SOME CONTRIBUTIONS TO SOIL-FERTILIZER RESPONSE MODELING

by

Gilberto Paez and L. A. Nelson

Institute of Statistics
Mimeograph Series No. 739
Raleigh - 1971

TABLE OF CONTENTS

	Page
LIST OF TABLES	vii
LIST OF APPENDIX TABLES	ix
LIST OF FIGURES	x
1. INTRODUCTION	1
1.1 The Problem	1
1.2 Formulation of the Problem	3
1.3 Objectives	4
2. REVIEW OF LITERATURE	6
3. MODEL DESCRIPTION AND RELATED POSTULATES	22
3.1 One Casual Factor	22
3.1.1 The Law of The Minimum or Liebig's Postulate	22
3.1.2 Parabolic Response Law	24
3.1.3 Asymptotic Response Law or Mitscherlich's Postulate	28
3.1.4 The Law of the Power Function or Cobb-Douglas Postulate	31
3.2 Extension of the Postulates to Three Input Factors	32
3.2.1 The Law of The Minimum for Three Input Factors	34
3.2.2 The Second-Order Polynomial Law for Three Input Factors	34
3.2.3 The Mitscherlich Law for Three Input Factors.	35
3.2.4 The Law of the Power Function for Three Input Factors	36
3.3 Point of Physical or Theoretical Maximum Rate (MR) for the Parabolic, Mitscherlich and the Cobb-Douglas Response Models. One Input Factor	37
3.3.1 Point of Theoretical Maximum for the Parabolic Model	37
3.3.2 Point of the Maximum Curvature for the Mitscherlich Response Model	38

Table of Contents (Continued)

Page

3.3.3 Point of the Maximum Curvature for the Power Function 39

3.4 Point of Optimum Rate (OR) for Parabolic and Mitscherlich and Cobb-Douglas Response Models for One Input Factor 39

3.5 Point of "Minimum Recommended Rate" (MRR) for the Parabolic, the Mitscherlich and the Cobb-Douglas Response Models for One Input Factor 43

4. MATERIALS AND METHODS 48

4.1 Fertilizer-Response Data 48

4.2 Criterion Used for Selecting Data for Numerical Estimation of the Initial Nitrogen Level . . . 51

4.3 Techniques and Procedures of Estimation 51

4.3.1 Principle of Transformation Searching for the Best Fitting Transformation Model 52

4.3.1.1 The Second Order Model and the Power Transformation of the Independent Variable 53

4.3.1.2 Special Kinds of Power Transformation of the Input Factor 55

4.3.2 The Non-linear Least Squares Method of Estimation of Parameters 57

5. RESULTS AND DISCUSSION 62

5.1 Estimation of the Parameter δ and the β Parameters by Fitting the Models: $E(Y) = \beta_1 (X + \delta) + \beta_2 (X + \delta)^p$ 63

5.2 Estimates of the Parameter δ and β by Fitting the Model: $E(Y) = \beta_1 (X + \delta)^p + \beta_2 (X + \delta)^{2p}$ . . 71

5.3 Relative Variance of the Estimated Parameters of the Response ($\hat{\delta}$, $\hat{\beta}_1$ and $\hat{\beta}_2$) 79

Table of Contents (Continued)

Page

5.4 Structure of the Asymptotic Variance-Covariance Matrix of the Estimated Parameters ($\hat{\delta}$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\rho}$) 84

5.5 Relationship Among the Estimated, $\hat{\delta}$ (lbs. per acre) the N(%), OM(%), NO3 (ppm) and the Average Yield at Zero Level of Application, y_o (Bu./acre) 85

5.6 Relationship Among the Estimated δ's by Several Models 90

5.7 Estimation of Point of Physical or Theoretical Maximum 91

5.8 Estimation of the Maximum Yield 95

5.9 Contour of the Standard Deviation from the Model . 96

5.10 General Comments on the Results 99

6. SUMMARY AND CONCLUSIONS 103

7. BIBLIOGRAPHY 106

8. APPENDIX 109

LIST OF TABLES

Page

1. Estimate of the parameter δ (lbs. per acre), obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$ for $p > 1$ . . 66
2. Estimate of the parameter δ (lbs. per acre), obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \alpha\{1 - \exp(-\gamma(X + \delta))\}$ 74
3. Relative variance components of the estimated $\hat{\delta}$, for locations and sites within location 81
4. Relative variance components of the estimated $\hat{\beta}_1$, for locations and sites within location 82
5. Relative variance components of the estimates $\hat{\beta}_2$, for locations and sites within location 83
6. Some estimates of the variance-covariance matrix of the estimated parameters $(\hat{\delta}, \hat{\beta}_1, \hat{\beta}_2)$, fitting a fixed power transformation model 87
7. Some estimates of the variance-covariance matrix of the estimated parameters $(\hat{\delta}, \hat{\beta}_1, \hat{\beta}_2$ and $\hat{p})$ 89
8. Correlation matrix of the soil test results: N, OM, NO3-production and the estimated amount of $N(\hat{\delta})$ by the square root model and the corn yield at zero level of N application (y_o) 90
9. Correlation matrix of the estimated parameter $\hat{\delta}$, fitting the models: $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$, for $p = .75$, .95, 1.25, 1.50, 1.75 and $\hat{p} = .93$ $(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4, \hat{\delta}_5$ and $\hat{\delta}_6)$; $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ for $p = .50$, 1.00 and $\hat{\hat{p}} = .62$ $(\hat{\delta}_7, \hat{\delta}_8$ and $\hat{\delta}_9)$ and $\hat{\delta}_{10}$, obtained using the Mitscherlich model 92

List of Tables (Continued)

Page

10. Average estimates of the stationary point, $\hat{X}$ (pounds per acre), which maximizes the response function given by the model $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ 93

11. Average estimates of the stationary point, $\hat{X}$ (pounds per acre) which maximizes the response function, given by the models, $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \alpha[1 - \exp.(\gamma(X + \delta))]$ 94

12. Average value of the estimate maximum yield (bushel per acre) obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$ for $p > 1$ 97

13. Average value of the estimate maximum yield (bushel per acre) obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \alpha[1 - \exp(-\gamma(X + \delta))]$ 98

LIST OF APPENDIX TABLES

Page

1. Estimates of the Regression Coefficient $\hat{\beta}_1$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$, for $p > 1$ 110

2. Estimates of the Regression Coefficient $\hat{\beta}_1$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \beta_0 + \beta_1 X + \beta_2 X^2$ 115

3. Estimates of the Regression Coefficient $\hat{\beta}_2$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$, for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$, for $p > 1$ 120

4. Estimates of the Regression Coefficients $\hat{\beta}_2$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \beta_0 + \beta_1 X + \beta_2 X^2$ 125

5. Standard Deviation from the model $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$ 130

6. Standard Deviation from the model $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ 135

7. Results of soil nitrogen tests, corn yield with zero application level of nitrogen and statistical estimates of the amount of nitrogen in the soil 140

8. Experimental results from 25 selected locations of the N.C. State TVA Cooperative Corn Project, (Norfolk soils). Yield response in bu. per acre 141

LIST OF FIGURES

Page

1. A graphical representation of the law of the minimum in the (X, η)-plane, $\beta > 0$ 25

2. A graphical representation of parabolic response law in the (X, η)-plane, $\beta > 0$ and $\beta_2 < 0$ 27

3. Asymptotic regression law in (X, η)-plane, $\alpha > 0$ and $\gamma > 0$ 30

4. Response curve of the Cobb-Douglas function, for $\beta_1 > 1$. 33

5. Relationship among the estimated parameters, δ, β_1, β_2 and the power transformation, p. Model: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ 65

6. Relationship among the estimated parameters, δ, β_1, β_2 and the power transformation, p. Model: $E(Y)$: $\beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ 73

1. INTRODUCTION

1.1 The Problem

In the decision-making framework of any farmer there always exists an economic end, that is the maximization of the profit function. If the farmer is concerned about the soil-plant-fertilizer response function, obviously he will need information to assist him in defining his own crop fertilizer plan. Any available information directly related to the farmer's problem will help him in decision making and will possibly substitute for years of on-farm experience. It is primarily for this reason that state and federal research agencies get involved in fertilizer response studies. In keeping with the original aim of the farmer, the aim of these agencies should be profit maximization for the individual farmer.

It is recognized that from a university experimental research set up, extrapolation to individual farm situations presents a number of knotty problems, which are not discussed here. Instead, listed below are some interesting questions related to fertilizer response functions which might arise within the university experimental research. The problems might be related to such questions as: How do you quantitatively describe a response function pattern since the nature of the true biological response behavior is dynamic and indeed unknown? What models should be fitted to describe more realistically a fertilizer response function? What is the role of soil test information in the response surface fitting and fertilizer response model selection?

An exact quantitative determination of the initial level of nutrient in the soil could be a great contribution in deriving a more realistic response function model. Unfortunately an exact diagnosis of the nutritional status of the soil is not yet well-known. Standard testing procedures which are acceptable to all workers have not been developed. For example, for the major nutrient, nitrogen, there are no reliable soil test procedures. Should soil nutrients be included in, or excluded from the model? In view of the fact that the response function depends directly on both effects, applied fertilizer as well as the initial level of nutrient present in the soil, it seems reasonable to think that initial level of nutrient should be included as a part of the response function model. However, it should be pointed out that very little consideration has been given to the latter as a relevant component of the response function model. Few attempts have been made for estimating it from the behavior of the observed response function (data), which essentially amounts to an extrapolation of the lower portion of the response curve.

For purposes of this study, the estimated amount of initial level of nutrient (δ) is defined by the segment of the input axis, bounded by the point of intersection of the response curve with the input axis and the point on the input axis at which no application of fertilizer is made ($X = 0$).

The problem is aggravated when interest is focused on combining results from several locations, since the response pattern might be quite variable from location to location, as well as from site to site within the same location. Even for a single input factor it is not clear how to treat the results from several locations, statistically, with the aim of obtaining a more general description of the response pattern. Of course, the complexity of the problem increases when multifactor considerations come into play.

1.2 Formulation of the Problem

In many scientific fields, the main aim of the research worker is the examination of the complex structure of a number of quantitative factors (x's) and their relationship with a quantity called the response (η) by way of a function called a response function (f). The simplified abstraction of the functional relationship between these two sets of complex phenomena is the so-called functional model, namely, $\eta = f(\beta, x)$. In the present case, η is the yield of some crop for an experimental plot. Both β and x may be vectors.

When data are collected on several plots, it is found that no single response function will fit all cases exactly; thus a statistical model is required. The statistical model is $\underline{Y} = \underline{\eta} + \underline{\epsilon}$, in which the terms are nx1 vectors, $\underline{Y}$ is observed yield, $\underline{\eta}$ is true yield, and $\underline{\epsilon}$ is random. Writing $\underline{\beta}$ and $\underline{x}_i$ for the i^{th} observed plot, $E(Y_i | x_i) = \eta_i = f(\underline{\beta}, \underline{x}_i)$. The matrix X whose n rows are the vectors $\underline{x}_i$ is called the design matrix and $\underline{\beta}$ is a vector of parameters.

The choice of the model is open to investigation, and any particular problem will require that several be examined. After having decided on the functional form of the relationship between response variables and causal factors a further task is to examine the functional equation in the presence of experimental error: *i.e.*, to determine the nature of ϵ in $Y = f(\beta,x) + \epsilon$. This equation is for a typical experimental plot. The presence of ϵ in the model reflects the uncertainty in the experiment, since the experimenter, in general, will be unable to observe the true response, $\eta = f(\underline{\beta},\underline{x})$, but will be observing the recorded response, $Y = f(\underline{\beta}, \underline{x}) + \epsilon$.

The investigator might be interested in comparing some of the forms that can be taken by the function f, when the response function is subject to experimental error. This problem essentially involves fitting curves, using different functional forms for f. On the other hand, the main interest may be concentrated on the determination of the attainable maximum value of the response function Y.

The specific problem which brought about this study is concerned with the estimation of the initial level of soil nutrient. This parameter could be dependent on the model used to represent the response curve. Therefore, several functional forms are considered herein. Since the number of models that could be used to describe the response pattern is large, this study deals only with the polynomial family and the Mitscherlich exponential curve.

1.3 Objectives

The primary objective of the present study was to develop a polynomial model, not merely as a close fitting curve, but to reflect current biological knowledge of the response function in the soil-plant-fertilizer relationship context. In particular, special attention was given to the use of the initial fertilizer level, δ.

Emphasis was placed on the estimation of parameters in fractional-power or p-power polynomial response functions, with one controllable factor.

The final aim of the present study was to compare the Mitscherlich model--a classical and widely used response function model--with the p-power polynomial model. The three criteria to be used in the evaluations of the models were: the closeness of fit to observed responses, the agreement between the estimate as obtained from using the model and that obtained by direct laboratory determination, and the reasonableness of the estimated point on the input axis which resulted in maximum response. For the most part, numerical comparisons based on actual data were used, because general algebraic expressions become cumbersome due to the essentially nonlinear nature of the models.

2. REVIEW OF LITERATURE

A long review of the literature dealing with biological response functions will not be given, but the reader is referred to a book by Heady and Dillon (1966) for an excellent review of many interesting papers. Also Mason (1956) gives a detailed description of many functional models used to characterize biological response functions. These authors have given a large number of references concerning the response function pattern and curve fitting in biological problems. However, most of the models have been developed on heuristic grounds and their theoretical basis is rather poor. Consequently the use of such models as productive tools is largely doubtful.

Probably the first attempt to describe the relationship between supply of nutrient and response measured, was made by Liebig (1855). He stated that if a soil contains all the essential nutrients for plant growth and development, except one, the crop growth is governed by the amount of this nutrient available in smallest supply. This postulate assumes that crop responses are directly proportional to the amount of nutrient available, when all soil nutrients are present in sufficient supply, addition of one or more would not increase the yield. Although Liebig did not suggest a specific albegraic form representing his postulated production function, nevertheless an acceptable functional form can be deducted.

Baule (1918) interpreted Liebig's postulate as a physiological limitation of the plant, which presumes that plants can take nutrients only in a given ratio, so that the yield will vary proportionally with the amount of nutrient available in smallest supply. Baule's interpretation of the Liebig Law of the Minimum or Liebig's postulate has been taken for one nutrient level as represented by the functional form of the straight line, whether passing through the origin or not.

$$\eta = \beta_1 X \qquad (2.1)$$

$$\eta = \beta_0 + \beta_1 X .$$

Where: η is the true response; X is the level of available nutrient in smallest supply; β_0 and β_1 are constants.

Extension of the Liebig's Law to more than one nutrient is relatively obvious from equation (2.1), for example, in the case of k nutrients the algebraic form of the response can be represented as follows:

$$\eta = \beta_1 X_1 X_2 X_3 \cdots X_k . \qquad (2.2)$$

Equation (2.2) suggests that response η would be proportional to input of any one of the available nutrients X_i.

Liebig's postulate created controversy among research workers of that era, since it fails to specify what happens for large values of the input factor. Nevertheless his work stimulated and greatly influenced further investigation on this topic.

After about a half-century of searching for a more realistic representation of the response measured, Mitscherlich's postulate appeared in 1909. This investigator formulated a nonlinear production function model, in an attempt to describe more realistically the relationship between applied fertilizer and crop response. His postulate states the following:

"The increment of the response function per unit of the lacking factor is proportional to the decrement from the maximum yield." The mathematical expression of his formulation is given by the equation of the form:

$$\eta = \alpha(1 - e^{-\gamma X}) \tag{2.3}$$

where: η is the true response; α the theoretical maximum yield; γ is a parameter defining the rate of decline of the yield (also termed efficiency factor) and X represents the level of available nutrient.

The Mitscherlich Function is referred to in the literature as the Law of Diminishing Returns or the First Order Reaction Law. It is a member of the group of Asymptotic Regressions.

Several criticisms have been made of Mitscherlich's postulate. One concerns the coefficient "γ". Mitscherlich maintained that "γ" the rate of diminution of the response from the asymptote is well-defined for a given nutrient, and constant under any condition of soil, climate or other environmental factors and it is independent of the nature of the crop. Another criticism of his original model was centered around the shape of the curve, that is, on the assumption that the curve is a nondecreasing function of X, which contradicts experimental evidence. To remedy this situation, Mitscherlich suggested a modified model which allows for a depression in yield at high rates of fertilizer application. Thus, he proposed the following alternative model:

$$\eta = \alpha(1 - e^{-\gamma X})e^{-kX^2} \qquad (2.4)$$

where k is the so called "schadigungsfaktor" which when translated from German means damage or injury factor.

Baule (1918) indicated an easy computational method for the parameters in the Mitscherlich equation $\ell n(\alpha - \eta) = \ell n\alpha - \gamma X$. This equation can be solved expressing the yields in relative terms as percentage of the maximum yield α, percentage yield for α is equal to 100. He assumed that there exists a quantity h of X such that an h units of X would always raise the yield by half the decrement from the maximum yield α. He pointed out that the soil contribution is included in the first, second or additional h units. For example, if η_0 is equal to $.5\alpha$, the soil contains 1h unit of X available nutrient.

Methods of fitting equation (2,3) to data are readily accessible in the literature. Stevens (1951) presented tables for computation of the parameters in the Mitscherlich model for a limited number of levels of the input factor. Many other tables have been constructed for the same purpose. However all of them have a number of restrictions, which limits their use in practical application. But with present computing capabilities, neither tables nor restrictions are needed for solution of the equation.

Gomez (1960, 1961) pointed out that the Mitscherlich equation, like other response equations, allows estimation of the most profitable level of fertilizer required, for which an approximate formula was given as follows:

$$X^* = \frac{1}{2} X_\mu + \frac{1}{\gamma} \log \frac{w \cdot \mu}{t \cdot X_\mu} \qquad (2.5)$$

where X* is the most profitable level of fertilizer required; μ is the increase in the yield with the application of X_μ amount of fertilizer; w is the price per unit of crop yield; t is the cost per unit of fertilizer and γ is the efficiency factor.

Gómes[†] (1968) also observed the relative constancy of the value of γ in the Mitscherlich equation, for a given crop.

Baule (1918) proposed a generalization of the Mitscherlich equation for more than one nutrient. His proposal was based on this "percentage sufficiency" for each individual nutrient, that is that the final "percentage sufficiency" in the multifactor case is the product of the individual sufficiencies of each of the nutrients involved in the process. According to this principle the multi-variable Mitscherlich function is given by the following equation.

$$\eta = \alpha(1 - e^{-\gamma_1 X_1})(1 - e^{-\gamma_2 X_2})\ldots(1 - e^{\gamma_k X_k}) \qquad (2.6)$$

An alternative response model, frequently used to estimate the response function, is the polynomial model. Many research workers prefer the use of polynomial functions to characterize the response curve, mainly because it can be fitted with relative ease and gives good fit in some regions of the factor space. Probably the chief advantages of the polynomial model are the easy computation of the optimum rate

---

† Personal communication. Escola Superior de Agricultura, Luisdi Queiroz, Piracicoba, Brazil

of fertilizer and the fact that exact confidence bounds for the optimum rate can be computed readily. For example, the second order polynomial, when only one input factor is considered, is as follows:

$$\eta = \beta_0 + \beta_1 X + \beta_2 X^2 \qquad (2.7)$$

where η is the true response function; X represents the amount of applied nutrient and β_0, β_1, β_2 are constants.

Heady and Dillon (1966) pointed out that the goodness of fit of the production surface by polynomial approximation depends on the choice of scale used for the response variable as well as on the choice of scale or of transformation for the input factor. The log transformation and square root transformation recently have been in common use for transforming the input factor. Box and Tidwell (1962) suggested an iterative procedure for deciding on transformations of single independent variables. Their results can be extended to the transformation of more than one independent variable.

In fertilizer experiments, it seems reasonable to consider in the model the initial level of nutrient in the soil. Mitscherlich's postulate considers the effect of the initial level of soil nutrient, since in his equation X stands for the total, available nutrient, that is, $X = X' + \delta$ when X' is the input factor and δ is the initial fertility level. He assumes that the input factor acts in an additive fashion to the initial level of nutrient δ. Some other writers have speculated that several methods of incorporating soil nutrient test measurement into yield response functions have been suggested.

Hildreth (1957) proposed a model, which he claimed is appropriate if the effect upon yields of a measured amount of nutrient in the soil is porportional to the effect of the same nutrient added artificially.

The functional form of Hildreth's model in the case of a single nutrient regards the response as a function of the amount of added nutrient plus some constant, λ, times the initial amount of nutrient in the soil so this last component would be measured by soil test and thus the constant multiplier serves as an efficiency factor. The functional model can be represented by the response pattern, $\eta = f(N)$ where $N = X + \lambda\delta$, N is the total available nutrient in the soil, both initial and added; X, the amount of added nutrient; λ, the proportionally or efficiency factor and δ, the initial soil nutrient as measured by soil test.

The form of the function of Hildreth's equation is, in general, non-linear, therefore, appropriate non-linear techniques must be used to estimate the λ parameter of the function. Hildreth suggested the maximum likelihood procedure to estimate the parameter λ. After λ has been estimated the other parameters of the function can be easily obtained by linear least squares techniques.

The representation of $(\gamma_1\delta + \gamma_2 X)$ was suggested as a pattern of the additive effect of initial level of nutrient and applied fertilizer, where γ_1 and γ_2 are called efficiency factors. Bray (1958), working with one input factor and making use of the Mitscherlich equation gave a procedure to estimate γ_1 and γ_2 in two steps. The first step consisted of fitting the model $\log(\alpha - y) = \log\alpha - \gamma_1\delta$ to obtain estimates of γ_1 where δ was replaced by the value of the soil test result. In the second step, the model $\log(\alpha - y) = \log \alpha - \gamma_1\delta - \gamma_2 X$ was fitted, the solution of which yielded the value of γ_2.

The author, working with phosphorous, reached the conclusions that the value of γ_1 and γ_2 do not represent just the efficiency of the soil and the fertilizer forms of the nutrient; they are also influenced by (a) the ability of a particular kind of plant to "forage for" a relatively immobile nutrient in the soil, (b) the degree of competition between roots of adjoining plants as caused by differences in planting rate and pattern, and (c) the distribution of the form in the soil in relation to the planting pattern and rooting habits of the plant. As these effects change, the γ_2 value should change. Many other research workers have used soil test results as an approximate value of δ, in the response function model.

Bray's main concern was in comparing the efficiency of soil nutrients with the efficiency of the added nutrient. His efficiency factor for soil nutrient assumes that soil test methods adequately characterize the available amount in the soil. Otherwise, the efficiency factor would be confounded with bias due to the soil test method.

If interest is focused on the total effect it does not matter which part of the response effect is attributable to the added factor "X" and which is due to the initial nutrient levels of the soil. Consequently, there is little interest in knowing the nature of the coefficients of such formulated linear combinations of X and δ, i.e., $\gamma_1 X + \gamma_2$.

Brown, Jackson and Petersen (1962) presented a method for incorporation of soil test measurements into the fertilizer response function. First of all, they established the relationship between the added nutrient and that existing in the soil (as measured by soil test). This process required two soil test analyses. The first one was carried out before fertilizer application and the second test was made after some time so that the applied nutrient had had time to become incorporated into the soil, but had not yet been significantly utilized by the crop. Thus the second soil test provided a direct estimate of the total nutrient available to the plant, hence the total nutrient, $N = X + \lambda\delta$ became an observable variable. The proportionality constant between nutrient in the soil and the applied nutrient can be estimated from the soil tests.

The proportionality constant λ can be estimated by means of two procedures: one of them consists of a direct estimation and the other fitting a straight line equation.

The direct estimation procedure established the following relationship:

$$X_{ij} - X_{kj} = \lambda\, N_{(i-k)j} \text{ or } \lambda = \frac{X_{ij} - X_{kj}}{N_{(i-k)j}} \qquad (2.8)$$

where X_{ij} = total nutrient as measured by soil test after nutrient application on the plot receiving the dose i at the location, X_{kj} = soil test reading on the plot receiving the same or lesser nutrient application than X_{ij} and $N_{(i-k)j}$ = difference in nutrient application between X_{ij} and X_{kj}.

The indirect method consists of fitting the equation as follows:

$$X_i = \beta_0 + \beta_1 N_i + \epsilon_i \qquad (2.9)$$

where X_i = total nutrient as measured by the second soil test, N_i = dose of nutrient applied and β_1 is a measure of λ.

After estimating λ, the initial soil nutrient δ, detected by the first test is added to the applied dose and a production function is fitted.

The main concern of the present study was in estimating the soil nutrient in units equivalent to the nutrient added; i.e., equal efficiency is assumed for both sources (additivity). Thus, if the estimates of δ are obtained by soil test methods, the second efficiency factor is needed, if they are obtained through mathematical estimation, only the one efficiency factor is needed.

Gomez (1960) discussed methods of estimating δ in the asymptotic regression model. He conceived the δ-parameter solely as an intersection of the extrapolated lower tail of the curve with the X-axis. He does not imply that the δ value necessarily estimates accurately the second order polynomial model. He was faced with the problem of unexplainable variation in the parameters which he suggested might have been caused by varying initial levels of nutrients.

Perhaps Anderson (1956) was the first to give extensive consideration to the problem of adjustment for available nutrients in the mathematical context using the second order polynomial model; therefore his work is given in some detail in the following discussion. Quoting directly from his work: "One of the major needs in the determination of fertilizer response surfaces is a method of adjusting for nutrients available in the soil before the experiment is started". He pointed out that probably a simple polynomial model will do sufficiently well for most practical purposes and that relatively little is to be gained by complicated formulations. Thus Anderson was concerned with maintaining the polynomial model because of its simplicity and advantages in estimating optimum nutrient level and exact confidence limits therefore, but at the same time in incorporating the initial level of nutrient into the model.

Considering the second order polynomial for an experiment involving only one nutrient, Anderson set up the following equation:

$$E(Y|X) = \beta_0 + \beta_1 X + \beta_2 X^2 \qquad (2.10)$$

where X is the added amount of fertilizer. If the actual amount of nutrient available in the soil is designated by $N = X + \delta$, then $X = N - \delta$, where δ is the initial level of nutrient available in soil. Now, substituting $N - \delta$ for X in (2.10) and re-arranging terms yields the following equation:

$$E(Y|N) = (\beta_0 + \beta_1\delta + \beta_2\delta^2) + (\beta_1 - 2\beta_2\delta)N + \beta_2 N^2 \qquad (2.11)$$

Applying this result to a particular farm condition in which the casual factor is $N^* = X + \delta_0$, the predicted response function will be:

$$E(Y|N = X + \delta_0) = (\beta_0 - \beta_1\delta + \beta_2\delta^2) + (\beta_1 - 2\beta_1\delta)(X + \delta_0)$$
$$+ \beta_2(X + \delta_0)^2. \qquad (2.12)$$

The bias incurred by neglecting the available soil nutrient is given by:

$$\text{Bias} = E(Y|X) - E(Y|N^* = X + \delta_0) = (\delta - \delta_0)(\beta_1 + 2\beta_2 X) - (\delta - \delta_0)^2\beta_2 \ . \qquad (2.13)$$

The bias will be zero only when the initial nutrient levels available at the farm and experimental plots are equal. Anderson then pointed out that not only the predicted response function is biased, but that the difference between the response function for two different levels of farm application will be biased. The bias in the expected response function, given X_2 instead of X_1 is $2\beta_2(X_2 - X_1)(\delta - \delta_0)$, which may be positive or negative depending upon the signs of the differences. As an example, suppose $X_2 > X_1$, and $\delta > \delta_0$, then the bias is negative since β_2 is expected to be negative, hence one would tend to underestimate the effect of the added nutrient if the available nutrients at the farm are less than at the experimental plots.

The problem becomes further aggravated when one attempts to combine results of experiments for two different agro-climatic conditions with $\delta = \delta_1$ for one location and $\delta = \delta_2$ for the other. If a quadratic model in one input factor is used, the development is as follows. Let:

$$E(Y|N) = \beta_0^* + \beta_1^* N + \beta_2^* N^2 \qquad (2.14)$$

be a general quadratic equation fitted for two different locations, then the parameters β_0^* and β_1^* will be quite different, unless

$\delta_1 = \delta_2$. Suppose for location 1, β_0' and β_1' have the following structure:

$$\beta_0' = \beta_0^* + \delta_1\beta_1^* + \delta_1^2\beta^*$$

$$\beta'_1 = \beta_1^* + 2\delta\beta_1^*$$

for location 2:

$$\beta_0'' = \beta_0^* + \delta_1\beta_1^* + \delta_2^2\beta_2^*$$

$$\beta_1'' = \beta_1^* + 2\delta_1\beta_2^*.$$

It is fairly obvious that an incorrect decision is made if the experimenter did not take into account the inequality in the available nutrients for the two experiments. Probably the failure of adjustment for the available nutrient level yields an unrealistic response function model for combined data. Hurst and Mason (1957), in line with Anderson's postulate, considered the second-order polynomial model for one input factor as follows:

$$E(Y|X) = \beta_0 + \beta_1(X + \delta) + \beta_2(X + \delta)^2$$

$$= \beta_0^* + \beta_1^*X + \beta_2^*X^2 \qquad (2.15)$$

where

$$\begin{pmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \end{pmatrix} = \begin{pmatrix} 1 & \delta & \delta^2 \\ 0 & 1 & 2\delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} . \qquad (2.15a)$$

The starred parameters are the customary regression coefficients and are biased as Anderson (1956) pointed out. The β_0^* and β_1^* should vary from location to location more than predicted by theory. On the contrary, β_2^* which is independent of δ, will show low variation from location to location. However, since the above equation does not afford an estimate of δ, an attempt to make this equation soluble for δ calls for the natural constraint of the Law of the Minimum, that is $Y = 0$ if $X + \delta = 0$ and $X + \delta = 0$ if and only if $\delta = 0$ and $X = 0$. These natural constraints simplify the response function model as follows:

$$E(Y|N = X + \delta) = \beta_1 N + \beta_2 N_2$$

$$= \beta_0^* + \beta_1^* X + \beta_2^* X^2 \qquad (2.16)$$

where

$$\begin{pmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \end{pmatrix} = \begin{pmatrix} \delta & \delta^2 \\ 1 & 2\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} . \qquad (2.16a)$$

The unstarred parameters then can be expressed in terms of the starred parameters. The solution yields ,

$$\beta_2 = \beta_2^*$$

$$B_1 = +\sqrt{\beta_1^{*2} - 4\beta_0^*\beta_2^*}$$

$$\delta = \frac{-\beta_1^* + \sqrt{\beta_1^{*2} - 4\beta_0^*\beta_2^*}}{2\beta_2^*} . \qquad (2.17)$$

Hurst (1962) discussed more extensively the estimation procedure for the starred parameters and their relationship to the unstarred ones. The starred parameters are obtained by ordinary linear least squares estimation procedures. Solution of the normal equations produces an estimate of $\underline{\beta}^*$, in the usual manner:

$$\hat{\underline{\beta}}^* = (X'X)^{-1}X'\underline{Y} \; . \qquad (2.18)$$

Equivalently, under the assumption that error in the model is normally and independently distributed with mean zero and common variance, $\hat{\underline{\beta}}^*$ is also the maximum likelihood estimate of $\underline{\beta}^*$. Then by the invariance property of maximum likelihood estimators, β_1, β_2 and δ can be estimated. These estimates, of course, have the usual properties of maximum likelihood estimates. Also, the sum of squares due to the model by using the unstarred parameters is preserved. Hurst also discussed simultaneous tests of hypotheses and simultaneous confidence bounds for the starred parameters in the second-order model.

