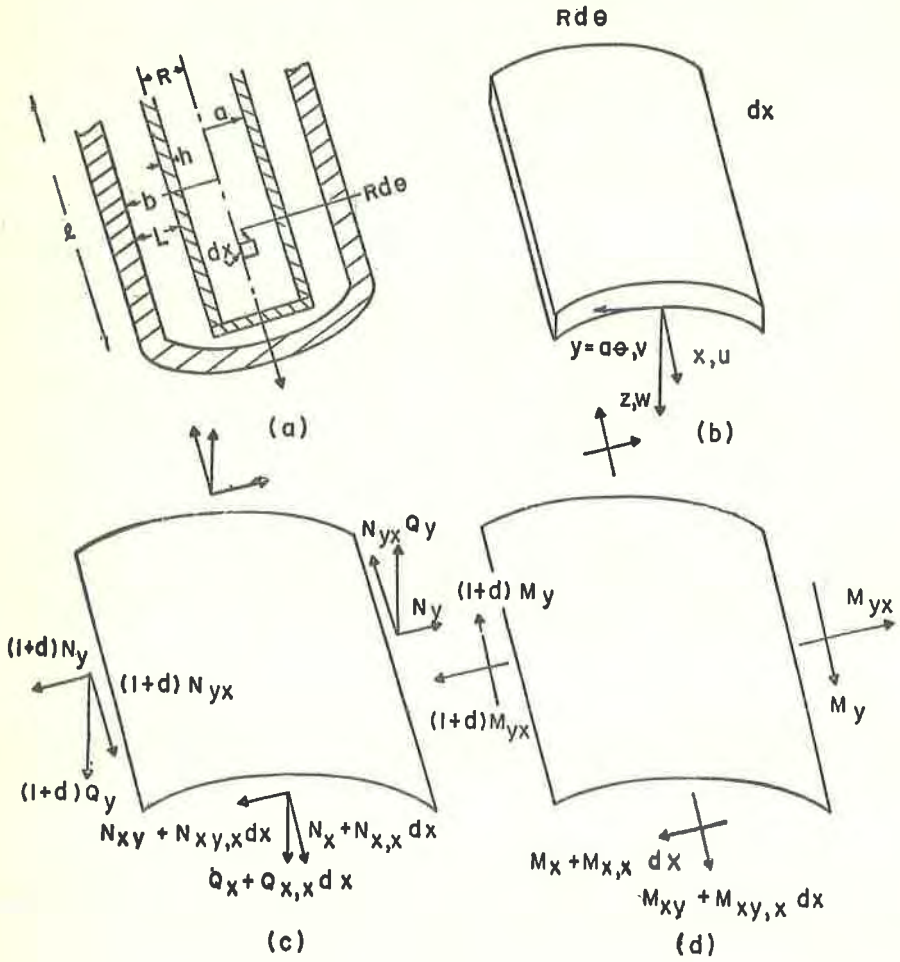


Fig. 5. Force resultants (c), (d) acting on shell element  $dxRd\theta$ , (a), (b).