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The evolution of rigid-plastic dynamic response by mathematical programming

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ABSTRACT: The paper presents a systematic approach to the automatic determination of the evolution of the rigid-plastic structural response caused by a proportional or single parameter pressure loading which varies with time in a monotonic nondecreasing fashion. The framed structure is envisaged as a network, and the fundamental vectorial conditions characterising its behaviour at any instant of time are combined in a consistent manner. By considering the level of the pressure loading to be a single load parameter ranging monotonically from zero to infinity, the structural governing system may be regarded as a parametric linear complementarity problem (PLCP).

1 INTRODUCTION

In a nuclear power plant, the rupture of a pipeline containing high energy fluid results in its high speed motion. To meet design requirements for plant safety, the study of this phenomenon, known as pipe whip, is important as the whipping pipe may cause damage to other pipes or equipment in the vicinity.

Since the plastic deformations in a pipe whip process are, in general, much larger than the elastic deformations, a rigid, perfectly plastic beam model for the pipe is widely used. The results of many investigations based upon this model have been published which clarify various aspects of the pipe whip behaviour.

Yu, Symonds and Johnson (1986) have considered the problem of a circular quadrant cantilever subjected to a radial impulsive load at its free end, the problem being a natural extension of that studied by Parkes (1955). Bent and semi-circular cantilevers struck normal to their plane by a step load applied at the tip have been studied (Hua, Yu and Johnson 1985; Yu, Hua and Johnson 1985). The dynamic behaviour of bent cantilevers loaded transversely to their plane by a step load at the free end has been also examined (Hua, Yu and Reid 1988; Reid, Hua and Yang 1990).

The main concern in all these works has been the heuristic determination of the shape of the velocity field with which motion is initiated. In order to overcome this difficulty, Sahlit and Lloyd Smith (1993) have proposed a mathematical programming formulation which is able to furnish automatically the initial deformation mechanism of a structural system subjected to a pulse loading, for any level of that loading.

In the present work, mathematical programming concepts are once again applied to study another possible aspect of the pipe whip behaviour. The problem consists now in calculating the dynamic response of a rigid-plastic framed structure (a pipe in the context of pipe whip) induced by a proportional pressure loading (blowdown force). The load-time relationship of the blowdown force is considered as monotone nondecreasing and with a null initial value.

2 BASIC STRUCTURAL RELATIONS

A framed structure can be envisaged as a network of assembled elements which interconnect at points called nodes. For an elementary dynamic modelling, it is assumed that the system is formed by discrete masses located at some or all of these nodes. The gravity centres of the masses can be assigned a number γ of independent displacements or dynamic degrees of freedom, collected for the complete set of masses into the vector $\boldsymbol{u}$ of inertia coordinates. Associated with them are the γ inertia forces

$$\boldsymbol{\mu} = -\boldsymbol{m}\ddot{\boldsymbol{u}}, \tag{1}$$

where $\boldsymbol{m}$ is a diagonal matrix of masses and mass moments of inertia and $\ddot{\boldsymbol{u}}$ denotes the second derivative of $\boldsymbol{u}$ with respect to time t.

The laws of kinetics and kinematics for the problem may be represented on a nodal basis as in (2) and (3), respectively,

$$\boldsymbol{Q} = \boldsymbol{0} = \begin{bmatrix} \boldsymbol{A}^T & \boldsymbol{A}_d^T & \boldsymbol{A}_o^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ -\boldsymbol{\mu} \\ -\boldsymbol{\lambda} \end{bmatrix}, \qquad \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\delta}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{A}_d \\ \boldsymbol{A}_o \end{bmatrix} \dot{\boldsymbol{q}}\,. \tag{2, 3}$$

under the restriction of small displacements.

