SERIES REPRESENTATIONS FOR GENERALIZED STOCHASTIC PROCESSES

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Abstract

Series representations are derived for bandlimited generalized functions and generalized stochastic processes. This work extends existing results concerning sampling representations of bandlimited functions and stochastic processes. The merit of such representations lies in the fact that a function (or process) may be exactly reconstructed using only a countable number of its values (or samples). These types of representations have found many applications in several areas of communication and information theory such as digital audio and visual recording, and satellite communications. In addition, random distributions have also been employed in a host of applied areas such as statistical mechanics, chemical reaction kinetics and neurophysiology.

1. Introduction. This paper is concerned with sampling representations for generalized functions (or distributions) and generalized stochastic processes (or random distributions). The terms distribution and generalized function will be used interchangeably throughout the text. The need to consider distributions (beyond classical functions) arises from the fact that in many physical situations it may be impossible to observe the instantaneous values f(t) (of a physical phenomenon) at the various values of t. For instance, if t represents time or a point in space, any measuring instrument would merely record the effect that f produces on it over non-vanishing intervals of time I: $\int_I f(t)\phi(t)dt$, where ϕ is a "smooth" function representing the measuring instrument, i.e. the physical phenomenon is specified as a functional rather than a function. Furthermore, it is becoming exceedingly clear that the tools and techniques of the theory of distributions are useful in investigating certain problems in many applied areas such as statistical mechanics (Holley and Strook, 1978), Chemical reaction kinetics (Kolelenez, 1982), and neurophysiology (Kallianpur and Wolpert, 1984a,b and Christenson, 1985). It is thus of interest to consider distributions beyond functions.

The sampling representations (expansion)

$$(1.1) \qquad f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2W - n)}{\pi(2Wt - t)}, \quad t \in IR^1,$$

was originated by E.T. Whittaker (1915). J.M. Whittaker (1929, 1935), Kotelnikov (1933), Shannon (1949), and others studied extensively the sampling theorem and its extensions in developing communication and information theory. For a review of the sampling theorem, see Jerri (1977). A function f which can be represented, for some $W_0 > 0$, by

(1.2) $$f(t) = \int_{-W_0}^{W_0} e^{2\pi itu} F(u)du \quad , \; t\varepsilon IR^1$$

is called L^1-bandlimited to W_0 if $F\varepsilon L^1[-W_0,W_0]$, and is called conventionally or L^2-bandlimited to W_0 if $F\varepsilon L^2[-W_0,W_0]$. In both cases the sampling representation (1.1) is valid for all $W \geq W_0$. The series in (1.1) converges uniformly on compact sets for L^1-bandlimited functions, and for conventionally bandlimited functions it converges in $L^2(IR^1)$ as well as uniformly on IR^1.

However, a function need not be bandlimited in the above sense to exhibit a sampling expansion of the form (1.1). Zakai (1965) extended the concept of "bandlimitedness" to a broader class in which functions need not be in the form (1.2). For a non-negative integer k, let $L^2(\mu_k)$ be the class of all complex valued functions defined on IR^1 that are square integrable with respect to the measure $d\mu_k(t) = \frac{dt}{(1 + t^2)^k}$. If $f\varepsilon L^2(\mu_k)$, then f defines a tempered generalized function (or tempered distribution) (denoted also by f) on the class S of rapidly decreasing functions by

$$f(\theta) = \int_{-\infty}^{\infty} f(t)\theta(t)dt \; , \; \theta\varepsilon S \; .$$

(See Section 2 for relevant definitions.) The distributional Fourier transform of f is the tempered distribution $\hat{f}$ defined by $\hat{f}(\theta) = f(\hat{\theta})$, $\theta\varepsilon S$. The spectrum of f is the support of $\hat{f}$. For k = 0,1,2,... and $W_0 > 0$, $B_k(W_0)$ is the class of all continuous functions $f\varepsilon L^2(\mu_k)$ whose (distributional) spectrum is contained in $[-W_0,W_0]$, and is called the class of W_0-bandlimited functions in $L^2(\mu_k)$. It is clear that $B_0(W_0)$ is the class of W_0-bandlimited functions in $L^2(IR^1)$, and $B_k(W_0) \subset B_{k+1}(W_0)$. Also, $B_0(W_0)$ is dense in $B_k(W_0)$ for every positive integer k (see Lee, 1976).

Zakai obtained a sampling representation for functions in $B_1(W_0)$. Cambanis and Masry (1976) characterized Zarkai's class $B_1(W_0)$ and as a consequence sharpened Zakai's sampling expansion. It was shown that if $f \epsilon B^1(W_0)$ and $W > W_0$ then f has a sampling representation of the form (1.1). Lee (1977) extended Zakai's result to functions in $B_k(W_0)$. He showed that if $f \epsilon B_k(W_0)$, $W > W_0$, $0 < \beta < W - W_0$ and ψ is an arbitrary but fixed C^∞-function with support in [-1,1] and $\int_{-\infty}^{\infty} \psi(t)dt = 1$, then

$$(1.3) \qquad f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n}{2W}) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} \hat{\psi}(\beta(t - \frac{n}{2W})) , \quad t \epsilon IR^1,$$

and the series converges uniformly on compact sets. It should be noted that the presence of the (damping) factor $\hat{\psi}$ in (1.3) cannot be eliminated, as (1.1) is not valid for $f \epsilon B_k$, $k \geq 2$. As a counter example consider $f(t)=t$ $(f \epsilon B_2(W_0))$; then $f(\frac{n}{2W}) = \frac{n}{2W}$ and the series in (1.1) does not converge.