In summarizing Hurst's approach to this problem, suppose the hypothesis to be tested is formulated as follows:

$$H_o: C\underline{\beta}^* = \underline{\beta}_0^*$$

where:

$$\underline{\beta}^* = \begin{pmatrix} \beta_0^* \\ \beta_1^* \\ B_2^* \end{pmatrix}$$

and C is the hypothesis matrix.

Then the test criterion is based on the usual F-test, which uses the statistic:

$$F(3,n-3) = \frac{(C\hat{\underline{\beta}}^{*}-\underline{\beta}_0^{*})'[C(X'X)^{-1}C']^{-1}(C\hat{\underline{\beta}}^{*}-\underline{\beta}_0^{*})/3}{\underline{Y}'[I-X(X'X)^{-1}X']\underline{Y}/n-3} . \qquad (2.19)$$

In addition, Hurst discussed an approximate t-test for $\hat{\beta}_1$ and $\hat{\delta}$, by directly computing the approximate variance of $\hat{\beta}_1$ and $\hat{\delta}$. Further he discussed an iterative procedure for combining information from several locations.

Hurst also presented a new formulation of the problem, assuming that δ could be a random parameter with expectation equal to δ and generating variance component σ_{δ}^2. However, he faced some difficulties in the estimation of parameters due to under-identification. Finally, he extended his results to the multifactor case, considering the δ's as random parameters in the second-order model with three input factors and then imposing a very artificial constraint, he obtained estimates of δ's in relative terms.

3. MODEL DESCRIPTION AND RELATED POSTULATES

In this chapter attention is focused upon the reformulation and description of some basic models, used to characterize response function, in the general soil-plant-fertilizer context. The biological implications also are described and discussed for each of the models.

The four models described herein are: the Law of The Minimum, the Parabolic Response Law, the Mitscherlich Equation and the Power Function. The first two models are members of the polynomial family of response curves, the third is a member of the exponential family, and the last one belongs to the class of logarithmic functions.

3.1 One Causal Factor

First of all the description of the response pattern η will be presented as a function of one controllable factor and then will be extended to the model description for more than one input factor.

3.1.1 The Law of The Minimum or Liebig's Postulate

The basic principle states: "The response function is directly proportional to the amount of nutrient available in smallest supply".

According to Heady and Dillon (1966) the Law of The Minimum can be mathematically justified as a simple expansion in a Taylor series of the unknown response $\eta = f(\beta, X)$. Clearly, the degree of the polynomial model depends on the step in the expansion at which truncation is made.

Thus, suppose $\eta = f(\beta,X)$, the function being a smooth curve of unknown algebraic form. Via a Taylor series expansion, it is possible to estimate the value of η for X values in the neighborhood of any point $X = X_o$.

$$\eta = f(X_o) + f'(X_o)\,\frac{(X - X_o)}{1!} + f''(X_o)\,\frac{(X - X_o)^2}{2!} \quad . \qquad (3.1)$$

Assuming that the second derivative of the response with respect to input, $f''(X_o)$ goes to zero, equation (3.1) becomes linear in X and by renaming the terms reduces to:

$$\eta = \beta\delta + \beta X$$
$$\eta = \beta(X + \delta), \qquad (3.2)$$

where X is the level of the input factor in smallest supply and δ is the initial level of this nutrient. Thus, the equation of the Law of The Minimum, as well as any other relationship, is represented by the usual linear regression form.

The biological justification may be stated roughly as follows. The multivariable function $\eta = f(\underline{\beta},\underline{X})$ can be expected to have a plateau for levels of $\underline{X}$ that may be called "adequate". These $\underline{X}$ values will be supposed to surround some particular value, say $\underline{X}_o$. As one retreats from $\underline{X}_o$ along one of the components, the yield may be expected to become zero also. Focusing on this one component, the value δ is taken to represent the current effect of part additions or depletions brought about by the farmer. The relation $\eta = \beta(X + \delta)$ then simply reflects the linear portions of what may be a much more complicated function, and can be expected to fit the true situation only for relatively narrow ranges of X values. Since δ is less than "adequate", β likely will be positive. The graphical representation of Liebig's Postulate is shown in Figure 1.

When data are collected to investigate the model $\eta = \beta(X + \delta)$, the model becomes $Y = \eta + \epsilon$. Now it is helpful to distinguish between two components of ϵ. One, say ϵ_1, is taken to represent the deviation of the true response from a linear form, whereas ϵ_2 may be taken to represent experimental error. From (3.2) is obtained the relation $\beta\delta = \beta_o^*$ or $\delta = \frac{\beta_o^*}{\beta}$. This relation indicates that the behavior of the response η is highly dependent on the initial level of soil nutrient. If the rate of increase of η per unit of X (linear regression coefficient β) is small, this will mean that δ is large. If this situation is true, it is unlikely that a straight line will fit the response function. Conversely, if the value of δ is small, the straight line probably is a good representation of the response function in a large range of X. This fact shows that δ imposes a natural boundary condition on the response η; namely, $\eta = 0$ if $X + \delta = 0$. On the other hand, if $X = 0$, then $\eta = \beta\delta$, the natural or potential productivity of the soil. If $\delta = 0$, then $\eta = \beta X$, which is the other extreme case in which η depends entirely on an artificial supply of nutrient (X).

3.1.2 Parabolic Response Law

The basic principle of the postulate states: "The rate of increase of the response η per unit of input X, declines parabolically with respect to the axis of symmetry".

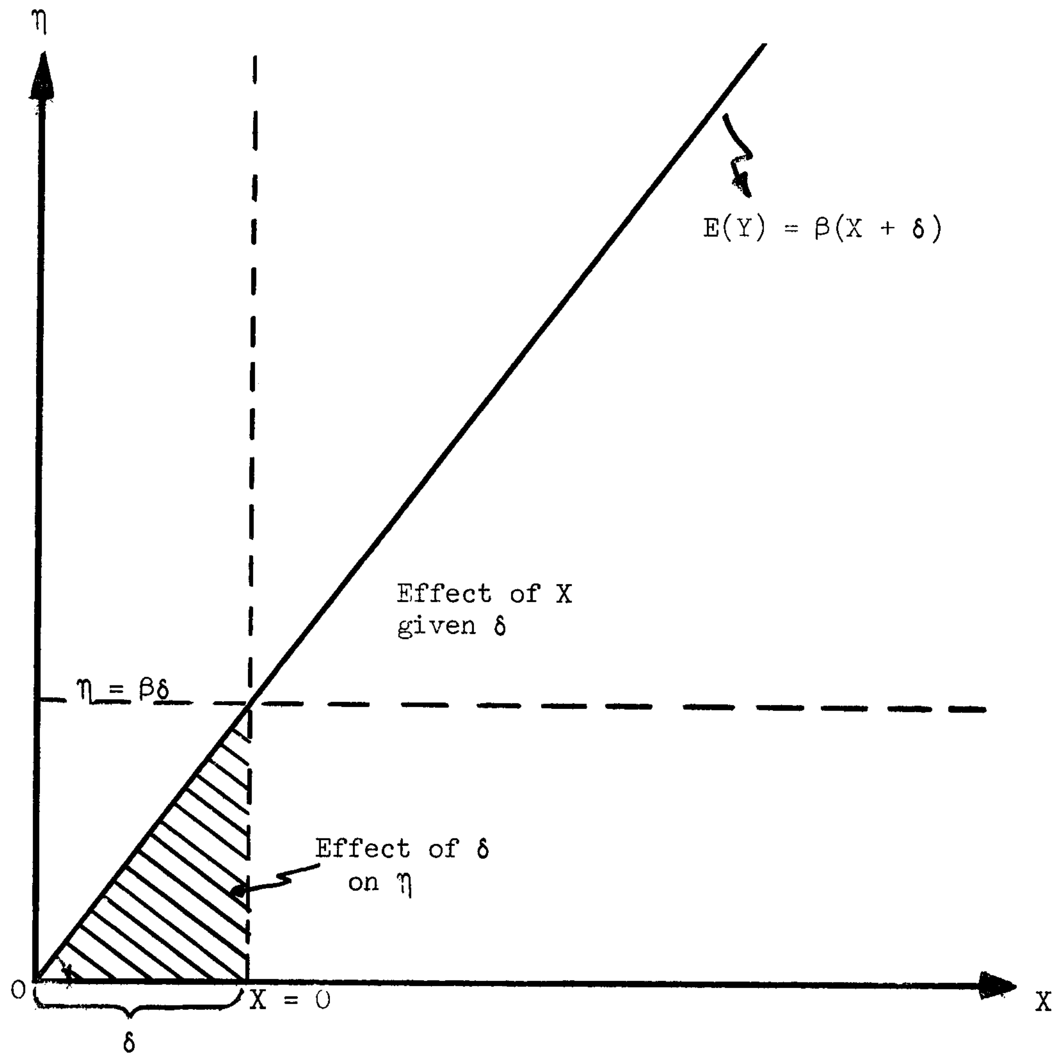

Figure 1: A graphical representation of the law of the minimum in the (X, η)-plane, $\beta > 0$

Following the same argument used to justify Liebig's postulate, one has the following:

$$\eta = f(X_o) + f'(X_o)\,\frac{(X-X_o)}{1!} + f''(X_o)\,\frac{(X-X_o)^2}{2!} + f'''(X_o)\,\frac{(X-X_o)^3}{3!}. \quad (3.3)$$

Assuming that $f'''(X_o)$ goes to zero and after grouping of X terms, equation (3.3) reduces to:

$$\eta = \beta_1(X + \delta) + \beta_2(X + \delta)^2 , \qquad (3.4)$$

where β_1 is the change of the response per unit of change of the input X and β_2 is the rate of change of β_1. The curve is expected to be opened downward, since β_1 must be greater than zero and β_2 smaller than zero.

The biological justification of this postulate is primarily based on the nature of the plant reaction to fertilizer application; that is, the response tends to increase with the supply of X, up to a certain point X_o. Then it is expected to decline as X gets larger. Of course, the term large is relative, because if δ is large enough, η may decline even for small values of X. A graphical interpretation of the postulate is given in Figure 2.

The nature of the parabolic response function indicates that $\eta = 0$ at the point $X + \delta = 0$. Since β_2 is expected to be negative and $\beta_1 > 0$, the response η also is equal to zero at the point $\beta_1(X + \delta) = - \beta_2(X + \delta)^2$. The two points at which the response $\eta = 0$ are equidistant from the point X_o at which η reaches its maximum. However, it has been noticed in practice that the parabolic response pattern

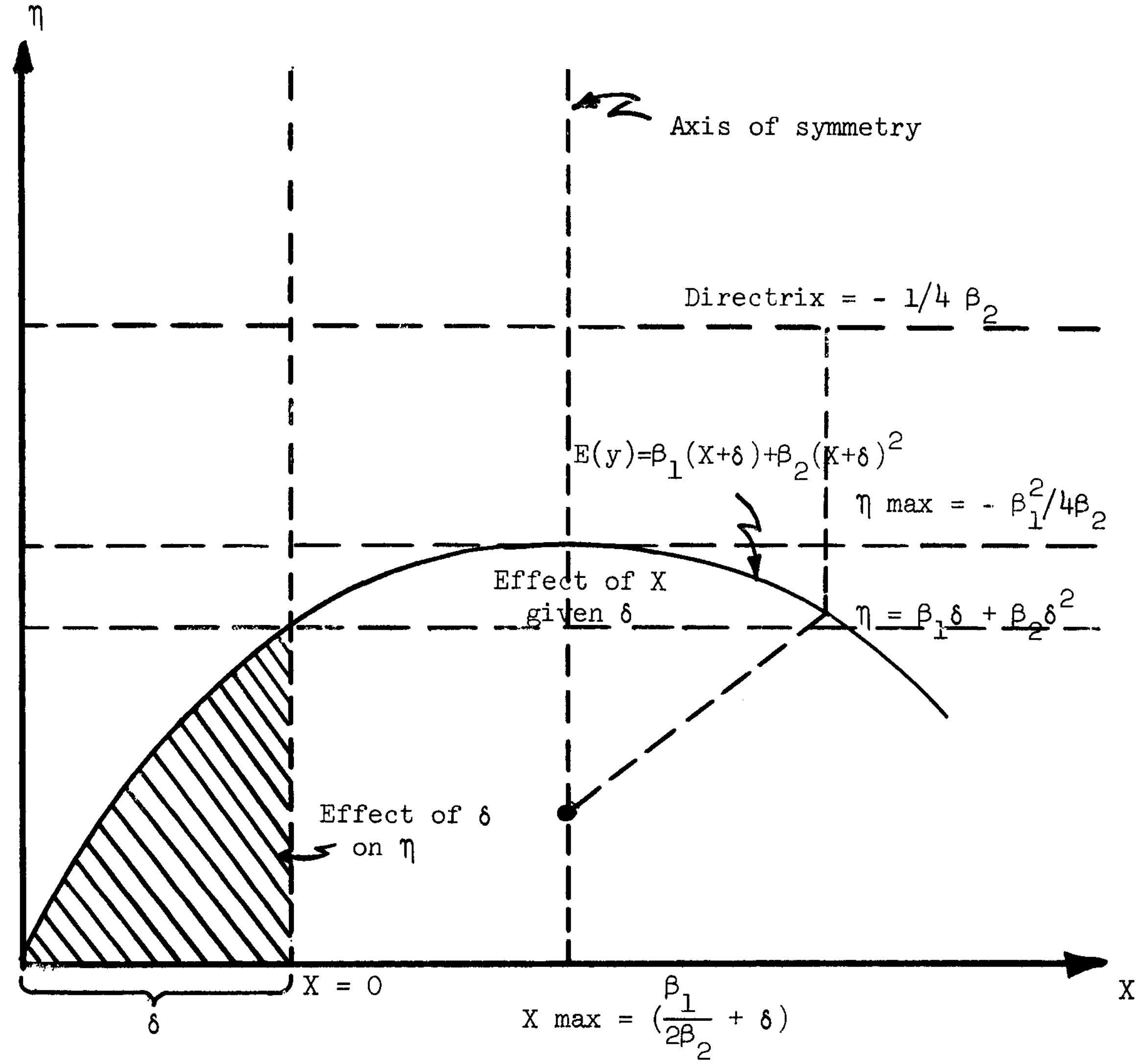

Figure 2: A graphical representation of parabolic response law in the (X, η)-plane, $\beta > 0$ and $\beta_2 < 0$

might be true for a small range of the input factor. Particularly it seems to be that a segment of the parabola fits the data reasonable well when $0 < X + \delta < X_o$; that is, the response η behaves as stated by the postulate, but apparently, for values of $X > X_o$, the response η does not decline parabolically, since η seems to be flat in the vicinity of X_o, even for a large value of X, and then declines slowly. This fact indicates that the response η does not decline symmetrically with respect to the axis of symmetry.

In spite of many dubious features of the Parabolic Response Law, $\eta = \beta_1(X + \delta) + \beta_2(X + \delta)^2$, it is considered to be a realistic function for describing the fertilizer response pattern in some region of the input factor; i.e., the increasing part of the segment of the parabola could fit fairly well the response η. In fact, most of the time researchers are more interested in the increasing portion of the response curve than in the declining portion.

3.1.3 Asymptotic Response Law or Mitscherlich's Postulate

This postulate states that: "The increment of the response η, per unit of the lacking factor X, is proportional to the decrement from the maximum", i.e., $\frac{d\eta}{dX} = (\alpha - \eta)\gamma$. In this case, the response function is $\eta = \alpha(1 - e^{-\gamma(X + \delta)})$.

The steps in the solution of the differential equations are as follows:

Let $\frac{d\eta}{dX}$ be the increment of the response function η per unit of X. Then:

$$\frac{d\eta}{dX} = (\alpha - \eta)\gamma \tag{3.5}$$

$$\int\frac{d\eta}{\alpha-\eta} = \int dX' \quad -\ell n(\alpha - \eta) = \gamma X' + k$$

$$(\alpha - \eta) = k'e^{-\gamma X'}. \tag{3.6}$$

Let $X' = X + \delta$, under the natural condition given by $X + \delta = 0$, then $\eta = 0$, which in turn implies that the constant of integration k' is equal to α, and therefore,

$$\eta = \alpha(1 - e^{-\gamma(X + \delta)}). \tag{3.7}$$

This equation is a member of the family of non-decreasing exponential functions with the following properties: If $X \to \infty$ or δ is large enough, then $\eta \to \infty$, where $\alpha = \max_{\{\eta\}} \{\eta_1, \eta_2, \dots \eta_i\}$ and γ is the rate of decline of the response function from the maximum and δ is the natural or initial level of the nutrient. Sometimes γ is called the efficiency factor, and frequently is expressed in acre/lb. Notice that X and δ are assumed to have the same efficiency; i.e., the coefficients of the linear combination $\gamma_1 X + \gamma_2 \delta$ are equal, $\gamma_1 = \gamma_3 = \gamma$ or $\gamma(X + \delta)$. This assumption may not be true, in general.

The asymptotic response function η has a simply geometrical representation, which is depicted in Figure 3. This function again presents some characteristics which are at variance with biologic fact, for example, the curve is never decreasing. However, in practical problems this aspect might not be very critical, since research workers most of the time are not very much interested in observing the response function for a very high value of the input factor, but

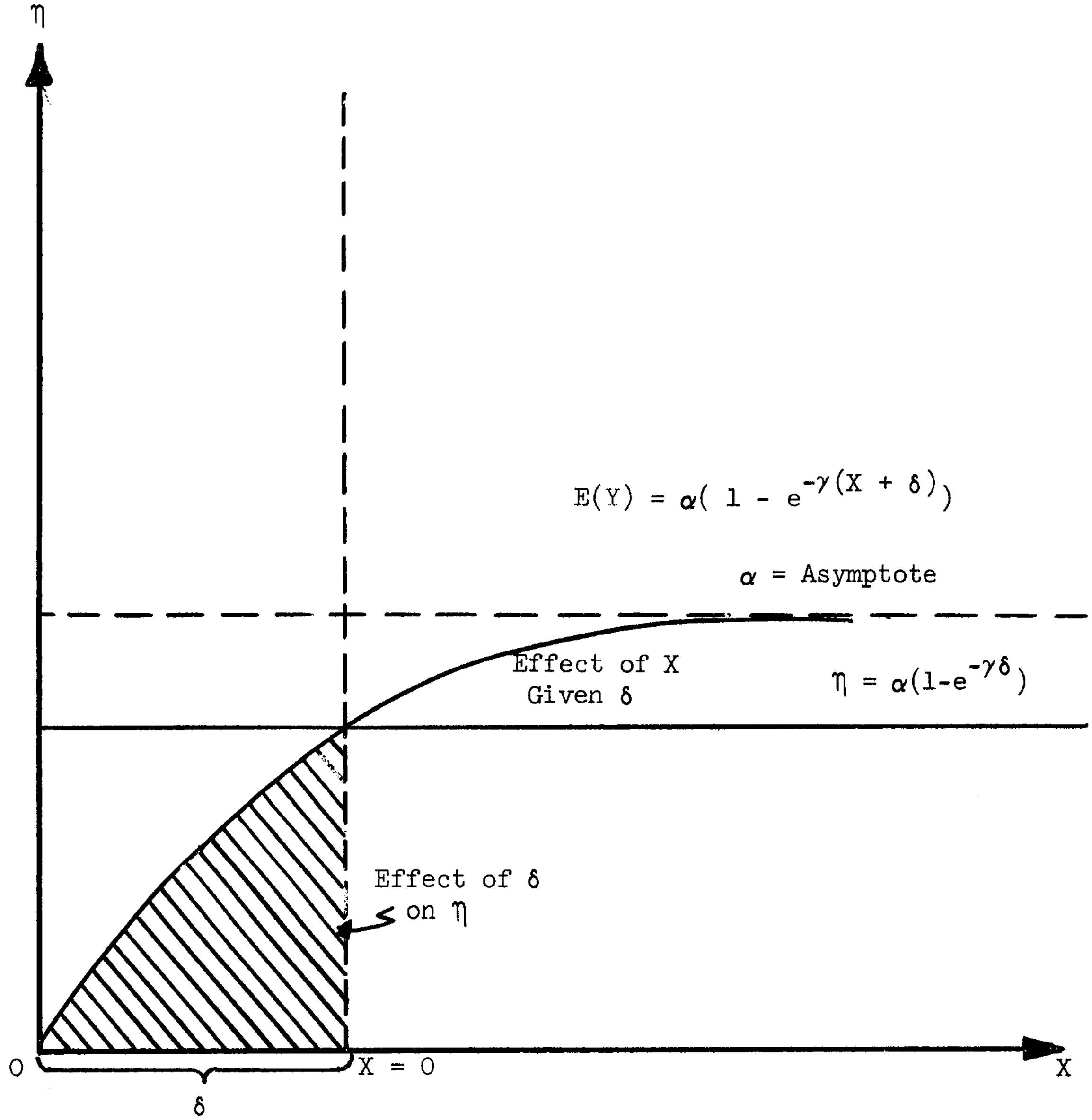

Figure 3: Asymptotic regression law in (X, η)-plane, $\alpha > 0$ and $\gamma > 0$

they might be interested in observing the behavior of η for rather few and small values of X. If this is the case, the Asymptotic Response Law could represent quite well many biological phenomenae.

3.1.4 The Law of the Power Function or Cobb-Douglas Postulate

The principle of this Law states: "The increment of the response η per unit of input X is proportional to the rate of transformation of the input (X) to the output (η)", *i.e.*, $\eta = \beta_o(X + \delta)^{\beta_1}$.

The theory behind the postulate is based on the concept of elasticity in economics, which is illustrated by the differential equation $\beta_1 = \frac{d\eta}{dX} \cdot \frac{X'}{\eta}$.

The solution of such an equation yields the well-known Cobb-Douglas function.

Let $\frac{d\eta}{dX'} = \beta_1 \frac{\eta}{X'}$ be the rate of increase of the production per unit of input. Integrating out both members of the equation:

$$\int\frac{d\eta}{\eta} = \int\beta_1\frac{dX}{X} \qquad (3.8)$$

$$\log \eta = \beta_1 \log X' + k$$

$$\eta = \beta_o(X + \delta)^{\beta'} \qquad (3.9)$$

where $X' = x + \delta$

The power function cannot satisfactorily be used for data with positive and negative marginal products. That is to say, a decline in the total yield will very likely be observed with an application of a large amount of nutrient. In other words, if the input increases very rapidly, the theoretical maximum production is not well defined, unless the economical optimum can be determined for a small dose of the input factor.

The Cobb-Douglas function could over-estimate the most profitable level of fertilizer to be used. This fact can be easily figured out by observing the equation of the marginal product:

$$\frac{d\eta}{dx} = \beta_o \beta_1 \frac{(X + \delta)^{\beta_1}}{(X + \delta)} . \qquad (3.10)$$

In the equation (3.10) if $\beta_1 = 1$, the marginal product is a constant at the level of β_o; if $\beta_1 > 1$ the marginal product increases with increasing value of X; if $\beta_1 < 1$, the marginal product will decrease for increasing value of X, since $(X + \delta)^{\beta_1 - 1} < 1$.

The graphical interpretation of the power function is depicted in Figure 4.

3.2 Extension of the Postulates to Three Input Factors

This section is concerned with the extension of the basic postulates, for a single casual factor described in 3.1, to more than one input factor although this study does not make use of them. To avoid complexity, the discussion focuses attention only upon the situation involving three input factors, i.e., $\eta = f(\underline{\beta}_1 X_1, X_2, X_3)$. For example, X_1 might represent the levels of Nitrogen, X_2 the levels of Phosphorus, and X_3 could be the doses of Potassium, which are the three major nutrients in the soil-plant-fertilizer context. Of course, if more variables are involved in the problem, the same principles can be extended and further extension does not present any difficulties.

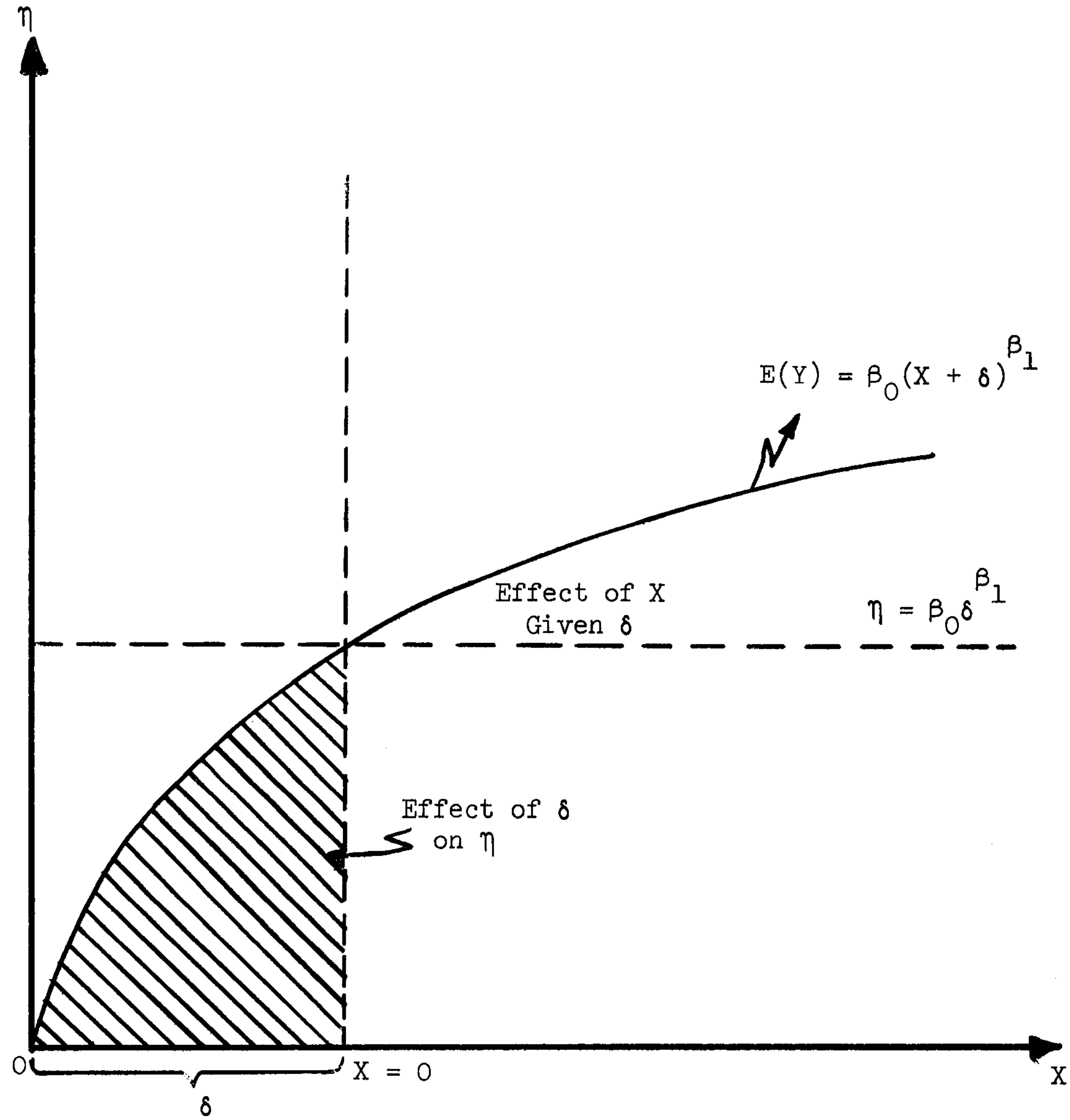

Figure 4: Response curve on the Cobb-Douglas function, for $\beta_1 > 1$

3.2.1 The Law of The Minimum for Three Input Factors

"The response surface η is directly and independently proportional to the amount of any of the factors available in smallest supply". This generalization of the Law of The Minimum has a simply mathematical form, as is shown below:

$$\eta = \beta(X_1 + \delta_1)(X_2 + \delta_2)(X_3 + \delta_3). \qquad (3.11)$$

The natural condition is again obvious here; i.e., if $(X_i + \delta_i) = 0$ for any i = 1, 2, 3 then $\eta = 0$, which means that $(X_i + \delta_i)$ is intrinsic limiting factor. This natural constraint is what is called in Plant Physiology, essentiality of a factor.

3.2.2 The Second-Order Polynomial Law for Three Input Factors

The polynomial response is specified by the surface $\eta = \underline{\beta}'(\underline{X} + \underline{\delta}) + (\underline{X} + \underline{\delta})' B (\underline{X} + \underline{\delta})$, where $\underline{\beta}' = [\beta_1, \beta_2, \beta_3]$ is the rate of change in the response surface η per unit of change in the input vector, $\underline{X} = [X_1, X_2, X_3]$, $\underline{\delta} = [\delta_1, \delta_2, \delta_3]$ is the vector of initial level of nutrient, and B is defined by the following matrix of constants:

$$B = \left\{ \begin{matrix} \beta_{11} & \frac{\beta_{12}}{2} & \frac{\beta_{13}}{2} \\ & \beta_{22} & \frac{\beta_{23}}{2} \\ & & \beta_{33} \end{matrix} \right.$$

The family of surfaces generated by the second-order model is a broad class. This family includes as a particular case the first-order model by putting B = 0.

Some of the typical contours are named: elliptic paraboloid, hyperbolic paraboloid, ellipsoid, hyperboloid of one or two sheets, elliptic cylinder, hyperbolic cylinder, parabolic cylinder and parallel plane. Preliminary information concerning the nature of the response pattern can be obtained by observing the canonical form of the surface, η.

The cross product terms of the surface, η, can be eliminated by rotating the principal axis of the fundamental equation; i.e., there exists an orthogonal transformation matrix, O, such that the surface, η, can be reduced to the following canonical form:

$$\eta = \lambda_1 W_1^2 + \lambda_2 W_2^2 + \lambda_3 W_3^2 \qquad (3.12)$$

where λ's are the characteristic roots of the matrix B, such that $O\ B\ O' = \Lambda$ is a diagonal matrix with the diagonal elements, λ's.

The sign and magnitude of characteristic roots play an important role in the specification of the nature of the response surface. For instance, in equation (3.12), if λ's are all negative, one has an ellipsoidal contour centered about a maximum. If the sign pattern of $[\lambda_1, \lambda_2, \lambda_3]$ is (-, +, 0) then it is describing the hyperbolic cylinder contour.

3.2.3 The Mitscherlich Law for Three Input Factors

"The increment of the response η per unit of any one of the lacking factors is independently proportional to the decrement from the maximum"; the equation of the response η is listed below:

$$\eta = \alpha(1 - e^{-\gamma_1(X_1 + \delta_1)})(1 - e^{-\gamma_2(X_2 + \delta_2)})(1 - e^{-\gamma_3(X_3 + \delta_3)}). \quad (3.13)$$

Equation (3.13) is similar to the Mitscherlich function for one input factor which has been described already.

3.2.4 The Law of the Power Function for Three Input Factors

"The increment of the response η, per unit of each input is proportional to the rate of transformation of each one of the factors".