The law of kinetics (2) relates the s independent member forces $\boldsymbol{X}(t)$ to the current applied loads $\lambda(t)$ and the current values of the inertia forces $\boldsymbol{\mu}(t)$ through β nodal constraint forces $\boldsymbol{Q}(t)$; for dynamic equilibrium, it is necessary that $\boldsymbol{Q}(t) = \boldsymbol{0}$. In the relations of compatibility (3), $\dot{\boldsymbol{\delta}}(t)$ are the load-point velocities which are dual to the applied loads $\lambda(t)$, the components of $\dot{\boldsymbol{x}}(t)$ are the s independent member deformation rates and $\dot{\boldsymbol{q}}(t)$ are the β velocities corresponding to the Lagrange or generalised coordinates. It may be noticed that, with the inclusion of the inertia forces $\boldsymbol{\mu}$, the nodal description sustains the duality (or contragredience) evidenced by the corresponding description (Lloyd Smith, 1974) for quasi-static structures.

For the proportional loading class considered herein, the dynamic pressure may be written in the form $\lambda(t) = \underline{\lambda}\lambda(t)$, where $\underline{\lambda}$ is a vector which defines the proportions among the applied loads and $\lambda(t)$ is the function which describes the time variation of the pressure. Hence, the matrix product $A_o^T\underline{\lambda}$ which appears in the kinetic relation (2) may be substituted by $\boldsymbol{a}_o\lambda(t)$, where $\boldsymbol{a}_o$ is a β-vector.

The structural material is assumed to be rigid, perfectly plastic and strain rate insensitive. Plastic deformations are considered to be concentrated at critical sections which are located at the termini of the structural members and a piecewise linear approximation of the yield surface is contemplated. The constitutive laws may then be written (Maier 1973) in the form of equations (4) where matrix $\boldsymbol{N}$ collects, as columns, the components of the unit vectors normal to the piecewise linearised yield surfaces.

$$\begin{bmatrix} 0 & N^T \\ N & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_* \\ S \end{bmatrix} + \begin{bmatrix} y_* \\ 0 \end{bmatrix} = \begin{bmatrix} X_* \\ \dot{s} \end{bmatrix} \qquad \begin{matrix} \text{(a)} \\ \text{(b)} \end{matrix} \qquad (4)$$

$$y_* \geq 0 \qquad y_*^T \dot{x}_* = 0 \qquad \dot{x}_* \geq 0 \qquad \text{(c–e)}$$

In (4), S and $\dot{s}$ contain respectively the stress resultants and the dual plastic strain resultant rates, which control plasticity at the critical sections. The plastic capacities are collected in array X_* and the vector of plastic multiplier rates $\dot{x}_*$ lists the contribution to $\dot{s}$ of each yield mode associated with the plastic potentials y_*.

The constitutive relations (4) must be modified so that the generalised stresses S and generalised strain rates $\dot{s}$, measured at the critical sections, are substituted by the independent member forces X and independent deformation rates $\dot{x}$, defined at the element extremities. For this purpose, transformations (5) and (6) may be used:

$$\dot{x} = T\dot{s}, \qquad S = T^T X. \qquad (5, 6)$$

3 THE GOVERNING SYSTEM

Attention is focused herein on the study of the dynamic behaviour of a rigid-plastic structural excited by a proportional pressure loading which varies with time in a monotonic nondecreasing fashion and has a null initial value. For this class of dynamic problems, the assumption that the plastic multiplier rates induced throughout the structure during the entire motion are also monotone nondecreasing is advanced. Consequently, the vector of plastic multiplier accelerations $\ddot{x}_*$ satisfies the following complementarity conditions (Tamuzh 1962):

$$y_* \geq 0, \qquad y_*^T \ddot{x}_* = 0, \qquad \ddot{x}_* \geq 0. \qquad (7a, b, c)$$

with respect to the vector of plastic potentials y_*. It must be said that the veracity of this conjecture, which is suggested by physical intuition, has been independently confirmed by complete time-stepping rigid-plastic analyses for general pulse loading, carried out in the manner described in LLoyd Smith and Sahlit (1990, 1991).