Campbell (1968) derived sampling expansions for the Fourier transforms (as functions)of tempered generalized functions with compact supports. If a tempered generalized function F has a compact support and $e_u(t) = e^{2\pi i t u}$, then $F(e_u)$ is well defined, since $e_u \epsilon C^\infty$ for all $u \epsilon IR^1$. In this case the Fourier transforms $\hat{F}$ of F may be thought of as a function defined on IR^1 by $\hat{F}(u) = F(e_u)$, $u \epsilon IR^1$ (see Section 2). Campbell showed that if F is a tempered generalized function with compact support and with Fourier transform f as a function on IR^1, i.e. $f(t) = F(e_t)$, $t \epsilon IR^1$, ψ is a test function such that $\psi(u) = 1$ on some open set containing supp(F), if $W > 0$ is such that the translates $\{supp(\psi) + 2nW\}$, $n \neq 0$, are disjoint from supp(F), then

(1.4) $$f(t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})\ K(t - \tfrac{n}{2W}) ,$$

where $K(t) = \frac{1}{2W} \int e^{2\pi i t u}\psi(u)du$, and the series converges for every $t \in IR^1$. Sampling expansions for functions which are Fourier transforms of generalized functions with compact support have also been considered by Hoskias and De Sousa Pinto (1984a,b).

Sampling representations of the types discussed above have also been established for stochastic processes. Let $X=\{X(t),\ t \in IR^1\}$ be a measurable stochastic processes with covariance function $R(k,s) = E[t)\overline{X}(s)]$, $t\ s \in IR^1$, which satisfies

(1.5) $$\int_{-\infty}^{\infty} R(t,t)d\mu_k(t) < \infty \quad k \geq 0.$$

The process X was defined by Lee (1976) to be bandlimited if almost every sample path of X was bandlimited, or equivalently, if the function R(t,.) was bandlimited. Let $BP_k(W_0)$ be the class of mean square continuous second order stochastic processes whose covariance functions satisfy (1.5). Zakai (1965) established a series representation similar to (1.1) for stochastic processes in $BP_1(W_0)$ (see also Cambanis and Masry, 1976). Indeed, it was shown that if $X = \{X(t), t \in IR^1\}$ belongs to $BP_1(W_0)$, then for any $W > W_0$ and $t \in IR^1$

(1.6) $$X(t) = \sum_{n=-\infty}^{\infty} X(\tfrac{n}{2W})\ \frac{\sin \pi(2W-n)}{k(2W-n)}$$

where the series converges in the mean square uniformly on compact sets

Lee (1976) established the following representation, which similar to (1.3), for processes X $X=\{X(t),\ t\in IR^1\}$ in $BP_k(W_0)$, $k\geq 1$

$$(1.7) \qquad X(t)=\sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right)\frac{\sin \pi(2Wt-n)}{\pi(2W-n)}\ \hat{\psi}\left(\beta\left(t-\frac{n}{2W}\right)\right),\ t\in IR^1$$

for any $W > W_0$ and $0 < \beta < W - W_0$ and where ψ is defined as in (1.3). The series in (1.7) convergences in the mean square uniformly on compact sets. See also Lee (1977) and Piranashvili (1967) for similar results. Campbell (1968) established a sampling representation similar to (1.7) for weakly stationary stochastic processes whose covariance functions are Fourier transforms of generalized functions with compact support. In this case the series converges in mean-square uniformly on compact set.

In this paper, series representations are derived for generalized functions and generalized stochastic processes which extend the sampling representations, of ordinary functions and stochastic processes, discussed above. In Section 2, notations and basic definitions needed in the sequel are given. In Section 3, the sampling representation (1.1) valid for functions in $B_0(W_0)$ and $B_1(W_0)$ and the representation (1.3) which is valid for functions in $B_k(W_0)$, $k \geq 2$ are extended to bandlimited generalized functions (Theorem 3.1). Examples which show how sampling representations of "ordinary" functions are recovered from Theorem 3.1 are given. In Section 4, series representations for bandlimited generalized stochastic processes are derived. These results extend sampling representation (1.7) for "ordinary" bandlimited stochastic process in $B_k(W_0)$, $k \geq 0$.

Theorem 4.1 derives sampling presentations for stochastic processes with sample paths which have symmetric spectrums as well as spectrums which are just compact sets in $\mathbb{R}^1$. Examples are also given to show how the classic results may be recovered from the ones presented in this section.

2. <u>Notation and basic definitions</u>. Let $C_c^\infty = C_c^\infty(\mathrm{IR}^1)$ be the class of all infinitely differentiable functions with compact support. A topology τ is introduced on the linear space C_c^∞ which makes it into a complete space; that is a sequence $\{\phi_n\}$ in C_c^∞ converges to zero in τ if there exists a compact $A \varepsilon \mathrm{IR}^1$ which contains the support of every ϕ_n, and for every non-negative integer k, $\phi_n^{(k)}(t) \to 0$ uniformly as $n \to \infty$. C_c^∞ with the topology τ is denoted by D, and its elements are called test functions. The members of the dual D' of D are called distributions, and the value of a distribution $f \varepsilon D'$ at a test function $\phi \varepsilon D$ is denoted by $f(\phi)$. A (weak-star) topology on D' is defined by the seminorms $|f(\phi)|$, $f \varepsilon D'$, as ϕ varies over all elements of D; thus for a sequence $\{f_n\}$ in D': $f_n \to 0$ weakly whenever $f_n(\phi) \to 0$ for all $\phi \varepsilon D$.