The Cobb-Douglas equation for three inputs has the following functional form,

$$= \beta_o(X_1 + \delta_1)^{\beta_1}(X_2 + \delta_2)^{\beta_2}(X_3 + \delta_3)^{\beta_3}. \quad (3.14)$$

The power function for the multivariable case presents many interesting features. Thus, if $\beta_1 + \beta_2 + \beta_3 = 1$, this implies that a percentage of the rates of increase of the input will yield the same proportion of increase as the response, η. On the other hand, if $\beta_1 + \beta_2 + \beta_3 < 1$, this indicates that the proportion of the rate of increase of the inputs will produce a smaller rate of increase in the response η and finally, if $\beta_1 + \beta_2 + \beta_3 > 1$, a greater increment of η will be obtained than the rate of increase of the inputs.

3.3 Point of Physical or Theoretical Maximum Rate (MR) for the Parabolic, Mitscherlich and the Cobb-Douglas Response Models.

One Input Factor

Definition 1. The Theoretical Maximum Rate (MR) of an input factor is the point X on the input factor axis which maximizes the response function η.

Consideration of the point of MR for Parabolic and an operationally defined MR for the Asymptotic Regression, and the Cobb-Douglas models is given subsequently. Knowledge of the point MR may play a relevant role in characterizing a response function model. Yet a more interesting application of MR is to find integrated errors in the estimated parameters (β's) in the model, although this in practical application is of little use.

3.3.1 Point of Theoretical Maximum for the Parabolic Model

Given the model $\eta = \beta_1(X + \delta) + \beta_2(X + \delta)^2$, the point $X = X_{MR(P)}$ at which η attains its maximum can be obtained straightforwardly by solving for X the equation $\frac{d\eta}{dX} = \beta_1 + 2\beta_2(X + \delta) = 0$, that is:

$$X_{MR(P)} = -\frac{\beta_1}{2\beta_2} - \delta. \qquad (3.15)$$

This maximum, in general, exists, as has been previously pointed out since $\beta_2 < 0$ and $\beta_1/2\beta_2 \geq \delta$. This implies $\frac{d\eta}{dX} > 0$, and $\frac{d2\eta}{dX2} = 2\beta_2 < 0$, therefore $X_{MR(P)}$, the stationary point, is the value of X which maximizes the response η.

This theoretical attainable maximum value of the response η at $X = X_{MR(P)}$ is given by:

$$\eta_{MR(P)} = - \beta_1^2/4\beta_2. \tag{3.16}$$

From (3.16) it is clear that $\eta_{(MR(P)}$ is independent of the value of the parameter δ.

3.3.2 Point of the Maximum Curvature for the Mitscherlich Response Model

For the Asymptotic Equation, the value of X on the input axis which maximizes the curvature of the response function η can be found. The method used for finding the point of maximum for the Parabolic function is inappropriate because $\frac{d\eta}{dX} = 0$, yields a trivial value of X as a solution. Since this method does not yield a solution, it is considered as an alternative method, the maximization of the curvature:

$$\alpha = \frac{d^2\eta}{dX^2} / [1 + (\frac{d\eta}{dX})^2]^{3/2}. \tag{3.17}$$

It is reasonable to think that by maximizing the curvature of the response, η, and taking that point on the input axis at which the maximum curvature occurs, then one has an equivalent definition of the point of "maximum".

The maximization of curvature is carried out in the usual manner:

$$\left.\begin{aligned} &\frac{d}{dX}\left\{\frac{d^2\eta}{dX^2}\Big/\left[1 + (\frac{d\eta}{dX})^2\right]^{3/2}\right\} = 0 \\ &\frac{d^3\eta}{dX^3}\left[1 + (\frac{d\eta}{dX})^2\right] - 3\frac{d\eta}{dX} \cdot (\frac{d^2\eta}{dX^2})^2 = 0 \end{aligned}\right\}. \tag{3.18}$$

Applying these criteria of maximization to the Asymptotic Response Law, after some algebraic manipulations produces the results:

$$X_{MR(M)} = \frac{1}{\gamma}\,(\ell n \gamma\alpha\sqrt{2}) - \delta. \tag{3.19}$$

The value of η evaluated at $X_{MR(M)}$ is given by:

$$\eta_{MR(M)} = \alpha(1 - e(-\ \ell n \gamma\alpha\sqrt{2})) = \alpha(1 - \frac{1}{\gamma\alpha\sqrt{2}}). \tag{3.20}$$

If the fraction $1/\gamma\alpha\sqrt{2}$ approaches zero, that is, for practical purposes, is negligible, the maximum value of the curvature of the response, η, approaches the asymptote. That is $\eta_{MR(H)} \doteq \alpha$.

3.3.3. Point of the Maximum Curvature for the Power Function

The Power Function or Cobb-Douglas Law, presents even more difficulties of maximization than the Mitscherlich equation. Therefore the maximization of the response, η, has to be carried out through maximizing the curvature. Applying formula (3.18) one gets:

$$X_{MR(PF)} = \sqrt[2(\beta_1-1)]{\frac{\beta_1-2}{\beta_0^2\beta_1^2(2\beta_1-1)}} - \delta. \tag{3.21}$$

Thus, the maximum value of the regression, η, at $X_{MR(PF)}$ is:

$$\eta_{MR(PR)} = \beta_0 \left(\frac{\beta_1-2}{\beta_0^2\beta_1^2(2\beta_1-1)} \right)^{\frac{\beta_1}{2(\beta_1-1)}} .$$

3.4. Point of Optimum Rate (OR) for Parabolic, Mitscherlich and Cobb-Douglas Response Models for One Input Factor

It has been pointed out previously that the farmer is more interested in the optimum rate (OR) of the factor input, instead of the rate producing the physical maximum, therefore, this point is treated very briefly in this section.

Definition 2. The optimum rate of X is that point of X which maximizes the net profit (return) per unit area (acre). Net profit is defined by the difference between the value of the total production and the total cost of fertilizer.

Definition 3. Total cost of fertilizer per unit of area (acre) is defined as a linear function of the amount of fertilizer X, thus the investment in fertilization is given by:

$$C = ¢_0 + ¢_1 X \tag{3.22}$$

where C = total cost (dollars/acre), $¢_0$ = cost of application of fertilizer per unit of area (dollars/acre), $¢_1$ = cost per unit of fertilizer (dollars/lb. of fertilizer) and X = input factor (lb. of fertilizer/acre). Formula (3.22) can be expressed in units of the response η as follows:

$$\eta_1 = \frac{¢_0}{p} + \frac{¢_1}{p} X \tag{3.23}$$

where η_1 = yield equivalent of fertilizer cost in lbs. of yield per acre and p = price per unit of crop yield (dollars/lb. of yield).

Now using definitions 2 and 3, one can obtain the OR of the input factor.

Then $(p\eta - p\eta_1) = p(\eta - \eta_1)$ represents the net profit. Under the assumption that the response $\eta = f(\underline{\beta}, X)$ is differentiable in some revelant region of the input factor, the maximum is obtained immediately by solving:

$$\left.\begin{array}{c}\frac{d}{dX}\left\{p[p(\underline{\beta},X) - \underline{p}_1(\beta_1,X)]\right\} = 0 \\ \text{or} \\ \frac{d}{dX}[p(f(\underline{\beta},X) - (¢_0 + ¢_1X))\} = 0\end{array}\right\} \qquad (3.24)$$

which leads to

$$\frac{df}{dX} = \frac{¢_1}{p} \; . \qquad (3.25)$$

Solving for X in equation (3.25) the stationary value, X which optimizes the response function at that point is obtained. Since expression (3.25) does not involve $¢_0$ then the OR of fertilizer is independent of the cost of the application of fertilizer.

The point of OR of the input factor for the Parabolic and the Mitscherlich response modelscan be produced by direct application of (3.25). Thus for the Parabolic model we have:

$$X_{OR(P)} = \frac{¢_1/p - \beta_1}{2\beta_2} - \delta = X_{MR(P)} + ¢_1/2\beta_2 \qquad (3.26)$$

and the optimum response at the point X_{OR} is

$$\eta_{OR(P)} = \frac{\beta_1^2}{4\beta_2} + \frac{(¢_1/p)^2}{4\beta_2} = -\eta_{MR(P)} + (¢_1/p)^2/4\beta_2 \; . \qquad (3.27)$$

As expected, both $X_{OR(P)} \leq X_{MR(P)}$ and $\eta_{OR(P)} \leq \eta_{MR(P)}$, since in equations (3.26) and (3.27) $\beta_2 \leq 0$. The formulas for obtaining X_{OR} and η_{OR} for the Mitscherlich model are:

$$X_{OR(M)} = X_{MR(M)} + \frac{1}{\gamma} \ln\left(\frac{p}{¢_1\sqrt{2}}\right) - \delta. \qquad (3.28)$$

Inserting $X_{OR(M)}$ into the model equation, $\eta_{OR(M)}$ is obtained.

$$\left.\begin{aligned} \eta_{OR(M)} &= \alpha(1 - \frac{1}{\sqrt{2\alpha\gamma}}) - \frac{1}{\alpha\gamma}(\frac{\sqrt{2}C_1 - p}{\sqrt{2p}}) \\ &= \eta_{MR(M)} - \frac{1}{\gamma}(\frac{\sqrt{2}C_1 - p}{\sqrt{2p}}). \end{aligned}\right\} \quad (3.29)$$

The optimum rate obtained by fitting the Mitscherlich model shows the following features:

$$\text{if } \frac{p}{¢_1\sqrt{2}} > 1 \text{ then}$$

$$X_{OR(M)} \geq X_{MR(M)} \text{ and}$$

$$\eta_{OR(M)} \geq X_{MR(M)}$$

$$\text{if } \frac{p}{¢_1\sqrt{2}} < 1 \text{ then}$$

$$X_{OR(M)} \leq X_{MR(M)} \text{ and}$$

$$OR(M) \leq MR(M).$$

The optimum rate of application, estimated by fitting the Cobb-Douglas equation is obtained in the manner as described above:

$$X_{OR(PF)} = \sqrt[\beta_1 - 1]{\frac{¢_1}{\beta_0\beta_1 p}} - \delta. \quad (3.30)$$

Inserting the formula $X_{PF}(OR)$ in the original equation, $\eta_{OR}(PF)$ is obtained, which is the expected optimum yield, given by:

$$\eta_{OR}(PF) = \beta_0(\frac{¢_1}{\beta_0\beta_1 p})^{\frac{\beta_1}{\beta_1 - 1}}. \quad (3.31)$$

3.5 Point of "Minimum Recommended Rate" (MRR) for the Parabolic, the Mitscherlich and the Cobb-Douglas Response Models for One Input Factor

Under limited capital situations the farmer may not be interested in the MR or in the OR, but rather in the "Minimum Recommended Rate" of the input factor. The concept of MRR was ellucidated by Pesek and Heady (1958) who were inspired by the assertion of Mitscherlich (1909) that the very low rates are unprofitable when cost of application is considered and that a certain minimum rate of fertilizer must be applied before the break-even point is reached.

<u>Definition 4.</u> The Minimum Recommended Rate (MRR) of input factor is that point X in the input factor which maximizes the net return per unit invested in fertilization. Net return per unit invested in fertilization is $p(\eta - \eta_1)/p\eta_1$.

Maximization of the net return per unit invested, $\max_{[X]}\left[\frac{\eta-\eta_1}{\eta_1}\right]$ is straightforward,

$$\frac{d}{dX}\left[\frac{p(f(\underline{\beta},X))}{¢_0 + ¢_1 X} - 1\right] = 0 \qquad (3.32)$$

$$\frac{f(\underline{\beta}_1,X)}{f'(\underline{\beta}_1,X)} = \frac{¢_0}{¢_1} + X. \qquad (3.33)$$

Solving for X in equation (3.33) gives the MRR of fertilizer. This formula is independent of the price per unit of crop yield (p).

The point of MRR of input factor for the Parabolic and Mitscherlich response models can be obtained by direct application of (3.33). Thus far the Parabolic model one has:

$$X_{MRR(P)} = -\frac{\phi_0}{\phi_1} + \sqrt{\left(\frac{\phi_0}{\phi_1}\right)^2 + \frac{\phi_1\delta(\beta_2\delta+\beta_1)}{\phi_1\beta_2} - \frac{\phi_0(2\beta_2\delta+\beta_1)}{\phi_1\beta_2}}$$

$$X_{MRP}(P) = +\sqrt{\left(\frac{\phi_0}{\phi_1}\right)^2 - 2\frac{\phi_0}{\phi_1}\left(\delta + \frac{\beta_1}{2\beta_2}\right) + \delta\left(\delta + \frac{\beta_1}{\beta_2}\right) - \frac{\phi_0}{\phi_1}} \qquad (3.34)$$

by completing the square under the radical and after some algebraic manipulation the following is obtained:

$$X_{MRR(P)} = +\sqrt{\left(\frac{\phi_0}{\phi_1} + X_{MR(P)}\right)^2 - \left(X_{MR(P)} + \delta\right)^2 - \left(\frac{\phi_0}{\phi_1}\right)}. \qquad (3.35)$$

Equation (3.35) yields meaningful results if the term under the square root of the expression is greater or equal to $\frac{\phi_0}{\phi_1}$.

The value of η at X_{MRR} does not yield a simple expression, therefore it will not be given. However, it is easy enough to obtain numerically.

For Mitscherlich's Response Model, the MRR of the input factor is as follows:

$$X + \delta = \frac{1}{\gamma}\left[\ell n\left(1 + \gamma\left(\frac{\phi_0}{\phi_1} + X\right)\right)\right]. \qquad (3.36)$$

Equation (3.36) has no explicit solution but can be solved approximately by iterative procedures or by expansion of the logarithmic function and truncating around some point X_0. The attainable value of η at MRR can also be obtained approximately.

The estimated Minimum Recommended Rate when the Power Function is fitted can be derived following the same scheme described above:

$$X + \delta = \beta_1\left(\frac{\phi_0}{\phi_1} + X\right)$$

$$X_{MRR}(PF) = \frac{\delta - \beta_1\frac{\phi_0}{\phi_1}}{\beta_1 - 1} . \qquad (3.37)$$

Therefore the expected yield at the point $X_{MRR}(PF)$ will be:

$$\eta_{MRR(PF)} = \beta_0\left(\frac{\beta_1(\delta\phi_1 - \phi_0)}{\phi_1(\beta_1 - 1)}\right)^{\beta_1} . \qquad (3.38)$$

It is interesting to point out that for all of the functions described here, the X_{MR}, X_{OR}, and X_{MRR} are dependent on the initial level of fertilizer in the soil. However the expected yield shows different behavior. Thus η_{MR} and η_{OR} are independent of the initial level of soil nutrient, δ, but η_{MRR} is a function of δ, for any of the models treated in this study.

3.6 Extension of the Concept of the Theoretical Maximum for Three Input Factors

In general, in multidimensional space the concept of maximization of some function is more complicated. Thus for the quadratic surface of the response η in the three variables described previously, one has:

$$\frac{\partial\eta}{\partial\underline{X}} = \underline{\beta} + 2B(\underline{X}+\underline{\delta}) = 0. \qquad (3.39)$$

The solution of equation (3.39) gives the factor combination $\underline{X}_0$, at which η is a local maximum or minimum, or a local stationary value where:

$$\underline{X}_0 = -\tfrac{1}{2}B^{-1}\underline{\beta} - \delta \ . \tag{3.40}$$

A necessary and sufficient condition for η to have a maximum at $\underline{X}_0$ is that the B matrix be negative definite, where:

$$B = \begin{bmatrix} B_{11} & \frac{B_{12}}{2} & \frac{B_{13}}{2} \\ & B_{22} & \frac{B_{23}}{2} \\ & & B_{33} \end{bmatrix} .$$

The value of the response η at $\underline{X}_0$ is given by:

$$\eta_o = -\tfrac{1}{4}\underline{\beta}' B^{-1}\underline{\beta} \tag{3.41}$$

From equation (3.41) it can be concluded that the maximum or minimum or a local stationary value attained by η is independent of the vector of initial levels of nutrient.

For the Mitscherlich response model the process of maximization is achieved by maximizing the curvature of η with respect to each input factor.

$$\begin{pmatrix} X_{10} \\ X_{20} \\ X_{30} \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma_1}\ell n(\sqrt{2\alpha\gamma_1}) \\ \frac{1}{\gamma_2}\ell n(\sqrt{2\alpha\gamma_2}) \\ \frac{1}{\gamma_3}\ell n(\sqrt{2\alpha\gamma_3}) \end{pmatrix} - \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} . \tag{3.42}$$

Thus the value of the response surface at $\underline{X}_0$ is as follows:

$$\eta_0 = \alpha(1-e^{-\ell n(\sqrt{2\alpha\gamma_1})})(1-e^{-\ell n(\sqrt{2\alpha\gamma_2})})(1-e^{-\ell n(\sqrt{2\alpha\gamma_3})})$$

$$= \alpha(1 - \frac{1}{\sqrt{2\alpha\gamma_1}})(1 - \frac{1}{\sqrt{2\alpha\gamma_2}})(1 - \frac{1}{\sqrt{2\alpha\gamma_3}}). \tag{3.43}$$

Suppose $\frac{1}{\sqrt{2\alpha\gamma_i}} \to 0$, then

$$\eta_0 \doteq \alpha$$

and the value of the response η at $\underline{X}_0$ is close to the asymptote of the surface α.

When the Cobb-Douglas equation is fitted with three input factors, the stationary points are given by:

$$\begin{bmatrix} X_{10} \\ X_{20} \\ X_{30} \end{bmatrix} = \begin{bmatrix} 2(\beta_1-1)\sqrt{\frac{\beta_1-2}{\beta_0^2\beta_1^2(2\beta_1-1)}} \\ 2(\beta_2-1)\sqrt{\frac{\beta_2-2}{\beta_0^2\beta_2^2(2\beta_2-1)}} \\ 2(\beta_3-1)\sqrt{\frac{\beta_3-2}{\beta_0^2\beta_3^2(2\beta_3-1)}} \end{bmatrix} - \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \qquad (3.44)$$

Inserting the stationary vector $\underline{X}_0$ into the equation of η, the value of η_0 is obtained.

4. MATERIALS AND METHODS

In this chapter the basic materials and methods utilized in the present study are described. It was preferred to use actual data, obtained from fertilizer experiments on soils which had been characterized, because of the possibility of comparing the results obtained by direct estimation of the parameter, δ, with the information on organic matter content, the percentage of nitrogen, and the nitrate production available on these soils. Although it may be tempting for convenience to use data generated by a simulation technique instead of experimental data, the scientific status of one's conclusions is more secure with actual data.

4.1 Fertilizer-Response Data

The basic data, corn yields, utilized in this study were obtained from the TVA - N. C. State Cooperative corn project described by Baird (1958). The experiment consisted of an N-P-K factorial arranged according to an augmented central composite design with 18 points and 4 replications. The project included 37 locations on Norfolk soils and 23 on Portsmouth soils during the years 1955, 1956, and 1957. About 20 experiments were conducted each year. As pointed out by Baird (1958), there were numerous variables which affected the yield and nutrient content of the corn produced in the field even though variability was reduced by using only one variety of corn at all locations.

One may attempt to explain the great variability observed in the corn responses as caused by local variation in numerous enviromental factors such as: light, temperature, rainfall, etc., which, in conjunction with the supply of nutrients and other physical and chemical factors, will be responsible for the crop responses. In addition, other factors, many times overlooked, can contribute to the variation observed in the crop response. For example, insects, nematodes and incidence of diseases may be very important components of the total variation in the crop growth and production. There may be enough within site variation is these external factors which were not evaluated to cause some confounding of response to nitrogen with these other factors. However it should be remembered that tests were conducted at a number of locations and that the effects of some of these auxiliary factors would not be expected to be manifested at all locations in the same manner.

Consequently, on the average, the effect of the individual factors not taken into consideration would be expected to be small. Although the effects of phosphorus and potassium were not considered to be large for the locations selected, there may have been some local variation both within and between sites caused by the variation in P and K and their interactions with nitrogen. Inspection of individual plot data reveal some evidence of P and K effects within certain locations.

All factors which conceivably might affect yield should be taken into consideration, where an evaluation of the biological response is made. However, usually it is not possible to quantitatively measure all these factors and thus it is difficult to separate the effects into contributions from all the various causal factors.

The results obtained by Baird, fitting the second order polynomial, $Y = f(N,P,K) + \epsilon$ indicated that response is strongly influenced by the nitrogen fertilization. Nitrogen fertilization generally increases the yield and the nitrogen content of the plants more than does corresponding fertilization with phosphourus and potassium. This is particularly true on Norfolk soil.

Welch (1960) utilized data from the TVA corn project and determined by chemical analysis, the percent of nitrogen the organic matter and the nitrate-production following nitrogen fertilization of Norfolk soil. These chemical analyses indicated that the yield responses from nitrogen application can be predicted from soil test data. He emphasized that the use of soil organic matter to express available nitrogen did not show significant influence on corn yields either with or without nitrogen application. The narrow range in soil organic matter and the wide range in corn yield and yield response no doubt accounted for the non-significant relationship.

On the other hand, Welch (1960) concluded that the results of NO^3-production tests gave a fair correlation with corn yields on the Norfolk soils. He pointed out, too, that one of the most important factors affecting responses to nitrogen fertilization on Coastal Plain soils is rainfall during the growing season. Finally he concluded that nitrogen measured directly by the Kjeldahl method or indirectly by organic matter gives only a crude estimate of the total nitrogen present in soil, but NO_3-production gives information on the quantity and rate at which nitrogen is mineralized to available forms.

4.2 Criterion Used for Selecting Data for Numerical Estimation of the Initial Nitrogen Level

Twenty-five different locations were chosen after evaluating the Baird and Welch results for the set of 60 locations. The choice of the 25 locations was based mainly on the criterion of good responses to nitrogen application, lack of response to phosphorus and potassium and absence of drought. Drought is an important factor strongly influencing corn yield. The great variation observed in the production of corn at many of the different locations, probably is caused by drought. This is one of the reasons for choosing only 25 carefully selected locations for this research.

In addition to the percentage of nitrogen, the percentage of organic matter and nitrate production are available for each location. Those quantities logically would be correlated to some extent among themselves and also with the estimated soil nutrient, $\hat{\delta}$, obtained by direct estimation procedures.

4.3 Techniques and Procedures of Estimation

In the present study interest is focused on the estimation of the parameter δ, the initial level of soil nutrient, and particularly on the initial level of nitrogen in the soil as estimated through corn response to fertilizer application. The procedure consisted of developing a response model by means of transformation of the input factor (applied fertilizer) and then making an actual estimation of the initial level of soil nutrient by the technique of non-linear least squares. These two procedures are described in detail in a subsequent section.

4.3.1 Principle of Transformation Searching for the Best Fitting Transformation Model

The word transformation is widely used in statistics to refer to replacing observed values by the values of some function of them. This change may involve the response variable, the parameters involved in the model or the transformation of one or more independent variables.

The transformation of the response η or the dependent variable is a well known and useful statistical technique for the purpose of stabilizing variances or fulfilling the assumption of normality or additivity, so that some of the techniques of statistical analysis (e.g. analysis of variance) can be more reliable on the transformed space or on a new scale. Another important use of the transformation is in the field of non-linear estimation. Thus, if $\eta = f(\underline{\beta}, X)$ is non-linear in the parameters but linearizable by means of a proper transformation then $\underline{\beta}$ can be estimated in the transformed space by a standard linear estimation procedure. With increasing interest in fitting non-linear models, there has been a need for techniques for transforming parameters.

Some methods of transformation of the parameter vector $\underline{\beta}$, have been suggested by Beale (1960) and other authors. Meeter (1964) discussed extensively the transformation of the parameter in the context of non-linear response function models. The purpose of his work mainly is to accelerate convergence in the non-linear least squares estimation technique.

The third place to apply transformations is on the independent variables. The transformation is on the independent variables, X, in the model, $f(\underline{\beta}, X)$ to the form $f(\underline{\beta}, T)$, where T is the model matrix in the transformed factor space, and the transform vector is $\underline{T}_i = f(\underline{X}_i, \underline{T}_i)$, where $\underline{T}_i$ is the vector of constants generating the transformation.

Box and Tidwell (1962), with the aim of fitting data to a single functional form in the space of the transformed variables or new variable space, developed the general concept of the function of X's under the assumption that the response function is at least approximately normal, $N(X\underline{\beta}, I\sigma^2)$. They focused attention on the fitting of a polynomial of low degree in the transformed variable rather than fitting a more complicated function in the original variables. They presented examples and detailed discussions on the first degree, second degree, and more general polynomial approximations.

4.3.1.1 The Second Order Model and the Power Transformation of the Independent Variance

In this study the basic model represents a combination of a power transformation of the independent variable and a second order model equation. The power transformation can be expressed as follows:

$$T = (X + \delta)^p \qquad (4.1)$$

The second order model becomes:

$$\eta = \beta_1 T + \beta_2 T^2. \qquad (4.2)$$

In the equation (4.1), if $p = \frac{1}{2}$, then equation (4.2) is the usual parabolic response function, fitted on the square root transformation, that is:

$$\eta = \beta_1 (X + \delta)^{\frac{1}{2}} + \beta_2 (X + \delta). \qquad (4.3)$$

If p = 1, equation (4.2) is converted to the identity transformation given by the following parabolic equation:

$$\eta = \beta_1 (X + \delta) + \beta_2 (X + \delta)^2. \qquad (4.4)$$

The third and more general type of transformation was obtained by allowing p to take any value in the range of .25 to 1.00. This type of transformation is what is called "the non-fixed" power transformation.

$$\eta = \beta_1 (X + \delta)^p + \beta_2 (X + \delta)^{2p}. \qquad (4.5)$$

Of course this model amounts to a specific fixed power transformation for a given p. Since p can take any value in a given interval, infinitely many transformations are possible.

Notice that p is really a parameter to be estimated from the data according to some method of estimation, which in this case is the least squares technique. Also it must be recalled that with the addition of p (power of the transformation) one has one more parameter to be estimated, in addition to β_1, β_2 and δ. This fact may be important if few degrees of freedom are available for estimating residual deviations from the model.

4.3.1.2 Special Kinds of Power Transformation of the Input Factor

In this section attention is focused on a special kind of transformation of the input factor. The technique used here is more or less equivalent to that of an empirical model building approach. Again use of a fixed and non-fixed power transformation is attempted. However, the model is no longer of the second order in the transformed variable, but a power transform in one variable, keeping always un-transformed one of the vector components of the design matrix X.

The fixed power transformation of the input factor used in the special model fitting, can be represented in the following form:

$$T = \left\{ \begin{array}{ll} T_1 = (X + \delta)^p & \text{for } p = .75 \text{ or } .95 \\ & \\ T_2 = (X + \delta) & \text{(identity)} \end{array} \right\} \tag{4.6}$$

then the functional forms become:

$$\eta = \beta_1 T_1 + \beta_2 T_2 = \beta_1 (X + \delta)^p + \beta_2 (X + \delta), \tag{4.7}$$

where p = .75 and .95.

The other fixed power transformation is given by:

$$T = \begin{cases} T_1 = (X + \delta) & \text{(identity)} \\ T_2 = (X + \delta)^p & \text{for } p = 1.25,\ 1.50,\ 1.75 \end{cases} \tag{4.8}$$

which produces the models:

$$\eta = \beta_1(X + \delta) + \beta_2(X + \delta)^p \quad p = 1.25,\ 1.50,\ 1.75.$$

The non-fixed special power transformation discussed here has the restriction on p, that $p \in [.25, 2.00]$. The boundary of p was arbitrarily established, but from a practical point of view, the boundary seems to be realistic, since for polynomials containing only two terms, $p > 2.00$ might occur in very few cases, particularly in dealing with fertilizer response functions. The two functional forms considered for this particular type of transformation are:

$$\eta = \beta_1(X + \delta)^p + \beta_2(X + \delta) \text{ for } p \in [.25, 1.00) \tag{4.9}$$

and

$$\eta = \beta_1(X + \delta) + \beta_2(X + \delta)^p \text{ for } p \in (1.00, 2.00]. \tag{4.10}$$

It turned out that there are strong reasons for bounding the powers of the transformation. On one hand the transformation used affects the estimate of δ directly; in fact $\hat{\delta}$ is highly positively correlated with $\hat{p}$. Bounding the powers can be used to eliminate non-sensical estimates of δ. On the other hand, since the model is essentially non-linear, the estimation of the parameters is tedious. The solution is expedited by setting lower and upper bounds upon the power which makes rapid convergence possible.

4.3.2 The Non-linear Least Squares Method of Estimation of Parameters

In many problems encountered in practice, for simplicity it is assumed that the model is additive and linear in the parameters. The assumption of additivity is reasonable in many cases and it may be true in a number of practical problems. The assumption of linearity in the parameters in many cases fails. In most of the physical situations, indeed, the linearity of the model is an exception and the non-linearity is more or less the rule. Nevertheless, in dealing with biological problems no model describes completely such complex phenomena.

A model is said to be linear in the parameter if $\frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}}$ is independent of $\underline{\beta}$, i.e., does not involve $\underline{\beta}$, that is, if $\frac{\partial}{\partial \underline{\beta}}\left[\frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}}\right] = 0$. Models not encompassed in this definition are called non-linear models.

From the estimation point of view, one may attempt to classify the models into three groups: linear models, non-linear but linearizable models and intrinsically non-linear models.

In this study, due to the nature of the parameter δ, the models turn out to be intrinsically non-linear. Therefore from the estimation point of view the linear least squares minimization of the error sum of squares does not exist, and an iterative procedure has to be used. There are two basic procedures in common usage in the field of non-linear estimation, the Gauss (1821) linearization method, and the steepest descent of Curry (1944). Many other modifications of these basic methods have been suggested due to difficulties in obtaining convergence as well as slowness of convergence. The useful modification developed by Morrison (1960) and Marquardt (1963) is in common use. This procedure is readily available in the literature.

The error in the observations and the experimental design could affect in some way the non-linearity. Beale (1960) has shown that where non-linearity is small, the linearization theory results in a valid approximation. In non-linear estimation, the estimated parameters, $\underline{\beta}$, are not in general, normally distributed. Also, $E(s^2) \neq \sigma^2$, but depends in some sense upon $(X'X)^{-1}\sigma^2$, i.e., $(X'X)^{-1}\sigma^2$ is not the proper variance-covariance matrix.

The technique of non-linear least squares used here is the Gauss Linearization Method, which is illustrated subsequently.

