HYBRID FINITE ELEMENT PROCEDURES FOR ANALYZING THROUGH FLAWS IN PLATES IN BENDING

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SUMMARY

This paper presents finite element solutions of the bending problem of a plate with a through-the-thickness crack. Both solutions are based on the use of special crack tip elements derived by the assumed stress hybrid model which involves an assumed equilibrating stress field within each element and compatible interelement boundary displacement field.

The first case consists of a sixth-order Reissner plate theory which includes the effects of transverse shear deformation. The loading on the plate is general such that all the three crack modes are present. Analytical results for asymptically singular bending moments, twisting moments, and shear force near the crack-tip, as derived by Sih (Int. J. of Fracture Mechanics, 7, pp. 39-61) and Wang (Int. J. of Fracture Mechanics, 6, 1970, pp. 307-308) are built-in to the assumed stresses in elements near the crack-tip. The present finite element procedure leads to matrix equations with the nodal displacements of the structure, as well as the three stress intensity factors for each mode of crack-tip behavior, as unknowns to be solved for directly. Results for bending stress intensity factors for various crack length/plate length ratios, and varous plate thickness/crack length ratios are obtained. The question of the optimum size of a finite element near the crack-tip is discussed. The relation of the optimum size of near-tip elements versus the thickness of the plate is obtained through careful numerical experimentation. Finite size correction factors for stress-intensity factors in a finite plate as compared to the solution for an infinite domain, are obtained.

The second case is concerned with thin plates with through-the-thickness cracks subjected to out-of-plane bending. Kirchhoff hypothesis can be employed here and only bending and twisting modes exist at the crack tip. The special elements are polygonal shaped with an imbedded crack, and in the matrix equations only the nodal displacements are unknowns. The stress intensity factors, however, can be evaluated directly when the nodal displacements of the special element are known. The formulation is based on the complex variable technique developed by Muskhelishvili. The analysis for pure cylindrical bending of center cracked plates of finite width is carried out for a number of crack length/plate width ratios. It was found by using only a single nine-node 27 d.o.f. element the calculated values of the bending stress intensity factor already correlate excellently with those of Wilson and Thompson [Engng. Fracture Mechanics, 3, 1971, pp. 97-102] which were obtained by using more than five hundred triangular bending elements.

Introduction: The nature of stresses near the tip of a through-the-thickness crack in a plate under bending loads has been analytically studied by several investigators. Williams [1] obtained the bending stress singularity near the tip of a straight-line crack, by using the method of eigenfunction expansion based on Kirchhoff plate theory, and found this to be of $1/\sqrt{r}$ (where r is the radial distance from the crack-tip) type. The results in [1] were incomplete in that the magnitudes of local stresses were left undetermined. Later, Sih and Rice [2] indicated a way of finding the coefficients in the eigenfunction expansions through the application of the complex-variable theory. However, the above results, based on Kirchhoff theory, have some discrepancies. In the formulation of [1,2] the three-physically distinct boundary conditions on the crack surface (viz., the vanishing of bending moment, twisting moment, and shear) are reduced to two approximate boundary conditions through the Kirchhoff hypothesis. On account of this, the stress distribution near the crack edge in [1,2] was found to be inaccurate. To overcome this difficulty, Knowles and Wang [3] employed a sixth-order Reissner's plate-bending theory [4], wherein the above mentioned three boundary conditions can be satisfied distinctly, to treat the problem of an infinite plate containing a finite through crack under constant bending moment applied at infinity. The results in [3] were good only for a vanishingly thin plate; and hence the work of [3] was later generalized by Hartranft et al [5] to include the effect of plate thickness. Recently Wang [6], with a treatment analogous to that in [3], solved the problem of a cracked plate subjected to remote external twisting moment, and also included the effects of plate thickness.

From the results of the aforementioned works, it is found that the stress distributions near the crack front, caused by bending as well as twisting, in a plate, are identical to those associated with the general opening, sliding, and tearing modes of crack extension. However it is noted that the solutions obtained by the above mentioned methods are based on some approximations to a fully three-dimensional theory, viz, the theory of bending of a plate, whether it is a 4th order classical or a 6th order advanced theory. Thus, through the above methods, the nonlinear disturbances near crack edges and plate surfaces, for cracks in very thick plates, cannot be accounted for [7]; for these cases, the use of a fully three-dimensional theory may be necessary. In the study of the effect of plate thickness on the stress-intensity factors for a crack in a plate in bending, Sih [8] obtained qualitative features of exact solutions by using a three-dimensional asymptotic expansion [9] of stresses and displacements.

As is now well known, the finite element technique, if suitably formulated can be used to solve for the stress-intensity factors for arbitrary shaped cracks in structural elements of arbitrary geometry (see Pian [10] for a recent comprehensive review). It is also known [10] that the most efficient finite element formulation for this class of problems is that of incorporating the asymptotic (singular) solution for stresses/strains/displacements, in finite-elements in the vicinity of the crack front. In elements far away from the crack front, arbitrary polynomial variation in stresses can be assumed. However, the conditions of interelement traction equilibrium and interelement displacement compatibility must be satisfied at the common boundaries of near-tip elements and far-field elements.

