MATRIX OF TRANSMISSION IN STRUCTURAL DYNAMICS

S. MUKHERJEE
Dept. TGK-SS, Brown, Boveri & Cie, Ltd., CH-5401 Baden, Switzerland

SUMMARY

The problem of close-coupled systems and cantilever type buildings can be treated efficiently by means of the very general and versatile method of transmission matrix.

The expression "matrix of transmission" is used to point out the fact that the method to be described differs fundamentally from another method related to matrix calculus, and also successfully used in vibration problem. In this method, forces and displacements are introduced as the "unknowns" of the problem. The "matrix of transmission" relates these quantities at one point of the structure to those at the neighbouring point. For example, consider a system analysed as a series of segments and such that specification of certain generalized forces and displacements at any division point, say the $(i-1)$th, between segments allows calculation of the corresponding quantities at i-th point. If these quantities are arranged into a column vector $\{z\}_{i-1}$ for $(i-1)$-th point, linear behaviour of the system ensures that $\{z\}_i$ can be obtained by premultiplying $\{z\}_{i-1}$ by a square transmission matrix and adding the effects of external forces acting between points $(i-1)$ and (i).
The quantities mentioned above are: W, φ, M, Q

where: W = Displacement, M = Moment
φ = Rotation, Q = Shear force

Thus

$$\{Z\}_i = [T]\{Z\}_{i-1}.$$

From an assumed $\{z\}$ that satisfies the boundary condition at point 0, we can obtain the entire configuration of the system by applying equation repeatedly.

The natural frequencies of a freely vibrating elastic system can be found by applying proper end conditions. The end conditions will yield the frequency determinate to zero. By using suitable numerical method, the natural frequencies and mode shapes are determined, by making a frequency sweep within the range of interest.

Results of analysis of a typical nuclear building by this method show very close agreement with the results obtained by using ASKA and SAP IV Program.

Introduction

The expression "matrix of transmission" is used to point out the fact that the method to be described differs fundamentally from another method related to matrix calculus, and also successfully used in vibration problem. In this method, forces and displacements are introduced as the "unknows" of the problem. The"matrix of transmission" relates these quantities at one point of the structure to those at the neighbouring point. For example, consider a system analysed as a series of segments and such that specification of certain generalised forces and displacements at any division point, say the $(i-1)^{th}$, between segments allows calculation of the corresponding quantities at i^{th} point. If these quantities are arranged in a colum vector $\{z\}_{i+1}$ for $(i+1)^{th}$ point, linear behaviour of the system ensures that $\{z\}_i$ can be obtained by premultiplying $\{z\}_{i+1}$ by a square transmission matrix and adding the effects of external forces acting between points(i+1) and (i).

The quantities mentioned above are: w, ϕ, M, Q

w = displacement M = moment

ϕ = rotation Q = shearforce

Thus: $\{z\}_i = [T]_i \{z\}_{i-1} + \{z\}_{s,i}$ (1)

Where: $\{z\}_{s,i}$ denotes the effects of external forces.

From an assumend $\{z\}_i$ that satisfies the boundary condition at point i, we can obtain the entire configuration of the system applying equation repeatedly. The above statement may be made clearer by turning our attention to a simple specific case:

Taking the quantities w, ϕ, M, Q at point i in Figure 1. and the corresponding quantities at the point i+1 of a beam, the matrix depending upon the governing differential equation of the field (i, i+1) (Fig.1) with materices written for each field we can by successive multiplication go from w_i ....Q_i to the values of w_{i+1}....Q_{i+1} at any point of the beam. If the conditions at point i are known (x = o), then the quantities at $_{i+1}$ be determined from the conditions at x = 1 and due to the matrix pattern it is convenient to formulate these conditions. Thus, the "matrices of transmission" provide a definite method for the determination of the quantities defining the behaviour of a beam or any structure.

Analytical Method

For a straight beam, the displacement at the point i are w_i and the slope ϕ_i. The internal forces are shaerforce Q_i corresponding to the displacement w_i and moment M_i corresponding to slope ϕ_i, According to figure 1.