By combining the kinetic, kinematic and constitutive laws (1-7), and letting $M_q = A_d^T m A_d$ denote the mass matrix referred to the accelerations $\ddot{q}$ of the generalised or Lagrange coordinates, the following nodal governing system is obtained:

$$\begin{bmatrix} -M_q & -A^T & 0 & 0 \\ -A & 0 & TN & 0 \\ 0 & N^T T^T & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{q} \\ X \\ \ddot{x}_* \\ y_* \end{bmatrix} = \begin{bmatrix} -a_o \lambda(t) \\ 0 \\ X_* \end{bmatrix} \qquad \begin{matrix} \text{(a)} \\ \text{(b)} \\ \text{(c)} \end{matrix} \qquad (8)$$

$$y_* \geq 0 \qquad y_*^T \ddot{x}_* = 0 \qquad \ddot{x}_* \geq 0 \qquad \text{(d–f)}$$

which gives an *exact* representation of the structural behaviour during the entire motion.

Since $\lambda(t)$ in system (8) is assumed to vary from zero in a monotonic nondecreasing fashion, this pressure function may be substituted by a single load parameter Λ, ranging monotonically from zero to infinity. Then the nodal formulation of the structural system, valid for any instant of time, may be expressed as

$$\begin{bmatrix} -M_q & -A^T & 0 & 0 & a_o \\ -A & 0 & TN & 0 & 0 \\ 0 & N^T T^T & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ X \\ \ddot{x}_* \\ y_* \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ X_* \end{bmatrix} \qquad \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \qquad (9)$$

$$y_* \geq 0 \qquad y_*^T \ddot{x}_* = 0 \qquad \ddot{x}_* \geq 0 \qquad (d\text{–}f)$$

The governing system (9) is a parametric linear complementarity problem (PLCP). The solution of this PLCP, which can be accomplished by a variant of Wolfe's algorithm in its long form (Sahlit 1992), furnishes the development of the accelerations and generalised stresses throughout the motion directly in terms of the load parameter. It also indicates the critical load levels at which the pattern of the velocity field changes from one form to another. Due to the inherent linearity of the PLCP solution, the variations of the accelerations and generalised stresses with the load factor are linear within each interval of the load parameter, no matter the force-time relationship of the loading pulse. If it is further assumed that the load parameter varies linearly between its critical values, then, by integration, the fields of velocities and displacements may also be determined, PLCP (9) yielding complete information about the structural dynamic response.

4 ILLUSTRATIVE EXAMPLE

A uniform cantilever is represented in discrete form by four point masses, as shown in Figure 1(a), where the critical sections are consecutively numbered. Motion is excited from stationary conditions by a pulse load $\lambda(t)$ which increases monotonically with time and is applied transversely at the tip of the cantilever. It is assumed that the beam is rigid-plastic, having plastic moment of resistance X_*.

The solution of PLCP (9) divides the evolution of the dynamic response into a sequence of contiguous intervals of the load parameter Λ for which the corresponding shapes of the associated velocity fields are those portrayed in Figure 1(b). It may be noticed that the activation of a plastic hinge, say that at critical section i, once commenced, does not cease. This notion is further suggested in the activation diagram, Figure 1(c), which illustrates the spreading of plasticity throughout the structure as the loading evolves and also indicates the intervals of the load parameter Λ given by the PLCP formulation.

For this example, the bending moment distribution for each of the transitional values of Λ is presented in Figure 2(a). As the load parameter is raised, it may be seen that a successively larger region of the cantilever will sustain a fully plastic bending moment. It is clear that, if the load parameter Λ is a linear function of time, then Figure (2a) will also represent the evolution of the response with time. The evolution of the transverse displacements of the lumped masses is pictured in Figure 2(b), which has been obtained under the further assumption that $4\Lambda L/X_* = t$.