In this paper we present two hybrid finite element models to treat the problem of a through-thickness crack in a plate under bending, using a 6th order Reissner-type, and a 4th order Kirchhoff plate theory, respectively. For this purpose we first note below the structures of asymptotic solutions for stresses, in the two theories, respectively.

Asymptotic Solutions:

(a) based on a Reissner-type plate theory [3,6,8]

$$M_{xx} = \frac{K_I}{\sqrt{2r}}\left(\cos\frac{\theta}{2} - \frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2r}}\left(7\sin\frac{\theta}{2} + \sin\frac{5\theta}{2}\right) - 2\sqrt{2r}\,K_{III}\sin\frac{\theta}{2} + O(r)$$

$$M_{yy} = \frac{K_I}{\sqrt{2r}}\left(\cos\frac{\theta}{2} + \frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2r}}\left(\sin\frac{\theta}{2} - \sin\frac{5\theta}{2}\right) + O(r)$$

$$M_{xy} = \frac{K_I}{\sqrt{2r}}\left(\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2r}}\left(3\cos\frac{\theta}{2} + \cos\frac{5\theta}{2}\right) + O(r) - \sqrt{2r}\,K_{III}\cos\frac{\theta}{2}$$

$$Q_x = \frac{K_{III}}{\sqrt{2r}}\sin\frac{\theta}{2} + O(r) \quad ; \quad Q_y = \frac{-K_{III}}{\sqrt{2r}}\cos\frac{\theta}{2} + O(r) \tag{1}$$

where r, θ are polar coordinates centered at the crack-tip (Figure 1) and in the plane of the plate. It is noted that terms with $\sqrt{r}$ in M_{xx} and M_{xy} were introduced, deliberately, to make the asymptotic stress field, as listed in Eq.(1), to satisfy the plate equilibrium equations (viz., $\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x = 0$, etc..,). It is also noted that in the present case, the functional form of only the basic singular part of the analytical solution is readily available in the literature [3,6,8]; also not readily available is the asymptotic solution for the displacement field.

(b) Based on Kirchhoff plate theory

The asymptotic behavior of the Kirchhoff stress-field in the vicinity of the crack tip has been obtained by Williams [1] and it takes the following form when expressed in terms of stress couples in polar coordinates:

$$M_r = -\frac{7+\nu}{24(3+\nu)}\frac{K_I^*}{\sqrt{2r}}h^2\left[\cos\frac{3\theta}{2} - \frac{3+5\nu}{7+\nu}\cos\frac{\theta}{2}\right] + \frac{5+3\nu}{24(3+\nu)}\frac{K_{II}^*}{\sqrt{2r}}h^2\left[\sin\frac{3\theta}{2} - \frac{3+5\nu}{5+3\nu}\sin\frac{\theta}{2}\right]$$

$$M_\theta = \frac{7+\nu}{24(3+\nu)}\frac{K_I^*}{\sqrt{2r}}h^2\left[\cos\frac{3\theta}{2} + \frac{5+3\nu}{7+\nu}\cos\frac{\theta}{2}\right] - \frac{5+3\nu}{24(3+\nu)}\frac{K_{II}^*}{\sqrt{2r}}h^2\left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right] \tag{2}$$

$$M_{\gamma\theta} = \frac{7+\nu}{12(3+\nu)}\frac{K_I^*}{\sqrt{2r}}h^2\left[\sin\frac{3\theta}{2} - \frac{1-\nu}{7+\nu}\sin\frac{\theta}{2}\right] - \frac{5+3\nu}{12(3+\nu)}\frac{K_{II}^*}{\sqrt{2r}}h^2\left[\cos\frac{3\theta}{2} - \frac{1-\nu}{5+3\nu}\cos\frac{\theta}{2}\right]$$

where K_I^* and K_{II}^* are the stress intensity factors for bending and twisting, respectively. It is recognized that the complete solution of such plate bending problems can be obtained by examining the equations of equilibrium in terms of displacements which take the form of a biharmonic equation,

$$\nabla^2\nabla^2 W = 0 \tag{3}$$

By applying the Muskhelishvili's method of complex variables, the general solution to equation 3 can be represented in terms of two analytic functions of the complex variable Z.

$$W(r,\theta) = \mathrm{Re}[\bar{Z}\,\varphi(Z) + \chi(Z)] \tag{4}$$

where $\bar{Z}$ is the complex conjugate of Z. The moments, and the Kirchhoff shear forces are expressed in terms of complex potentials φ(Z) and χ(Z) by

$$M_x + M_y = -4D(1+\nu)\,\mathrm{Re}[\varphi'(Z)]$$

$$M_y - M_x + 2i\,M_{xy} = 2D(1-\nu)\,\{\bar{Z}\,\varphi''(Z) + \chi''(Z)\}$$

$$Q_x - iQ_y = -4D\,\varphi''(Z) \tag{5}$$

where D is the bending rigidity of the plate. To express the singularities of all order, the following conformal mapping function is selected.

$$Z = \xi^2 \tag{6}$$

The arguments of Z and ξ are limited to the range $-\pi \le \arg Z \le \pi$, and $-\frac{\pi}{2} \le \arg \xi \le \frac{\pi}{2}$. If the two crack surfaces are on the negative real axis of the Z plane, they are mapped on the imaginary axis of the ξ plane and the domain lies on the region where the real part of ξ is positive.