The state vector at end $i = \{z\}_i$

and $\{z\}_i = \begin{Vmatrix} w \\ \phi \\ M \\ V \end{Vmatrix}_i$ (2)

Applying the equilibrium equations refering to figure 1.

and noting that

$$Q_i^R = Q_{i+1}^L \quad (3)$$

$$M_{i+1}^L = M_i^L - Q_{i+1}^L \; l_{i+1} \quad (4)$$

in matrix notation, the equilibrium equation can be expressed in the following form

$$\begin{Vmatrix} -w \\ \phi \\ M \\ Q \end{Vmatrix}_{i+1}^L = \begin{bmatrix} 1 & l & \frac{l^2}{2EJ} & \frac{l^3}{6EJ} \\ 0 & 1 & \frac{l}{EJ} & \frac{l^2}{2EJ} \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Vmatrix} -w \\ \phi \\ M \\ Q \end{Vmatrix}_i^R \quad (5)$$

Here w_{i+1}^L $_{i+1}^L$ M_{i+1}^L Q_{i+1}^L refer to the displacement, and forces for the left of point i+1, and w_i^R, ϕ_i^R, M_i^R, Q_i^R are displacements and forces to the right of point i.

The above matrix formulation can be written as follows:

$$\{z\}_{i+1}^L = [F]_{i+1} \{z\}_i^R \quad (6)$$

Where F_{i+1} is the field transfer matrix of massless flexular beam.

If a mass m_{i+1} is attached to one of the vibrating beam, the vibrating mass will introduce a discontinuity in the shear. Then form the simple equilibrium consideration, the following relation is derived:

$$Q_{i+}^R = Q_{i+1}^L - m_{i+1}\omega^2 \; w_{i+1} \quad (7)$$

In addition to above, if the rotary inertia in required to be Considered, the point matrix can be written in the following form:

$$\begin{Vmatrix} -w \\ \phi \\ M \\ Q \end{Vmatrix}_{i+1}^L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -J_p\omega^2 & 1 & 0 \\ m_{i+1}\omega^2 & 0 & 0 & 1 \end{bmatrix}_{i+1} \cdot \begin{Vmatrix} -w \\ \phi \\ M \\ Q \end{Vmatrix}_{i+1}^R \quad (8)$$

Or $$\{z\}_{i+1}^R = [P]_{i+1} \{z\}_{i+1}^L \quad (9)$$

Substituting for $\{z\}^L_{i+1}$ from equation (6) we get

$$\{z\}^R_{i+1} = [P]_{i+1} \, [F]_{i+1} \, z^R_i \qquad (10)$$

Or

$$\{z\}^R_{i+1} = [T]_{i+1} \, z^R_i \qquad (11)$$

Where T_{i+1} is the final matrix of transmission connecting the two state vectors $\{z\}^R_{i+1}$ and $\{z\}^R_i$

2. Field transfer matrix for flexural vibration of beam as continuum including the effect of shear deflection and rotary inertia.

The Governing equation for the mentioned above is a fourth order differential equation in W:

$$\frac{d^4 w}{dx^2} + \frac{\mu\omega^2}{EJ_y}\left(\frac{EJ_y}{GA_s} + i_y^2\right)\frac{d^2 w}{dx^2} - \frac{\mu\omega^2}{EJ_y}\left(1 - \frac{\mu i_y^2 \omega^2}{G A_s}\right) w = 0 \qquad (12)$$

With the substitution

$$\sigma = \frac{\mu\omega^2 l^2}{GA_s} \qquad \tau = \frac{\mu i_y^2 \omega^2 l^2}{EJ_y}$$

$$\beta^4 = \frac{\mu\omega^2 l^4}{EJ_y}$$

Equation (12) becomes

$$\frac{d^4\omega}{dx^4} + \frac{\sigma+\tau}{l^2}\frac{d^2\omega}{dx^2} - \frac{\beta^4 - \sigma\tau}{l^4} w = 0 \qquad (13)$$

It is from this equation that the field transfer matrix is determined. The solution $W = \bar{c}\, e^{\lambda_1 x/l}$ is now substituted in the above equation, which leads to the characteristic equation in λ

$$\lambda^4 + (\sigma+\tau)\lambda^2 - (\beta^4 - \sigma\tau) = 0 \qquad (14)$$

$$\lambda_{1,2} = \sqrt{\sqrt{\beta^4 + \frac{1}{4}(\sigma-\tau)^2} \mp \frac{1}{2}(\sigma+\tau)} \qquad (15)$$