By definition, linear approximation amounts to the following form:

$$\eta = f(\underline{\beta},X)\Big|_{\underline{\beta} = \underline{\beta}_o} + \frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}}\Big|_{\underline{\beta}=\underline{\beta}_o} (\underline{\beta}-\underline{\beta}_o) + O\{(\underline{\beta}-\underline{\beta}_o)^2\}. \qquad (4.11)$$

This method consists in essence in the expansion and truncation of the function $\eta = f(\underline{\beta},X)$ through the first derivative, then evaluation of $\frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}}$ at the initial guess or preliminary value of the parameters, $\underline{\beta}_0$, is made, thus the design matrix:

$$\frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}}\Big|_{\underline{\beta} = \underline{\beta}_o}$$

is generated, and the procedure is equivalent to that of the linear least squares method of estimation. The estimates of parameters are improved by Newton's iteration procedures until convergence is reached. The procedure is illustrated on the top of the following page:

Let $\underline{Y} = f(\underline{\beta},X) + \underline{e}$ be the observation vector in the response space, and it is desired to have:

$$\min_{\{\underline{\beta}\}} \underline{e}'\underline{e} = \min_{\{\underline{\beta}\}} \{\underline{Y} - f(\underline{\beta},X)\}'\{\underline{Y} - f(\underline{\beta},X)\}.$$

The linear approximation can be written as:

$$\underline{Y} = f(\underline{\beta}_0,X) + Z(\underline{\beta} - \underline{\beta}_0) + \underline{\epsilon}^* \tag{4.12}$$

where $\underline{\beta}_0$ is the first guess for the least squares estimate of $\underline{\beta}$ then, $Z = \frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}} \Big|_{\underline{\beta} = \underline{\beta}_0}$.

Letting $\underline{Y}^* = \underline{Y} - f(\underline{\beta}_0 X)$ the above expression can be written as

$$\underline{Y}^* = Z(\underline{\beta} - \underline{\beta}_0) + \underline{e}^*. \tag{4.13}$$

Now, since the approximation amounts to a linear least squares problem, then the task is to find that value $\underline{\Delta}_\beta = (\underline{\beta} - \underline{\beta}_0)$ such that $\underline{e}^{*\prime}\underline{e}^*$ is a minimum, that is to say:

$$\min_{\{\underline{\beta}\}} \{\underline{e}^{*\prime}\underline{e}^*\} = \min_{\{\underline{\beta}\}} \{Y^* - Z(\underline{\beta} - \underline{\beta}^o)\}'\{Y^* - Z(\underline{\beta} - \underline{\beta}^o)\} \tag{4.14}$$

$$\frac{\partial \underline{e}^{*\prime}\underline{e}^*}{\partial \underline{\Delta}_\beta} = -\, 2Z'\underline{Y}^* + ZZ'Z\underline{\Delta}_\beta = 0$$

$$\hat{\underline{\Delta}}_\beta = (Z'Z)^{-1}Z'Y^*$$

$$\hat{\underline{\Delta}}_\beta = \underline{\beta} = \underline{\beta}_0$$

$$\hat{\underline{\beta}}^{(1)} = \underline{\Delta}_\beta + \underline{\beta}_0$$

and, in general, this process can be written as:

$$\hat{\underline{\beta}}^{(i+1)} = \underline{\Delta}_{\underline{\beta}}^{(i)} + \underline{\beta}^{(i)} = [Z^{(i)\prime}Z^{(i)}]^{-1}Z^{(i)}\underline{Y}*^{(i)} + \underline{\beta}^{(i)}. \qquad (4.15)$$

Iteration will continue until convergence, that is until

$$|\beta^{(i+1)} - \beta^{(i)}|$$

is "small enough." In reality it is not necessary to know the partial derivation $\frac{\partial f(\underline{\beta},X)}{\partial \underline{\beta}}$ in order to evaluate the design matrix Z. It is usually easiest to approximate the derivative by the difference quotient

$$\frac{\partial f(\underline{\beta},X)}{\partial \beta_i} = \frac{f(\beta_1,\ldots,\beta_1+\Delta_{\beta_1},\ldots,X) - f(\underline{\beta}X)}{(\beta_i + \Delta_{\beta_i}) - \beta_i}. \qquad (4.16)$$

Thus the value of the parameter $\underline{p}$ which yields the min$\{\underline{ee}\}$, exactly occurs by using $\underline{\beta}$ and only approximately min$\{\underline{e}*'\underline{e}*\}$ by using $\underline{\beta}^{(\ell+1)}$.

Hartley (1961) showed that under certain general assumptions the $\underline{e}'\underline{e}_{\beta}$ decreased initially along the vector $\underline{\beta}^1 - \underline{\beta}^0$; therefore, the Gauss method must converge (theoretically) since for small $\lambda > 0$, the $SS(\underline{\beta}_0 + \lambda\{\underline{\beta}^1 - \underline{\beta}_0\} < SS(\underline{\beta}_0)$, where $\underline{\beta}^1 - \underline{\beta}_0$ is the correction vector predicted by the Gauss method.

Data were processed in the Triangle Universities Computation Center with an IBM 360/75. The computer program used for carrying out the computations described on the previous pages was the Statistical Analysis System (SAS), established at the Computer Center by Mr. J. Goodnight and Mr. A. J. Barr.

The program used essentially consisted of a iterative multiple regression package. Using this package, the process can be repeated any number of times, as specified in the subroutine.

Once the model is specified and the design matrix is produced by giving the initial guessed values of the parameters and the input data, then, the iterative process continues until convergence. The output consists of the bivariate statistics, the parameter estimates, analysis of variance and the variance-covariance matrix of the estimated parameters.

5. RESULTS AND DISCUSSION

In Chapter 4, the basic material to be used in this study, as well as the methodological treatment to be applied to the data, such as transformations of input factors, and the estimation procedure of the parameters involved in each model were described. This chapter is devoted to the actual estimation of the parameters involved in the model. The main interest is focused upon the estimation of the initial level of soil nutrient (δ), detected by means of the crop response. It is also of interest to know the behavior of the parameter δ and its relationship with the other parameters describing the model, etc.

Dr. L. A. Nelson* (1968) pointed out that the second order model fitted to the data, using the input factor without any transformation tends to produce an overestimation of the value of δ, i.e., δ is biased upward. On the other hand, the use of very small values of the power in the transformation of the input, seems to result in an under estimate of the parameter δ, i.e., the estimate is biased downward.

When dealing with an intrinsic non-linear model, the estimates of parameters are only approximate. However, in the case treated in this research, the approximation is "quite good", so that the results can be used to compare the goodness of the models described in a previous chapter. Although an approximate algebraic comparison might be possible the algebra is quite messy. Numerical comparisons, on the other hand, are quite feasible.

---

* Personal communication, North Carolina State University, Raleigh, N.C.

In view of this fact, only numerical results will be given and in light of the results one will be able to judge the goodness of the models, at least approximately.

5.1 Estimation of the Parameter δ and the β Parameters by Fitting the Models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$

In Table 1 are reported the individual estimates of the parameter δ. The estimation was carried out by individual replication, then, the average (over 4 replications) on a per location basis, was computed. Both pre-fixed and non-fixed power transformation models were used. The powers were set at .75, .95, 1.25, 1.50 and 1.75 for the pre-fixed power model, and the power was allowed to vary from .25 to 2.00 for the non-fixed power case. The domain of p was decided upon the basis of experience. The values of $p < .25$ detected only a trace of $\hat{\delta}$ which is known not to be the case for Norfolk soils. On the other hand, values of $p > 2.00$ were found to yield extremely high values of $\hat{\delta}$, which also contradict the facts.

The effect of the power transformation on the parameter δ is given by 0.1 unit of increase in the pre-fixed power p, the value of the estimate $\hat{\delta}$ increases approximately 3.6 lbs. (35.87 lbs./1 unit of power). This very fact shows the great importance of the choice of model for estimating initial level of nutrient content in the soil.

The behavior of the estimated parameter $\hat{\beta}_1$ is very closely related to the variation of the power. In Table 1 of the Appendix is reported the estimates, $\hat{\beta}_1$, obtained for sites within location. It is very interesting to notice that the rate of increase in the response per unit of input factor ($\hat{\beta}_1$) tends to increase up to a certain point, as the power increases, and when p is less than 1. Theoretically, $\hat{\beta}_1$ should attain its maximum at $p = 1$, which is the case of the simple straight line. After reaching the maximum, the estimated parameter, $\hat{\beta}_1$, tends to decrease when the power p is greater than 1. This fact is depicted very clearly in Figure 5. This Figure was constructed taking the overall mean of the estimated parameters, but the tendency is fairly constant for each replication within location. So that the curve drawn on the average basis, gives a good idea of the behavior of the estimate of the parameter β_1, represented in the model.

The curve of the estimated parameter $\hat{\beta}_2$, shows the opposite tendency to that described for $\hat{\beta}_1$. In other words, the curve of $\hat{\beta}_2$ decreases down to the minimum when the power p of the transformation tends to increase up to 1. The theoretical minimum of $\hat{\beta}_2$ will occur at the point $p = 1$, but unfortunately this value is a singularity point, in fact $\hat{\beta}_2$ does not exist when $p = 1$, since it is confounded with $\hat{\beta}_1$. After $\hat{\beta}_2$ has reached the minimum, it starts rising very rapidly up to $p = 1.25$, approximately, and then for $p > 1.25$ the value of $\hat{\beta}_2$ shows very little variation, although the tendency continues to decrease very slowly. More detailed information about the behavior of $\hat{\beta}_2$ can be found in Table 3 of the Appendix.

The estimates $\hat{\delta}$, $\hat{\beta}_1$ and $\hat{\beta}_2$, obtained by fitting the model with the non-fixed power transformation were not considered in drawing the

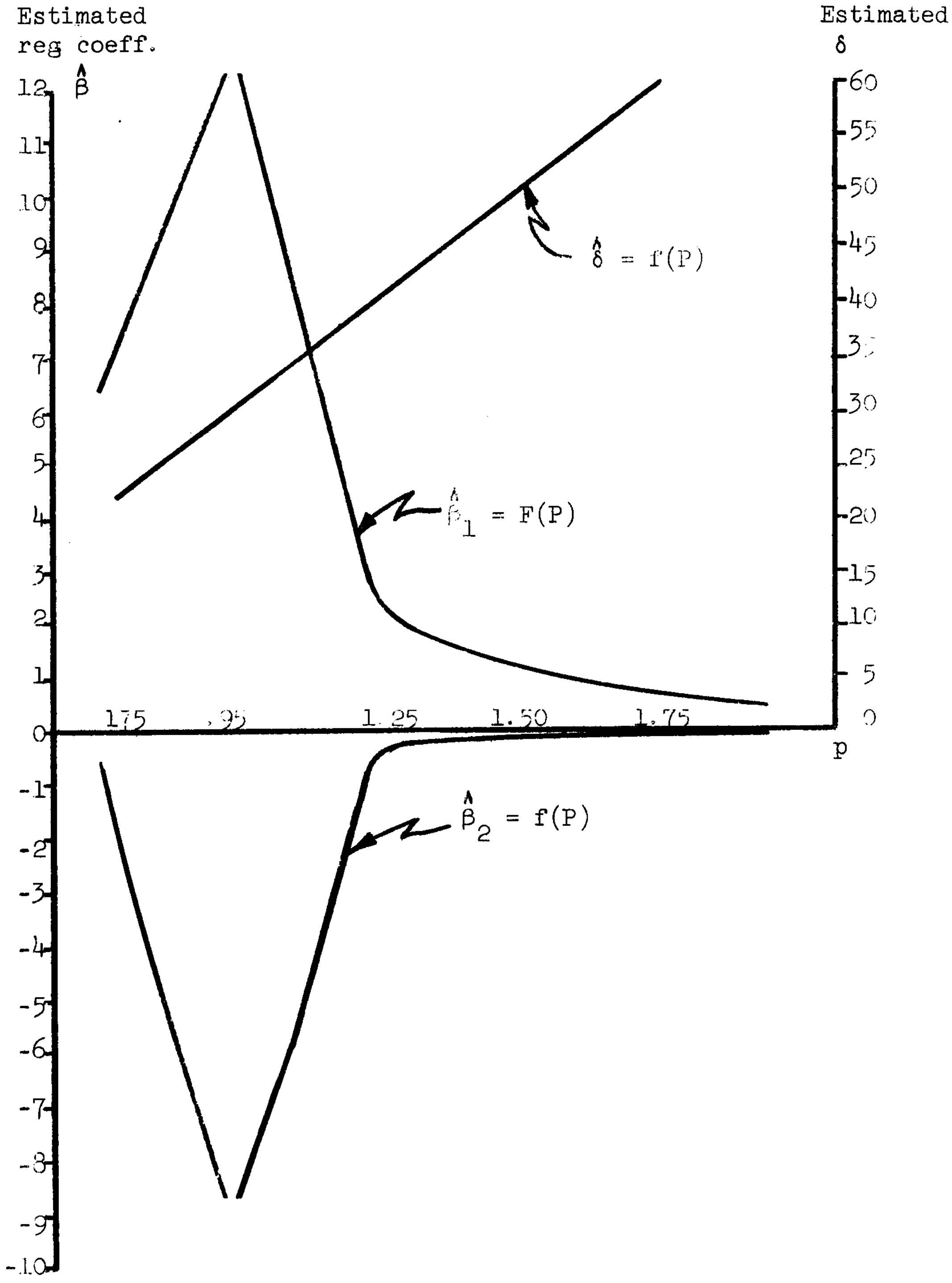

Figure 5: Relationship among the estimated parameters, δ, β_1, β_2 and the power transformation, p. Model: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$

Table 1. Estimates of the parameter δ (lbs. per acre), obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$ for $p > 1$

Baird Loc. No.	$p = .75$	$p = .95$	$p = 1.25$	$p = 1.50$	$p = 1.75$	$\{.25 \leq \hat{p} \leq 2.00\}$
551.1	22.85	30.51	41.33	49.75	57.69	23.56(.77)*
2	6.09	8.83	12.94	16.42	19.96	20.60(1.79)
3	20.64	29.97	44.11	55.62	66.67	1.61(.25)
4	10.49	15.52	23.41	30.11	36.79	3.29(.43)
Mean	15.02	21.21	30.45	37.97	45.28	12.26(.81)
552.1	7.02	10.87	17.12	22.59	28.13	6.33(.71)
2	51.36	67.16	89.02	105.51	121.24	9.43(.25)
3	56.68	71.31	91.15	106.02	120.03	47.96(.64)
4	-	-	-	-	-	- -
Mean	38.35	49.78	65.76	78.04	89.80	21.24(.53)
553.1	6.72	10.22	15.81	20.63	25.48	6.00(.71)
2	15.06	21.05	29.88	36.99	43.83	6.98(.48)
3	20.73	28.83	40.64	50.01	58.90	2.13(.25)
4	27.03	35.68	47.88	57.39	66.35	33.59(.90)
Mean	17.38	23.94	33.55	41.25	48.64	12.17(.58)
554.1	22.18	37.39	62.88	84.62	105.64	.74(.25)
2	7.46	11.58	18.44	24.57	30.88	2.79(.48)
3	9.71	16.85	30.14	42.68	55.89	.23(.25)
4	6.18	9.80	15.88	21.29	26.86	2.13(.51)
Mean	11.38	18.90	31.83	43.29	54.82	1.47(.37)
555.1	12.66	17.83	25.43	31.52	37.36	17.82(.95)
2	11.31	15.23	20.69	24.99	29.13	33.15(2.00)
3	4.02	6.82	9.73	12.23	14.86	14.23(1.69)
4	10.43	14.49	20.38	25.10	29.65	21.05(1.28)
Mean	9.60	13.59	19.06	23.46	27.75	21.56(1.48)
556.1	30.64	41.04	55.62	66.80	77.18	4.34(.25)
2	26.25	35.04	47.38	56.90	65.78	11.58(.43)
3	43.81	55.31	70.78	82.36	92.96	64.87(1.13)
4	33.74	45.60	62.64	76.06	88.79	4.80(.25)
Mean	33.61	44.25	59.10	70.53	81.17	21.40(.52)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \le \hat{p} \le 2.00\}$
557.1	20.78	29.47	42.76	53.87	64.87	6.74(.40)*
2	9.70	14.88	23.34	30.71	38.16	1.49(.35)
3	29.57	39.95	54.69	66.16	76.91	4.01(.25)
4	5.85	9.09	14.42	19.21	24.21	12.66(1.15)
Mean	16.47	23.38	33.80	42.49	51.04	6.22(.54)
559.1	29.07	38.54	51.74	61.84	71.21	9.04(.35)
2	12.16	18.13	27.51	35.41	43.21	.69(.25)
3	8.16	12.40	19.15	24.90	30.63	3.86(.53)
4	4.16	7.11	12.48	17.59	23.02	1.61(.53)
Mean	13.39	19.04	27.72	34.94	42.02	3.80(.41)
650.1	19.06	25.63	35.02	42.45	49.56	33.31(1.19)
2	29.73	39.27	52.79	63.36	73.36	15.11(.46)
3	15.58	21.36	29.74	36.43	42.83	18.38(.85)
4	21.38	28.19	37.59	44.79	51.48	30.10(1.01)
Mean	21.44	28.61	38.78	46.76	54.31	24.22(.88)
651.1	16.18	23.20	33.85	42.62	51.16	3.20(.34)
2	23.71	29.68	37.52	43.30	48.57	53.42(2.00)
3	28.46	37.75	50.91	61.21	70.94	14.07(.45)
4	62.76	76.27	93.82	106.47	117.98	115.50(1.69)
Mean	32.78	41.72	54.02	63.40	72.16	46.55(1.12)
652.1	41.48	52.52	67.52	78.79	82.15	21.53(.43)
2	11.64	15.40	20.55	24.57	28.41	32.12(2.00)
3	14.75	20.15	27.93	34.13	40.08	20.67(.97)
4	21.58	27.69	35.93	42.14	47.87	49.17(1.81)
Mean	22.36	28.94	37.98	44.91	49.63	30.87(1.30)
653.1	7.22	10.99	17.12	22.52	28.06	7.69(.78)
2	31.05	40.40	53.46	63.56	73.06	41.80(.98)
3	17.63	24.11	33.48	40.95	48.10	27.81(1.07)
4	19.28	27.39	39.19	48.44	57.07	1.60(.25)
Mean	18.79	25.72	35.81	43.87	51.57	19.72(.77)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤p̂≤2.00}
654.1	11.38	15.80	22.24	27.42	32.43	17.50(1.03)*
2	5.80	8.69	13.16	16.92	20.65	8.89(.96)
3	9.71	13.79	19.74	24.54	29.15	20.75(1.30)
4	15.63	21.57	30.19	37.07	43.64	19.03(.86)
Mean	10.63	14.96	21.33	26.49	31.47	16.54(1.04)
655.1	19.70	26.42	35.91	43.32	50.32	25.48(.92)
2	41.33	51.73	65.32	75.19	83.99	12.23(.29)
3	54.23	67.84	85.94	99.32	111.44	13.38(.25)
4	42.67	54.14	69.53	81.00	91.46	36.67(.65)
Mean	39.48	50.03	64.17	74.71	84.30	21.94(.53)
656.1	33.94	44.31	58.69	69.73	80.01	21.42(.52)
2	25.54	34.50	47.19	57.02	66.21	3.29(.25)
3	20.88	28.26	38.85	47.21	55.13	15.76(.61)
4	29.06	38.71	52.36	62.99	73.00	11.04(.39)
Mean	27.36	36.44	49.27	59.24	68.59	12.88(.44)
657.1	15.35	20.63	28.09	33.94	39.47	30.02(1.33)
2	9.88	13.91	19.81	24.56	29.15	15.73(1.04)
3	9.65	13.52	19.17	23.74	28.18	18.38(1.21)
4	14.31	19.11	25.85	31.13	36.16	27.42(1.32)
Mean	12.30	16.79	23.23	28.34	33.24	22.89(1.23)
658.1	125.43	164.94	208.40	240.96	242.05	36.43(.25)
2	71.08	85.32	103.85	117.37	129.55	140.63(2.00)
3	31.68	40.99	53.93	63.89	73.22	72.28(1.72)
4	66.91	83.87	106.53	123.36	138.69	16.52(.25)
Mean	73.77	93.78	118.18	136.39	145.88	66.49(1.06)
659.1	16.02	22.13	30.97	37.98	44.65	10.11(.56)
2	9.15	13.98	21.63	28.08	34.39	.37(.25)
3	8.46	12.13	17.59	22.09	26.51	13.30(1.01)
4	15.90	19.88	24.97	28.67	32.04	35.17(2.00)
Mean	12.38	17.03	23.79	29.21	34.40	14.74(.96)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \le \hat{p} \le 2.00\}$
660.1	11.88	16.73	23.86	29.58	35.09	13.99(.84)*
2	11.58	15.66	21.42	25.97	30.37	33.75(1.95)
3	16.70	22.56	30.89	37.44	43.66	21.01(.90)
4	66.74	78.86	94.39	105.52	115.49	124.46(2.00)
Mean	26.72	33.45	42.64	49.63	56.15	48.30(1.42)
661.1	16.51	22.24	30.39	36.80	42.90	24.40(1.03)
2	8.84	12.62	18.21	22.76	27.18	17.57(1.22)
3	10.48	14.46	20.27	24.99	29.63	19.79(1.22)
4	10.24	14.80	21.71	27.40	32.93	4.54(.49)
Mean	11.52	16.03	22.64	27.99	33.16	16.57(.99)
662.1	23.16	30.53	40.72	48.52	55.76	62.51(2.00)
2	30.14	37.98	48.42	56.17	63.23	64.62(1.80)
3	16.82	22.01	29.14	34.61	39.72	36.69(1.60)
4	22.50	28.75	37.18	43.52	49.37	50.40(1.80)
Mean	23.40	29.82	38.86	45.70	52.02	53.56(1.80)
751.1	21.77	27.82	36.02	42.85	48.06	53.31(2.00)
2	25.68	32.91	42.79	50.53	57.29	40.98(1.19)
3	17.31	22.69	30.12	35.84	41.22	30.65(1.25)
4	19.79	25.59	33.39	39.22	44.55	45.58(1.80)
Mean	21.14	27.25	35.58	41.96	47.78	42.68(1.56)
752.1	17.37	23.39	31.89	38.52	44.76	23.57(.96)
2	13.37	18.99	27.10	33.39	39.24	.98(.25)
3	2.41	3.41	4.72	5.88	7.33	9.10(2.00)
4	23.31	34.82	52.86	67.94	82.75	1.65(.25)
Mean	14.11	20.15	29.14	36.43	43.52	8.82(.86)
753.1	11.07	15.55	22.07	27.28	32.27	13.43(.86)
2	12.30	16.18	21.39	25.32	28.94	29.18(1.77)
3	2.43	3.44	4.67	5.65	6.79	8.18(2.00)
4	1.50	2.46	4.22	6.17	8.62	4.39(1.28)
Mean	6.82	9.41	13.09	16.11	19.16	13.79(1.47)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \le \hat{p} \le 2.00\}$
754.1	28.61	38.08	51.43	61.80	71.74	25.08(.68)*
2	30.78	41.15	55.81	67.11	77.80	4.56(.25)
3	52.52	68.26	90.27	106.97	123.22	10.46(.25)
4	35.99	47.13	62.98	75.47	87.37	29.31(.63)
Mean	36.98	48.65	65.12	77.89	90.03	17.35(.45)
Grand Mean	22.69	30.11	40.60	48.83	56.33	32.12(.93)

* Number in parentheses following $\hat{\delta}$ is estimated power

curves depicted in Figure 5, because the variation of any such estimated parameters is very large, consequently, the average will not yield accurate information about the tendency of the estimated parameter.

5.2 Estimates of the Parameters δ and β by Fitting the Model: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$

The models considered in this section are essentially those which are second degree polynomials in the transformed input variable. The transformations carried out, account for the pre-fixed and non-fixed power models and again the value of p was bounded by the interval [.25, 1.00], i.e., the value of the power must be at least .25, and at the most 2.00. This strategy was followed for the non-fixed power transformation procedure. For the fixed power, the transformations included were the square root transformation and the full second order.

In Table 2 are presented the estimated values of the parameter δ, computed for each replication with location. Also, as a matter of comparison the estimates of the parameter δ by fitting the well-known Mitscherlich equation were obtained.

The estimate of δ yielded by the Mitscherlich equation is more or less comparable with that of the polynomial model of low degree. The magnitude of $\hat{\delta}$ yielded by the Mitscherlich equation is far below the value estimated using the polynomial model with power transformation greater than 1.75 for the special transformation or p near 1, for the second order model.

The values of the estimates δ, obtained by using the model described above are presented in Table 2. Also in Tables 2 and 4 of the Appendix are reported the estimated values of $\hat{\beta}_1$ and $\hat{\beta}_2$, computed on the basis of individual replications. In order to make clearer the behavior of these estimated parameters, curves were drawn relating the power p with the estimatedvalues of $\hat{\delta}$, $\hat{\beta}_1$ and $\hat{\beta}_2$, which are depicted in Figure 6.

Under the second order power transformation model, the estimated value, $\hat{\delta}$, increases 5.24 lbs. for each unit of increase in power p by .1 (52.39 lb./1 unit of power). The power transformation is linearly related to the estimated parameter, $\hat{\delta}$. It should be also noticed that under the second order transformation, the value of $\hat{\delta}$ increases faster with an increase in p, than does the $\hat{\delta}$ obtained using the special power transformation.

The regularity observed in the change of the estimated $\hat{\delta}$, as a function of the power, is very significant from the estimation point of view. Not only $\hat{\delta}$, but also $\hat{\beta}_1$ and $\hat{\beta}_2$ present a very well defined tendency, which was not observed when the transformation conforms to the special power model.

Since the estimated $\hat{\delta}$, should be non-negative real values, with very high probability, the second order transformation detects a very small or near zero value of δ, when the power of the second order model approaches the value of .35, whereas the special order power transformation model produces a negligible value of $\hat{\delta}$ when p approaches .20.

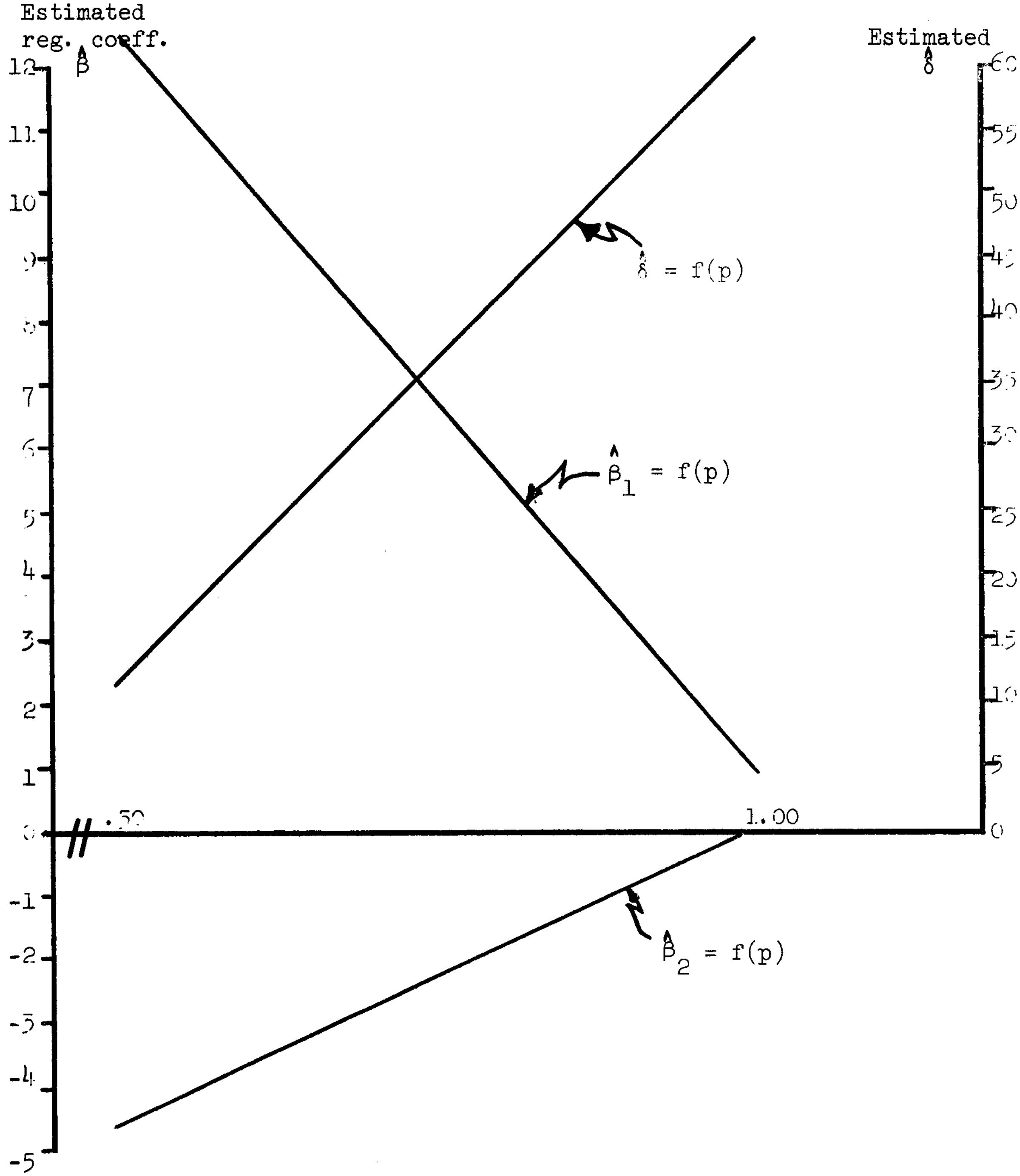

Figure 6: Relationship among the estimated parameters, δ, β_1, β_2 and the power transformation, p. Model: E(Y): $\beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$

Table 2. Estimates of the parameter δ (lbs. per acre) obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \alpha\{1 - \exp(-\gamma(X + \delta))\}$

Baird Loc. No.	p = .50	$\{.25 \le \hat{p} \le 1.00\}$	p = 1.00	Mitscherlich
551.1	12.83	21.52(.62)*	65.17	28.10
2	2.72	19.47(.94)	23.53	12.14
3	9.72	.46(.25)	77.27	6.01
4	4.74	5.82(.52)	43.31	14.55
Mean	7.52	11.82(.58)	52.32	15.20
552.1	2.82	5.63(.60)	33.62	8.15
2	30.37	4.04(.25)	135.71	19.51
3	36.48	46.72(.56)	132.94	41.15
4	-	- -	-	-
Mean	23.22	18.76(.47)	100.76	22.99
553.1	2.78	5.23(.59)	30.27	11.59
2	7.64	6.86(.48)	50.36	21.56
3	10.71	1.41(.26)	67.31	22.18
4	15.62	30.40(.66)	74.83	28.85
Mean	9.19	10.97(.50)	55.69	21.04
554.1	7.82	.13(.25)	126.66	5.00
2	3.03	2.68(.48)	37.22	10.03
3	3.31	.06(.25)	69.42	4.00
4	2.37	2.47(.50)	32.42	8.84
Mean	4.13	1.34(.39)	66.43	6.97
555.1	6.24	15.73(.69)	42.94	18.26
2	6.06	33.15(1.00)	33.15	20.28
3	2.23	11.40(.74)	17.62	10.41
4	5.24	18.65(.80)	34.04	18.72
Mean	4.94	19.73(.81)	31.94	16.92
556.1	17.10	2.22(.25)	86.84	28.11
2	14.74	12.53(.47)	74.09	28.27
3	27.81	59.41(.73)	102.75	47.54
4	18.68	1.71(.25)	100.84	10.65
Mean	19.58	18.97(.42)	91.13	28.64

Table 2. (continued)