The complex potential φ and χ, which are analytic functions of Z, are then also analytic functions of ξ and the regular polynomial function of ξ can be used for their representation.

$$\varphi(\xi) = \sum_{j=1}^{N} \left[\left(\beta_{2j-1} + i\beta_{2N+2j-1}\right)\xi^{2j-1} + \left(\beta_{2j} + i\beta_{2N+2j}\right)\xi^{2j} \right]$$

$$\chi(\xi) = \sum_{j=1}^{N} \left[\left(\gamma_{2j+1} + i\gamma_{2N+2j+1}\right)\xi^{2j+1} + \left(\gamma_{2j+2} + i\gamma_{2N+2j+2}\right)\xi^{2j+2} \right] \tag{7}$$

By applying the Kirchhoff stress-free condition over the crack surfaces, all γ's can be expressed in terms of β's. Hence all the field variables W, M_x, M_{xy}, Q_x, and Q_y can be expressed in terms of β's only. The stress intensity factors K_I^* and K_{II}^* are simply,

$$K_I^* = -\frac{6\sqrt{2}\,D(3+\nu)}{h^2}\,\beta_1 \quad ; \quad K_{II}^* = \frac{6\sqrt{2}\,D(3+\nu)}{h^2}\,\beta_{2N+1} \tag{8}$$

Finite-Element Formulations

(I) Hybrid-Stress Model for the Crack-element Using Reissner-type Theory:

As noted in Eq.(1), only the basic singular part of the asymptotic solution for moments and shear near the crack-tip is readily available, and this will be sought to be embedded in elements adjoining the crack-tip. However, to keep the size of the crack-tip elements to be reasonably large, a regular polynomial variation of the stress-field will also be assumed in the crack-tip element, in addition to the basic singular terms. Compatibility of displacements and equilibrium of the tractions between these crack-tip elements and the surrounding regular elements (with only a polynomial variation in stresses) will be maintained through a Lagrange Multiplier technique, based on the hybrid-stress finite element model [11]. Briefly this model states that if the assumed stresses in each element satisfy a priori the conditions, (a) $\sigma_{ij,j} + E_i = 0$ in V_m and (6) $\sigma_{ij}\eta_j \equiv T_i = \bar{T}_i$ on S_{σ_m}, then the vanishing of the first variation of the functional,

$$\pi_{HS}\left(\sigma_{ij}, \tilde{u}_{i\rho}\right) = \sum \left\{ \int_{V_m} B\left(\sigma_{ij}\right) dV - \int_{S_{u_m}} T_i \bar{u}_i ds - \int_{\rho_m} T_i \tilde{u}_{i\rho} ds \right\} \tag{9}$$

leads to the Euler equations: (i) $\epsilon_{ij} = \frac{\partial B}{\partial \sigma_{ij}} = \frac{1}{2}\left(f_{i,j} + f_{j,i}\right)$ in V_m, (ii) $f_i = \bar{f}_i$ on S_{u_m}, (iii) $f_i^+ = f_i^- = \tilde{u}_{i\rho}$ at ρ_m; and (iv) $T_i^+ + T_i^- = 0$ at ρ_m(where V_m is the domain, S_{u_m} is the displacement-prescribed boundary, and S_{σ_m} is traction-prescribed boundary, respectively, of the mth element, ρ_m is that portion of mth element boundary where neighboring elements interface; B is the complementary-energy density, and the superscripts (+) and (-) denote, arbitrarily, the left and right sides of ρ_m as ρ_m is approached. If traction boundary conditions

$T_i = \bar{T}_i$ on S_{σ_m} cannot be met a priori, they can be introduced as additional conditions of constraint, and this modification to the functional in Eq.(9) results in,

$$\pi_{HS}\left(\sigma_{ij}, \tilde{u}_{i\partial V}\right) = \sum \left\{ \int_{V_m} B\left(\sigma_{ij}\right) dV - \int_{\partial V_m} T_i \tilde{u}_{\partial V} ds + \int_{S_{\sigma_m}} \bar{T}_i \tilde{u}_{i\partial V} ds \right. \qquad (10)$$

where $\tilde{u}_{\partial V}$ is introduced all along the boundary ∂V_m of V_m, such that, a priori, $\tilde{u}_{i\partial V} = \bar{u}_i$ on S_{u_m}. For elements which share the crack surface as part of their boundaries, S_{σ_m} can be identified with the crack-surface where, without loss of generality, traction-free conditions are assumed. In the present problem of a straight-line crack, along the x-axis, it is simple to satisfy traction b·C a priori, and thus base the finite element formulation on Eq.(9). However, to study the effects of precisely satisfying traction b·C on the computed stress-intensity factors, the problem was also solved based on the formulation of Eq.(10).