The field transfer matrix now finally has the following form: $[F] =$

$$\begin{bmatrix} \sigma_o - \sigma C_2 & l\,[C_1 - (\sigma+\tau) C_3] & a\, C_2 & \frac{al}{\beta^4}\left[-(\sigma C_1 + (\beta^4 + \sigma^2) C_3)\right] \\ \frac{\beta^4 C_3}{l} & C_o - \tau C_2 & \frac{a}{l}[C_1 - \tau C_3] & a\, C_2 \\ \frac{\beta^4 C_2}{a} & \frac{l}{a}\left[-\tau C_1 + (\beta^4 + \tau^2) C_3\right] & C_o - \tau C_2 & l\,[C_1 - (\sigma+\tau) C_3] \\ \frac{\beta^4}{al}(C_1 - \sigma C_3) & \frac{\beta^4 C_2}{a} & \frac{\beta^4 C_3}{l} & C_o - \sigma C_2 \end{bmatrix} \qquad (16)$$

Where: $C_o = \Lambda \left[\lambda_2^2 \operatorname{Cosh} \lambda_1 + \lambda_1^2 \operatorname{Cos} \lambda_2 \right]$

$$C_1 = \Lambda \left[\frac{\lambda_2^2}{\lambda_1} \sinh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sin \lambda_2 \right]$$

$$C_2 = \Lambda \left[\operatorname{Cosh} \lambda_1 - \operatorname{Cos} \lambda_2 \right]$$

$$C_3 = \Lambda \left[\frac{\sin \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2} \right]$$

And:

$$a = \frac{l^2}{EJ_y} \qquad \Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

3. Determination of Eigenfrequencies and Mode Shapes

If $\{z\}_i$ and $\{z\}_o$ are the state vectors at point i, and o, and the transfer matrix connecting these vectors is T then from the relation previously stated, one can write down the following equation:

$$\{z\}_i = [T]_i \{z\}_o$$

Where $\{z\}_i = \begin{vmatrix} -w \\ \phi \\ M \\ Q \end{vmatrix}_i \qquad \{z\}_o = \begin{vmatrix} -w \\ \phi \\ M \\ Q \end{vmatrix}_o$

By applying proper boundary conditions we can eventually formulate the frequency condition. For example, in case of a cantelever type structure, with one end fixed and the other free:

$M_i = 0 \qquad Q_i = 0 \qquad$ (free end)

$w_o = 0 \qquad \phi_o = 0 \qquad$ (fixed end)

The above end conditions will now yield the frequency determinate

$$\Delta(\omega) = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} = 0 \qquad (17)$$

The natural frequencies of an elastic vibrating system can be obtained by numerical method, by making a frequency sweep within the range of interest.

4. Determination of Mode Shape

When the natural frequencies of an elastic system are determined, it will be a comparatively easy matter to compute the vibration modes. For example with boundary conditions $\phi_o = o$ $w_o = o$, we can obtain the relation between V_o and M_o; i.e.

$$M_o \quad a_{33} + Q_o \quad a_{34} = 0 \tag{18}$$

$$M_o \quad a_{43} + Q_o \quad a_{44} = 0 \tag{19}$$

From the above relation, we obtain

$$Q_o = - \frac{M_o \quad a_{33}}{a_{34}} \tag{20}$$

Now all the state vectors can be written in terms of only, M_o which however, remains undetermined. This can be resolved by choosing arbitrarly equal to unity.

5. Response Calculation

The mode shapes which are defined by the Eigenvalues and Eigenvectors are now formed, by means of matrix of transmission,the damping terms were omitted for this purpose. For each mode there is a natural frequency and mode shape or eigenvectors.

Having established the normal modes, the uncoupled equation of motion may be written in matrix formulation:

$$\{\ddot{q}\}_n + 2[\beta_n][\omega_n]\{\dot{q}\}_n + [\omega_n^2]\{q\}_n = -R\{\Gamma\}_n \ddot{G}_o(t) \tag{21}$$

Where $\{\ddot{q}\}$, $\{\dot{q}\}$ and $\{q\}$ are vector of acceleration, velocity and displacement,