Baird Loc. No.	p = .50	$\{.25 \leq \hat{p} \leq 1.00\}$	p = 1.00	Mitscherlich
557.1	10.49	7.33(.45)*	75.70	9.00
2	4.07	1.94(.42)	45.55	8.59
3	16.28	1.82(.25)	87.00	23.76
4	2.31	10.41(.75)	29.31	5.15
Mean	8.29	5.40(.47)	59.39	11.62
559.1	16.59	10.74(.42)	79.93	29.95
2	5.33	1.26(.35)	50.79	12.15
3	3.42	3.82(.50)	36.23	11.14
4	1.36	1.61(.50)	28.58	42.19
Mean	6.67	4.36(.44)	48.88	23.86
650.1	10.55	24.83(.71)	56.36	25.56
2	17.26	14.94(.47)	82.81	31.91
3	8.21	16.25(.65)	48.95	22.64
4	12.26	26.80(.71)	57.71	31.75
Mean	12.07	20.70(.63)	61.46	27.96
651.1	7.86	4.00(.41)	59.43	13.88
2	15.12	53.42(1.00)	53.42	38.91
3	16.36	14.01(.46)	80.14	30.79
4	43.06	128.33(1.00)	128.33	77.99
Mean	20.60	49.95(.72)	80.33	40.39
652.1	26.33	21.45(.45)	98.73	51.50
2	6.47	32.12(1.00)	32.12	21.58
3	7.82	18.17(.69)	45.78	22.74
4	13.08	48.48(.95)	83.21	34.66
Mean	13.40	30.06(.77)	64.96	32.62
653.1	3.02	6.48(.62)	33.64	10.21
2	18.56	37.91(.68)	82.02	33.35
3	9.39	24.66(.72)	54.95	21.15
4	9.43	.92(.25)	65.09	21.33
Mean	10.10	17.49(.57)	58.92	21.51

Table 2. (continued)

Baird Loc. No.	p = .50	{.25≤$\hat{p}$≤1.00}	p = 1.00	Mitscherlich
654.1	5.77	15.14(.72)*	37.25	19.58
2	2.43	7.54(.71)	24.27	12.41
3	4.66	18.11(.80)	33.56	17.15
4	8.12	16.93(.61)	49.93	21.29
Mean	5.24	14.43(.71)	36.25	17.61
655.1	10.93	22.80(.67)	56.92	27.17
2	26.46	12.78(.32)	91.90	59.84
3	35.06	11.59(.28)	122.53	61.87
4	26.72	31.14(.36)	101.05	46.76
Mean	24.79	19.58(.41)	93.10	48.91
656.1	20.13	21.37(.51)	89.64	32.98
2	13.99	4.34(.34)	74.79	26.05
3	11.41	14.74(.55)	62.25	25.54
4	16.52	11.98(.44)	82.43	27.53
Mean	15.51	13.11(.46)	77.28	28.03
657.1	8.40	27.18(.81)	44.75	24.67
2	4.84	13.64(.73)	33.55	18.01
3	4.76	16.10(.78)	32.47	17.66
4	7.92	24.92(.81)	40.95	24.34
Mean	6.48	20.46(.78)	37.93	21.17
658.1	87.64	16.48(.25)	298.94	21.59
2	50.29	140.63(1.00)	140.63	86.81
3	19.12	68.66(.91)	82.02	32.69
4	43.13	9.25(.25)	152.77	52.77
Mean	50.04	58.76(.60)	168.59	48.46
659.1	8.32	9.64(.53)	50.97	23.19
2	3.82	.38(.28)	40.47	13.02
3	3.97	11.40(.72)	30.81	15.76
4	9.99	35.17(1.00)	35.17	28.80
Mean	6.52	14.15(.62)	39.36	20.19

Table 2. (continued)

Baird Loc. No.	p = .50	{.25≤$\hat{p}$≤1.00}	p = 1.00	Mitscherlich
660.1	5.85	12.36(.65)*	40.36	19.03
2	6.17	33.94(.99)	34.61	20.59
3	9.09	18.61(.67)	49.56	25.48
4	48.66	124.46(1.00)	124.46	91.64
Mean	17.44	47.34(.83)	62.25	39.19
661.1	9.03	21.57(.71)	48.71	25.28
2	4.18	15.27(.78)	31.45	16.14
3	5.37	17.51(.78)	34.16	19.00
4	4.78	4.40(.48)	38.25	17.56
Mean	5.84	14.69(.69)	38.14	19.50
662.1	13.29	64.92(1.02)	62.51	29.60
2	19.06	63.15(.95)	69.70	44.88
3	9.72	35.34(.90)	44.54	28.31
4	13.77	49.74(.95)	54.81	35.87
Mean	13.96	53.29(.96)	57.89	34.67
751.1	13.34	58.20(1.04)	53.51	34.42
2	15.73	37.49(.77)	63.83	37.91
3	9.99	27.91(.80)	46.28	28.72
4	11.73	43.88(.95)	49.47	33.19
Mean	12.70	41.87(.89)	53.27	33.56
752.1	9.49	21.00(.69)	50.64	26.32
2	6.44	1.01(.25)	44.64	21.91
3	.99	9.10(1.00)	9.10	6.27
4	10.57	.32(.25)	97.17	4.87
Mean	6.87	7.86(.54)	50.39	14.84
753.1	5.47	11.85(.66)	37.02	19.48
2	6.92	28.50(.95)	32.33	23.59
3	.99	8.17(1.00)	8.17	6.97
4	.47	3.54(.81)	11.45	3.89
Mean	3.46	13.02(.86)	22.24	13.48

Table 2. (continued)

Baird Loc. No.	p = .50	$\{.25 \leq \hat{p} \leq 1.00\}$	p = 1.00	Mitscherlich
754.1	16.25	24.23(.58)*	80.71	21.22
2	17.35	2.61(.25)	87.76	31.11
3	31.68	4.46(.25)	138.25	25.68
4	21.40	27.31(.55)	98.75	27.48
Mean	21.67	14.65(.41)	101.32	26.37
Grand Mean	13.21	21.71(.62)	64.04	25.43

* Number in parentheses following $\hat{\delta}$ is estimated power

As may be expected, the transformation of the independent variable considerably affects the rate of change in Y per unit change of the input factor (i.e., the effect of p on $\hat{\beta}_1$) as well as the decline of the rate of change (effect of $\hat{\beta}_2$). Consequently, $\hat{\delta}$ is greatly affected by the transformation since it is related to both $\hat{\beta}_1$ and $\hat{\beta}_2$. The following relationship follows directly from the full second order model in view of the fact that $\beta_1 \geq 0$, $\beta_2 \leq 0$ and $\delta \geq 0$.

$$\beta_1 = \max\{\pm\sqrt{\beta_1^{*2} - 4\beta_0^*\beta_2}\} = +\sqrt{\beta_1^{*2} - 4\beta_0^*\beta_2} \qquad (5.1)$$

$$\delta = \min \frac{\{\beta_1^* \pm\sqrt{\beta_1^{*2} - 4\beta_0^*\beta_2}\}}{2\beta_2} = \frac{-\beta_1^* + \sqrt{\beta_1^{*2} - 4\beta_0^*\beta_2}}{2|\beta_2|} \qquad (5.2)$$

or

$$\delta = \frac{\beta_1 - \beta_1^*}{2|\beta_2|} .$$

The above holds true since the parameter estimate, $\hat{\delta}$ is highly positively correlated with the power of the transform ($r \doteq .99$). This indicates strongly that the transformation used controls to a high degree the behavior of the estimate of the parameter δ.

5.3 Relative Variation of the Estimated Parameters of the Response ($\hat{\delta}$, $\hat{\beta}_1$ and $\hat{\beta}_2$)

In this section are reported the relative variance components of $\hat{\delta}$, $\hat{\beta}_1$ and $\hat{\beta}_2$, estimated for locations and sites (replications) within location. From Table 3, it can be inferred that the estimate of parameter, $\hat{\delta}$ shows a little more variability within a location than among locations. This fact seems to be very illogical, since, from simple reasoning one should expect to find more variation among locations than within locations. However, considering this fact from another point of view, it might be possible to think that local conditions affect strongly the variation of the response pattern, since soil variability can occur from one centimeter to another.

When a pre-fixed power transformation is carried out on the independent variable and then fits any of the models used in this study, it is observed that about 54% of the variation is accounted for by the site condition within locations and about 46% of the observed variation is attributable to the location to location variability. The differential contribution of the locations and sites to the total variation of $\hat{\delta}$ is much more dramatic when the non-fixed power transformation is fitted.

In Table 4 are reported the variance components of the estimated parameter $\hat{\beta}_1$. On the average, it can be concluded that from the total variation of $\hat{\beta}_1$ about 50% is attributed to the location-to-location variability and the site effect is responsible for the other 50%.

The variability of the estimated $\hat{\beta}_2$ seems to be even more stable among locations than within locations, as indicated in Table 5, and the same trend already mentioned for the pre-fixed and non-fixed power transformation is observed in the behavior of this estimated parameter. It is very interesting to notice in Tables 4 and 5, that the magnitudes of the variations of $\hat{\beta}_1$ and $\hat{\beta}_2$ are the same whether or not $\hat{\delta}$ is considered in the model.

Table 3. Relative variance components of the estimated $\hat{\delta}$, for locations and sites within location

Model		Location* $\frac{\sigma_\ell^2}{\sigma_\ell^2 + \sigma_s^2} \times 100$	Site $\frac{\sigma_s^2}{\sigma_\ell^2 + \sigma_s^2} \times 100$
$E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$	$p = .75$	47.139	52.861
"	$p = .95$	45.897	54.103
$E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$	$p = 1.25$	44.946	55.054
"	$p = 1.50$	44.142	55.858
"	$p = 1.75$	45.145	54.855
Non-fixed power	$\bar{\hat{p}} = .93$	24.596	75.404
$E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$	$p = .50$	46.312	53.688
"	$p = 1.00$	45.103	54.897
Non-fixed power	$\bar{\hat{p}} = .62$	19.588	80.412

* σ_ℓ^2 = variance component for locations

σ_s^2 = variance component for sites

Table 4. Relative variance components of the estimated $\hat{\beta}_1$, for locations and sites within location

Model		Location * $\frac{\sigma^2_\ell}{\sigma^2_\ell + \sigma^2_s} \times 100$	Site $\frac{\sigma^2_s}{\sigma^2_\ell + \sigma^2_s} \times 100$
$E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$	$p = .75$	52.268	47.732
"	$p = .95$	43.649	56.351
$E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$	$p = 1.25$	49.231	50.769
"	$p = 1.50$	53.639	46.361
"	$p = 1.75$	56.459	43.541
Non-fixed power	$\bar{\hat{p}} = .93$	17.457	82.543
$E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$	$p = .50$	64.025	35.975
"	$p = 1.00$	58.500	41.500
Non-fixed power	$\bar{\hat{p}} = .62$	11.624	88.376
$E(Y) = \beta_o + \beta_1 X + \beta_2 X^2$ (without δ)	-	51.450	47.54

* σ^2_ℓ = variance component for locations

σ^2_s = variance component for sites

Table 5. Relative variance components of the estimated $\hat{\beta}_2$, for locations and sites within location

Model		Location* $\frac{\sigma_\ell^2}{\sigma_\ell^2 + \sigma_s^2} \times 100$	Site $\frac{\sigma_s^2}{\sigma_\ell^2 + \sigma_s^2} \times 100$
$E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$	$p = .75$	30.769	69.231
"	$p = .95$	40.099	59.901
$E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$	$p = 1.25$	35.484	64.516
"	$p = 1.50$	32.275	67.725
"	$p = 1.75$	31.156	68.844
Non-fixed power	$\bar{\hat{p}} = .93$	9.539	90.461
$E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$	$p = .50$	26.009	73.991
"	$p = 1.00$	30.616	69.384
Non-fixed power	$\bar{\hat{p}} = .62$	4.269	95.731
$E(Y) = \beta_o + \beta_1 X + \beta_2 X^2$ (without δ)	-	30.150	69.850

* σ_ℓ^2 = variance component for locations

σ_s^2 = variance component for site

5.4 Structure of the Asymptotic Variance - Covariance Matrix of the Estimated Parameters ($\hat{\delta}$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{p}$)

For purposes of illustration, the variance-covariance matrix of the estimated parameters corresponding to low, medium and large values of $\hat{\delta}$, has been considered in this section. It must be recalled once again, that one is dealing with an intrinsic non-linear model, therefore, the estimated variance covariance matrix is not a proper one, but it is an approximation to $(X'X)^{-1}\sigma^2$. However, in most of the polynomial models attempted in this research, the convergence was relatively fast and completed over all the parameters. In general, when fixed power transformation is used, 4 to 6 iterations were sufficient for complete convergence up to 5 decimal places.

In Table 6 are reported some values of the variance-covariance matrices of the estimated parameters obtained when fixed power transformation is used. Most of the models used here have been included in the small example shown in Table 6. The figures show that the estimated parameter $\hat{\delta}$ is negatively correlated with the rate of change of the response Y, per unit of change of the input factor $X(\hat{\beta}_1)$. The reason for the negative covariance between $\hat{\delta}$ and $\hat{\beta}_1$ [$\mathrm{cov}(\hat{\delta}, \hat{\beta}_1) < 0$)] is obvious, since for a large value of $\hat{\delta}$, an overestimate of δ, one expects that the rate of increase of Y per unit of applied fertilizer will be underestimated as compensation.

The correlation between $\hat{\delta}$ and $\hat{\beta}_2$, (rate of decline of the response function) is positive [($\mathrm{cov}(\hat{\delta}, \hat{\beta}_2) > 0$)], i.e., for a large amount of $\hat{\delta}$ (the initial level of input factor), the response curve declines faster. The other relationship remaining to be specified is the covariance between $\hat{\beta}_1$ and $\hat{\beta}_2$ which is negative $[(Cov(\hat{\beta}_1,\hat{\beta}_2) < 0)]$.

Table 7 illustrates the complete asymptotic variance-covariance matrix, when the power p is estimated. The peculiarity of the figures shown in Table 7 is very interesting since they indicate how the structure of the variance-covariance matrix changed from one model to another.

5.5 Relationship Among the Estimated, $\hat{\delta}$ (lbs. per acre) the N(%), OM(%), NO_3 (ppm) and the Average Yield at Zero Level of Application, y_0 (Bu./acre)

The correlation matrix presented in Table 8, indicates that the estimated amount of initial nitrogen in the soil, $\hat{\delta}$, (using the square root model) is not highly correlated with the laboratory determination (N, NO_3, and OM). However, the estimated correlation coefficient between $\hat{\delta}$ and the corn yield at zero application of N (y_0) is very high. The other correlation coefficients, appear to be unimportant in magnitude, except, the correlation coefficient between OM and N and also between NO_3 and the average yield at zero level of application (y_0) which show some association.

From this result it seems to be evident that neither soil test analysis nor statistical estimation of the amount of nutrient in the soil test analysis could explain satisfactorily the response curve of the corp yield. Probably many other factors are operative in regulating or limiting the yield. One of the important environmental factors affecting responses to fertilization is rainfall during the growing season. The distribution of rainfall is a relevant factor because the rate of biomass production is not constant throughout the growing season.

It is important to emphasize that total nitrogen measured directly by the Kjeldahl method or indirectly as organic matter gives only a very crude indication of the estimate of total nitrogen present in the soil. Although the NO_3-production could yield information on the quantity and rate at which nitrogen is mineralized to available forms, it is very much influenced by many other external factors that can completely confound the true nutritional condition of the soil.

On the other hand, considering the amount of the initial level of nutrient in the soil estimated from the information supplied by the response curve, it is probable that the actual value of δ is the result of a complex interaction among several other factors. However, the value of $\hat{\delta}$, estimated from the actual response curve could be much more reliable from the practical standpoint, because this estimated value might indicate reasonably well, the general reaction of the crop to the production factor.

Table 8 was constructed on the basis of the average (per location basis) of the parameter estimate, $\hat{\delta}$, and also the yield at zero level of N application. This procedure was followed because the laboratory determinations for N, OM and NO_3-production are only available on a per location basis. The original values of all variables used in the correlation studies can be obtained from Table 7 of the Appendix. The yields by individual replications are reported in Table 8 of the Appendix.

Table 6. Some estimates of the variance-covariance matrix of the estimated parameters ($\hat{\delta}$, $\hat{\beta}_1$, $\hat{\beta}_2$), fitting a fixed power transformation model (for selected location replications)

Baird Loc. No.	Var($\hat{\delta}$)	Var($\hat{\beta}_1$)	Var($\hat{\beta}_2$)	Cov($\hat{\beta}_1$,$\hat{\delta}$)
654.2(2.43)*	1.7995	.2995	.0016	-.2632
650.1(10.55)	13.3057	1.1178	.0059	-2.2645
655.4(26.72)	93.7376	1.7498	.0082	-9.9325
659.3(9.10)	43.8393	.0044	.0000+	-.3364
553.2(69.42)	170.4759	.0081	.0000+	-1.0279
658.2(140.63)	1399.2099	.0113	.0000+	-3.8135
659.3(8.46	7.5428	.2331	.0166	-.7315
553.3(20.73)	78.8920	.8376	.0579	-5.7722
552.3(56.68	788.4188	1.5315	.1003	-29.9409
753.3(2.46)	10.8607	1.2488	.7382	-1.3498
533.3(28.83)	126.8193	3.3595	1.9584	-15.8613
658.2(85.32	833.6787	3.7080	2.0823	-50.9055
553.1(15.81)	36.3983	.0552	.0398	-.9709
655.1(37.59)	65.0531	.0560	.0369	-1.5463
658.4(106.53)	1103.5890	.1116	.0064	-10.3936
555.3(12.23)	40.1804	.0164	.0001-	-.5690
556.1(38.52)	115.1775	.0259	.0001+	-1.1435
651.4(106.47)	1451.8200	.0442	.0001+	-7.4902
752.3(7.33)	122.2431	.0186	.0000+	-1.0042
556.1(77.18)	579.4685	.0212	.0000+	-3.1918
651.4(117.98)	1862.2128	.0240	.0000+	-6.3317

Table 6. (continued)

Baird Loc. No.	$Cov(\hat{\beta}_2,\hat{\delta})$	$Cov(\hat{\beta}_1,\hat{\beta}_2)$	Model	
654.2(2.43)*	.0180	-.0270	$E(Y)=\beta_1(X+\delta)^p+\beta_1(x+\delta)^{2p}$	p= .5
650.1(10.55)	.1592	-.0799	"	p= .5
655.4(26.72)	.6613	-.1189	"	p= .5
659.3(9.10)	.0015	-.000018	"	p=1.0
553.2(69.42)	.0040	-.000031	"	p=1.0
658.2(140.63)	.0109	-.000033	"	p=1.0
659.3(8.46)	.1902	-.0619	$E(Y)=\beta_1(X+\delta)^p+\beta_2(X+\delta)$	p= .75
553.3(20.73)	1.5004	-.2198	"	p= .75
552.3(56.58)	7.6456	-.3916	"	p= .75
753.3(2.46)	1.0353	-.9600	"	p= .95
553.3(28.83)	12.0898	-.2565	"	p= .95
658.2(85.32)	38.2959	-2.7646	"	p= .95
553.1(15.81)	.2533	-.0142	$E(Y)=\beta_1(X+\delta)+\beta_2(X+\delta)^p$	p=1.25
655.1(37.59)	.3949	-.0144	"	p=1.25
658.4(106.53)	2.4775	-.0267	"	p=1.25
555.3(12.23)	.0381	-.0011	"	p=1.50
556.1(38.52)	.0918	-.0017	"	p=1.50
651.4(106.47)	.4248	-.0025	"	p=1.50
752.3(7.33)	.0019	-.0003	"	p=1.75
556.1(77.18)	.0468	-.0003	"	p=1.75
651.4(117.98)	.0828	-.0003	"	p=1.75

* Number in parentheses following location number is estimated δ

Table 7. Some estimates of the variance-covariance matrix of the estimated parameters ($\hat{\delta}$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{p}$)

Location	Var($\hat{\delta}$)	Var($\hat{\beta}_1$)	Var($\hat{\beta}_2$)	Var($\hat{p}$)
554.4(.50)*	35.7764	93.3756	.3556	.1741
552.1(.71)	79.0117	42.8285	10.1759	.2015
650.3(.85)	156.3541	14.5469	83.3257	.1624
551.2(1.79)	420.5774	1.0189	.0055	2.6243
652.4(1.81)	1446.5489	1.1376	.0040	1.3901

Location	Cov($\hat{\delta}$,$\hat{\beta}_1$)	Cov($\hat{\delta}$,$\hat{\beta}_2$)	Cov($\hat{\delta}$,$\hat{p}$)	Cov($\hat{\beta}_1\hat{\beta}_2$)
554.4(.50)*	-53.8877	-3.1487	2.3104	5.5093
552.1(.71)	-53.3673	-25.1861	3.5934	20.5300
650.3(.85)	42.6766	-106.3687	4.7293	-34.5766
551.2(1.79)	-18.0149	1.2613	19.7765	-.0744
652.4(1.81)	-38.8504	2.2426	57.2087	-.0670

Location	Cov($\hat{\beta}_1\hat{p}$)	Cov($\hat{\beta}_2\hat{p}$)
554.4(.50)*	-4.0204	-.2405
552.1(.71)	-2.9145	-1.4296
650.3(.85)	1.5214	-3.6778
551.2(1.79)	-1.1787	.0874
652.4(1.81)	-1.7180	.1022

* Number in parentheses following location numbers is the power transformation

Table 8. Correlation matrix of the soil test results: N, OM, NO_3-production and the estimated amount of N($\hat{\delta}$) by the square root model and the corn yield at zero level of N application (y_0)

	N	OM	NO_3	$\hat{\delta}$	y_0
N	1.000	.555	.106	.398	.339
OM		1.000	.113	.350	.488
NO_3			1.000	.474	.601
$\hat{\delta}$				1.000	.817
y_0					1.000

5.6 Relationship Among the Estimated δ's by Several Models

In general terms, the $\hat{\delta}$ values estimated by the various polynomial models were quite highly correlated. However, considerably lower correlation coefficients are observed when the estimated value was obtained by the non-fixed power technique and correlated with the estimates $\hat{\delta}$ obtained from the pre-fixed power transforms. This behavior or effect of the model might be expected if it is considered that non-fixed power transformations only have the constraint of minimizing the sum of squares of deviations from the model, consequently, the power as well as the other parameters can take any value within their domains. So, the $\hat{\delta}$ estimated for this particular model may not exactly follow the same pattern of that yielded by the fixed power transform.

On the other hand, the correlation coefficients for the relationship between the estimates of δ from both non-fixed power transformation models is about 1, which implies that although estimates are slightly different in magnitude for each model, they follow the same direction.

The estimates of $\hat{\delta}$ from the Mitscherlich equation are correlated fairly well with the estimates of δ by all of the polynomial models. Although the amount of $\hat{\delta}$ estimated is quite different, the estimates apparently follow the same directions as the polynomial estimates.

Actually the differences in the magnitude of the $\hat{\delta}$ values estimated by the Mitscherlich and the Polynomial model are greatest when p is greater than .80, for the special model and about .70 for the second order model. But for low values of p, the polynomial models yielded values of $\hat{\delta}$ very close in magnitude to those given by the Mitscherlich equation.

In Table 9 are reported the correlation coefficients for δ's obtained by fitting the several models studied in this research.

5.7 Estimation of Point of Physical or Theoretical Maximum

In this section is treated very briefly the effect of the transformation model on the theoretical estimated maximum point on the input axis.

With the lower power transformation of the polynomial the point of theoretical maximum is attained at a slightly higher level than that attained with a higher order power transformation (Tables 10 and 11). However, there are a few exceptions to this statement, since in some locations, apparently the transformations do not affect the point of physical maximum. In some of these locations a slight displacement of the point $\hat{X}$, to the right can be noticed as the power increases, but, this fact is unimportant and one would concentrate attention on the general trends. From the practical standpoint there is no differences among models, in terms of the estimates of $\hat{X}$.

Table 9. Correlation matrix of the estimate parameter $\hat{\delta}$, fitting the models: $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$, for p = .75, .95, 1.25, 1.50, 1.75 and $\hat{p}$ = .93 ($\hat{\delta}_1$, $\hat{\delta}_2$, $\hat{\delta}_3$, $\hat{\delta}_4$, $\hat{\delta}_5$ and $\hat{\delta}_6$); $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ for p = .50, 1.00 and $\hat{p}$ = .62 ($\hat{\delta}_7$, $\hat{\delta}_8$ and $\hat{\delta}_9$) and $\hat{\delta}_{10}$, obtained using the Mitscherlich model

	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_3$	$\hat{\delta}_4$	$\hat{\delta}_5$	$\hat{\delta}_6$	$\hat{\delta}_7$	$\hat{\delta}_8$	$\hat{\delta}_9$	$\hat{\delta}_{10}$
$\hat{\delta}_1$	1.000	.999	.993	.986	.974	.633	.997	.969	.581	.761
$\hat{\delta}_2$		1.000	.999	.993	.984	.603	.992	.990	.550	.733
$\hat{\delta}_3$			1.000	.999	.994	.562	.982	.991	.507	.697
$\hat{\delta}_4$				1.000	.998	.531	.972	.996	.470	.670
$\hat{\delta}_5$					1.000	.490	.956	.997	.436	.642
$\hat{\delta}_6$						1.000	.676	.483	.993	.789
$\hat{\delta}_7$							1.000	.951	.625	.783
$\hat{\delta}_8$								1.000	.427	.626
$\hat{\delta}_9$									1.000	.777
$\hat{\delta}_{10}$										1.000

Table 10. Average estimates of the stationary point, $\hat{X}$ (pounds per acre), which maximizes the response function, given by the model, $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \le \hat{p} \le 2.00\}$
551	173.80	170.20	170.77	172.07	173.60	173.61(.81)*
552	154.44	157.04	160.44	162.01	165.18	153.00(.53)
553	188.00	183.99	182.45	182.70	183.47	213.16(.58)
554	146.02	149.94	155.46	159.47	163.07	129.63(.37)
555	202.94	191.30	184.53	182.58	182.08	180.80(1.48)
556	173.61	173.41	174.48	175.60	176.80	185.92(.52)
557	137.14	141.26	146.12	149.21	152.52	138.75(.54)
559	159.84	161.08	163.71	162.56	168.18	167.68(.41)
650	192.29	187.33	185.25	184.39	184.71	190.02(.88)
651	201.68	194.16	189.53	188.01	187.38	184.51(1.12)
652	226.52	213.29	204.47	201.14	199.39	225.36(1.30)
653	168.54	168.38	169.76	171.35	173.09	196.21(.77)
654	214.23	201.50	193.77	191.29	188.92	199.09(1.04)
655	242.50	228.95	219.48	215.44	212.86	280.33(.53)
656	174.02	173.55	174.33	175.52	173.57	189.40(.44)
657	229.74	213.85	202.91	199.10	202.83	204.34(1.23)
658	191.31	186.85	186.95	187.04	190.97	189.92(1.06)
659	242.32	221.07	207.99	203.27	202.85	250.24(.96)
660	221.97	209.81	201.54	198.19	196.48	202.58(1.42)
661	224.12	209.88	200.61	197.95	196.63	236.51(.99)
662	230.82	215.23	204.69	200.51	198.31	198.70(1.80)
751	238.02	229.50	216.12	210.37	207.17	211.26(1.57)
752	188.81	181.99	178.39	177.69	177.79	203.40(.86)
753	234.82	220.78	204.21	198.46	195.68	202.88(1.47)
754	159.45	161.07	163.70	165.52	167.63	181.02(.46)

* Number in parentheses following $\hat{X}$ is the average power

Table 11. Average estimates of the stationary point, $\hat{X}$ (pounds per acre) which maximizes the response function, given by the models, $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \alpha[1 - \exp.(-\gamma(X + \delta))]$

Baird Loc. No.	p = .50	{.25≤$\hat{p}$≤1.00}	p = 1.00	Mits. Point of Max. Curv.
551	180	181(.58)*	175	189
552	154	160(.47)	167	122
553	205	201(.50)	184	220
554	141	192(.39)	167	125
555	222	203(.81)	182	265
556	177	210(.42)	180	190
557	131	158(.47)	155	123
559	161	171(.44)	170	197
650	211	211(.63)	185	250
651	311	189(.72)	187	242
652	246	263(.77)	199	310
653	174	200(.57)	174	179
654	265	213(.71)	162	266
655	268	282(.41)	211	311
656	179	205(.46)	178	201
657	292	225(.78)	196	308
658	193	203(.60)	188	198
659	287	238(.69)	200	315
660	263	218(.83)	196	307
661	279	248(.69)	195	294
662	289	252(.96)	197	314
751	292	253(.89)	205	365
752	216	286(.54)	178	268
753	275	236(.86)	194	297
754	159	188(.41)	169	144

* Number in parentheses following $\hat{X}$ is the average power

The small displacement of $\hat{X}$ is justified in terms of biological fact, since for higher powers one expects to obtain higher values of the estimates of $\hat{\delta}$, which in turn implies that the response function may decline at lower rates of application. Consequently, the maximum should be attained at lower values of X. Again the Mitscherlich model gives a value for the physical maximum which is closer to the value given by lower ordered polynomials.

The estimate of the maximum rate has relevant application since both the optimum rate or the minimum recommended rate can be expressed as a function or the maximum rate. Therefore, an overestimation or an underestimation of the point of maximum directly affects the profit function of the farmer.

Although the comparisons of models on the basis of the point of maximum is difficult, the knowledge of it gives the ideas about the effect, (if any) of the transformations on the behavior of the response function. Also it must be pointed out that in some of the lower power transformation models, as well as in the Mitscherlich equation, some of the points of maximum in the input axis are ruled out, since they are outside the range investigated. As indicated previously the five levels used by Baird (1958) in his experiments were 0 to 250 lb./acre of N, equally spaced at 62.5 pound intervals.

5.8 Estimation of the Maximum Yield

In Chapter 3, was introduced the actual derivation of the maximum attainable values of the response function. It was showen that the maximum value of the response function is independent of the value of the parameter δ or initial level of nutrient present in the soil. Observing Tables 12 and 13 one can infer that there exists a slight upward displacement of the maximum attained by the response function, when the power increases; although the figures show that differences are unimportant from a practical standpoint.

In any case, the maximum value of the response function seems to be fairly invariant under various models; this applies to all polynomial models and to the Mitscherlich equation. The numerical values of the asymptote and the attainable maximum in the Mitscherlich function are given in the last two columns of Table 13. The figures of these two criteria are very close to each other and therefore in practice the asymptote can be taken as an estimate of the attainable maximum yield by the Mitscherlich equation.

5.9 Contour of the Standard Deviation from the Model

The standard deviations of the lack of fit of the models are reported in Tables 5 and 6 in the Appendix. The standard deviation from the model appears to be fairly uniform among models. However, it shows a large variability not only among locations but also among sites within locations. This indicates very clearly the great heterogeneity of the response pattern, even from one site to another within each location. Of course, heterogeneity can be found also in some locations considered to be of homogeneous sites.