As applied to the plate-bending problem where transverse shear is accounted for, the functional in Eq.(9) can be written as (see [14] for further details),

$$\pi_{HS} = \sum_{m=1}^{N} \left\{ \frac{1}{2} \int_{V_m} \left(\underset{\approx}{M}^T \underset{\approx}{D}\, \underset{\approx}{M} + D_1\, \underset{\sim}{Q}^T \underset{\sim}{Q} - D_2\, \underset{\approx}{M}^T \underset{\approx}{P}_M \right) dV - \int_{S_{u_m}} \underset{\sim}{T}^T \bar{\underset{\sim}{U}}\, ds - \int_{\rho_m} \underset{\sim}{T}^T \underset{\sim}{U}\, ds \right\} \qquad (11)$$

where $[M] = [M_{xx}, M_{yy}, M_{xy}]$, $[Q] = [Q_x, Q_y]$, and $\underset{\sim}{U}$ is the vector of generalized displacements (consisting of the normal displacement W; its derivatives $W_{,x}$; $W_{,y}$; and the rotations Φ_x, Φ_y) at ρ_m. Also, $\underset{\approx}{D}$ is the compliance property matrix for the plate (assumed to be isotropic); $D_1 = 12(1+\nu)/Eh$; $D_2 = 12\nu/5Eh$, and $\underset{\approx}{P}_M = [P,P,0]$ where P is the applied normal force distribution on the plate.

In the class of problems considered here, effects of transverse shear strain/stress are considered only in each of the four singular elements surrounding the crack tip (Fig. 2). In each singular element, the bending and twisting moments, and the shear forces are assumed as follows.

$$\underset{\sim}{M} = \underset{\approx}{N}\, \underset{\sim}{\beta} + \underset{\approx}{N}_s\, \underset{\sim}{\beta}_s + \underset{\approx}{N}_t\, \underset{\sim}{\beta}_t + \underset{\sim}{N}_p$$

and (12)

$$Q = \underset{\approx}{N}_Q\, \underset{\sim}{\beta} + \underset{\approx}{N}_{Qt}\, \underset{\sim}{\beta}_t + \underset{\sim}{Q}_p$$

wherein, the functions $\underset{\approx}{N}$ and $\underset{\approx}{N}_Q$ with undetermined parameters $\underset{\sim}{\beta}$ correspond to a self-equilibrated moment/shear field, and are obtained from 2 stress functions, using a static-geometric analogy. Further, $[\beta_s] = [K_I, K_{II}]$, $\beta_t = K_{III}$, and thus $\underset{\approx}{N}_s$, $\underset{\approx}{N}_t$ and $\underset{\approx}{N}_{Qt}$ are the corresponding asymptotically correct singular functions as given in Eq.(1). Finally $\underset{\sim}{N}_p$ and $\underset{\sim}{Q}_p$ are particular-solution functions corresponding to the applied external pressure on the plate.

Since the asymptotic solution for the displacement field near the crack-tip is not readily available, the boundary displacement field for the singular element is assumed, based on considerations of the qualitative features of the asymptotic solution in a fully three-dimensional case. In the 3-D case all the 3 displacements vary as $\sqrt{r}$ near the crack front. But in a plate bending problem, components of inplane displacements, u_α, are linearly proportional to the total rotations Φ_α. Further, $\Phi_\alpha = (\partial W/\partial x_\alpha) + \gamma_\alpha$ where γ_α are shear strains. Thus, in parallel to the three dimensional situation, the displacement W at the boundary for the present element is assumed to vary as $r^{3/2}$ and the total rotation to vary as $r^{\frac{1}{2}}$ near the crack front. Thus, along the line 12 for instance (see Fig. 1), where 2 is the crack-tip, the displacement field is assumed as,

$$
\begin{aligned}
W_{12} &= (1-s)^{3/2}\, W_1 + [1 - (1-s)^{3/2}]\, W_2 \\
\Phi_{y12} &= - \sqrt{1-s}\ \Phi_{y1} - (1 - \sqrt{1-s})\ \Phi_{y2} \\
\Phi_{x12} &= \sqrt{1-s}\ \Phi_{x1} + (1 - \sqrt{1-s})\ \Phi_{x2}
\end{aligned}
\tag{13}
$$

similar assumptions are made for side 23; where as, at sides (14) and (54) which adjoin the surrounding regular elements, the displacement field is assumed such that W is a cubic and the normal slope is a linear function, in terms of the generalized nodal coordinates at points 1,4 and 5, respectively as indicated in Fig. 2. However, this implies the following approximation: the nodal coordinates at node 1, for instance, are interpreted as W_1; Φ_{y1}; Φ_{x1} for purposes of interpolation along side 12; where as they are interpreted as W_1; $W,_{y1}$; $W,_{x1}$ for purposes of interpolation along side 1,4. This approximation was made with the aim of keeping the total number of nodal (generalized) displacement coordinates as small as possible, and yet obtain meaningful engineering results. Lastly, for all the regular elements the boundary displacement functions are such that, W is cubic and the normal slope is linear at any boundary-segment.