R is an earthquake scaling factor

Γ_n = participation factors arranged in a column

$\ddot{G}(t)$ Time varying earthquake input

$$\Gamma_n = \frac{\{\phi\}_n^T [m]}{\{\phi\}_n^T [m] \{\phi\}_n}$$

Where $[m]$ = diagonal mass matrix

$\{\phi\}$ = mode shape vector

$\{\phi\}^T$ = Transpose of mode Shape vector

$\{D\}$ = Column matrix governing the direction of earthquake excitation

The displacement response is therefore:

$$q_n(t) = \frac{\Gamma_n R}{\omega_n \sqrt{1-\beta_n^2}} \int_0^t \ddot{G}_o(t)\, e^{-\beta_n \omega_n (t-\tau)} \sin[\omega_n \sqrt{1-\beta_n^2}\,(t-\tau)]\, d\tau \tag{22}$$

Finally, the acceleration response, and earthquake loads are calculated from the displacement response.

6. Application of Matrix of Transmission in the Study of Primary-Secondary Structure Interaction Problem

A Power Plant contains so many inter-connected items such as pressure vessels, pipes, and fuel handling machinery etc, that a lumped mass model of the entire system can become unmanageably large. In such a case it is necessary to reduce the model so that it describes smaller components and appendages rigidly attached with the larger components. However, this approach can lead to erroneous results when the motions of the primary system are greatly amplified in the appendage and dynamic feedback becomes significant.

For a clear understanding of the applied method we start with a simple example of a two mass system representing the primary and secondary structure, and analyse the feedback effect by means of matrix of transmission. (Fig.2).

The main problem now is to set-up a transfer matrix connecting the State Vectors $\{z\}_i^L$ and $\{z\}_i^R$ which will include the effects of the secondary mass m, upon the primary system M, and vice-versa. The effects of wheel friction and damping are neglected.

The first step is to treat this secondary system separately and establish necessary state vectors.

It follows that: $\{z\}_j^L = [F]_j \{z\}_o$

and $\{z\}_j^R = [P]_j \{z\}_j^L$

Or $\{z\}_j^R = [P]_j [F]_j \{z\}_o$ (23)

Or $\{z\}_j^R = [T]_j \{z\}_o$

$[T]_j$ = Transfermatrix for the secondary system.

Let us assume that the complete system vibrates at a frequency Ω_n (system frequency), and the inertia force acting on mass m is:

$$m\, \Omega_n^2\, x_j^R$$

Where $x^R{}_j$ is the displacement of the mass m (Displacement component of the state vectors $\{z\}_j^R$)

Wherby it should be noted that at point j, $x_j^R = x_j^L$

Therefore the Transfermatrix for the Secondary System can be written as follows:

$$\begin{Bmatrix} x \\ Q \end{Bmatrix}_j^R = \begin{bmatrix} 1 & \frac{1}{K} \\ -m\Omega^2{}_n & \frac{1-m\Omega^2{}_n}{K} \end{bmatrix} \cdot \begin{Bmatrix} x \\ Q \end{Bmatrix}_o \qquad (24)$$

From the above matrix equation we can derive the following linear equations:

$$x_i^R = x_o + \frac{Q_o}{K} \qquad (25)$$

$$Q_i^R = -m\Omega_n^2 x_o + Q_o \left[1 - \frac{m\Omega_n^2}{K}\right] \qquad (26)$$

Now from the end condition $Q_i^R = 0$ follows

that, $$Q_o = \frac{m\Omega_n^2 x_o}{1 - \frac{m\Omega_n^2}{K}} \qquad (27)$$

or $$\frac{Q_o}{\Omega_n^2 x_o} = \frac{m}{1 - \frac{\Omega_n^2}{\omega_s^2}} \qquad (28)$$

Where $\frac{m}{K} = \frac{1}{\omega_s^2}$ (ω_s = Natural circular frequency of the secondary system)

Let $$\frac{Q_o}{\Omega_n^2 x_o} = \delta_m(\Omega_n) \qquad (29)$$

then $$\delta_m(\Omega_n) = \frac{m}{1 - \frac{\Omega_n^2}{\omega_s^2}}$$

$\delta_m(\Omega_n)$ is the frequency dependent dynamic mass.

We may note here that when $\omega_s \to \Omega_n$ $\delta_m(\Omega_n) \to \infty$

This however, is not the case, because the secondary system has damping, and therefore the max. response is limited to $1/(2\beta)$.