The class S of rapidly decreasing functions consists of all infinitely differentiable functons ($\phi \varepsilon C^\infty$) for which

$$|t^m \phi^{(k)}(t)| \leq C_{m,k} \ , \ -\infty < t < \infty$$

for all non-negative integers m,k. A topology on S is defined by the seminorms

$$||\phi||_{m,k} = \sup_{0 \leq n \leq m} \ \sup_{t \varepsilon \mathrm{IR}^1} \{(1+|t|)^k |\phi^{(n)}(t)|\} \quad , \quad m,k = 0,1,2,\ldots,$$

i.e., a sequence $\{\phi_n\}_{n=1}^\infty$ is of functions in S is said to converge in S, if for every set of non-negative integers, the sequence $\{(1+|t|)^m \phi_n^{(k)}(t)\}_{n=1}^\infty$ converges uniformly on IR^1. S is complete, and the dual S' of S is called the class of tempered distributions. Similarly, a (weak-star) topology is defined on S' by the seminorms $|f(\phi)|$, $f \varepsilon S'$, as ϕ varies over all elements of S', i.e., f_n converges in S' if $f_n(\phi)$ converges for all $\phi \varepsilon S$. The space $D'(S')$ is (weak-star) sequentially complete, that is, if $\{f_n\}_n$ is a sequence in $D'(S')$ such that $\{f_n(\phi)\}_n$ is a Cauchy sequence for every $\phi \varepsilon D(S)$, then there exists a distribution

$f \in \mathcal{D}'$ ($\mathcal{S}'$) such that $f_n \to f$ in $\mathcal{D}'(\mathcal{S})$.

Finally, the space C^∞ with the topology defined by the seminorms

$$P_{m,A}(\phi) = \sum_{0 \leq n \leq m} \sup_{t \in A} |\phi^{(n)}(t)| \quad , \quad \phi \in C^\infty ,$$

where A ranges over all compact sets in $\mathbb{R}^1$ and m over all non-negative integers, is denoted by E.

The Fourier transform $F(F(\phi) = \hat{\phi}, \phi \in \mathcal{S})$ is a one-to-one biocontinuous mapping from $\mathcal{S}$ onto itself. If $f \in \mathcal{S}'$, the Fourier transform $\hat{f}$ of f is defined by $f(\phi) = f(\hat{\phi})$, $\phi \in \mathcal{S}$, and is a tempered distribution. If $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$, the convolution $f * \phi$ is defined as a function on $\mathbb{R}^1$ by

$$(f*\phi)(t) = f(\tau_t \check{\phi}) \ , \ t \in \mathbb{R}^1 ,$$

where $\check{\phi}(t) = \phi(-t)$ and the shift operator τ_t is defined by $(\tau_t \phi)(u) = \phi(u-t)$. $f*\phi \in C^\infty$ has a polynomial growth and thus determines a tempered distribution.

Suppose $f \in \mathcal{D}'$, f is said to vanish in an open set $U \subset \mathbb{R}^1$ if $f(\phi) = 0$ for every $\phi \in \mathcal{D}$ with $\text{supp}(\phi) \subset U$. Let V be the union of all open sets $U \in \mathbb{R}^1$ in which f vanishes. The complement of V is the support of f. Distributions with compact supports are tempered distributions. Now, if f is a distribution with compact support (i.e., $f \in \mathcal{S}'$), then f extends uniquely to a continuous linear functional on E. If $\psi \in \mathcal{D}$ is such that $\psi(u) = 1$ on some open set containing supp(f), then $\psi f = f$, i.e. $(\psi f)(\phi) = f(\psi\phi) = f(\phi)$ for all $\phi \in \mathcal{S}$, but since $e_t(u) = e^{2\pi i t u}$ is a C^∞-function, $f(e_t) = f(\psi e_t)$ exists, and the distribution $\hat{f}$ is generated by the function $\hat{f}(t)$ defined on $\mathbb{R}^1$ by

(2.1) $$\hat{f}(t) = f(e_t) .$$

Indeed,

(2.2) $$\hat{f} = (\psi f)^{\wedge} ,$$

and $(\psi f)^{\wedge}$ (and therefore $\hat{f}$) is generated by the C^{∞}-function $(\hat{f}*\hat{\psi})(t)$ which has a polynomial growth (see Rudin, 1973, p.179). By choosing $\phi \varepsilon S$ such that $\hat{\phi} = \psi$, we have

$$(\hat{f}*\hat{\psi})(t) = (\hat{f}*\check{\phi})(t) = \hat{f}(\tau_t\phi) = f((\tau_t\phi)^{\wedge})$$

$$= f(e_t\hat{\phi}) = f(\psi e_t) = f(e_t) ,$$

and from (2.2), (2.1) is justified. Hence the Fourier transform of a distribution with compact support may be thought of as a function defined by IR^1 by (2.1).

Let (Ω,F,P) be a probability space. A random distribution (or a generalized stochastic process) is a continuous linear operator from D (or S) into a topological vector space of random variables. Specifically, a second order random distribution is a continuous linear operator from D (or S) onto $L^2(\Omega) = L^2(\Omega,F,P)$, the Hilbert space of all finite second moment random variables. For example, let $\{X(t),t\varepsilon IR^1\}$ be a measurable second order zero-mean stochastic process with covariance function $R(t,s) = E[X(t)\overline{X}(s)]$. Assume that R is locally integrable (i.e., R is integrable over every compact subset of IR^2). The process defined by

$$X(\phi) = \int_{-\infty}^{\infty} X(t)\ \phi(t)\ dt\ ,\ \phi\varepsilon D$$

is a generalized stochastic process, i.e. X defines a continuous linear mapping from D to $L_2(\Omega)$. Let R be the covariance functional of the generalized process X defined on $D \times D$ by $R(\phi,\psi) = E[X(\phi)\overline{X}(\psi)]$. R is given by $R(\phi,\psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t,s)\, \phi(t)\, \overline{\psi}(s)\,dt\, ds.$

3. Sampling representations for bandlimited distributions. In this section a sampling theorem for tempered distributions whose Fourier transforms have compact supports is established. A distribution $f \in \mathcal{S}'$ is said to be W-bandlimited, $W > 0$, if $\mathrm{supp}(\hat{f}) \subset (-W,W)$. The class of all W-bandlimited distributions will be denoted by $B^d(W)$.