By substituting the assumptions for singular elements given in Eq.(12), and similar assumptions for the far-field elements (with only simple polynomial type stresses), into Eq.(11) and following the usual procedure of the hybrid stress model [11], final algebraic equations can be obtained from which the system displacements $\underset{\sim}{q}$*and the three stress-intensity factors K_I, K_{II}, and K_{III} are <u>directly</u> solved. The procedure to obtain, and the form of these algebraic equations is analogous to that in Pian, Tong and Luk [12] who treat the plane stress crack problem, and in Luk [13] who also formulates a crack element based on Reissner-type plate theory but without implementation. Thus further details of matrix algebra are ommitted for want of space, and can be found in [13] and in the thesis by Rhee [14].

(II) <u>Generalized Hybrid Model for Crack-Element Using Kirchhoff Theory</u>

The series expansion given in Eq.(7), represents solution functions which satisfy equilibrium as well as compatibility conditions in the general boundary value problem of a plate (under Kirchhoff theory) containing a crack. This solution is sought to be embedded in the crack-element, and further, compatibility of displacements and equilibrium of tractions are sought to be maintained between this singular element (which completely surrounds the crack tip i.e., $-\pi \le \theta \le \pi$ for general loading situations) and the surrounding regular elements. Thus when the constraint condition, $\partial B/\partial\sigma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \partial u_j/\partial x_i\right)$ which is satisfied a priori, is removed from the functional corresponding to the singular element (designated as V_1), the total functional in Eq.(10) becomes,

$$
\pi_{HS}\left(\sigma_{ij}, \tilde{u}_{i\partial V}\right) = \int_{\partial V_1} T_i\left(\frac{1}{2}\, u_i - \tilde{u}_{i\partial V}\right) ds + \int_{S_{\sigma_1}} \bar{T}_i\, \tilde{u}_{i\partial V} ds + \sum_{m=2}^{N} \left\{ \int_{V_m} B\, dV - \int_{\partial V_m} T_i \tilde{u}_{i\partial V} ds + \int_{S_{\sigma_m}} \bar{T}_i \tilde{u}_{i\partial V} ds \right\} \tag{14}
$$

Of course, the regular elements, m=2, ..N can be simply generated through the usual finite element displacement method provided, at the interface between these and V_1, the displacement field is the same, viz., $\tilde{u}_{i\partial V}$ of V_1. The formulation of this special crack element follows exactly the same procedure as that for plane-elements given in Ref. [16]. Using the displacements and moments derived from the potential functions $\varphi(Z)$ and $\chi(Z)$, one can assume

$$\tilde{T} = \{V_Z, -M_\nu, -M_{\nu s}\} = \tilde{R}\,\tilde{\beta}$$

$$\tilde{u} = \{W, W,_\nu, W,_s\} = \tilde{U}\,\tilde{\beta}$$

$$\tilde{\tilde{u}} = \{W, W,_\nu, W,_s\} = \tilde{L}\,\tilde{q}$$

where $\tilde{\beta} = \{\beta_1, \beta_2, \ldots, \beta_{4N}\}$, $\tilde{L}$ contains the interpolation functions of the boundary displacements along the element boundaries. Here, for a segment between two neighboring nodes, the boundary displacements $\tilde{W}$ and $\tilde{W},_\nu$ are assumed by the cubic and linear functions of s respectively.

By substituting Eq.(15) into Eq.(14) and through some algebraic manipulations, the stiffness matrix $\tilde{K}$ for the special crack element can be written as

$$\tilde{K} = \tilde{G}^T \tilde{H}^{-1} \tilde{G} \qquad (16)$$

where

$$\tilde{G} = \int_{\partial V_m} \tilde{R}^T \tilde{L}\, ds$$

$$\tilde{H} = \frac{1}{2}\int_{\partial V_m} (\tilde{R}^T\tilde{U} + \tilde{U}^T\tilde{R})\, ds \qquad (17)$$

It is seen that in constructing the crack element only two line integrals along element boundary need be evaluated. It should be remarked here that the line integral of $\tilde{T}^T\tilde{u}$ over the crack surfaces can be written as

$$\int \tilde{T}^T\tilde{u}\, ds = M_{\nu s}\, W \Big|_{B+}^{C-}$$

where B and C are the nodes at the intersection of the crack surfaces and the side of the special element. The symbols B+ and C- are used to indicate that the values of $M_{\nu s}$ are that on the crack surface side of these nodes. These terms are separated by concentrated loads at the nodes and should be added to the matrices $\tilde{G}$ and $\tilde{H}$. It is, thus, seen that the boundary displacements $\tilde{W}$ and $\tilde{W},_\nu$ need not be assumed over the crack surfaces. The process for integrating the $\tilde{G}$ and $\tilde{H}$ matrices does not involve any singular terms, hence, can be carried out numerically without any special treatment. Details for the development of this element can be found in the thesis by Moriya [17]. The computing effort required for this special crack element is quite small. For example, for a nine-node element the CPU time required for generating the stiffness matrix is less than one second by an IBM 370/065 computer.