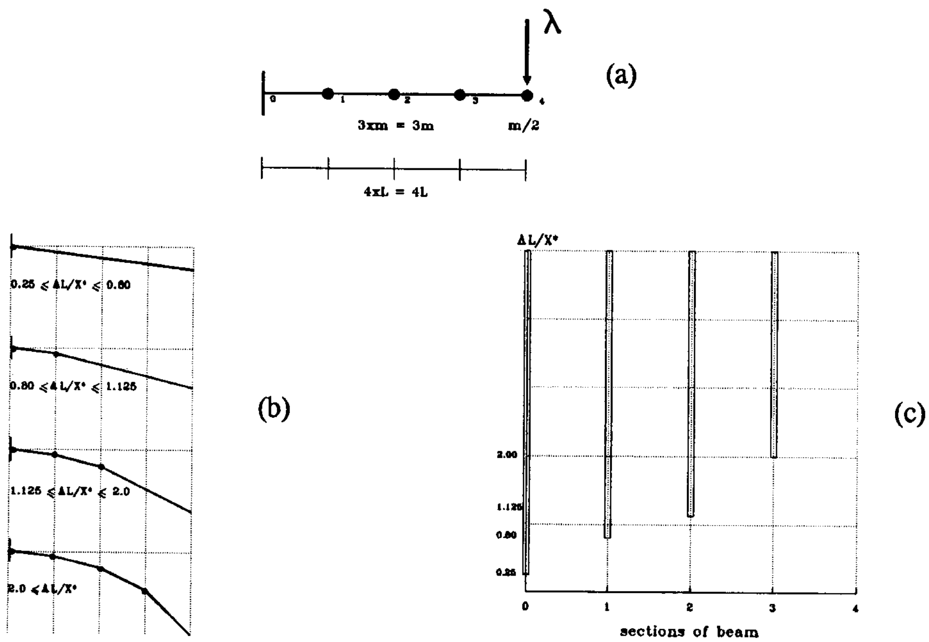

Figure 1. Geometrical and loading scheme (a); Sequence of velocity profiles (b); Active negative critical sections v. load parameter (c).

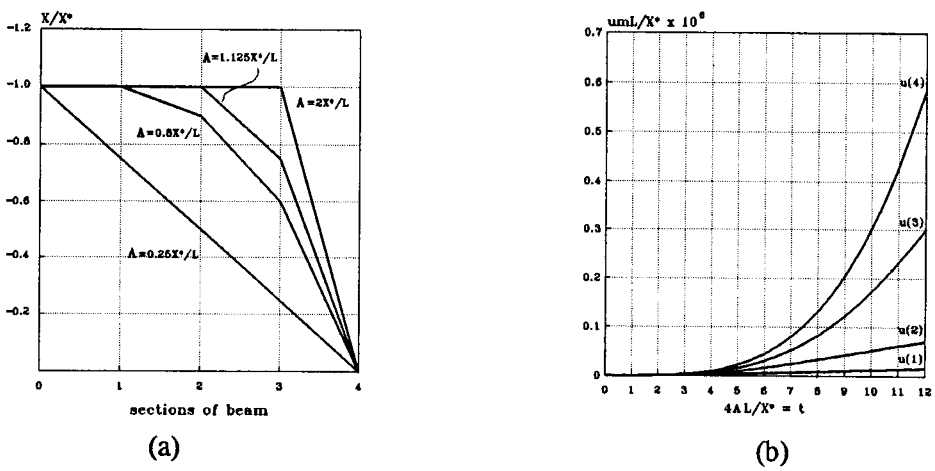

Figure 2. Bending moment distributions (a); Evolution of nodal displacements (b).

The validity of the present PLCP approach depends on the conjecture (7c) that $\dot{\boldsymbol{x}}_* \geq \boldsymbol{0}$. This condition, as well as the results just presented, were independently confirmed (Sahlit 1992) by carrying out a full time-stepping, rigid-plastic dynamic analysis.

5 CLOSURE

A systematic and consistent procedure has been given for formulating and solving the problem of a rigid-plastic framed structure (a pipe in the context of pipe whip), excited by a proportional pressure loading (the blowdown force) which varies from a null initial value in a monotonic nondecreasing manner. It is shown that the mathematical system governing the behaviour of such a structure is a PLCP, the solution of which yields the evolution of the dynamic response.

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