Let $\mathcal{D}[-W,W]$, $W > 0$, be the class of all C^∞-functions ϕ with $\mathrm{supp}(\phi) \subset [-W,W]$, and define $Z(W) \triangleq \hat{\mathcal{D}}[-W,W] = \{\hat{\phi} \in \mathcal{S}: \hat{\phi} \in \mathcal{D}[-W,W]\}$. Pfaffelhuber (1971) stated that if $H \in B^d(W)$ and h is its Fourier transform (defined as a function on $\mathbb{R}^1$), then

$$(3.1) \qquad h(t) = \sum_{n=-\infty}^{\infty} h\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

and the series converges absolutely in $Z'(W)$ (the dual of $Z(W)$). Equation (3.1) means precisely that, for every $\phi \in Z(W)$,

$$\int_{-\infty}^{\infty} h(t)\phi(t)dt = \sum_{n=-\infty}^{\infty} h\left(\frac{n}{2W}\right) \int_{-\infty}^{\infty} \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} \phi(t)dt,$$

and the series converges absolutely. Campbell (1968) had already noted that (3.1) does not hold pointwise for arbitrary bandlimited distributions. Though (3.1) is correct, the arguments presented in its proof are not convincing.

The following lemma is a modification of Lemma 1 of Pfaffelhuber (1971) and will be needed in the proof of theorem 3.1.

Lemma 3.1. Let $f \in \mathcal{S}'$ be such that $\hat{f}$ has compact support. Let E be a closed set properly containing $\mathrm{supp}(\hat{f})$, and ψ any test function with support E and $\psi = 1$ on some open set containing $\mathrm{supp}(\hat{f})$. Then f is uniquely determined by its restriction to $\hat{\mathcal{D}}(E)$, i.e., the values $f(\theta), \theta \in \hat{\mathcal{D}}(f)$, by

(3.2) $$f(\phi) = f(\hat{\psi}_* \phi) \ , \quad \phi \varepsilon S \ .$$

The shift operator τ_ℓ is defined on $D'(S')$, for every $\ell \varepsilon IR^1$, by

$$(\tau_\ell f)(\phi) = f(\tau_{-\ell}\phi) \ , \ \phi \varepsilon D(S) \ .$$

A distribution $f \varepsilon D'(S')$ is said to be periodic with period $T > 0$, if

(3.3) $$(\tau_T f)(\phi) = f(\phi) \ , \text{ for every } \phi \varepsilon D(S) \ ,$$

and T is the smallest positive number for which (3.3) holds.

<u>THEOREM 3.1</u>. Let $f \varepsilon S'$ be a tempered distribution such that $\hat{f}$ has compact support, and let the closed set E and $W > 0$ be such that $\text{supp}(f) \subsetneq E$ and the translates $\{E+2nW\}$, $n \neq 0$, are disjoint from $\text{supp}(\hat{f})$. Let α and ψ be any test functions such that ψ has support E, and $\alpha = 1$, $\psi = 1$ each on some open set containing $\text{supp}(\hat{f})$. Then

(3.4) $$f(\phi) = \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} K_W)(\phi) \ , \quad \phi \varepsilon S \ ,$$

where $K_W(t) = \frac{1}{2W} \int_E e^{2\pi i t u} \psi(u) du$, and $K_W(\phi) = \int_{-\infty}^{\infty} K_W(t)\phi(t)dt$, $\phi \varepsilon S$.

If $f \varepsilon B^d(W)$, then

(3.5) $$f(\phi) = \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} G_W)(\hat{\psi} * \phi) \ , \ \phi \varepsilon S \ ,$$

where $G_W(t) = \frac{\sin 2\pi Wt}{2\pi Wt}$, and $G_W(\phi) = \int_{-\infty}^{\infty} G_W(t)\phi(t)dt$, $\phi \varepsilon \mathcal{S}$.

PROOF. It will first be shown that the sequence of partial sums $S_N = \Sigma_{n=-n}^{N} \tau_{-2nW} \hat{f}$, $N \geq 1$, converges in $\mathcal{S}'$. For any $\phi \varepsilon \mathcal{S}$,

$$
\begin{aligned}
S_N(\phi) &= \sum_{n=-N}^{N} (\tau_{-2nW} \hat{f})(\phi) \\
&= \sum_{n=-N}^{N} \hat{f}(\tau_{2nW} \phi) \\
&= \hat{f}(\sum_{n=-N}^{N} \tau_{2nW} \phi) \\
&= f(\xi \sum_{n=-N}^{N} \tau_{2nW} \phi) \qquad (3.6)
\end{aligned}
$$

where $\xi \varepsilon \mathcal{D}$ is a test function such that $\xi(t) = 1$ on some open set containing $\text{supp}(\hat{f})$. It will be shown that the sequence $\phi_N(t) = \xi(t)\Sigma_{n=-N}^{N} \phi(t - 2nW)$, $N \geq 1$, converges in S. Since $\phi \varepsilon \mathcal{S}$, there exists a constant $B > 0$ such that $|\phi(t)| \leq B(1+t^2)^{-1}$ for all $t \varepsilon IR^1$, and thus

$$|\phi(t-2nW)| \leq \frac{B}{1+(t-2nW)^2} \leq \frac{2B(1+t^2)}{1+(2nW)^2} .$$

Since $\xi \varepsilon \mathcal{D}$, it follows that $\text{supp}(\xi) \subset [-C,C]$ for some $C > 0$ and $|\xi(t)| \leq A$ for some $A > 0$. It then follows that for all $t \varepsilon IR^1$ and non-negative integers m,

$$
\begin{aligned}
&(1+|t|)^m |\xi(t)| \sum_{n=-N}^{N} |\phi(t-2nW)| \qquad (3.7) \\
&\leq 2AB(1+C)^m(1+C^2) \sum_{n=-\infty}^{\infty} \frac{1}{1+(2nW)^2} < \infty ,
\end{aligned}
$$