The additional dynamic mass $\delta_m(\Omega_n)$ is now added to mass M, and hence the final transfer matrix connecting the state vectors $\{z\}_i^L$ and $\{z\}_i^R$ has the following from

$$\{z\}_i^R = \begin{bmatrix} 1 & 0 \\ -(M + \delta_m(\Omega_n))\Omega_n^2 & 1 \end{bmatrix} \{z\}_i^L \qquad (30)$$

Where Ω_n is the System frequency due to interaction effect between the primary and secondary system.

We will now extend this idea to flexular beam model of a Reactor building supporting multiple secondary systemsat different nodal points. Fig (4).

For the purpose of simplicity, we will only concentrate on the secondary system having mass m_{zi}

From the figure (3), the following relation may be written for the secondary s+ructure: $$\{z\}_{i+1}^L = [P]_{zi+1}[F]_{zi+1}\{z\}_i^L \qquad (31)$$

Applying proper end conditions, in $\{z\}_1^L$, that is $Q_1^L = o$

It can be shown that at the point of support the additional frequency dependent dynamic mass

$$m(\Omega_n) \sim \frac{m_{zi}}{1+\frac{m_{zi}\,\Omega_n^2\, l_{i+1,z}^3}{6(EJ)_{z,i+1}}} \qquad (32)$$

A more exact procedure of analysis would be to expand the final transfer matrix of the secondary system under consideration by prescribing proper end conditions. The expanded matrix will render two simultaneous equations with elements M_{i+1} Q_{i+1} of the vector as unknowns., From these two equations the component vectors M_{i+1} and Q_{i+1} are cumputed in terms of given Support montion (Primary structure) and end conditions $M_1 = Q_1 = o$ M_{i+1} and Q_{i+1} are the resultant shearforce and moment at the base of the secondary system at the point of attachement.

M_{i+1} will contribute to the additional rotation of the nodal point of the primary system, and $Q_{i+1}/\Omega^2 w_i$ will contribute to the nodal point mass to which the secondary system in question is attached.

In practice however, the initial values of additional dynamic mass and rotation of the nodal point due to interaction effect are obtained as a first approximation by using primary system frequency, without the effect of the secondary system.

These values are then introduced in the primary system transfer matrix. Finally, from the frequency determinate the eigenfrequency for each mode can be determined. This process is repeated several cycles until the system frequency is stabilised. Once the exact system frequency is found, it is a comparatively easy matter to find the mode shape, modal moment and forces for both primary and secondary system.

Discussion

A number of Nuclear Reactor buildings have been analysied by this method, including interaction problems between structures and the influence of installed heavy equipment. For the purpose of Comparison, some of the most interesting results obtained from the dynamic analysis of a typical concrete Shield building are presented in Table 1. The Shield building has a diameter of 41.5 m, is 68.0 m high, and has a wall thickness of 1.0 m. The results shows a fairly good agreement with the conventional methods. The response calculation is based on response spectra derived from five artificially generated time-history aculeration.

Conclusion

The problems in structural dynamics are conveniently and readily solved by means of this very general method of Transmission matrix. As there is no necessity to form stiffness matrices, the computation time is shorter. The nature of the Transmission matrix, together with the arrangement of elements in the coloumn vector, allows the nodal displacements, rotation, Moments, and shearforces to be conveniently obtained without requiring the stiffness matrices. In some complex problems, such as close coupled Systems, or Structures with varibale end conditions such as elastic support, fixed end support, or fixed-free end conditions can be solved very conveniently by this method. Eigensolutions of systems with very large degrees of freedom, do not pose any special problems except at higher freequencies ($f \approx 25$ C/S), where the accuracy of eigenvalues are questionable. This can be handeled effectively by using "shifted Transfer matrix". However as far as the Structural response is concerned, an Eigenfrequency of 25 C/S is in most cases a cut-off frequency in the response spectra diagram, this limitation has therefore no practical significance. The method discussed in this paper therefore can be used for static and dynamic analysis of structures speedely and accurately.

Table I

Results of dynamic analysis of a typical concrete Shield building dia = 4,1 m, h = 68,0 m, t = 1,0 m.