Results: Even though the formulations presented are valid for general mixed-mode loading conditions, example problems are presented here, pertaining to Mode I loadings only (specifically for a uniformly distributed M_{yy} and M_{xx} all around plate see Fig. 3), wherein, due to symmetry only a quarter of the plate needs to be analyzed. It is natural that the computed solutions for K_I, using the hybrid-stress model, should vary with the size of a singular element in the finite element grid, since the singular nature of moments and shear is predominant in a small but finite region near the crack-tip. That an optimum size exists f for singular elements based on the hybrid stress model and the reasons for it are discussed in [15]. It is also seen that the three-dimensional effects (and thus the transverse shear stresses) become more important as the ratio h/a ('h' is plate thickness and 'a' is semi-crack length) increases. Since transverse shear effects (and the transverse shear singularities) are property accounted for only in the singular elements in the present formulation, it can be heuristically seen that the optimum size-ratio c/a (c is characteristic size of singular element as in Fig. 2) for the singular element must increase as h/a increases. This has been confirmed by the numerical results for computed K_I values for various h/a ratios, as shown in Fig. 3. The size ratio c/a which gives the peak value for K_I for each h/a ratio is taken to be the optimum for the particular h/a ratio. In subsequent computations for different h/a and 2a/L (crack-length to plate with ratio), the singular element size was taken to correspond to the above optimum size. Thus, for instance, Fig. 4 shows the variation of K_I for different h/a ratios for the case when (2a/L) = 0.1. In this figure K_I is the analytical solution for the infinite domain obtained by Sih [8], K_F is the computed solution for the finite domain when the stress free conditions on the crack face are satisfied a priori and the formulation is based on Eq.(9); and K_{FP} is the computed solution for the finite domain when the stress free conditions on the crack face are satisfied a posteriori through the variational principle based on Eq.(10). From Fig. 4 it can be concluded that the a priori satisfaction of traction b·C on the crack face is mandatory to obtain meaningful results for K_I, based on the hybrid stress model. Finally, by using the optimum singular element size c/a, stress intensity factors K_F are computed for various 2a/L ratios and h/a ratios. These results are summarized in Fig. 5, which shows the factor K_F/K_I (K_F and K_I as defined above) for cracks in finite plates, as the ratio of crack length to plate width varies. It is seen that the finite size correction factor K_F/K_I varies significantly, as expected, with the 2a/L ratio; however this variation with the h/a ratio is not as significant for any particular 2a/L ratio. Further detailed numerical results are available in [14].

The special crack element formulated under the Kirchhoff plate bending assumption has been utilized to analyze the problem of pure cylindrical bending of a thin plate with finite width and centrally located through-the-thickness crack. A number of cases were analyzed with different ratios of 2a/L, where 2a as the total crack length and L, the width of the plate. In each case only a single nine-node special element with 27 degrees-of-freedom was used . Figure 6 presents the results of stress intensity-factor K_F^* versus 2a/L. The solid line in this figure represents the results obtained by Wilson and Thompson [18] which were computed using five hundred conventional 3-node triangular plate elements based on assumed displacements. The results indicated by circles are those obtained by the present special element. It is seen that results obtained by a single special element already correlate very well with the other results. Extensions of the Kirchhoff-type formulation have been made to bending analysis of an anisotropic plate with through-the-thickness crack, an isotropic plate with wedge shaped notch, and bi-material plates with through-the-thickness crack located normal to the interface [17].

It is noted that the two sets of results, using the Reissner-type theory and Kirchhoff theory, respectively, are for slightly different loading conditions (see Figs. 3 and 6); and hence no direct comparison of the two results is attempted. Also the present study did not attempt to consider a degenerate case such that the Reissner-type theory and Kirchhoff theory can be expected to coincide.