i.e., the sequence of partial sums on the left hand side of (3.7) converges uniformly on IR^1. Hence the sequence $(1+|t|)^m \phi_N(t)$, $N \geq 1$, converges uniformly on IR^1 for every $m \geq 0$. Similarly, it can be shown that for every $m,k \geq 0$, the sequence $(1+|t|)^m \phi_N^{(k)}(t)$, $N \geq 1$, converges uniformly on IR^1, i.e. $\{\phi_N\}$, $N \geq 1$, converges in $\mathcal{S}$, and since $\mathcal{S}$ is complete, its limit ϕ belongs to $\mathcal{S}$, and $\phi_N \to \phi$ in $\mathcal{S}$. It follows from (3.6) that

$$S_N(\phi) = \hat{f}(\phi_N) \to \hat{f}(\phi) \text{ , as } N \to \infty$$

and since $\mathcal{S}'$ is (weak-star) sequentially complete, then there exists a tempered distribution $F \epsilon \mathcal{S}'$ such that $S_N \to F$ in $\mathcal{S}'$.

Therefore, $F = \lim_{N\to\infty} S_N = \Sigma_{n=-\infty}^{\infty} \tau_{-2nW} \hat{f}$ is a periodic tempered distribution with period 2W. It follows that F has the Schwartz-Fourier series (Zemanian, 1965, p. 332)

$$\text{(3.8)} \qquad F = \sum_{n=-\infty}^{\infty} \tau_{-2nW} \hat{f} = \sum_{n=-\infty}^{\infty} a_{\frac{n}{2W}} e_{\frac{n}{2W}} \text{ , in } \mathcal{S}' \text{ ,}$$

where $e_t(u) = e^{2\pi i tu}$, and

$$a_{\frac{n}{2W}} = \frac{1}{2W} F(Ue_{-\frac{n}{2W}}) \quad ,$$

where $U \epsilon \mathcal{U}_{2W}$ is a unitary function (Zemanian, 1965, p. 315), i.e. $U \epsilon \mathcal{D}$ and $\Sigma_{n=-\infty}^{\infty} U(t-2nW) = 1$ for all $t \epsilon IR^1$. From (3.8) it follows that

$$2W\, a_{\frac{n}{2W}} = \sum_{m=-\infty}^{\infty} (\tau_{-2mW}\hat{f})\,(Ue_{-\frac{n}{2W}})$$

$$= \sum_{m=-\infty}^{\infty} \hat{f}([\tau_{2mW}\, U]e_{-\frac{n}{2W}})\ .$$

Since $\hat{f}$ has a compact support and $U \varepsilon \mathcal{D}$, then there is only a finite number of non-zero terms in the last summation, and hence

$$2W\, a_{\frac{n}{2W}} = \hat{f}([\sum_{m=-\infty}^{\infty} \tau_{-2mW}\, U]e_{-\frac{n}{2W}})$$

(3.9) $$= \hat{f}(e_{-\frac{n}{2W}}) = \hat{f}(\alpha e_{-\frac{n}{2W}}) = f(\tau_{-\frac{n}{2W}}\hat{\alpha}).$$

From (3.8) and (3.9) it follows that

(3.10) $$\hat{f}(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2W}\, f(\tau_{-\frac{n}{2W}}\hat{\alpha})e_{\frac{n}{2W}}(\theta)\ ,\quad \theta \varepsilon \mathcal{D}(E)\ ,$$

where $e_{\frac{n}{2W}}(\theta) = \int_{-\infty}^{\infty} e^{\pi i \frac{n}{W} u}\, \theta(u)du = \hat{\theta}(-\frac{n}{2W})$. Thus

(3.11) $$\hat{f}(\theta) - \hat{f}(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2W}\, f(\tau_{\frac{n}{2W}}\hat{\alpha})\hat{\theta}(\frac{n}{2W})\ ,\quad \hat{\theta}\varepsilon\hat{\mathcal{D}}(E)\ ,$$

and by Lemma 3.1 it follows that for every $\phi \varepsilon \mathcal{S}$

(3.12) $$f(\phi) = f(\hat{\psi}*\phi) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}} \hat{\alpha})(\hat{\psi}*\phi)(\tfrac{n}{2W})$$

(since $\hat{\psi}*\phi = (\psi\check{\hat{\phi}})^\wedge \; \varepsilon \; \hat{\mathcal{D}}(E)$). But

$$(\hat{\psi}*\phi)(\tfrac{n}{2W}) = \int_{-\infty}^{\infty} \hat{\psi}(\tfrac{n}{2W} - t)\phi(t)dt$$

$$= 2W \int_{-\infty}^{\infty} K_W(t - \tfrac{n}{2W})\phi(t)dt$$

(3.13) $$= 2W(\tau_{\frac{n}{2W}} K_W)(\phi) \;, \quad \phi\varepsilon\mathcal{S},$$

and (3.4) follows from (3.12) and (3.13).

To prove (3.5) notice that when $\theta\varepsilon\mathcal{D}[-W,W]$,

$$e_{\frac{n}{2W}}(\theta) = \int_{-W}^{W} e^{\pi i \frac{n}{W} u}\theta(u)du$$

$$= 2W \int_{-W}^{W} \frac{\sin \pi(2Wt + n)}{\pi(2Wt + n)} \hat{\theta}(t)dt$$

$$= 2W(\tau_{-\frac{n}{2W}} G_W)(\hat{\theta}) \;.$$

It follows from (3.10) that for $\hat{\theta}\varepsilon\hat{\mathcal{D}}[-W,W]$,

$$f(\hat{\theta}) = \hat{f}(\theta) = \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} G_W)(\hat{\theta}) \;,$$

and (3.5) follows by Lemma 3.1. □

Theorem 3.1 shows that a tempered generalized function f with compact spectrum can be reconstructed via (3.4) from its values (samples) evaluated at the translates of an arbitrary, but fixed test function α which equals one on some open set containing supp($\hat{f}$). On the other hand, if we denote $\hat{f}(e_t)$ by f(t) then from (3.9) if follows that $f(\tau_{\frac{n}{2W}}\hat{\alpha}) = \hat{f}(e_{\frac{n}{2W}}) = f(\frac{n}{2W})$, and (3.4) reads

$$f(\phi) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})(\tau_{\frac{n}{2W}} K_W)(\phi) \ , \qquad \phi\varepsilon\mathcal{S}$$

so that a tempered distribution f with compact spectrum can be reconstructed using the samples of the function $f(t) = \hat{f}(e_t)$.