However, this does not happen if one observes the contour of the sum of squares of deviations. But in considering the standard deviation from the model one takes into consideration the degrees of

Table 13. Average value of the estimated maximum yield (bushels per acre) obtained by fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \alpha[\ 1 - \exp(-\gamma(X + \delta))\ ]$

Baird Loc. No.	p = .50	$\{.25 \le \hat{p} \le 1.00\}$	p = 1.00	Mitscherlich Attain. Max. Curvature	Mitscherlich Asymptote
551	77	80(.58)*	80	77	77
552	85	85(.47)	87	82	83
553	96	96(.50)	99	95	96
554	41	40(.39)	42	40	40
555	94	94(.81)	96	94	95
556	88	87(.42)	91	87	87
557	61	61(.47)	63	59	60
559	76	75(.44)	79	79	80
650	113	114(.63)	116	113	114
651	88	88(.72)	89	87	88
652	109	108(.77)	110	111	112
653	78	79(.57)	80	77	77
654	85	85(.71)	87	85	85
655	116	113(.41)	115	115	116
656	118	118(.46)	122	117	117
657	105	104(.78)	105	105	106
658	96	95(.60)	97	95	95
659	119	118(.62)	119	118	121
660	106	106(.83)	108	106	109
661	106	106(.69)	107	107	107
662	106	105(.96)	106	108	108
751	142	139(.89)	141	144	145
752	88	89(.54)	90	86	87
753	104	104(.86)	106	106	107
754	95	94(.41)	97	93	93

* Number in parentheses following the estimated maximum yield is the average power

freedom and since the non-fixed power transformation has one degree less than the pre-fixed power transformation, this difference in the degrees of freedom may be responsible for the differences in standard deviation.

Generally speaking, again the lower power transformation ($p = .50$), yielded the smaller sum of deviations in most locations. On the other hand, the distance square $|\underline{Y} - \hat{\underline{Y}}|^2$ in the Mitscherlich model is always larger than that produced by the non-fixed power model.

The criteria of judging the goodness of a model by the magnitude of the standard deviations (or the sum of squares of deviation) from the model, or judging it by the value of the multiple coefficient of determination probably is not the best procedure to be followed in actual practice. All depends upon the objectives of the study. Probably if interest is focused merely on a best fitting curve, the minimum distance between the observed and expected response curve criteria is a reasonable decision rule in the absence of any other information on the loss of the wrong decision.

5.10 General Comments on the Results

If the decision rule to judge the goodness of the models is established <u>only</u> on the basis of the minimality of the loss function, taken as the sum of squares of deviations from the model, then the non-fixed transformation power minimizes the loss of the wrong decision, meaning that non-fixed power transformations are "superior" to fixed power transformations. However this statement is not necessarily true. Using the same criterion of loss function, to compare among the pre-fixed power transformations, the conclusion is that lower power transformations produce a better fit than higher power transformations with $p \geq .70$ for the second order model and $p \geq .80$ for the special power transformation model. This is a general evaluation and it may not be true for a particular case. The judgments of the models from the point of view of curve fitting, probably is valid, for fertilizer response functions, in which the response pattern increases up to some level of the applied fertilizer, then tends to be more or less constant (flat) and then decreases for very large values of the input. So the "best" model will be that which fits the data more closely. The asymmetry of the response might occur in most of the cases in which more than 4 levels of input factor and large enough doses of fertilizer are used. With less than 4 levels of the input factor essential features of the response curve may not be detected.

Comparing the two non-fixed power transformation models used (second order and the special one) on the grounds of "best fitting" criteria, there is not much difference between them and either choice is satisfactory.

Use of the decision rule to judge goodness of models based only upon the magnitude of the estimated initial soil fertility, $\hat{\delta}$, is not clear too; because exact quantitative determinations of δ do not exist. However, the loss incurred by using the wrong δ in the estimation of X_{MR}, X_{OR}, X_{MRR} seems to be unimportant. Consequently, this does not affect significantly the total of fertilizer that the farmer is advised to apply.

In the light of the results and considering the value of $\hat{\delta}$ solely, we can say in general terms that the polynomial model using any kind of power transformation detects very low values of $\hat{\delta}$ if the power transformation p is smaller than .50 and .40 for the special and second order transformations, respectively. On the other hand, if the power transformation is greater than or equal to .80 and .70 for both schemes then the $\hat{\delta}$ value seems to be highly inflated.

The estimated value of δ from the actual response curve, shows a great variability within the same enviromental conditions more than can be expected for a given soil. Also the other parameters of the second order-like-models show a great instability. This feature makes the description of the response pattern very difficult.

The correlation between $\hat{\delta}$ and the yield at zero level of application, indicates that $\hat{\delta}$ explains relatively well the complex phenomena of the soil-plant-response. Although the correlations of the estimated of the initial amount of N in the soil, $\hat{\delta}$, with the laboratory determinations of nitrogen (OM, NO_3, N) is not great, all of them gave good indications that the value of δ, estimated from the yield measure, can be considered as a reliable quantity.

The great advantage of the estimated initial nutrient level in the soil $\hat{\delta}$ is its quantitative nature. Since most of the laboratory determinations do not necessarily express quantitatively in pounds of added mutrient the amount of nutrient detected by the test.

The lack of a meaningful loss function to judge the goodness of the value of $\hat{\delta}$ also makes difficult the establishment of a reasonable decision rule for choosing the "best model". For the purpose of judging the "best model" to fit fertilizer response data, it is possible to establish an operational decision rule based on the minimality of the deviations from the model and at the same time on the closeness of the initial value of soil nutrient $\hat{\delta}$, estimated by statistical technique to some other good indicator of δ.

The loss function involves the size of $\hat{\delta} - \delta$. When $\hat{\delta}$ is an estimated value from the response and δ a pre-fixed quantity, the resulting decision rule is as follows: choose the model which minimizes the lack of fit and also minimizes the differences $|\hat{\delta} - \delta|$. If one follows the decision rule just pointed out above, probably he will be inclined to choose as the "best model" the lower power transformation, e.g. $p \doteq 50$ for the second order transformation model. Because the deviation from the model is small enough although not necessarily the smallest, and the value of $\hat{\delta}$ seems to be consistent with estimates of δ's obtained by nitrate production, organic matter content, percent of N determined by Kjeldahl and the yield at zero level of application.

6. SUMMARY AND CONCLUSIONS

In order to estimate the initial soil fertility level, from fertilizer response data, two types of transformations of the input factor were carried out, on the following models:

$$E(Y) = \beta_1(X+\delta) + \beta_2(X+\delta)^p \qquad (6.1)$$

where: Y = Response data, X = Amount of the applied fertilizer, δ = Parameter, representing the initial soil fertility level and p = Power of the transformation. This power was pre-fixed at .75, .95, 1.25, 1.50 and 1.75 and also p was allowed to take any value in the interval [.25, 2.00]. The other model used in this study is a combination of the power transformation and the second order model.

$$E(Y) = \beta_1(X+\delta)^p + \beta_2(X+\delta)^{2p} \qquad (6.2)$$

In equation (6.2) the power p, was pre-fixed at .50 and 1.00 and for the non-fixed power transformation, the value of p was bounded by the interval [.25, 1.00].

For the purpose of comparing the results yielded by models (6.1) and (6.2) with a familiar model, the Mitscherlich response model (6.3) was included.

$$E(Y) = \alpha(1 - e^{-\gamma(X+\delta)}) \qquad (6.3)$$

Since all those models are essentially non-linear, the non-linear least squares technique was used to estimate the parameters of the response models. The main interest is focused on the estimation of the parameter δ, the initial soil fertility level.

From the results obtained in this study, the following conclusions can be drawn:

1. The initial soil fertility levels estimated from the response data (such as N in this study) present wide horizons for developing calibration methods for laboratory soil test procedures.
2. The estimated parameter, $\hat{\delta}$ shows great sensitivity to the effect of local conditions. This appears to be indicative of the fact that a fertilizer response pattern is the result of the complex effect of multiple factors.
3. A high positive correlation exists between $\hat{\delta}$ (initial nitrogen level), estimated by statistical procedures and the corn yield at zero level application of N. Also there is a close correlation between $\hat{\delta}$ and the estimated nitrogen of laboratory tests such as: N, NO_3-production and OM.
4. For estimating the parameter δ by statistical procedures, the polynomial model of fixed power transformation $p \doteq .50$ (square root model) yields a reasonably "good" value of $\hat{\delta}$.
5. There are no convergence problems with the non-linear estimation technique used in this study when a polynomial-like model is fitted. The convergence is very fast and the degree of non-linearity seems to be small. Consequently, there are no difficulties in using the linearization method in the non-linear least square procedure.

6. If multifactor non-linear problems have to be handled, fixing the power transformation at about .50, and using direct search techniques to obtain the initial values of $\hat{\delta}$ make the convergence easier and faster.

7. BIBLIOGRAPHY

Anderson, R. L. 1956. Comparison of discrete and continuous models in agricultural production analysis. Chapter 3 of Methodological Procedures in the Economic Analysis of Fertilizer Use Data. Iowa State College Press, Ames, Iowa.

Baird, B. L. 1958. Factors affecting the nutrient content and yield response of corn to nitrogen, phosphorus, and potassium fertilization. Ph.D. Thesis, North Carolina State College, Dept. of Soil Science, (Univ. Microfilms, Ann Arbor), Raleigh.

Baule, B 1918. Zu Mitscherlich's Gesets der Physiologischen Beziechungen. Landw. Janh. 51:363-85.

Beale, E. M. L. 1960. Confidence regions in nonlinear estimations. Journ. Roy. Stat. Soc. (B) 22:41-76.

Box, G. E. P., and P. W. Tidwell. 1962. Transformation of the independent variables. Technometrics 4:531-550.

Bray, R. H. 1958. The correlation of a phosphorus soil test with the response of wheat through a modified Mitscherlich equation. Soil Science Society of America Proceedings 22:314-317.

Brown, W. G., T. L. Jackson, and R. G. Petersen. 1962. A method for incorporating soil test measurements into fertilizer response function. Agronomy Journal 54:152-154.

Curry, H. B. 1944. The method of steepest descent for nonlinear minimization problems. Quart. Appl. Math. 2:258-261.

Gauss, C. F. 1821. Theory of least squares, English translation by H. F. Trotter, Princeton University, Stat. Tech. Tes. Group, Tech. Report No. 5.

Gomez, F. D. 1960. Curso de estatistica experimental Escola Superior de Agricultura "Luis de Queiroz", Piracicaba, Publicacão Didactica No. 2, 229 p.

Gomez, F. D. 1961. On a formula for estimation of the optimum dressing of a fertilizer. Biometrics 17:492-494.

Hartley, H. O. and A. Booker. 1965. Nonlinear least squares estimation Anals. Math. Stat. 36:638-650.

Hartley, H. O. 1961. The modified Gauss-Newton method for fitting of nonlinear regression functions by least squares; Technometrics 3:269-280.

Heady, E. O. and J. L. Dillon. 1966. Agricultural production functions. Iowa State University Press, Ames, Iowa.

Hildreth, C. 1957. Possible Models for Agronomic-economic research. Chapter 16 of Economic and Technocial Analysis of Fertilizer Innovations and Resource Use. Iowa State College Press, Ames, Iowa.

Hurst, D. C. 1962. Modifications of response surface techniques for biological use. Ph.D. Thesis, Dept. of Statistics, N. C. College Library, Raleigh, N. C.

Hurst, D. C., and D. D. Mason. 1957. Some statistical aspects of the TVA North Carolina Cooperative Project on determination of yield response surfaces for corn. Chapter 18 of Economic and Technical Analysis of Fertilizer Innovation and Resource Use. Iowa State College Press, Ames, Iowa.

Liebig, J. W. 1855. Die Grundsatze der Agriculturchemie Mitriiskscht auf die in England angestellten Untersuchungen. Frederick Viewing und Sohn.

Marquardt, D. L. 1963. An algorithm for least squares estimation of nonlinear parameters. Jour. Soc. Indust. Appl. Math. 2:431-441.

Mason, D. D. 1956. Functional models and experimental designs for characterizing response curves and surfaces. Chapter 5 of Methodological Procedures in the Economic Analysis of Fertilizer Use Data. Iowa State College Press, Ames, Iowa.

Meeter, D. A. 1964. Problems in the analysis of nonlinear models by least squares. Ph.D. Thesis, University of Wisconsin, Dept. of Statistics, Madison, Wisc., (Univ. Microfilm, Ann Arbor).

Mitscherlich, E. A. 1909. Das Gesetz des Minimums und das Gesetz des Abnehmenden Bodenertrages. Landw. Jahr. 38:537-552.

Morrison, D. D. 1960. Methods for nonlinear least squares problems and convergence proofs. Proc. Jet Propulsion Lab. Seminar: Tracking Problems and Orbit Determination. Space Technology Laboratory, Redondo Beach, Calif. 9 p.

Pesek, J. T. and E. O. Heady. 1958. Derivation and application of a method for determining minimum recommended rates of fertilization. Soil Sci. Soc. of Amer. Proc. 22:419-23.

Saaty, T. L. and J. Bramm. 1964. Nonlinear mathematics. McGraw-Hill Book Co., New York, New York.

Stevens, W. L. 1951. Asymptotic regression. Biometrics 7:247-267.

Sukhatme, P. V. 1941. Economics of manuring. Indian Journal of Agricultural Science 11:325-33.

Welch, C. D. 1960. A study of the mineralizable nitrogen fraction of selected North Carolina soils and its relation to crop response to applied fertilizer nitrogen. Ph.D. Thesis, Dept. of Soil Science, N. C. State College Library, Raleigh, N.C.

8. A P P E N D I X

Table 1. Estimates of the Regression Coefficient $\hat{\beta}_1$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$ for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$, for $p > 1$

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
551.1	5.284	8.959	1.645	0.952	.715	5.169(.77)*
2	7.217	13.008	2.511	1.491	1.142	1.102(1.79)
3	5.765	9.663	1.688	0.944	.690	25.683(.25)
4	6.366	11.066	2.072	1.207	.910	14.457(.43)
Mean	6.158	10.674	1.979	1.148	.864	11.603(.81)
552.1	8.456	15.033	2.766	1.589	1.185	8.910(.71)
2	5.759	9.100	1.569	0.878	.645	29.671(.25)
3	6.182	9.900	1.723	0.970	.714	7.986(.64)
4	-	-	-	-	-	- -
Mean	6.799	11.344	2.019	1.146	.848	15.522(.53)
553.1	7.700	13.656	2.617	1.546	1.179	8.183(.71)
2	6.837	11.721	2.213	1.302	.989	13.577(.48)
3	7.189	12.077	2.203	1.270	.951	32.292(.25)
4	6.667	11.246	2.025	1.157	.861	7.458(.90)
Mean	7.098	12.175	2.265	1.319	.995	15.377(.58)
554.1	3.000	4.740	0.762	0.404	.283	14.208(.25)
2	3.087	5.444	1.008	0.581	.434	5.756(.48)
3	4.470	7.666	1.295	0.697	.492	18.789(.25)
4	4.592	8.146	1.523	0.883	.664	7.843(.51)
Mean	3.787	6.499	1.147	0.641	.468	11.649(.37)
555.1	6.269	10.936	2.048	1.196	.905	10.821(.95)
2	6.854	12.212	2.319	1.366	1.041	.874(2.00)
3	6.300	11.431	2.334	1.435	1.126	1.180(1.69)
4	7.383	12.997	2.537	1.522	1.174	2.284(1.28)
Mean	6.702	11.894	2.310	1.380	1.062	3.790(1.48)
556.1	6.840	11.243	2.022	1.159	.866	32.401(.25)
2	7.301	12.231	2.222	1.279	.958	17.409(.43)
3	5.834	9.515	1.711	0.984	.737	3.023(1.13)
4	6.013	9.874	1.714	0.958	.700	28.219(.25)
Mean	6.497	10.716	1.917	1.095	.815	20.263(.52)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \leq \hat{p} \leq 2.00\}$
557.1	5.474	9.352	1.610	0.886	.639	13.926(.40)*
2	6.039	10.558	1.906	1.081	.798	17.605(.35)
3	5.007	8.262	1.472	0.838	.622	23.410(.25)
4	6.479	11.655	2.118	1.204	.891	3.246(1.15)
Mean	5.750	9.957	1.777	1.002	.737	14.547(.54)
559.1	5.758	9.378	1.733	0.996	.745	18.530(.35)
2	6.872	11.866	2.159	1.234	.917	28.906(.25)
3	6.678	11.759	2.200	1.280	.965	11.061(.53)
4	6.616	11.873	2.166	1.232	.912	10.683(.53)
Mean	6.489	11.219	2.064	1.186	.885	17.295(.41)
650.1	8.818	15.154	2.802	1.625	1.223	7.785(1.19)
2	7.867	13.070	2.360	1.353	1.009	17.337(.46)
3	8.076	13.947	2.622	1.537	1.166	7.891(.85)
4	7.752	13.176	2.499	1.479	1.131	63.846(1.01)
Mean	8.128	13.837	2.571	1.499	1.132	24.215(.88)
651.1	7.549	12.945	2.323	1.318	.974	23.746(.34)
2	5.198	8.904	1.744	1.054	.819	.699(2.00)
3	6.249	10.409	1.883	1.081	.806	13.933(.45)
4	5.111	8.117	1.471	0.855	.646	.680(1.69)
Mean	6.027	10.094	1.855	1.077	.812	9.764(1.12)
652.1	6.461	10.439	1.926	1.128	.855	16.585(.43)
2	7.887	14.012	2.740	1.645	1.272	1.079(2.00)
3	7.444	12.902	2.452	1.447	1.103	19.155(.97)
4	6.164	10.542	2.047	1.230	.951	.909(1.81)
Mean	6.989	11.974	2.291	1.362	1.045	9.432(1.30)
653.1	7.227	12.848	2.380	1.372	1.026	7.061(.78)
2	6.496	10.877	1.954	1.116	.830	24.191(.98)
3	6.635	11.460	2.091	1.201	.897	6.894(1.07)
4	5.039	8.413	1.549	0.900	.679	22.198(.25)
Mean	6.349	10.900	1.993	1.147	.858	15.086(.77)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.24≤$\hat{p}$≤2.00}
654.1	5.690	9.957	1.936	1.158	.892	14.299(1.03)*
2	5.082	9.055	1.813	1.103	.860	11.578(.96)
3	5.930	10.459	2.017	1.200	.921	1.738(1.30)
4	6.636	11.473	2.133	1.241	.936	6.675(.86)
Mean	5.834	10.236	1.975	1.176	.902	8.572(1.04)
655.1	8.249	14.105	2.630	1.537	1.163	10.505(.92)
2	5.664	9.090	1.741	1.047	.811	25.330(.29)
3	6.372	10.062	1.825	1.061	.801	33.890(.25)
4	7.057	11.485	2.074	1.197	.898	8.713(.65)
Mean	6.835	11.185	2.068	1.211	.918	19.610(.53)
656.1	8.782	14.528	2.600	1.485	1.105	15.717(.52)
2	8.936	14.924	2.706	1.557	1.165	40.904(.25)
3	8.978	15.257	2.803	1.622	1.218	11.944(.61)
4	9.011	15.008	2.688	1.533	1.139	24.903(.39)
Mean	8.927	14.929	2.699	1.549	1.157	23.367(.44)
657.1	6.970	12.099	2.316	1.375	1.053	1.858(1.33)
2	6.533	11.481	2.254	1.357	1.080	11.202(1.04)
3	7.151	12.624	2.475	1.488	1.150	2.877(1.21)
4	6.230	10.843	2.122	1.277	.988	1.743(1.32)
Mean	6.723	11.762	2.292	1.374	1.060	4.420(1.23)
658.1	4.974	6.999	1.150	0.634	.455	32.149(.25)
2	5.586	8.797	1.593	0.926	.700	.583(2.00)
3	7.864	13.223	2.362	1.343	.997	1.022(1.72)
4	5.918	9.192	1.609	.916	.680	32.436(.25)
Mean	6.086	9.553	1.679	.955	.708	16.548(1.06)
659.1	8.153	13.973	2.639	1.554	1.182	12.609(.56)
2	8.318	14.450	2.726	1.598	1.211	34.819(.25)
3	7.562	13.361	2.624	1.578	1.220	37.812(1.01)
4	6.188	10.870	2.236	1.390	1.102	.954(2.00)
Mean	7.555	13.163	2.556	1.530	1.179	21.548(.96)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
660.1	7.459	13.006	2.483	1.469	1.122	7.250(.84)*
2	8.075	14.272	2.762	1.648	1.267	1.102(1.95)
3	6.659	11.455	2.181	1.291	.986	7.365(.90)
4	5.474	8.710	1.627	0.965	.740	.776(2.00)
Mean	6.917	11.861	2.263	1.343	1.029	4.123(1.42)
661.1	7.480	12.905	2.454	1.450	1.107	18.419(1.03)
2	7.444	13.166	2.557	1.528	1.176	2.883(1.22)
3	7.024	12.382	2.418	1.449	1.117	2.630(1.22)
4	6.101	10.592	2.069	1.241	.957	11.526(.49)
Mean	7.012	12.261	2.374	1.417	1.089	8.864(.99)
662.1	8.069	13.767	2.530	1.466	1.104	.918(2.00)
2	6.477	10.839	2.068	1.233	.948	.910(1.80)
3	6.399	11.075	2.169	1.308	1.014	1.132(1.60)
4	5.708	9.741	1.890	1.135	.878	.847(1.80)
Mean	6.663	11.355	2.164	1.286	.986	.952(1.80)
751.1	8.711	14.973	2.873	1.711	1.314	1.110(2.00)
2	8.366	14.121	2.690	1.598	1.225	3.334(1.19)
3	8.532	14.688	2.886	1.744	1.352	2.699(1.25)
4	7.818	13.379	2.633	1.597	1.243	1.197(1.80)
Mean	8.357	14.290	2.771	1.662	1.284	2.085(1.56)
752.1	7.579	13.021	2.473	1.462	1.116	14.328(.96)
2	5.892	10.112	1.944	1.160	.892	25.383(.25)
3	5.554	10.287	2.061	1.252	.976	.832(2.00)
4	7.321	12.137	2.039	1.107	.789	32.891(.25)
Mean	6.586	11.389	2.129	1.245	.943	18.359(.86)
753.1	7.288	12.719	2.484	1.492	1.152	7.266(.86)
2	5.781	10.099	2.112	1.325	1.056	1.044(1.77)
3	8.279	15.365	3.034	1.828	1.420	1.209(2.00)
4	5.978	10.996	2.190	1.321	1.021	2.028(1.28)
Mean	6.831	12.294	2.455	1.492	1.162	2.887(1.47)

Table 1. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{$.25 \leq \hat{p} \leq 2.00$}
754.1	8.053	13.528	2.399	1.358	1.003	9.219(.68)*
2	7.659	12.552	2.267	1.304	.975	36.477(.25)
3	6.268	9.947	1.707	.952	.695	31.910(.25)
4	7.376	12.232	2.134	1.194	.873	9.445(.63)
Mean	7.339	12.065	2.127	1.202	.886	21.763(.45)
Grand Mean	6.738	11.506	2.150	1.258	.953	12.864(.93)

* Number in parentheses is estimated power

Table 2. Estimates of the Regression Coefficient $\hat{\beta}_1$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \beta_o + \beta_1 X + \beta_2 X^2$

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
551.1	9.889	.594	5.379(.62)*	27.184
2	12.606	.962	1.328(.94)	52.829
3	10.849	.559	39.683(.25)	23.736
4	11.529	.755	10.562(.52)	38.315
Mean	11.218	.717	14.238(.58)	35.500
552.1	14.949	.977	9.027(.60)	50.369
2	11.696	.525	43.920(.25)	18.673
3	12.397	.583	8.584(.56)	20.059
4	-	-	- -	-
Mean	13.014	.695	20.510(.47)	29.700
553.1	13.663	.988	8.546(.59)	52.998
2	12.613	.827	13.793(.48)	41.227
3	13.557	.787	39.616(.26)	36.310
4	12.546	.709	5.235(.66)	30.541
Mean	13.095	.828	16.798(.50)	40.269
554.1	5.901	.221	24.076(.25)	8.110
2	5.498	.358	5.915(.48)	18.372
3	8.129	.386	25.071(.25)	16.242
4	8.134	.550	7.943(.50)	28.912
Mean	6.915	.379	15.751(.39)	17.909
555.1	11.347	.756	4.343(.69)	37.933
2	12.138	.874	.128(1.00)	45.449
3	10.937	.966	3.025(.74)	55.350
4	13.253	.995	2.981(.80)	52.835
Mean	11.918	.897	2.619(.81)	47.892
556.1	13.267	.715	43.674(.25)	30.639
2	13.858	.793	16.195(.47)	35.098
3	11.476	.610	3.024(.73)	24.304
4	11.630	.568	43.134(.25)	21.841
Mean	12.558	.671	26.507(.42)	27.970

Table 2. (continued)

Baird Loc. No.	p = .50	p = 1.00	{.25≤$\hat{p}$≤1.00}	Second order Model without δ
557.1	10.077	.512	13.247(.45)*	20.467
2	10.849	.652	16.488(.42)	31.733
3	9.651	.511	32.801(.25)	21.537
4	11.311	.730	3.080(.75)	37.907
Mean	10.472	.601	16.404(.47)	27.911
559.1	11.025	.617	16.369(.42)	16.740
2	12.528	.754	26.550(.35)	36.409
3	11.933	.802	11.061(.50)	41.509
4	11.571	.746	10.549(.50)	39.229
Mean	11.764	.730	16.132(.44)	35.972
650.1	16.243	1.016	5.509(.71)	47.993
2	15.061	.832	17.479(.47)	35.781
3	14.791	.974	7.027(.65)	48.133
4	14.513	.952	5.066(.71)	46.059
Mean	15.152	.943	8.770(.63)	44.491
651.1	13.874	.797	21.840(.41)	36.565
2	9.689	.699	.699(1.00)	34.598
3	11.928	.665	13.938(.46)	28.898
4	10.439	.539	.539(1.00)	20.465
Mean	11.483	.675	9.254(.72)	30.131
652.1	12.864	.714	16.497(.45)	30.790
2	14.032	1.079	1.079(1.00)	57.307
3	13.588	.925	5.137(.69)	46.526
4	11.487	.807	1.044(.95)	40.019
Mean	12.993	.882	5.939(.77)	43.660
653.1	12.775	.847	7.045(.62)	43.853
2	12.337	.683	4.397(.68)	28.542
3	12.137	.741	3.771(.72)	34.628
4	9.570	.565	24.624(.25)	26.796
Mean	11.705	.709	9.959(.57)	33.455

Table 2. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
654.1	10.276	.754	3.566(.72)*	39.603
2	8.991	.734	3.449(.71)	40.935
3	10.616	.778	2.340(.80)	41.114
4	12.134	.779	5.492(.61)	37.950
Mean	10.504	.761	3.712(.71)	39.900
655.1	15.300	.971	6.198(.67)	46.326
2	11.435	.690	24.661(.32)	31.664
3	13.064	.668	37.901(.28)	26.815
4	13.922	.745	30.557(.36)	30.269
Mean	13.430	.769	24.829(.41)	33.768
656.1	16.922	.910	15.801(.51)	37.498
2	17.010	.964	38.474(.34)	42.775
3	16.743	1.010	12.857(.55)	46.821
4	17.197	.937	23.229(.44)	32.932
Mean	16.968	.955	22.590(.46)	40.006
657.1	12.719	.887	2.572(.81)	44.815
2	11.746	.891	3.950(.73)	47.640
3	12.819	.976	3.281(.78)	52.253
4	11.345	.838	2.355(.81)	43.515
Mean	12.157	.898	3.040(.78)	47.056
658.1	11.249	.370	49.883(.25)	9.151
2	11.529	.583	.583(1.00)	21.513
3	14.867	.819	1.406(.91)	33.658
4	12.351	.559	45.279(.25)	19.709
Mean	12.499	.583	24.288(.60)	21.008
659.1	15.062	.990	13.143(.53)	49.074
2	15.071	1.011	41.988(.25)	52.213
3	13.504	1.034	4.790(.72)	55.847
4	11.217	.954	.954(1.00)	51.181
Mean	13.714	.997	15.219(.62)	52.079

Table 2. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
660.1	13.514	.943	6.464(.65)*	48.531
2	14.437	1.071	1.119(.99)	56.205
3	12.269	.829	5.294(.67)	41.285
4	11.196	.624	.624(1.00)	24.806
Mean	12.854	.867	3.375(.83)	42.707
661.1	13.736	.929	4.667(.71)	46.275
2	13.281	.994	3.348(.78)	53.209
3	12.584	.944	3.116(.78)	50.134
4	11.095	.809	11.792(.48)	42.857
Mean	12.674	.919	5.731(.69)	48.119
662.1	15.019	.918	.819(1.02)	41.988
2	12.397	.802	1.091(.95)	44.187
3	11.747	.862	1.502(.90)	37.503
4	10.667	.745	.978(.95)	36.756
Mean	12.457	.832	1.097(.96)	40.108
751.1	16.121	1.110	.865(1.04)	54.234
2	15.811	1.032	3.891(.77)	49.143
3	15.745	1.150	3.517(.80)	58.912
4	14.567	1.061	1.460(.95)	53.818
Mean	15.562	1.088	2.433(.89)	54.027
752.1	13.992	.938	5.338(.69)	46.434
2	10.878	.755	34.423(.25)	38.775
3	9.426	.832	.832(1.00)	49.367
4	13.873	.627	43.691(.25)	23.359
Mean	12.043	.788	21.071(.54)	39.484
753.1	13.204	.976	6.112(.66)	51.560
2	10.528	.918	1.180(.95)	50.418
3	14.025	1.209	1.209(1.00)	71.884
4	10.220	.863	2.415(.81)	50.686
Mean	11.994	.992	2.729(.86)	56.137

Table 2. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \leq \hat{p} \leq 1.00\}$	Second order Model without δ
754.1	15.213	.821	9.583(.58)*	33.925
2	14.899	.805	47.361(.25)	34.906
3	12.648	.563	48.028(.25)	19.378
4	14.121	.709	10.512(.55)	26.825
Mean	14.220	.725	28.871(.41)	28.758
Grand Mean	12.535	.796	12.875(.62)	38.321

* Number in parentheses is estimated power

Table 3. Estimates of the Regression Coeffecient $\hat{\beta}_2$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)$, for $p < 1$ and $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$, for $p > 1$