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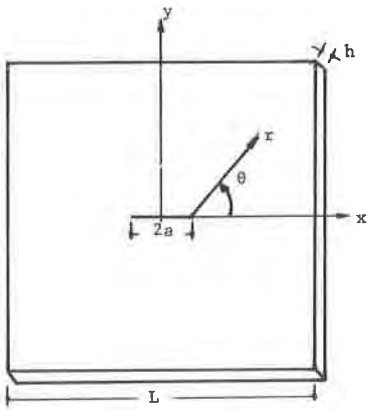

Fig. 1 Nomenclature for a Cracked Plate

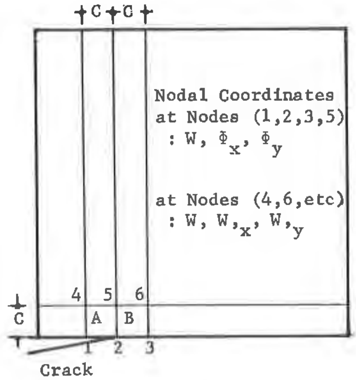

Fig. 2 A Typical Finite Element Gride of a Quarter Plate

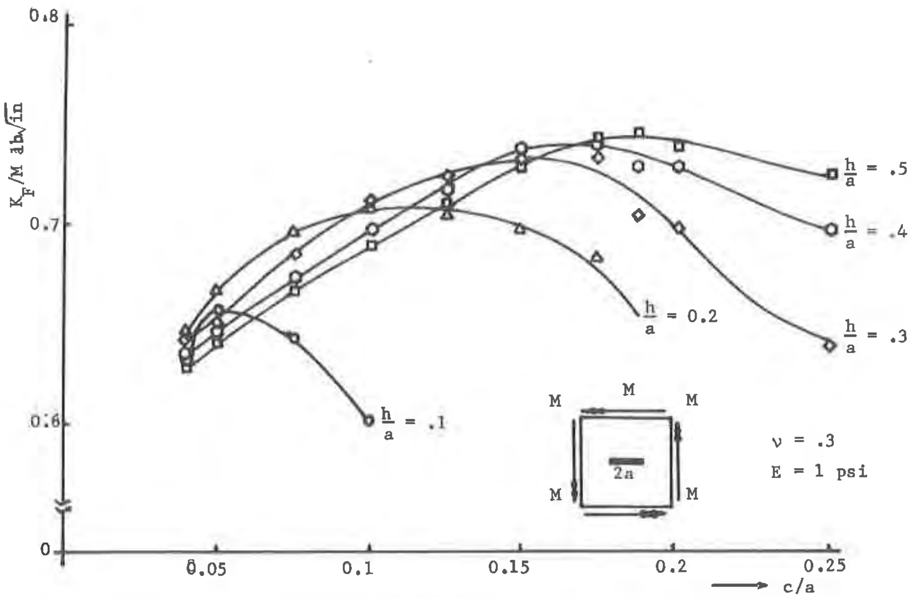

Fig. 3 Bending Stress Intensity Factors with Variations of Thickness and Singular Element Size (Using Reissner's Plate Theory).

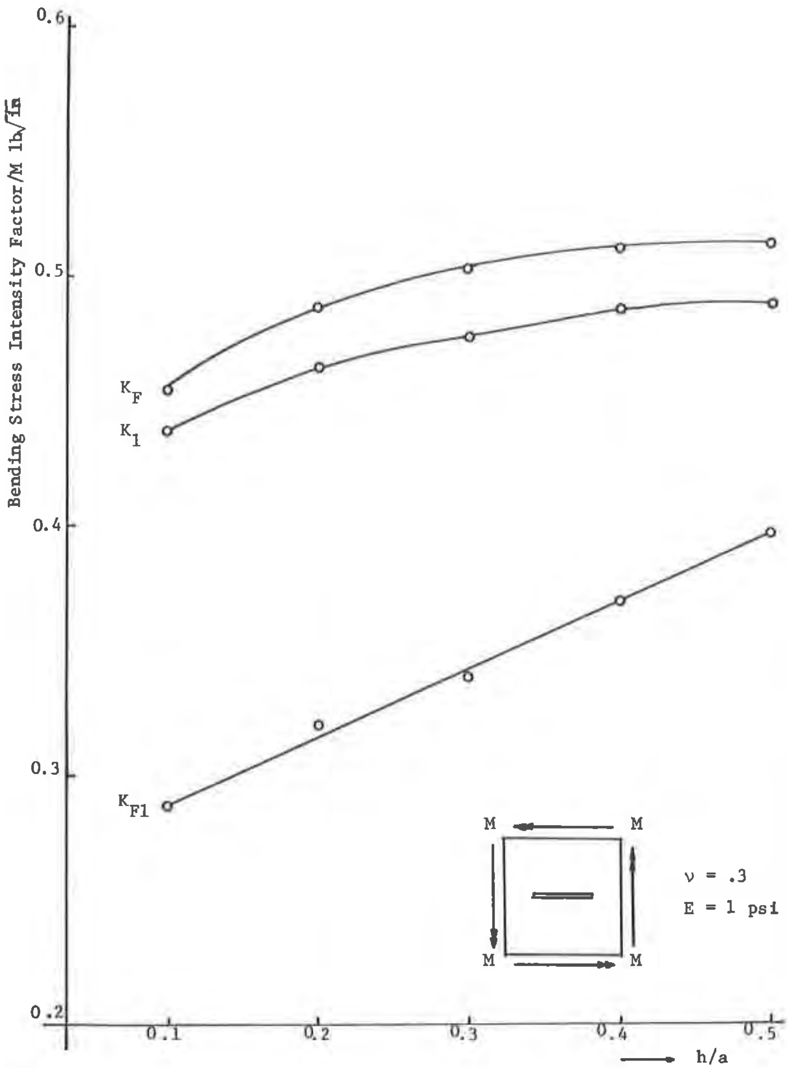

Fig. 4 Variation of Bending Stress Intensity Factor with the Thickness of Plate : 2a/L = .1 (Using Reissner's Plate Theory)