Now it is shown that the sampling theorem for tempered generalized functions with compact spectrum includes as special cases the sampling theorems for conventionally bandlimited functions (Example 3.1) as well as for bandlimited functions in $L^2(\mu_k)$ (Example 3.2).

<u>EXAMPLE 3.1</u>. (Conventionally bandlimited functions). Let $f\varepsilon L^2(IR^1)$ be a continuous function such that $\hat{f}$ has compact support E. Then f determines a tempered generalized function:

$$(3.14) \qquad f(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)dt \ , \ \phi\varepsilon\mathcal{S} \ ,$$

and its distributional Fourier transform (denoted also by $\hat{f}$) is defined by $f(\phi) = f(\hat{\phi})$, $\phi\varepsilon\mathcal{S}$, or equivalently by

$$\hat{f}(\phi) = \int_{-\infty}^{\infty} \hat{f}(u)\phi(u)du \ , \ \phi\varepsilon\mathcal{S} \ .$$

$\hat{f}$ (as a tempered generalized function) is supported by E. Hence (3.4) applies and if W > 0 is defined as in Theorem 3.1, we have from (3.14)

$$f(\tau_{\frac{n}{2W}}\hat{\alpha}) = \hat{f}(e_{\frac{n}{2W}}) = \int_{-W}^{W} \hat{f}(u)e^{\pi i \frac{n}{W} u} du = f(\tfrac{n}{2W}) .$$

For $\nu > 0$, define the function

$$\phi_\nu(t) = \begin{cases} C_\nu^{-1} \exp \dfrac{-1}{1-(t/\nu)^2} & \text{for } |\frac{t}{\nu}| \leq 1 \\ 0 & \text{for } |\frac{t}{\nu}| \quad 1 , \end{cases}$$

where $C_\nu = \int_{-\infty}^{\infty} \exp\{\frac{-1}{1-(t/\nu)^2}\}dt$. For each $\nu > 0$, $\phi_\nu \varepsilon D$ and $\int_{-\infty}^{\infty}\phi_\nu(t)dt = 1$ and for each continuous function g and every $t \varepsilon IR^1$ $\int_{-\infty}^{\infty} g(u)\phi_\nu(t-u)du \to g(t)$ as $\nu \downarrow 0$. From (3.4) it follows that for each $t \varepsilon IR^1$ and $\nu > 0$

(3.15) $$\int_{-\infty}^{\infty} f(u)\phi_\nu(t-u)dt = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W}) \int_{-\infty}^{\infty} K_W(u-\tfrac{n}{2W})\phi_\nu(t-u)dt.$$

Since f and K_W are uniformly continuous, we have for each fixed $t \varepsilon IR^1$ and $n \varepsilon IN$

$$\int_{-\infty}^{\infty} f(u)\phi_\nu(t-u)du \to f(t) \text{ as } \nu \downarrow 0 ,$$

$$\int_{-\infty}^{\infty} K_W(u - \tfrac{n}{2W})\phi_\nu(t-u)du \to K_W(t - \tfrac{n}{2W}) .$$

Now by Theorem 24 of Lighthill (1958, p.64), if for any sequence $\{b_n\}$ which is $O(n)$ as $n \to \infty$, $\sum_{n=-\infty}^{\infty} b_n a_{n,\nu}$ is absolutely convergent and tends to a finite limit as $\nu \to 0$, then

$$(3.16) \qquad \lim_{\nu\to 0} \sum_{n=-\infty}^{\infty} a_{n,\nu} = \sum_{n=-\infty}^{\infty} \lim_{\nu\to 0} a_{n,\nu} .$$

But, for each fixed $t\varepsilon IR^1$,

$$\sum_{n=-\infty}^{\infty} \left| b_n f(\tfrac{n}{2W}) \int_{-\infty}^{\infty} K_W(u - \tfrac{n}{2W})\phi_\nu(t-u)du \right|$$

$$\leq 2^k C_k B \left\{ \sum_{n=-\infty}^{\infty} \frac{|n|}{(1+ (\frac{n}{2W})^2)^k} \right\} \int_{-\infty}^{\infty} (1+u^2)^k \phi_\nu(t-u)du$$

$$\underset{\nu\to 0^+}{\longrightarrow} \; 2^k C_k B(1+t^2)^k \left\{ \sum_{n=-\infty}^{\infty} \frac{|n|}{(1+ (\frac{n}{2W})^2)^k} \right\} < \infty \quad ,$$

since f is bounded, $|b_n| \leq B|n|$, and for $k > 1$,

$$|K_W(u - \tfrac{n}{2W}) \leq \frac{C_k}{(1+(u - \frac{n}{2W})^2)^k} \leq 2^k C_k \frac{(+u^2)^k}{(1+(\frac{n}{2W})^2)^k} \quad .$$

It follows that the right hand side of (3.15) satisfies the conditions leading to (3.16), and hence by letting $\nu \downarrow 0$, we obtain

$$(3.17) \qquad f(t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})K_W(t - \tfrac{n}{2W}) \; , \quad t\varepsilon IR^1 ,$$

which is the sampling theorem for a conventionally bandlimited function with compact spectrum.