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
551.1	-1.052	-6.519	-0.343	-0.042	-0.007	-1.160(.77)*
2	-1.478	-9.536	-0.547	-0.073	-0.013	-0.009(1.79)
3	-1.204	-7.081	-0.360	-0.043	-0.007	-0.167(.25)
4	-1.271	-8.065	-0.437	-0.055	-0.009	-0.283(.43)
Mean	-1.251	-7.800	-0.421	-0.053	-0.009	-0.405(.81)
552.1	-1.816	-11.095	-0.616	-0.080	-0.013	-1.508(.71)
2	-1.122	-6.576	-0.312	-0.035	-0.005	-0.154(.25)
3	-1.218	-7.171	-0.347	-0.039	-0.006	-0.751(.64)
4	-	-	-	-	-	- -
Mean	-1.385	-8.280	-0.425	-0.051	-0.008	-0.804(.53)
553.1	-1.540	-9.966	-0.558	-0.072	-0.012	-1.229(.71)
2	-1.318	-8.491	-0.455	-0.057	-0.009	-0.351(.48)
3	-1.416	-8.769	-0.454	-0.055	-0.008	-0.119(.25)
4	-1.357	-8.209	-0.426	-0.052	-0.008	-3.962(.90)
Mean	-1.407	-8.859	-0.473	-0.059	-0.009	-1.415(.58)
554.1	-0.627	-3.465	-0.157	-0.017	-0.002	-0.105(.25)
2	-0.643	-3.997	-0.219	-0.028	-0.004	-0.200(.48)
3	-1.000	-5.687	-0.291	-0.035	-0.005	-0.176(.25)
4	-0.955	-5.980	-0.332	-0.043	-0.007	-0.328(.51)
Mean	-0.806	-4.783	-0.250	-0.031	-0.004	-0.205(.37)
555.1	-1.270	-7.990	-0.437	-0.055	-0.009	-7.870(.95)
2	-1.416	-8.959	-0.505	-0.066	-0.011	-0.002(2.00)
3	-1.180	-8.274	-0.487	-0.065	-0.011	-0.017(1.69)
4	-1.417	-9.424	-0.528	-0.068	-0.011	-0.384(1.28)
Mean	-1.321	-8.661	-0.489	-0.064	-0.011	-2.068(1.48)
556.1	-1.334	-8.142	-0.410	-0.048	-0.007	-0.130(.25)
2	-1.449	-8.890	-0.459	-0.055	-0.009	-0.370(.43)
3	-1.128	-6.878	-0.344	-0.040	-0.006	-1.301(1.13)
4	-1.231	-7.206	-0.357	-0.042	-0.006	-0.177(.25)
Mean	-1.285	-7.779	-0.392	-0.046	-0.007	-0.494(.52)

Table 3. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
557.1	-1.207	-6.920	-0.358	-0.043	-0.007	-0.345(.40)*
2	-1.290	-7.780	-0.419	-0.053	-0.008	-0.255(.35)
3	-0.995	-6.002	-0.302	-0.036	-0.005	-0.110(.25)
4	-1.451	-8.664	-0.486	-0.064	-0.011	-1.309(1.15)
Mean	-1.236	-7.342	-0.392	-0.049	-0.008	-0.505(.54)
559.1	-1.137	-6.954	-0.356	-0.043	-0.006	-0.203(.35)
2	-1.420	-8.693	-0.463	-0.057	-0.009	-0.150(.25)
3	-1.371	-8.613	-0.474	-0.061	-0.010	-0.495(.53)
4	-1.460	-8.805	-0.492	-0.064	-0.011	-0.581(.53)
Mean	-1.347	-8.266	-0.446	-0.056	-0.009	-0.357(.41)
650.1	-1.775	-11.053	-0.591	-0.074	-0.012	-3.053(1.19)
2	-1.549	-9.485	-0.483	-0.058	-0.009	-0.443(.46)
3	-1.594	-10.145	-0.458	-0.069	-0.011	-2.939(.85)
4	-1.471	-9.518	-0.507	-0.063	-0.010	-60.656(1.01)
Mean	-1.598	-10.050	-0.532	-0.066	-0.010	-13.555(.88)
651.1	-1.578	-9.498	-0.500	-0.062	-0.010	-0.295(.34)
2	-0.950	-6.400	-0.349	-0.044	-0.007	-0.001(2.00)
3	-1.233	-7.557	-0.386	-0.046	-0.007	-0.345(.45)
4	-0.937	-5.811	-0.283	-0.032	-0.005	-0.007(1.69)
Mean	-1.174	-7.316	-0.380	-0.046	-0.007	-0.162(1.12)
652.1	-1.175	-7.472	-0.372	-0.043	-0.006	-0.253(.43)
2	-1.543	-10.193	-0.578	-0.076	-0.013	-0.002(2.00)
3	-1.453	-9.371	-0.510	-0.065	-0.011	-15.784(.97)
4	-1.137	-7.587	-0.411	-0.052	-0.008	-0.005(1.81)
Mean	-1.327	-8.656	-0.468	-0.059	-0.009	-4.011(1.30)
653.1	-1.534	-9.465	-0.526	-0.068	-0.011	-1.764(.78)
2	-1.311	-7.926	-0.408	-0.049	-0.008	-0.214(.98)
3	-1.376	-8.400	-0.450	-0.056	-0.009	-4.559(1.07)
4	-0.968	-6.083	-0.313	-0.038	-0.006	-0.062(.25)
Mean	-1.297	-7.969	-0.424	-0.053	-0.009	-1.650(.77)

Table 3. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
654.1	-1.086	-7.212	-0.400	-0.051	-0.009	-11.988(1.03)*
2	-0.950	-6.544	-0.374	-0.049	-0.008	-9.142(.96)
3	-1.167	-7.613	-0.425	-0.055	-0.009	-0.267(1.30)
4	-1.338	-8.373	-0.452	-0.057	-0.009	-2.819(.86)
Mean	-1.135	-7.435	-0.413	-0.053	-0.009	-6.054(1.04)
655.1	-1.625	-10.251	-0.547	-0.068	-0.011	-6.372(.92)
2	-0.959	-6.435	-0.323	-0.037	-0.006	-0.019(.29)
3	-1.151	-7.185	-0.347	-0.039	-0.006	-0.081(.25)
4	-1.351	-8.289	-0.414	-0.048	-0.007	-0.856(.65)
Mean	-1.271	-8.040	-0.408	-0.048	-0.007	-1.832(.53)
656.1	-1.747	-10.557	-0.535	-0.064	-0.010	-0.669(.52)
2	-1.766	-10.838	-0.557	-0.067	-0.011	-0.169(.25)
3	-1.789	-11.105	-0.585	-0.072	-0.012	-0.954(.64)
4	-1.807	-10.923	-0.557	-0.067	-0.010	-0.410(.39)
Mean	-1.777	-10.856	-0.558	-0.067	-0.011	-0.550(.44)
657.1	-1.350	-8.778	-0.481	-0.061	-0.010	-0.233(1.33)
2	-1.235	-8.305	-0.464	-0.060	-0.010	-8.599(1.04)
3	-1.367	-9.147	-0.514	-0.067	-0.011	-0.778(1.21)
4	-1.165	-7.827	-0.433	-0.055	-0.009	-0.225(1.32)
Mean	-1.279	-8.514	-0.473	-0.061	-0.010	-2.459(1.23)
658.1	-0.881	-4.961	-0.206	-0.020	-0.003	-0.148(.25)
2	-1.010	-6.282	-0.303	-0.034	-0.005	-0.001(2.00)
3	-1.614	-9.664	-0.499	-0.061	-0.009	-0.012(1.72)
4	-1.100	-6.587	-0.308	-0.034	-0.005	-0.129(.25)
Mean	-1.151	-6.873	-0.329	-0.037	-0.005	-0.072(1.06)
659.1	-1.575	-10.126	-0.543	-0.068	-0.011	-0.625(.56)
2	-1.638	-10.510	-0.570	-0.072	-0.012	-0.090(.25)
3	-1.442	-9.680	-0.454	-0.071	-0.012	-34.614(1.01)
4	-1.079	-7.773	-0.443	-0.058	-0.010	-0.002(2.00)
Mean	-1.433	-9.522	-0.525	-0.067	-0.011	-8.832(.96)

Table 3. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
660.1	-1.460	-9.453	-0.519	-0.066	-0.011	-2.529(.84)*
2	-1.589	-10.389	-0.583	-0.076	-0.013	-0.003(1.95)
3	-1.279	-8.295	-0.448	-0.056	-0.009	-3.756(.90)
4	-0.959	-6.194	-0.306	-0.035	-0.005	-0.008(2.00)
Mean	-1.322	-8.583	-0.464	-0.058	-0.009	-1.574(1.42)
661.1	-1.448	-9.359	-0.507	-0.064	-0.010	-15.450(1.03)
2	-1.453	-9.572	-0.538	-0.070	-0.012	-0.749(1.22)
3	-1.349	-8.979	-0.503	-0.065	-0.011	-0.639(1.22)
4	-1.131	-7.635	-0.419	-0.053	-0.008	-0.274(.49)
Mean	-1.345	-8.886	-0.492	-0.063	-0.010	-4.278(.99)
662.1	-1.626	-10.040	-0.533	-0.066	-0.011	-0.002(2.00)
2	-1.188	-7.784	-0.409	-0.050	-0.008	-0.006(1.80)
3	-1.186	-7.980	-0.439	-0.056	-0.009	-0.017(1.60)
4	-1.050	-7.006	-0.378	-0.047	-0.008	-0.005(1.80)
Mean	-1.262	-8.202	-0.440	-0.055	-0.009	-0.007(1.80)
751.1	-1.657	-10.827	-0.587	-0.074	-0.012	-0.002(2.00)
2	-1.556	-10.167	-0.538	-0.066	-0.010	-0.961(1.19)
3	-1.557	-10.566	-0.578	-0.073	-0.012	-0.469(1.25)
4	-1.049	-9.597	-0.523	-0.066	-0.011	-0.008(1.80)
Mean	-1.545	-10.289	-0.557	-0.070	-0.011	-0.360(1.56)
752.1	-1.460	-9.434	-0.509	-0.064	-0.010	-10.792(.96)
2	-1.108	-7.299	-0.395	-0.049	-0.008	-0.025(.25)
3	-1.131	-7.544	-0.454	-0.063	-0.011	-0.002(2.00)
4	-1.582	-8.939	-0.441	-0.051	-0.008	-0.277(.25)
Mean	-1.320	-8.304	-0.450	-0.057	-0.009	-2.774(.86)
753.1	-1.373	-9.192	-0.509	-0.065	-0.011	-2.834(.86)
2	-0.957	-7.169	-0.408	-0.053	-0.009	-0.008(1.77)
3	-1.741	-11.324	-0.681	-0.095	-0.017	-0.003(2.00)
4	-1.205	-8.049	-0.478	-0.065	-0.012	-0.378(1.28)
Mean	-1.319	-8.933	-0.519	-0.069	-0.012	-0.806(1.47)

Table 3. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \le \hat{p} \le 2.00\}$
754.1	-1.666	-9.899	-0.509	-0.061	-0.010	-1.197(.68)*
2	-1.472	-9.068	-0.455	-0.053	-0.008	-0.125(.25)
3	-1.236	-7.204	-0.342	-0.038	-0.005	-0.178(.25)
4	-1.527	-8.946	-0.450	-0.053	-0.008	-0.929(.63)
Mean	-1.475	-8.779	-0.439	-0.051	-0.008	-0.607(.45)
Grand Mean	-1.343	-8.359	-0.446	-0.056	-0.009	-2.233(.93)

* Number in parentheses is estimated power

Table 4. Estimates of the Regression Coefficient $\hat{\beta}_2$, fitting the models: $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$ and $E(Y) = \beta_0 + \beta_1 X + \beta_2 X^2$

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
551.1	-0.349	-0.0012	-0.102(.62)*	-4.751
2	-0.455	-0.0024	-0.004(.94)	-9.633
3	-0.445	-0.0011	-6.523(.25)	-4.535
4	-0.397	-0.0016	-0.334(.52)	-6.397
Mean	-0.412	-0.0016	-1.741(.58)	-6.329
552.1	-0.632	-0.0025	-0.226(.60)	-9.938
2	-0.418	-0.0008	-6.008(.25)	-3.250
3	-0.453	-0.0010	-0.216(.56)	-3.851
4	-	-	- -	-
Mean	-0.501	-0.0014	-2.150(.47)	-5.679
553.1	-0.463	-0.0023	-0.180(.59)	-9.022
2	-0.394	-0.0016	-0.470(.48)	-6.498
3	-0.466	-0.0015	-3.865(.26)	-5.966
4	-0.477	-0.0015	-0.082(.66)	-5.742
Mean	-0.450	-0.0017	-1.149(.50)	-6.807
554.1	-0.251	-0.0003	-4.152(.25)	-1.419
2	-0.213	-0.0008	-0.248(.48)	-3.350
3	-0.393	-0.0009	-2.989(.25)	-3.558
4	-0.312	-0.0013	-0.297(.50)	-5.253
Mean	-0.292	-0.0008	-1.921(.39)	-3.395
555.1	-0.409	-0.0017	-0.059(.69)	-6.750
2	-0.457	-0.0022	-0.009(1.00)	-8.615
3	-0.281	-0.0022	-0.267(.74)	-8.846
4	-0.394	-0.0022	-0.020(.80)	-8.544
Mean	-0.385	-0.0021	-0.087(.81)	-8.188
556.1	-0.455	-0.0013	-4.860(.25)	-5.052
2	-0.491	-0.0015	-0.673(.47)	-6.096
3	-0.389	-0.0011	-0.266(.73)	-4.195
4	-0.461	-0.0011	-6.505(.25)	-4.226
Mean	-0.449	-0.0012	-3.076(.42)	-4.892

Table 4. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
557.1	-0.473	-0.0012	-0.820(.45)*	-4.750
2	-0.457	-0.0016	-1.070(.42)	-6.177
3	-0.349	-0.0010	-4.080(.25)	-3.727
4	-0.534	-0.0021	-0.038(.75)	-8.218
Mean	-0.453	-0.0014	-1.502(.47)	-5.718
559.1	-0.388	-0.0012	-0.860(.42)	-4.615
2	-0.485	-0.0017	-2.221(.35)	-6.579
3	-0.444	-0.0019	-0.380(.50)	-7.412
4	-0.526	-0.0020	-0.435(.50)	-8.083
Mean	-0.460	-0.0017	-0.974(.44)	-6.672
650.1	-0.585	-0.0022	-0.066(.71)	-8.568
2	-0.526	-0.0015	-0.710(.47)	-6.109
3	-0.497	-0.0021	-0.111(.65)	-8.129
4	-0.437	-0.0019	-0.053(.71)	-7.262
Mean	-0.511	-0.0019	-0.235(.63)	-7.517
651.1	-0.560	-0.0018	-1.405(.41)	-6.960
2	-0.252	-0.0013	-0.001(1.00)	-5.297
3	-0.418	-0.0012	-0.572(.46)	-4.940
4	-0.311	-0.0008	-0.001(1.00)	-3.219
Mean	-0.385	-0.0013	-0.495(.72)	-5.104
652.1	-0.362	-0.0011	-0.589(.45)	-4.382
2	-0.442	-0.0025	-0.002(1.00)	-9.839
3	-0.438	-0.0020	-0.062(.69)	-7.704
4	-0.310	-0.0015	-0.003(.95)	-6.106
Mean	-0.388	-0.0018	-0.164(.77)	-7.007
653.1	-0.522	-0.0021	-0.155(.62)	-8.431
2	-0.461	-0.0013	-0.057(.68)	-5.351
3	-0.477	-0.0017	-0.045(.72)	-6.656
4	-0.306	-0.0010	-3.547(.25)	-4.085
Mean	-0.441	-0.0015	-0.951(.57)	-6.130

Table 4. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
654.1	-0.302	-0.0016	-0.036(.72)*	-6.293
2	-0.236	-0.0016	-0.036(.71)	-6.316
3	-0.344	-0.0018	-0.016(.80)	-6.941
4	-0.436	-0.0017	-0.088(.61)	-6.716
Mean	-0.329	-0.0017	-0.044(.71)	-6.566
655.1	-0.517	-0.0020	-0.084(.67)	-7.867
2	-0.252	-0.0010	-0.624(.32)	-3.887
3	-0.372	-0.0010	-2.817(.28)	-3.802
4	-0.459	-0.0013	-2.223(.36)	-5.044
Mean	-0.400	-0.0013	-1.437(.41)	-5.150
656.1	-0.613	-0.0017	-0.534(.51)	-6.736
2	-0.597	-0.0018	-3.072(.34)	-7.275
3	-0.591	-0.0021	-0.347(.55)	-8.109
4	-0.633	-0.0018	-1.162(.44)	-5.861
Mean	-0.609	-0.0018	-1.279(.46)	-6.995
657.1	-0.399	-0.0019	-0.016(.81)	-7.389
2	-0.331	-0.0019	-0.038(.73)	-7.461
3	-0.372	-0.0021	-0.025(.78)	-8.361
4	-0.312	-0.0017	-0.014(.81)	-6.747
Mean	-0.354	-0.0019	-0.023(.78)	-7.489
658.1	-0.346	-0.0004	-6.854(.25)	-1.461
2	-0.333	-0.0008	-0.001(1.00)	-6.663
3	-0.582	-0.0017	-0.005(.91)	-3.320
4	-0.393	-0.0008	-5.255(.25)	-3.117
Mean	-0.413	-0.0009	-3.028(.60)	-3.640
659.1	-0.475	-0.0020	-0.562(.53)	-7.842
2	-0.496	-0.0022	-3.186(.25)	-8.434
3	-0.389	-0.0022	-0.050(.72)	-8.867
4	-0.226	-0.0019	-0.000(1.00)	-7.496
Mean	-0.396	-0.0021	-0.949(.62)	-8.159

Table 4. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second Order Model without δ
660.1	-0.436	-0.0020	-0.099(.65)*	-8.029
2	-0.467	-0.0024	-0.003(.99)	-9.663
3	-0.378	-0.0017	-0.070(.67)	-6.616
4	-0.286	-0.0009	-0.001(1.00)	-3.566
Mean	-0.392	-0.0017	-0.043(.83)	-6.968
661.1	-0.433	-0.0019	-0.049(.71)	-7.565
2	-0.417	-0.0023	-0.026(.78)	-8.837
3	-0.374	-0.0021	-0.023(.78)	-8.110
4	-0.295	-0.0016	-0.033(.48)	-6.289
Mean	-0.380	-0.0019	-0.033(.69)	-7.700
662.1	-0.549	-0.0020	-0.001(1.02)	-7.673
2	-0.340	-0.0014	-0.003(.95)	-6.780
3	-0.315	-0.0017	-0.005(.90)	-5.641
4	-0.286	-0.0014	-0.002(.95)	-5.580
Mean	-0.372	-0.0016	-0.003(.96)	-6.418
751.1	-0.482	-0.0023	-0.001(1.04)	-8.819
2	-0.449	-0.0019	-0.027(.77)	-7.511
3	-0.402	-0.0022	-0.021(.80)	-8.736
4	-0.359	-0.0020	-0.004(.95)	-7.867
Mean	-0.423	-0.0021	-0.013(.89)	-8.233
752.1	-0.457	-0.0019	-0.063(.69)	-7.515
2	-0.311	-0.0015	-4.286(.25)	-5.866
3	-0.323	-0.0023	-0.002(1.00)	-9.033
4	-0.629	-0.0013	-6.474(.25)	-5.090
Mean	-0.425	-0.0017	-2.706(.54)	-6.876
753.1	-0.371	-0.0020	-0.081(.66)	-7.970
2	-0.262	-0.0017	-0.003(.95)	-6.678
3	-0.536	-0.0035	-0.003(1.00)	-13.882
4	-0.342	-0.0023	-0.018(.81)	-8.900
Mean	-0.377	-0.0024	-0.026(.86)	-9.357

Table 4. (continued)

Baird Loc. No.	p = .50	p = 1.00	$\{.25 \le \hat{p} \le 1.00\}$	Second order Model without δ
754.1	-0.609	-0.0017	-0.239(.58)*	-6.731
2	-0.490	-0.0014	-5.682(.25)	-5.487
3	-0.466	-0.0009	-6.858(.25)	-3.572
4	-0.573	-0.0014	-0.316(.35)	-5.528
Mean	-0.534	-0.0013	-3.274(.36)	-5.329
Grand Mean	-0.421	-0.0016	-1.090(.62)	-6.492

* Number in parentheses is estimated power

Table 5. Standard Deviation from the model $E(Y) = \beta_1(X + \delta) + \beta_2(X + \delta)^p$

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \leq \hat{p} \leq 2.00\}$
551.1	6.4908	6.5623	6.5623	6.6230	6.6930	6.4857(.77)*
2	14.2964	14.0284	13.7839	13.6845	13.6527	13.6502(1.79)
3	7.6463	7.9455	8.2902	8.5125	8.6946	6.6380(.25)
4	8.5301	8.7976	9.2425	9.5007	9.8012	8.3744(.43)
Mean	9.2409	9.3334	9.4697	9.5802	9.7103	8.7870(.81)
552.1	10.9298	11.0028	11.2337	11.4687	11.7101	10.9267(.71)
2	8.0664	8.1276	8.3385	8.4299	8.5045	7.5275(.25)
3	3.5933	3.5954	3.6000	3.6037	3.6083	3.5926(.64)
4	-	-	-	-	-	- -
Mean	7.5298	7.5752	7.7240	7.8341	7.9409	7.3489(.53)
553.1	8.9479	9.0405	9.3272	9.6298	9.9164	8.9404(.71)
2	7.7416	7.9287	8.2502	8.5242	8.7898	7.6112(.48)
3	10.9267	11.1443	11.4454	11.6618	11.8518	10.4433(.25)
4	6.4571	6.5316	6.5776	6.6380	6.7132	6.5316(..90)
Mean	8.5408	8.6613	8.9001	9.1134	9.3178	8.3816(.58)
554.1	9.1614	9.3414	9.5076	9.5949	9.6574	8.2582(.25)
2	10.8931	10.9329	11.0116	11.0814	11.1472	10.8658(.48)
3	10.2923	10.6174	11.0119	11.2603	11.4542	9.3700(.25)
4	6.6331	6.7576	6.9946	7.2012	7.3896	6.5660(.51)
Mean	9.2450	9.4123	9.6314	9.7844	9.9121	8.7650(.37)
555.1	5.9256	5.8921	5.9437	6.0315	6.1425	5.8921(.95)
2	11.4046	11.0178	10.5861	10.3279	10.1389	10.0031(2.00)
3	10.1947	9.7805	9.3914	9.2409	9.2123	9.2086(1.69)
4	9.5288	9.3594	9.2734	9.2985	9.3773	9.2734(1.28)
Mean	9.2634	9.0124	8.7986	8.7247	8.7177	8.5943(1.48)
556.1	10.1717	10.3246	10.5198	10.6520	10.7671	9.7021(.25)
2	5.7328	5.8650	7.5453	6.2393	6.3927	5.6096(.43)
3	8.0950	8.0829	8.0829	8.0867	8.0992	8.0787(1.13)
4	5.0330	5.2724	5.5616	5.7560	5.9161	4.2559(.25)
Mean	7.2581	7.3862	7.9274	7.6835	7.7937	6.9115(.52)

Table 5. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤$\hat{p}$≤2.00}
557.1	7.3212	7.4431	7.6287	7.7715	7.9034	7.2019(.40)*
2	7.7630	8.0121	8.3982	8.6984	8.9664	7.4741(.35)
3	10.4689	10.5513	10.6613	10.7392	10.8041	10.2306(.25)
4	14.5027	14.4289	14.4126	14.4498	14.5118	14.4080(1.15)
Mean	10.0140	10.1082	10.2752	10.4147	10.5464	9.8286(.54)
559.1	6.9130	6.9900	7.0990	7.1780	7.2570	6.8160(.35)
2	7.3980	7.7630	8.2822	8.6602	8.9925	6.7670(.25)
3	7.4430	7.6200	7.9497	8.2381	8.5087	7.3573(.53)
4	13.3514	13.4757	13.7186	13.9355	14.1444	13.2941(.53)
Mean	8.7635	8.9621	9.2624	9.5029	9.7256	8.5586(.41)
650.1	7.9540	7.8655	7.9034	8.0207	8.1813	7.7886(1.19)
2	12.3855	12.4818	12.6411	12.7749	12.9022	12.3096(.46)
3	6.4237	6.4237	6.5977	6.8115	7.0568	6.4082(.85)
4	8.4100	8.3705	8.3984	8.4655	8.5518	8.3705(1.01)
Mean	8.7933	8.7854	8.8851	9.0181	9.1730	8.7192(.88)
651.1	8.6868	8.9367	9.3057	9.5915	9.8454	8.3625(.34)
2	7.8908	7.6940	7.4565	7.3439	7.2432	7.1691(2.00)
3	10.6520	10.7237	10.8441	10.9422	11.0362	10.5923(.45)
4	9.8386	9.8815	9.7842	9.7671	9.7431	9.7568(1.69)
Mean	9.2670	9.2915	9.3476	9.4112	9.4670	8.9701(1.12)
652.1	8.6254	8.6984	8.8165	8.8164	9.0036	8.5595(.43)
2	12.4924	11.1444	10.5828	10.2567	10.0300	9.8826(2.00)
3	6.5063	6.4343	6.5112	6.6682	6.8603	6.4343(.97)
4	9.6434	9.5113	9.3651	9.3486	9.3308	9.4268(1.81)
Mean	9.3169	8.9471	8.8189	8.7725	8.8061	8.5758(1.30)
653.1	6.8505	6.9089	7.1924	7.5099	7.8443	6.8505(.78)
2	7.9747	7.9540	7.9706	8.0080	8.0540	7.9747(.93)
3	9.1505	9.1028	9.1103	9.1505	9.2086	9.0992(1.07)
4	7.8314	8.0496	8.3145	8.4891	8.6370	7.0994(.25)
Mean	7.9518	8.0038	8.1469	8.2893	8.4360	7.7560(.77)

Table 5. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤p̂≤2.00}
654.1	9.8250	9.7808	9.8012	9.8658	9.9530	9.7772(1.03)*
2	4.4870	4.4271	4.5017	4.6476	4.8303	4.4870(.96)
3	5.2409	5.0793	4.9933	5.0132	5.0726	5.0793(1.30)
4	5.8250	5.8136	5.9161	6.0442	6.1913	5.8079(.86)
Mean	6.3445	6.2752	6.3031	6.3927	6.5118	6.2878(1.04)
655.1	8.1402	8.1116	8.1813	8.2905	8.4219	8.1402(.92)
2	5.8250	5.9047	6.0220	6.1046	6.1859	5.6863(.29)
3	8.2014	8.2704	8.3625	8.4299	8.4891	8.0248(.25)
4	8.3507	8.3625	8.3904	8.4181	8.4418	8.3465(.65)
Mean	7.6293	7.6623	7.7390	7.8108	7.8847	7.5494(.53)
656.1	6.5877	6.6881	6.8652	7.0142	7.1552	6.5267(.52)
2	8.3943	8.6515	8.9886	9.2340	9.4516	7.8652(.25)
3	4.6687	4.8717	5.2977	5.6625	6.0220	4.6187(.61)
4	5.2851	5.5556	5.9608	6.2662	6.5368	4.9597(.39)
Mean	6.2339	6.4417	6.7781	7.0442	7.2914	5.9926(.44)
657.1	5.7676	5.5495	5.4284	5.4405	5.5195	5.4222(1.33)
2	6.5877	6.5908	6.5216	6.6380	6.7971	6.4857(1.04)
3	5.7095	5.4649	5.3726	5.4526	5.6272	5.3664(1.21)
4	9.5743	9.4480	9.3773	9.3878	9.4374	9.3736(1.32)
Mean	6.9098	6.7383	6.6750	6.7280	6.8453	6.6620(1.23)
658.1	10.9755	10.9998	11.0331	11.0514	11.0695	10.7919(.25)
2	8.0664	8.0248	7.9832	7.9582	7.9414	7.9288(2.00)
3	10.4307	10.3440	10.1423	10.2536	10.2469	10.2469(1.72)
4	7.9455	8.0080	8.0829	8.1322	8.1730	6.2502(.25)
Mean	9.3545	9.3441	9.31037	9.3488	9.3577	8.8045(1.06)
659.1	9.3773	9.5149	9.7772	10.0088	10.2502	9.3236(.56)
2	7.1691	7.7929	8.6450	9.2520	9.7705	5.9832(.25)
3	6.5317	5.8593	6.4343	6.6734	6.9329	6.3979(1.01)
4	12.8426	12.5164	12.1271	11.8771	11.6732	11.5123(2.00)
Mean	8.9802	8.9209	9.2459	9.4528	9.6567	8.3042(.96)

Table 5. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	{.25≤p̂≤2.00}
660.1	5.1640	5.1704	5.3664	5.6094	5.8763	5.1511(.84)*
2	7.9373	7.4116	6.8882	6.6432	6.5267	6.5012(1.95)
3	6.7741	6.7527	6.8361	6.9615	7.0994	6.7477(.90)
4	7.1134	7.0142	6.9089	6.8410	6.7873	6.7377(2.00)
Mean	6.7472	6.5872	6.4999	6.5138	6.5724	6.2844(1.42)
661.1	7.5761	7.4965	7.5278	7.6331	7.7715	7.4922(1.03)
2	8.7633	8.6099	8.5477	8.5983	8.6946	8.6099(1.22)
3	8.8051	8.6176	8.5399	8.5944	8.7176	8.5399(1.22)
4	7.6853	7.8696	8.2056	8.5007	8.7824	7.5673(.49)
Mean	8.2074	8.1484	8.2053	8.3316	8.4915	8.0523(.99)
662.1	8.5983	8.4498	8.3145	8.3127	8.2136	8.1973(2.00)
2	9.9799	9.8960	9.9631	9.8012	9.7875	9.7875(1.80)
3	9.0111	8.8392	8.7023	8.7023	8.6602	8.5828(1.60)
4	10.8165	10.7176	10.6300	10.6300	10.5861	10.5861(1.80)
Mean	9.6015	9.4757	9.4025	9.3616	9.3119	9.2884(1.80)
751.1	10.3760	10.0367	9.7057	9.5464	9.4586	9.4232(2.00)
2	9.4552	9.3450	9.3094	9.3486	9.4268	9.3057(1.19)
3	8.7483	8.5399	8.4379	8.4733	8.5789	8.4379(1.25)
4	8.1813	7.9623	7.7759	7.6985	7.6724	7.6680(1.80)
Mean	9.1902	8.9709	8.8072	8.7667	8.7842	8.7087(1.56)
752.1	9.8725	9.8387	9.8826	9.9598	10.0597	9.8725(.96)
2	16.0375	16.1101	16.2110	16.2890	16.3564	15.9100(.25)
3	15.4553	15.0576	14.6058	14.3481	14.1844	14.0994(2.00)
4	17.6161	17.8885	18.1932	18.3830	18.5327	16.6153(.25)
Mean	14.7454	14.7237	14.7231	14.7450	14.7833	14.1243(.86)
753.1	12.0442	12.0414	12.1242	12.2337	12.3610	12.0359(.86)
2	9.1831	9.0036	8.8467	8.7893	8.7635	8.7635(1.77)
3	21.4615	20.7491	20.6309	19.2629	18.7987	18.4462(2.00)
4	9.1831	8.8730	8.6946	8.7558	8.9479	8.6910(1.28)
Mean	12.9680	12.6667	12.5741	12.2604	12.2178	11.9842(1.47)

Table 5. (continued)

Baird Loc. No.	p = .75	p = .95	p = 1.25	p = 1.50	p = 1.75	$\{.25 \leq \hat{p} \leq 2.00\}$
754.1	8.0664	8.0950	8.1689	8.2381	8.3106	8.0623(.68)*
2	13.8996	14.0664	14.2851	14.4405	14.5784	13.4684(.25)
3	6.7080	6.8603	7.0380	7.1552	7.2569	6.1859(.25)
4	8.4299	8.4774	8.5750	8.6641	8.7558	8.4219(.63)
Mean	9.2760	9.3748	9.5167	9.6244	9.7254	9.0346(.45)