Transmission matrix mehtod Program HT/173 BBC	Conventional method Program ASKA-Dynan
Natural Frequency	
MODE 1 2.318 C/S	2.309 C/S
MODE 2 6.017 C/S	6.014 C/S
MODE 3 13.647 C/S	13.532 C/S
MODE 4 21.104 C/S	20.933 C/S
MODE 5 25.367 C/S	25.132 C/S
Response Calculation (SRSS values)	
Max Base Moments 592800 (mt)	Max Base Moment 594000 (mt)
Base Shear 13650 (t)	Base Shear 13000 (t)
Max displacement 5.91 cm	Max displacement 5.86 cm
Max accln. 0.68 g	Max accln. 0.7 g

References

[1]. Newmark und Rosenblueth - Fundamentals of Earthquake Engineering. Prentice-Hall Inc.

[2]. Hansen, Robert J, editor. Seismic Design for Nuclear Power Plants. The M.I.T. Press, Cambridge, Mass.

[3]. Wesley D.A. Response Spectrum Approach to Earthquake Response of Nuclear Power Station Appendages. Gulf General Atomic Inc., San Diego, California, USA. No. GA-9232, April '69.

[4]. Newmark N. Earthquake Response Analysis of Reactor Structure. Paper K1/1. First International Conference Structural Mechanics in Reaktor Technology. 20-24 Step. 1971.

[5]. Seismic Analysis of Pressure Vessel and Pipping Components. ASME First National Congress on Pressure Vessels and Piping May 10, 1971. The ASMA, United Engineering Centre, New York, N.Y. 10017.

[6]. Marguerre K. Metrices of Transmission in Beam Problems. Technische Hochschule Darmstadt.

[7]. Pestel E. und Leckie F. Matrix Methods in Elastomechanics. McGraw-Hill Book Company, Inc.

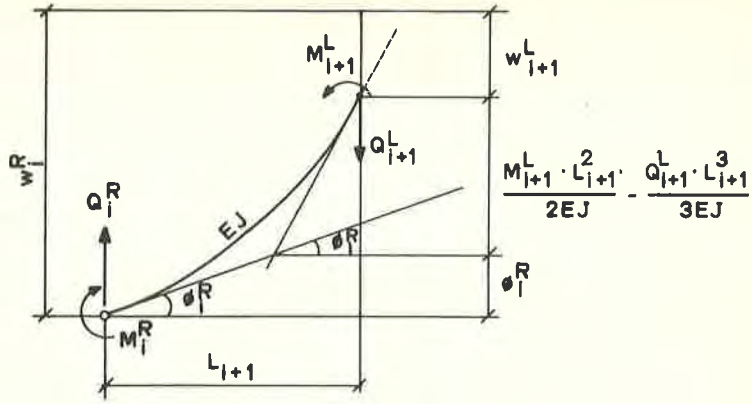

Fig. 1

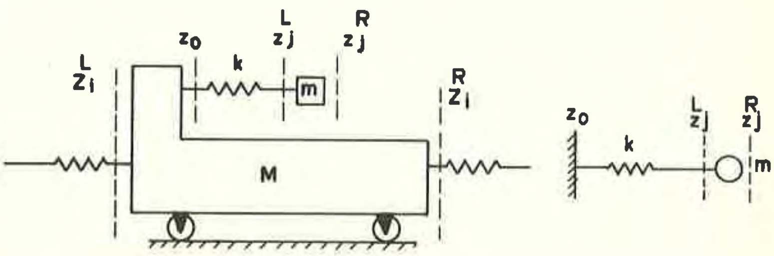

Fig. 2

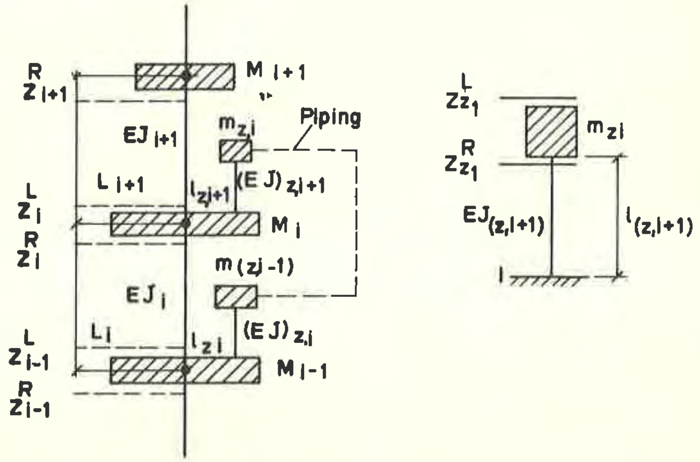

Fig. 3