Example 3.2. (Bandlimited functions in $L^2(\mu_k)$). Let $f \varepsilon L^2(\mu_k)$, $k \geq 0$, be a continuous function. Then f determines a tempered distribution by (3.14). If its distributional Fourier transform $\hat{f}$ has a compact support, then (3.4) applies and we have

$$f(\tau_{\frac{n}{2W}} \hat{\alpha}) = \hat{f}(e_{\frac{n}{2W}}) = f(\frac{n}{2W})$$

Since f is a C^∞-function and $|f(t)| \leq C_k(1+|t|^k$,
for $C_k > 0$ (Lee, 1977), then (3.15) holds and following the arguments used in Example 3.1, one obtains (3.17) which is similar to (2.3) and is identical to (2.4). It should be noted, though, that (3.4) cannot be obtained from Campbell's result (1.4), since the convergence in (2.4) is not uniform on compact sets.

4. Series expansions for random distributions. In this section sampling expansions for stationary random distributions are derived. Let $X = \{X(\phi), \phi \in S\}$ be a second order random distribution. X is said to be weakly stationary, if for every h > 0 and $\phi, \psi \in S$,

$$E(\tau_h X(\phi) \cdot \tau_h X(\psi)) = E(X(\phi) \cdot X(\psi)) \quad .$$

If X is a weakly stationary random distribution (WSRD), then there exists a unique tempered distribution $\rho \in S'$ such that for every $\phi, \psi \in S$,

(4.1) $$R(\phi,\psi) = E(X(\phi) \cdot X(\psi)) = \rho(\phi * \check{\psi}) \quad ,$$

where $\check{\psi}(t) = \psi(-t)$ (Itô, 1954) and ρ has the spectral representation

(4.2) $$\rho(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(u) d\mu(u) \ , \ \phi \in S \ ,$$

where μ is a non-negative measure on IR^1 such that $\int_{-\infty}^{\infty} \frac{d\mu(u)}{(1+u^2)^k} < \infty$

for some integer k. In this case X is said to be of type k, and μ is called the spectral measure of X.

Let B^* be the set of all Borel sets with finite μ-measure. An $L^2(\Omega)$-valued function Z defined on B^* is called a random measure with respect to μ if

$$E(Z(B_1) \cdot Z(B_2)) = \mu(B_1 \ B_2) \ , \ B_1, B_2 \in B^* \quad .$$

Hence $E(Z^2(b)) = \mu(B)$, and $Z(B_1) \perp Z(B_2)$ if B_1 and B_2 are disjoint. Since μ is σ-additive, then $Z(B) = \Sigma_{n=1}^{\infty} Z(B_n)$, whenever $B_1, B_2, \ldots$ are disjoint sets in $\mathcal{B}^*$ with $\cup_{n=1}^{\infty} B_n = B$. It follows by (4.1) and (4.2) that there exists a random measure Z with respect to μ such that

$$X(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(u)dZ(u) , \ \phi\varepsilon\mathcal{S} .$$

If H(X) is the linear subspace of $L^2(\Omega)$ generated by $\{X(\phi), \phi\varepsilon\mathcal{S}\}$, then H(X) and $L^2(\mu)$ are isometrically isomorphic under the correspondence $X(\phi) \longmapsto \hat{\phi}$, $\phi\varepsilon\mathcal{S}$. A WSRD X is said to be W_0-bandlimited, $W_0 > 0$, if $\mu\{[-W_0, W_0]^c\} = 0$.

THEOREM 4.1. (a) If $X = \{X(\phi), \ \phi\varepsilon\mathcal{S}\}$ is a W_0-bandlimited WSRD, $W > W_0$, $\alpha\varepsilon\mathcal{D}$ and $\psi\varepsilon\mathcal{D}[-W,W]$ with $\alpha(t) = 1 = \psi(t)$ on $[-W_0, W_0]$, then for every $\phi\varepsilon\mathcal{S}'$,

$$(4.3) \qquad X(\phi) = \sum_{n=-\infty}^{\infty} X(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} G_W)(\hat{\psi}*\phi)$$

in a mean-square, where $G_W(\phi) = \int_{-\infty}^{\infty} \frac{\sin\pi 2 \ Wt}{2\pi Wt} \ \phi(t)dt$.

(b) Let $X = \{X(\phi), \ \phi\varepsilon\mathcal{S}\}$ be a WSRD with spectral measure μ which has compact support. Let the closed set E and $W > 0$ be such that $\text{supp}(\mu) \subsetneq E$ and the translates $\{E+2nW\}$, $n \neq 0$, are disjoint from $\text{supp}(\mu)$. Let α and ψ be any test functions such that ψ has support E, and $\alpha(t) = 1 = \psi(t)$ on $\text{supp}(\mu)$. Then

$$(4.4) \qquad X(\phi) = \sum_{n=-\infty}^{\infty} X(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} K_W)(\phi) , \quad \phi\varepsilon\mathcal{S} ,$$

in mean-square, where $K_W(t) = \frac{1}{2W} \int_E \psi(u)e^{2\pi i tu}du$.

Proof. To prove (a), first let $\phi \epsilon S$ be such that $\hat{\phi}\epsilon S[-W,W]$. Then $\Phi(u) = \Sigma_{n=-\infty}^{\infty}\hat{\phi}(u+2nW)$ is a C^{∞}-function which is periodic with period 2W and has the Fourier series

$$\Phi(u) = \sum_{n=-\infty}^{\infty} \frac{1}{2W}\phi\left(\frac{n}{2W}\right)e^{\pi i \frac{n}{W} u}, \quad u \epsilon IR^1, \tag{4.5}$$

which converges uniformly on IR^1. Since $\hat{\phi}\epsilon D[-W,W]$,

$$(\tau_{\frac{n}{2W}} G_W)(\phi) = \int_{-\infty}^{\infty} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \phi(t)dt$$

$$= \frac{1}{2W}\int_{-W}^{W} e^{\pi i \frac{n}{W} u} \phi(u)du$$

$$= \frac{1}{2W}\phi\left(\frac{n}{2W}\right),$$

Consider the mean square error

$$e_N^2(\phi) = E\left|X(\phi) - \sum_{n=-N}^{N} X(\tau_{\frac{n}{2W}}\hat{\alpha})(\tau_{\frac{n}{2W}}G_W)(\phi)\right|^2$$

$$= \int_{-W_0}^{W_0} \left|\hat{\phi}(u) - \sum_{n=-N}^{N}\frac{1}{2W}\phi\left(\frac{n}{2W}\right)e^{\pi i \frac{n}{W} u}\right|^2 d\mu(u).$$