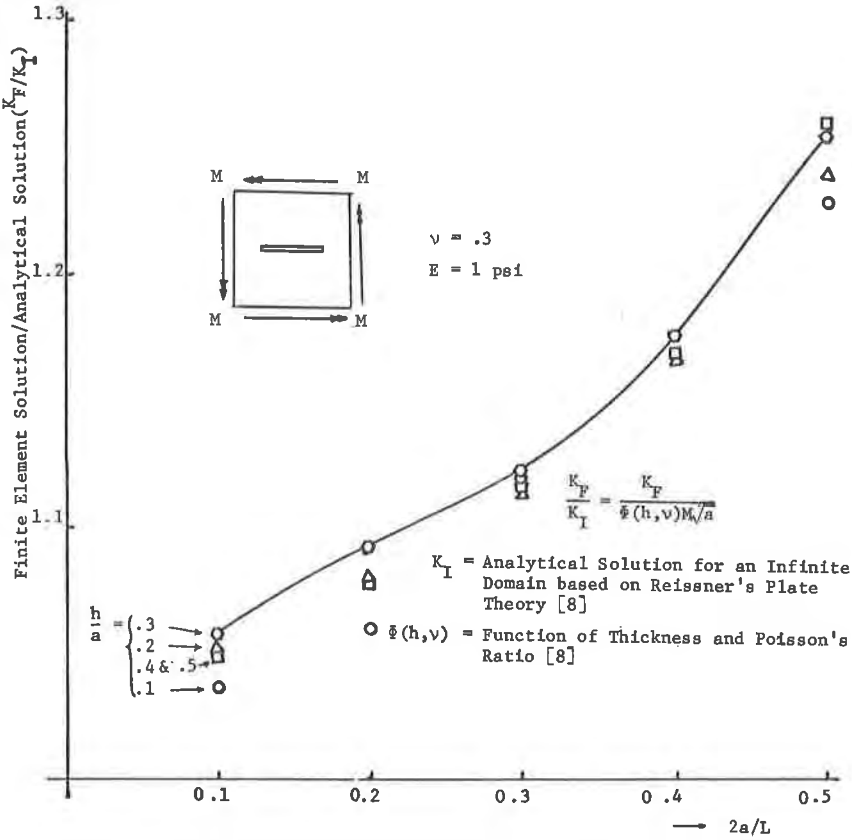

Fig. 5 Finite Dimension Correction Factors (Using Reissner's Plate Theory)

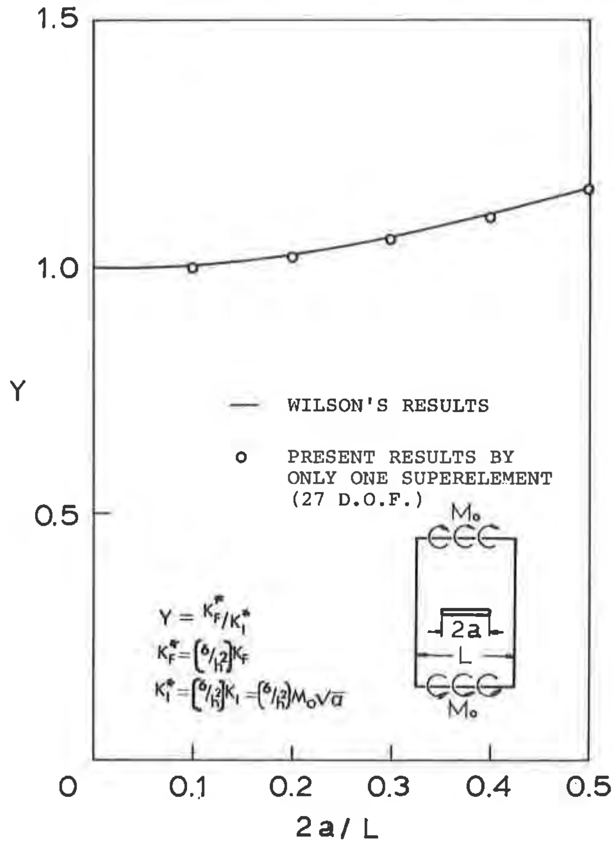

FIG. 6 BENDING STRESS INTENSITY FACTORS FOR CENTER CRACKED PLATES SUBJECT TO CYLINDRICAL BENDING (USING KIRCHHOFF'S PLATE THEORY)