* Number in parentheses is estimated power

Table 6. Standard Deviation from the model $E(Y) = \beta_1(X + \delta)^p + \beta_2(X + \delta)^{2p}$

Baird Loc. No.	p = .50	{.25≤$\hat{p}$≤1.00}	p = 1.00	Mitscherlich
551.1	6.5368	6.7163(.62)*	6.7674	6.6131
2	14.7940	14.1293(.94)	13.6623	14.9601
3	7.1831	6.1053(.25)	8.8467	6.3034
4	8.3863	8.6131(.52)	10.0762	8.1690
Mean	9.2251	8.8911(.58)	9.8382	9.0114
552.1	11.0150	11.3002(.60)	11.9440	12.0168
2	7.8526	7.4592(.25)	8.5673	7.2567
3	3.5934	11.7595(.56)	3.6119	11.5442
4	-	- -	-	-
Mean	7.4870	10.1730(.47)	8.0411	10.2725
553.1	9.0442	9.2575(.59)	10.2469	9.0517
2	7.6156	7.8793(.48)	9.0331	7.4654
3	10.6458	10.7956(.26)	12.0220	10.4657
4	6.6582	6.7616(.66)	6.7873	7.8179
Mean	8.4910	8.6735(.50)	9.5223	8.7001
554.1	8.8012	7.2922(.25)	9.7023	8.0249
2	10.8658	11.2483(.48)	11.2128	10.8688
3	9.9962	10.1050(.25)	11.6130	9.9766
4	6.5660	6.7984(.50)	7.5885	6.6384
Mean	9.0593	8.8610(.39)	10.0292	8.8771
555.1	6.1142	6.0848(.69)	6.2587	6.4653
2	12.0388	10.3565(1.00)	10.0031	12.5541
3	10.9208	8.3625(.74)	9.2699	9.9269
4	9.9329	9.6015(.80)	9.4865	9.6781
Mean	9.7517	8.6013(.81)	8.7546	9.6561
556.1	9.9430	9.9598(.25)	10.8658	9.7911
2	5.6156	5.8071(.47)	6.5317	5.7793
3	8.1402	8.3625(.73)	8.1116	8.2865
4	4.6618	4.2820(.25)	6.0550	4.2347
Mean	7.0902	7.1029(.42)	7.8910	7.0229

Table 6. (continued)

Baird Loc. No.	p = .50	{.25≤$\hat{p}$≤1.00}	p = 1.00	Mitscherlich
557.1	7.2156	7.4510(.45)*	8.0207	8.7939
2	7.5231	7.7318(.42)	9.2123	7.7588
3	10.3471	10.5616(.25)	10.8658	10.2274
4	14.7037	14.9058(.75)	14.5851	16.0850
Mean	9.9474	10.1625(.47)	10.6710	10.7162
559.1	6.8361	6.9655(.42)	7.3256	6.8944
2	6.9760	6.9977(.35)	9.2843	6.7280
3	7.3574	7.6145(.50)	8.1635	7.5277
4	13.3065	13.7601(.50)	14.3365	13.3419
Mean	8.6190	8.8349(.44)	9.7775	8.6230
650.1	8.2985	8.1482(.71)	8.3587	9.4162
2	12.3125	12.7389(.47)	13.0230	12.2555
3	6.6730	6.6453(.65)	7.3119	6.6683
4	8.5944	8.6628(.71)	8.6486	8.5400
Mean	8.9667	9.0488(.63)	9.3355	9.2200
651.1	8.4299	8.6545(.41)	10.0729	8.1935
2	8.2340	7.4224(1.00)	7.1692	7.9498
3	10.5955	10.9654(.46)	11.1263	10.3432
4	9.8826	10.0835(1.00)	9.7431	9.9431
Mean	9.2855	9.2814(.72)	9.5279	9.1074
652.1	8.5629	8.8559(.45)	9.0920	8.4852
2	12.4924	10.2299(1.00)	9.8826	12.1651
3	6.8263	6.6757(.69)	7.0710	6.6832
4	9.9161	9.6602(.95)	9.3344	9.6850
Mean	9.4494	8.8554(.77)	8.8450	9.2546
653.1	7.0519	7.1000(.62)	8.1730	7.9035
2	8.0747	8.2356(.68)	8.1074	8.4735
3	9.3094	9.4139(.72)	9.2735	9.8657
4	7.4922	6.5355(.25)	8.7597	7.4833
Mean	7.9820	7.8213(.57)	8.5784	8.4315

Table 6. (continued)

Baird Loc. No.	p = .50	$\{.25 \le \hat{p} \le 1.00\}$	p = 1.00	Mitscherlich
654.1	9.9933	10.1277(.72)*	10.0530	9.8318
2	4.7540	4.5705(.71)	5.0264	4.5825
3	5.6272	5.1449(.80)	5.1640	5.7096
4	5.9943	6.0105(.61)	6.3455	6.2770
Mean	6.5922	6.4634(.71)	6.6472	6.6002
655.1	8.3427	8.3977(.67)	8.5634	8.4655
2	5.7328	5.8868(.32)	6.2662	5.8480
3	8.1074	8.3040(.28)	8.5477	8.0870
4	8.3545	9.5030(.36)	8.4694	8.4852
Mean	7.6343	8.0233(.41)	7.9617	7.7214
656.1	6.5319	6.7599(.51)	7.2892	6.8604
2	8.0705	8.1526(.34)	9.6471	8.0000
3	4.6688	4.7834(.55)	6.3507	4.8785
4	4.9998	5.1321(.44)	6.7824	5.1704
Mean	6.0678	6.2070(.46	7.5174	6.2273
657.1	6.2608	5.6226(.81)	17.8062	6.1046
2	6.9089	6.7145(.73)	6.9760	6.6281
3	6.3086	5.5657(.78)	5.8363	5.8366
4	9.8725	9.7163(.81)	9.5113	9.5463
Mean	7.3377	6.9048(.78)	10.0324	7.0289
658.1	10.9117	3.2126(.25)	11.0785	10.4594
2	8.1495	8.2074(1.00)	7.9288	8.1975
3	10.6332	10.6049(.91)	10.2536	11.4950
4	7.8400	7.8847(.25)	8.2149	7.7071
Mean	9.3836	7.4774(.60)	9.3689	9.4647
659.1	9.3310	9.6520(.53)	10.4720	9.2736
2	6.4186	6.1865(.28)	10.2208	6.3770
3	6.9954	6.6335(.72)	7.2156	6.5675
4	13.3615	11.9173(1.00)	11.5123	12.6119
Mean	9.0266	8.5973(.62)	9.8552	8.7075

Table 6. (continued)

Baird Loc. No.	p = .50	$\{.25 \leq \hat{p} \leq 1.00\}$	p = 1.00	Mitscherlich
660.1	5.4222	5.3292(.65)*	6.1480	5.4406
2	8.8542	6.7290(.99)	6.5012	8.6445
3	5.9241	6.9953(.67)	7.2709	6.8068
4	7.2753	6.9746(1.00)	6.7377	7.2203
Mean	6.8689	6.5070(.83)	6.6645	7.0280
661.1	7.8738	7.7552(.71)	7.9321	7.7201
2	9.1614	7.8114(.78)	8.8242	9.0774
3	9.2626	8.8661(.78)	8.8844	8.8090
4	7.5673	7.8318(.48)	9.0481	7.5100
Mean	8.4663	8.0661(.69)	8.6722	8.2791
662.1	8.9068	8.4851(1.02)	8.1973	9.5394
2	10.1619	10.1339(.95)	9.7909	10.0560
3	9.3667	8.9661(.90)	8.6910	9.0185
4	11.0210	3.4654(.95)	10.5862	10.8410
Mean	9.8641	7.7626(.96)	9.3163	9.8637
751.1	11.0029	9.7483(1.04)	9.4232	10.7110
2	9.7604	9.6528(.77)	9.5252	9.4586
3	9.2554	8.7607(.80)	8.7253	8.5868
4	8.6215	7.9406(.95)	7.6809	8.2016
Mean	9.6600	9.0256(.89)	8.8386	9.2395
752.1	10.0367	10.1858(.69)	10.1687	10.0000
2	15.9541	16.3712(.25)	16.4195	16.0973
3	16.0688	14.5941(1.00)	14.0995	15.7984
4	17.1715	16.8165(.25)	18.6546	15.9307
Mean	14.8078	14.4919(.54)	14.8355	14.4566
753.1	12.1681	12.4574(.66)	12.4979	11.9859
2	9.5288	9.0723(.95)	8.7749	8.9516
3	22.4974	19.0939(1.00)	18.4463	22.8005
4	9.8079	9.0463(.81)	9.2123	9.3274
Mean	13.5006	12.4175(.86)	12.2328	13.2663

Table 6. (continued)

Baird Loc. No.	p = .50	$\{.25 \le \hat{p} \le 1.00\}$	p = 1.00	Mitscherlich
754.1	8.0871	8.3384(.58)*	8.3786	8.8955
2	13.6722	13.9077(.25)	14.7013	13.4779
3	6.4756	6.4433(.25)	7.3440	6.0332
4	8.4379	8.7202(.55)	8.8469	9.3381
Mean	9.1682	9.3524(.41)	9.8177	9.4361

* Number in parentheses is estimated power

Table 7. Results of soil nitrogen tests, corn yield with zero application level of nitrogen and statistical estimates of the amount of nitrogen in the soil

Baird Loc. No.	N (%)+	O.M. (%)++	Nitrate(ppm)++	Estimated Nitrogen lbs/acre (p = .50)	Average yield at zero level of N. application (bu/acre)
551	.034	.94	10.50	7.52	25.84
552	.031	.94	14.20	23.22	43.96
553	.034	1.25	8.00	9.19	33.75
554	.043	1.19	10.00	4.13	11.65
555	.021	.73	7.00	4.94	25.07
556	.040	.87	22.50	19.58	45.62
557	.031	.78	10.00	8.29	24.09
559	.035	.78	9.20	6.67	24.41
650	.034	1.58	9.00	12.07	46.55
651	.056	2.42	6.70	20.60	41.70
652	.036	1.28	14.00	13.40	41.35
653	.031	.84	9.10	10.10	31.20
654	.032	1.44	11.50	5.24	22.56
655	.039	1.74	18.00	24.79	55.15
656	.039	1.89	19.60	15.51	56.80
657	.027	.98	12.60	6.48	29.12
658	.040	1.39	15.40	50.04	64.05
659	.026	1.19	8.70	6.52	32.07
660	.028	.98	14.90	17.44	41.35
661	.031	1.25	7.40	5.84	28.80
662	.054	.07	12.60	13.96	42.15
751	.028	1.28	14.90	12.69	51.55
752	.026	1.03	14.70	6.87	27.80
753	.021	1.12	13.90	3.46	19.62
754	.034	1.57	20.40	21.67	53.20
Mean	.034	1.181	12.571	13.209	36.776
Standard Deviation	.009	.4561	4.328	10.004	13.314

\+ From Welch

++ From Baird

Table 8. Experimental results of 25 selected locations of the N.C. State TVA Cooperative Corn Project, (Norfolk Soils). Yield response in bu. per acre

Baird Loc. No.	Level of nitrogen (lbs. per acre)* 0.0	0.0	62.5	62.5	62.5	62.5
551.1	28.9	33.5	57.0	67.8	59.2	50.1
2	17.5	23.2	66.5	75.9	66.4	54.3
3	23.7	33.4	68.2	62.0	59.8	70.3
4	21.6	24.9	68.6	64.2	70.5	62.5
552.1	34.6	12.3	70.6	82.5	89.7	74.5
2	64.3	35.9	82.0	78.0	79.3	74.2
3	72.0	44.7	68.4	72.3	84.3	90.5
553.1	29.4	14.8	75.2	80.8	70.7	87.0
2	39.8	24.1	68.7	80.9	80.4	76.5
3	46.2	31.3	84.8	97.1	80.0	74.5
4	38.9	46.3	75.2	70.6	73.9	68.5
554.1	12.9	13.0	47.2	39.6	38.5	34.9
2	4.6	13.3	33.0	33.0	21.8	31.2
3	6.0	19.8	49.2	46.4	41.0	39.5
4	16.1	7.5	39.2	52.9	41.7	46.2
555.1	23.2	28.9	67.9	64.8	60.9	67.1
2	38.4	18.6	51.7	66.0	78.5	51.3
3	15.8	17.6	54.5	54.6	73.6	57.3
4	21.8	36.3	74.3	91.3	82.9	63.0
556.1	44.3	47.8	66.3	80.3	85.6	82.7
2	42.4	49.1	68.8	63.2	87.3	81.3
3	58.7	41.2	76.0	70.5	75.3	73.2
4	42.6	39.0	70.5	66.4	72.2	71.6
557.1	35.4	19.8	58.8	57.3	51.2	38.8
2	10.8	29.0	60.8	62.7	47.6	63.5
3	30.0	35.2	60.5	51.1	57.7	72.9
4	11.7	20.8	33.4	68.1	32.5	62.2
559.1	30.8	45.5	69.4	66.1	67.7	72.6
2	31.0	21.4	71.8	60.2	76.4	75.3
3	22.7	18.4	68.1	65.5	64.0	74.2
4	9.8	15.7	52.0	56.4	70.1	70.0

Table 8. (continued)

Baird Loc. No.	125.0	125.0	125.0	125.0	125.0
551.1	82.9	60.1	72.5	63.0	63.8
2	86.7	96.1	84.6	86.6	84.1
3	70.8	53.2	58.7	65.3	58.9
4	83.5	72.2	92.7	90.6	83.9
552.1	93.0	78.0	85.7	81.7	89.2
2	81.3	81.0	68.6	75.3	80.6
3	76.6	89.5	91.1	65.1	93.4
553.1	105.9	99.3	88.5	100.9	88.2
2	91.0	100.5	103.1	87.5	92.2
3	90.9	94.0	82.9	106.3	96.2
4	83.6	81.3	84.7	72.1	91.4
554.1	35.4	32.9	23.0	33.1	18.7
2	53.7	47.7	17.8	36.0	29.6
3	28.7	46.6	50.2	34.2	51.8
4	54.8	40.5	65.3	56.2	55.3
555.1	73.8	66.8	77.4	79.7	75.5
2	92.4	75.3	73.6	93.8	75.3
3	106.0	98.0	100.9	90.7	92.7
4	101.1	89.3	116.0	101.0	108.0
556.1	68.4	94.0	91.6	107.4	85.9
2	92.1	92.7	100.3	95.2	82.9
3	78.9	65.2	91.8	96.6	72.9
4	69.4	70.6	70.3	75.2	70.4
557.1	55.3	54.5	55.7	52.6	52.8
2	57.3	62.1	65.4	69.3	70.6
3	71.7	56.9	74.1	70.3	44.0
4	41.0	55.5	74.8	69.9	55.3
559.1	66.2	67.0	80.3	80.5	73.6
2	83.9	72.7	85.4	79.1	77.9
3	81.3	77.3	76.5	74.9	75.3
4	50.0	91.6	66.5	48.6	49.7

Table 8. (continued)

Baird Loc. No.	187.5	187.5	187.5	187.5	250.0	250.0	250.0
551.1	67.6	75.5	74.5	64.3	64.6	65.9	72.2
2	89.0	95.5	77.0	111.4	94.4	43.8	97.4
3	66.8	57.0	61.0	72.5	54.6	69.2	60.7
4	72.8	83.5	85.1	62.1	87.7	87.4	85.9
552.1	92.6	92.8	97.8	85.1	51.6	93.7	76.1
2	77.4	84.1	72.5	88.1	87.1	78.7	76.8
3	88.2	83.6	84.7	87.7	73.9	64.4	98.7
553.1	82.4	114.2	109.4	98.2	92.0	99.7	104.9
2	93.2	99.7	91.4	91.0	99.1	103.2	108.5
3	91.6	90.2	79.9	118.4	91.6	103.5	105.8
4	95.0	80.2	77.0	79.9	65.2	83.3	80.8
554.1	38.3	22.3	27.9	30.6	41.6	27.1	39.0
2	36.0	30.2	28.3	33.9	56.7	21.4	26.8
3	26.3	30.1	30.4	23.7	46.3	26.3	40.6
4	63.5	55.3	49.2	40.4	54.3	51.0	47.0
555.1	81.8	82.9	87.6	81.9	75.3	83.1	61.2
2	86.4	78.1	92.2	97.5	76.8	62.2	67.1
3	90.0	98.7	88.3	105.9	89.3	97.4	111.1
4	104.5	114.2	120.5	102.7	109.9	98.3	114.7
556.1	84.3	112.0	101.5	92.4	98.4	100.9	89.0
2	93.1	107.9	105.7	95.3	92.4	97.4	92.7
3	89.8	91.0	89.8	80.5	82.1	78.6	82.9
4	75.9	75.7	69.9	61.2	69.1	62.7	74.1
557.1	54.6	46.1	48.9	33.3	35.1	37.2	49.8
2	62.4	64.2	65.4	47.5	54.0	53.6	68.6
3	61.0	54.7	77.2	72.0	72.8	50.9	73.5
4	42.6	61.8	85.2	51.0	21.8	47.6	51.1
559.1	89.4	72.6	73.5	88.5	80.9	71.9	73.7
2	75.0	68.7	86.2	78.6	81.0	66.3	90.1
3	96.9	65.7	90.6	75.1	83.8	72.8	72.7
4	64.5	77.2	53.9	58.0	41.8	50.7	55.5

Table 8. (continued)

Baird Loc. No.	0.0	0.0	62.5	62.5	62.5	62.5
650.1	43.8	51.3	87.3	89.8	90.8	97.7
2	49.9	57.4	87.7	95.0	87.6	92.8
3	47.6	30.2	78.6	90.7	91.3	83.1
4	44.8	47.4	94.4	92.3	82.6	92.1
651.1	30.5	38.0	84.0	69.5	80.8	77.8
2	37.0	34.0	58.0	61.3	57.3	64.8
3	46.5	36.5	65.8	78.0	69.8	74.0
4	53.3	57.8	72.0	81.0	75.0	69.0
652.1	62.3	50.9	88.1	77.4	99.9	83.6
2	41.8	27.7	74.4	70.3	82.0	77.3
3	37.2	33.4	84.8	74.3	76.1	81.0
4	39.1	38.5	88.5	65.0	62.5	71.6
653.1	20.6	21.2	64.6	63.3	72.5	65.3
2	35.9	54.5	68.0	72.3	74.8	71.0
3	22.1	44.3	63.3	62.6	61.2	79.6
4	27.1	24.0	71.4	63.1	52.2	68.3
654.1	19.6	27.4	69.4	63.7	63.2	47.0
2	13.3	14.0	60.5	55.1	55.6	51.5
3	27.4	16.0	62.3	65.7	60.5	61.0
4	39.1	23.7	69.1	69.5	70.6	69.2
655.1	44.5	46.6	98.3	86.7	88.2	88.7
2	49.1	54.5	86.7	100.8	80.8	77.1
3	48.7	78.7	89.5	95.5	99.0	91.7
4	60.0	59.1	101.1	87.9	86.3	91.9
656.1	64.2	62.9	106.7	98.4	108.1	100.6
2	54.7	54.4	98.2	103.5	107.6	115.7
3	51.8	54.4	102.1	100.4	101.6	90.4
4	65.4	53.0	100.6	104.2	103.5	105.9
657.1	28.2	40.6	72.4	78.6	77.0	69.8
2	28.2	21.3	64.1	75.3	77.7	66.7
3	26.3	27.5	78.9	68.6	77.9	77.0
4	17.0	44.0	68.6	63.1	71.5	67.4
658.1	63.8	80.6	83.9	101.5	99.4	75.8
2	66.4	66.2	87.4	88.7	86.9	66.2
3	50.4	59.3	89.9	89.9	62.2	94.4
4	53.1	72.6	86.7	97.3	88.7	82.0

Table 8. (continued)

Baird Loc. No.	125.0	125.0	125.0	125.0	125.0
650.1	123.8	103.2	127.5	106.8	112.0
2	116.4	81.0	117.1	113.5	118.5
3	112.4	101.8	113.9	109.4	103.2
4	97.3	117.1	105.1	130.6	106.8
651.1	99.0	69.0	72.3	100.3	86.3
2	87.0	91.8	76.8	67.0	90.0
3	90.5	82.3	85.8	92.3	77.8
4	76.3	67.0	81.5	88.5	90.5
652.1	102.1	110.2	112.8	117.0	98.5
2	120.2	105.4	119.0	97.0	106.5
3	110.1	111.6	90.7	101.5	100.7
4	92.1	104.7	86.0	107.6	90.3
653.1	71.6	89.0	96.1	72.8	74.4
2	78.6	81.8	88.4	74.6	93.8
3	64.6	78.4	85.4	78.4	78.6
4	73.7	55.6	69.4	60.8	66.7
654.1	77.1	72.8	95.6	74.9	88.2
2	72.1	69.6	78.2	62.8	73.6
3	72.7	74.7	71.5	73.0	84.1
4	75.8	81.7	81.9	95.6	77.7
655.1	103.6	98.0	111.9	111.1	125.4
2	110.4	99.1	98.3	99.3	103.4
3	108.1	112.3	89.1	110.9	107.0
4	91.6	101.8	111.8	88.0	98.8
656.1	109.8	102.5	123.2	120.8	113.0
2	104.7	112.3	112.3	117.6	120.8
3	118.8	112.8	118.1	115.9	120.0
4	111.1	112.3	112.5	117.9	120.0
657.1	99.9	97.0	96.1	96.6	95.6
2	82.7	89.9	102.3	98.9	92.7
3	91.1	107.3	107.3	103.7	96.8
4	98.0	86.8	91.1	99.2	100.9
658.1	97.0	89.8	93.1	81.8	79.4
2	96.3	103.2	93.9	77.5	106.5
3	101.0	96.8	92.1	88.2	97.3
4	82.0	101.5	90.9	84.7	95.6

Table 8. (continued)

Baird Loc. No.	187.5	187.5	187.5	187.5	250.0	250.0	250.0
650.1	108.0	122.2	103.0	111.6	104.3	115.6	105.5
2	100.7	89.6	103.2	110.9	128.5	104.0	93.9
3	105.5	104.7	113.8	108.7	108.3	101.3	116.4
4	116.0	126.7	113.0	122.7	107.9	126.8	120.3
651.1	84.0	84.0	81.8	80.8	81.5	75.0	83.5
2	90.3	85.8	99.5	86.0	85.0	79.0	95.0
3	82.3	103.0	59.5	78.0	80.8	99.8	76.3
4	77.3	97.5	86.0	106.0	82.0	72.0	99.3
652.1	92.5	110.1	107.4	106.1	119.8	116.3	115.4
2	101.2	110.7	117.8	131.9	90.2	104.8	115.1
3	102.0	110.0	97.6	100.8	101.3	100.5	111.4
4	92.1	95.1	102.3	116.5	111.4	102.7	91.4
653.1	73.9	71.9	73.9	83.4	75.5	67.3	68.5
2	86.8	83.6	64.9	93.3	75.7	83.4	72.8
3	77.5	92.4	69.6	77.7	59.2	81.4	66.7
4	78.4	79.8	77.1	63.5	71.2	76.6	81.1
654.1	88.7	69.3	74.2	101.7	90.6	82.1	85.8
2	78.7	82.7	82.2	86.3	85.5	81.4	74.5
3	88.4	86.9	81.5	90.7	78.2	72.8	80.7
4	92.4	83.8	83.0	86.0	72.9	84.8	83.8
655.1	117.7	106.0	113.2	118.2	125.7	106.0	99.4
2	114.9	110.1	112.8	113.2	124.7	122.1	116.9
3	105.6	107.1	112.1	116.1	120.1	115.8	111.9
4	101.8	120.7	116.5	101.8	109.7	95.6	100.7
656.1	118.8	124.2	119.1	109.4	117.9	101.3	115.7
2	119.1	121.3	127.8	124.7	136.3	111.8	107.6
3	115.2	120.3	115.7	115.7	120.0	122.2	107.2
4	119.6	105.9	120.8	115.9	119.1	112.8	106.7
657.1	100.6	90.8	100.1	113.0	103.0	94.4	98.2
2	100.4	98.5	96.3	106.8	101.3	91.5	110.2
3	105.6	103.7	111.4	109.7	112.8	98.5	109.0
4	102.1	101.3	85.1	94.6	88.4	116.2	96.1
658.1	67.9	96.5	100.6	88.6	95.5	83.3	100.3
2	105.0	95.1	103.2	98.4	99.8	94.1	99.5
3	101.8	102.8	99.8	85.7	90.9	64.2	95.6
4	100.5	103.7	85.7	100.3	102.5	97.1	91.1

Table 8. (continued)

Baird Loc. No.	0.0	0.0	62.5	62.5	62.5	62.5
659.1	31.0	48.3	89.1	81.4	87.2	110.5
2	24.0	30.5	93.5	94.8	98.2	84.1
3	24.2	27.7	76.4	79.0	80.3	84.9
4	32.0	39.2	73.0	61.8	66.9	77.5
660.1	29.2	31.8	83.1	71.2	82.0	82.2
2	34.4	34.4	69.5	93.9	77.7	78.4
3	31.5	36.8	79.1	71.7	70.0	73.6
4	68.1	64.8	87.7	74.3	84.8	79.1
661.1	38.3	38.1	97.3	77.0	68.7	80.6
2	26.0	25.9	82.0	61.5	95.8	70.8
3	15.6	40.7	73.6	77.1	65.6	69.7
4	30.9	15.0	62.5	75.3	69.2	71.7
662.1	50.7	45.6	85.9	89.9	81.7	93.8
2	52.5	45.1	80.9	82.6	68.5	90.5
3	52.6	17.0	72.3	76.1	74.2	63.5
4	22.4	51.4	62.2	62.0	61.7	80.2
751.1	56.4	53.2	94.5	96.6	105.2	82.2
2	63.8	50.7	108.9	93.7	97.6	92.2
3	44.6	50.0	100.9	95.9	97.0	96.1
4	48.9	44.9	85.3	107.0	89.9	96.0
752.1	45.6	33.5	93.5	81.9	80.4	82.9
2	33.3	17.1	69.3	82.7	71.1	65.8
3	12.6	7.1	46.4	51.9	31.0	44.6
4	59.1	14.1	90.0	52.4	70.6	111.1
753.1	24.2	34.3	78.4	80.1	83.4	83.9
2	25.2	30.8	77.9	74.8	61.0	72.8
3	16.9	12.1	56.4	58.7	84.7	55.4
4	5.0	9.3	54.7	53.9	46.9	50.4
754.1	41.7	61.1	83.8	95.0	91.4	82.0
2	59.5	45.8	111.4	79.3	106.6	87.4
3	51.5	59.1	83.9	75.7	88.7	76.6
4	57.4	49.6	85.0	86.3	75.7	70.1

Table 8. (continued)

Baird Loc. No.	125.0	125.0	125.0	125.0	125.0
659.1	110.0	110.4	127.4	113.8	100.4
2	92.8	101.8	105.9	108.8	116.9
3	99.7	119.1	107.9	105.7	101.6
4	102.0	101.6	103.3	115.4	104.7
660.1	92.7	101.3	97.3	96.1	103.5
2	112.6	113.0	112.6	107.6	101.1
3	100.9	93.7	101.3	84.1	95.1
4	94.6	96.8	92.7	104.9	112.6
661.1	102.8	105.3	113.5	103.8	102.3
2	99.4	93.8	93.1	100.0	102.8
3	108.5	105.9	94.9	97.4	107.3
4	95.6	94.2	92.7	91.2	86.7
662.1	98.8	101.3	86.4	94.5	107.9
2	112.3	86.6	113.3	91.8	105.9
3	100.6	100.0	98.2	93.7	101.4
4	98.2	87.1	87.2	96.7	80.6
751.1	125.7	133.1	136.6	135.7	123.0
2	127.5	118.9	146.2	130.6	138.5
3	130.3	142.2	147.5	138.5	127.0
4	137.5	121.7	109.2	122.1	124.7
752.1	122.5	116.9	105.8	87.4	93.7
2	70.8	61.7	85.9	65.0	104.3
3	96.3	67.8	84.7	59.2	64.0
4	75.6	73.6	58.5	73.1	70.3
753.1	96.3	115.9	90.5	93.5	123.0
2	112.4	113.1	91.7	95.3	101.8
3	62.0	108.1	90.7	103.6	114.2
4	86.9	75.9	83.4	80.1	79.4
754.1	101.9	88.8	84.0	84.7	96.2
2	93.7	109.5	116.9	112.1	111.6
3	85.1	83.4	78.4	92.3	86.2
4	78.2	97.8	96.3	92.3	86.5

Table 8. (continued)

Baird Loc. No.	187.5	187.5	187.5	187.5	250.0	250.0	250.0
659.1	110.9	123.3	110.0	119.7	111.8	127.6	120.7
2	114.9	110.0	105.1	116.2	115.4	119.5	114.7
3	108.0	110.6	112.6	119.7	121.0	119.9	101.7
4	103.7	130.3	142.3	101.8	113.7	126.5	102.0
660.1	111.6	107.1	98.9	105.6	96.3	111.1	99.7
2	118.8	115.9	114.2	109.9	115.4	98.9	101.8
3	89.1	99.2	86.8	109.2	94.4	102.5	100.6
4	112.6	103.2	109.2	103.5	105.9	95.4	112.6
661.1	97.8	102.1	112.8	110.7	109.6	112.6	101.3
2	117.3	112.3	95.8	110.7	107.1	101.4	95.3
3	96.2	104.2	90.3	106.4	106.2	99.8	107.8
4	97.7	80.9	86.9	100.0	96.5	107.2	110.7
662.1	109.7	104.5	116.8	115.7	96.0	102.4	80.7
2	111.9	92.3	113.4	121.7	115.7	101.5	110.0
3	111.3	109.8	93.4	98.5	109.2	98.9	106.2
4	98.0	105.4	102.5	71.2	94.0	95.9	96.0
751.1	118.9	143.6	123.7	143.4	121.5	139.6	129.3
2	137.0	120.7	128.9	132.1	135.1	140.5	139.4
3	132.5	131.3	147.1	142.5	158.1	140.8	146.0
4	136.9	135.2	134.8	147.2	140.9	129.0	141.1
752.1	115.9	124.2	108.1	97.8	111.9	109.1	109.1
2	105.6	68.8	117.9	99.3	82.2	110.4	82.2
3	63.0	74.8	67.8	79.4	49.9	42.3	91.2
4	58.0	51.2	78.1	80.4	64.3	71.3	62.5
753.1	100.8	105.6	122.0	124.7	127.5	91.0	124.7
2	111.6	121.0	103.8	134.8	127.5	125.2	117.4
3	118.4	109.9	80.1	124.0	92.7	44.1	75.1
4	71.3	90.2	60.2	66.3	78.1	76.4	69.8
754.1	100.9	92.3	101.3	100.7	81.1	75.2	92.2
2	77.8	120.0	107.0	122.4	111.4	111.1	128.8
3	86.5	80.0	83.1	74.6	94.0	74.8	84.0
4	84.4	80.2	90.4	69.9	69.5	91.3	74.0

* Heading of Table 8 refers to nitrogen levels. The different columns under one nitrogen level were obtained at different combinations of p and k treatment levels