There exists a constant $M > 0$ such that for all N and $u \epsilon IR^1$,

$$|\hat{\phi}(u) - \sum_{n=-N}^{N} \frac{1}{2W} \phi(\frac{n}{2W}) e^{\pi i \frac{n}{W} u}| \leq M .$$

Since, by (4.5), $\Sigma_{n=-N}^{N} \frac{1}{2W} \phi(\frac{n}{2W})e^{\pi i \frac{n}{W} u}$ converges to $\hat{\phi}(u)$ on $[-W_0,W_0]$, by the dominated convergence theorem, $e_N^2(\phi) \to 0$ as $N \to \infty$. Thus for every $\phi \varepsilon \hat{\mathcal{D}}[-W,W]$, we have

$$X(\phi) = \sum_{n=-\infty}^{\infty} X(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} G_W)(\phi) . \tag{4.6}$$

Now for every $\phi \varepsilon \mathcal{S}$ and ψ as in part (a) of the statement of the theorem, it follows

$$X(\phi) = \int_{-W_0}^{W_0} \hat{\phi}(u)dZ(u) = \int_{-W_0}^{W_0} \psi(u)\hat{\phi}(u)dZ(u)$$

$$= \int_{-W_0}^{W_0} (\hat{\psi}*\phi)^{\wedge}(u)dZ(u) = X(\hat{\psi}*\phi) ,$$

where $\hat{\psi}*\phi = (\psi\hat{\phi})^{\vee\wedge} \phi\hat{\mathcal{D}}$ $[-W,W]$, and (4.3) follows from (4.6) and (4.7). The proof of part (b) is similar to that of (a) with the obvious modification and hence is omitted. □

It should be noted that, since $\alpha = 1$ on $[-W_0,W_0]$,

$$X(\tau_{\frac{n}{2W}} \hat{\alpha}) = \int_{-W_0}^{W_0} e^{-\pi i \frac{n}{W} u} dZ(u) , n \varepsilon \mathrm{IN} .$$

Define

$$x(\tau) = \int_{-W_0}^{W_0} e^{-2\pi i tu} dz(u) , t\varepsilon IR^1 ,$$

then $\{x(t), t\varepsilon IR^1\}$ is a weakly stationary W_0-bandlimited stochastic process, $X(\tau_{\frac{n}{2W}} \hat{\alpha}) = x(\frac{n}{2W})$, and (4.3) reads

$$(4.8) \qquad x(\phi) = \sum_{n=-\infty}^{\infty} x(\frac{n}{2W})(\tau_{\frac{n}{2W}} G_W)(\hat{\psi}*\phi) , \phi\varepsilon S ,$$

i.e., the random distribution X is reconstructed using the samples of the ordinary stochastic process x. Hence there is a one-to-one correspondence between W_0-bandlimited weakly stationary random distributions X and W_0-bandlimited weakly stationary processes x determined by $X(\phi) = \int_{-W_0}^{W_0} \hat{\phi}(u)dZ(u)$ and $x(t) = \int_{-W_0}^{W_0} e^{2\pi i tu} dZ(u)$ and satisfying (4.8).

Now it is shown that the sampling theorem for bandlimited weakly stationary random distributions includes as a particular case the sampling theorem for bandlimited weakly stationary processes.

Example 4.1. Let $x = \{x(t), t\varepsilon IR^1\}$ be a measurable, mean-square continuous, weakly stationary process which is W_0-bandlimited, i.e.,

$$(4.9) \qquad x(t) = \int_{-W_0}^{W_0} e^{-2\pi i tu} dZ(u) ,$$

where Z is a random measure with respect to the spectral measure μ of x with $\mu\{[-W_0,W_0]^c\} = 0$. Then x determines a W_0-bandlimited WSRD by

$$X(\phi) = \int_{-W_0}^{W_0} \hat{\phi}(u)dZ(u) \ , \quad \phi\varepsilon\mathcal{S} \ ,$$

which can also be written as

$$x(\phi) = \int_{-W_0}^{W_0} \left(\int_{-\infty}^{\infty} e^{-2\pi itu}\phi(t)dt\right)dZ(u)$$

$$= \int_{-\infty}^{\infty} x(t)\hat{\phi}(t)dt \ ,$$

where the latter integral exists both with probability one as well as in quadratic mean. Then by (4.4) if follows that for each $t\varepsilon IR^1$ and $\nu > 0$,

$$\int_{-\infty}^{\infty} x(u)\phi_\nu(t-u)du = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W}) \int_{-\infty}^{\infty} K_W(u - \tfrac{n}{2W})\phi_\nu(t-u)du \tag{4.10}$$

in quadratic mean. As in example 3.1,

$$\int_{-\infty}^{\infty} x(u)\phi_\nu(t-u)du \to x(t) \quad \text{as } x\downarrow 0$$

in quadratic mean, $\int_{-\infty}^{\infty} K_W(u - \frac{n}{2W})\phi_\nu(t-u)du \to K_W(t - \frac{n}{2W})$ as $\nu\downarrow 0$, and the right hand side of (4.10) converges in quadratic mean to $\Sigma_{n=-\infty}^{\infty} x(\frac{n}{2W})K_W(t - \frac{n}{2W})$. It follows that

$$x(t) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W})K_W(t - \tfrac{n}{2W}) \ , \quad t\varepsilon IR^1$$

in quadratic mean.

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