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ON THE ELEMENTARY RENEWAL THEOREM FOR NON-IDENTICALLY DISTRIBUTED VARIABLES. (Revised version)

by

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1. Introduction

Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with $0 < EX_n < \infty$; write $S_n = X_1 + X_2 + \dots + X_n$; let N_x be the number of partial sums $S_n \leq x$; write $H(x) = EN_x$. The Elementary Renewal Theorem states that under certain conditions $H(x)/x \longrightarrow \{EX_n\}^{-1}$ as $x \longrightarrow \infty$

Kawata (1956) has proved a result which, as we shall see below, is equivalent to a generalization of the Elementary Renewal Theorem to the case in which the $\{X_n\}$ are non-identically distributed. Unfortunately, he found it necessary to impose quite heavy restrictions upon the distribution functions involved. In this note we shall also be concerned with the proof of the Elementary Renewal Theorem for non-identically distributed random variables, but under substantially weaker conditions than Kawata's. This renewal theorem, essentially, provides an asymptotic estimate to the sum $\Sigma_{n=1}^{\infty} P\{S_n \leq x\}$; actually, we shall discuss in this paper the asymptotic behavior of more general sums $\Sigma_{n=1}^{\infty} a_n P\{S_n \leq x\}$, for certain general classes of positive coefficient - sequences $\{a_n\}$.

It is well if we point out that there is another line of inquiry which could be pursued in the present context, one with which the present investigation must not be confused. Instead of considering N_x, one could define a random variable

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M_x as the least m for which $S_m > x$, and then study the asymptotic behavior of EM_x/x. The latter problem (also for non-identically distributed $\{X_n\}$) has been tackled in recent work announced by Robbins and Chow (1962). However, as might be expected, the problem we consider and the problem considered by Robbins and Chow differ in important respects, in general. Indeed, a reference to Theorem A, which we quote below, will show that one can construct a sequence of independent and identically distributed random variables with a finite first moment, for which EM_x is finite but EN_x is infinite. Evidently conditions which are adequate for a study of M_x may prove inadequate for a similar study of N_x. However, when all the $\{X_n\}$ are non-negative we have an exceptional case; for then $M_x = N_x + 1$, and the distinction between the two lines of inquiry disappears. Our main result, Theorem 1 announced below, is more general (albeit much less simple) than the one announced by Robbins and Chow, for the case of non-negative random variables.

Let us write $F_n(x) \equiv P\{X_n \leq x\}$ and $G_n(x) \equiv P\{S_n \leq x\}$; we shall also need the unit function $U(x) \equiv P\{0 \leq x\}$.

The function $L(x)$, defined for all sufficiently large x, is said to be a function of slow growth if, for every $c > 0$

$$(1.1) \qquad \frac{L(cx)}{L(x)} \longrightarrow 1 \quad , \quad \text{as} \quad x \longrightarrow \infty \quad .$$

It follows from the work of Karamata (1930), that a non-negative function of slow growth can always be represented thus:

$$(1.2) \qquad L(x) = \frac{a(x)}{x} \; e^{\int_1^x \frac{a(u)}{u} du} \quad ,$$

where $a(x)$ is a function which tends to unity as x tends to infinity. An easy consequence of this respresentation (1.2) is that the convergence (1.1) takes place uniformly with respect to c in any interval not containing the origin.

As a final preliminary we must say a word about the non-negative coefficient-sequences $\{a_n\}$ which we consider. For such a sequence we shall suppose there exist numbers $\alpha > 0$, $\gamma \geq 0$, and some non-negative function of slow growth $L(x)$, such that

$$(1.3) \qquad \sum_{n=1}^{\infty} a_n x^n \sim \frac{\alpha}{(1-x)^{\gamma}} L\left(\frac{1}{1-x}\right), \qquad \text{as } x \longrightarrow 1-0 .$$

By an appeal to a Tauberian theorem due to Karamata (Hardy, 1949, p. 166) it is possible to deduce from (1.3) that

$$(1.4) \qquad \sum_{n=1}^{N} a_n \sim \frac{\alpha N^{\gamma} L(N)}{\Gamma(1+\gamma)}, \qquad \text{as } N \longrightarrow \infty ,$$

although we shall omit details of this deduction. Conversely, if one starts from (1.4) then an appeal to an appropriate Abelian theorem will show that (1.3) follows. Thus (1.3) and (1.4) are equivalent assumptions on the non-negative $\{a_n\}$. We also note, as an easy deduction from (1.4), that

$$(1.5) \qquad a_n = o(n^{\gamma} L(n)), \quad \text{as } n \longrightarrow \infty .$$

In connection with these sequences $\{a_n\}$ we need to define an index:-

<u>Definition.</u> <u>The index k of the sequence $\{a_n\}$ is the least integer k such that $a_n = O(n^k)$.</u>

From (1.5) it is clear that $k \leq$ i.p. $[\gamma] + 1$, where i.p. $[\gamma]$ is to mean the integer part of γ. On the other hand, we can infer from (1.4) that $k \geq \gamma - 1$, and, when $L(x)$ is an unbounded function, we can even infer the stronger inequality $k > \gamma - 1$. We have, therefore,

<u>Lemma 1.</u> <u>If $L(x)$ is bounded and γ is an integer then the index k may have one of three possible values: $\gamma - 1$, γ, $\gamma + 1$. In all other cases k may have one of only two possible values : i.p. $[\gamma]$, i.p. $[\gamma + 1]$.</u>

The main result of this paper can now be stated.

<u>Theorem 1.</u> <u>Suppose the following conditions hold.</u>

(T1) <u>$\{X_n\}$ is a sequence of independent random variables with distribution functions $\{F_n(x)\}$ and finite expectations $\mu_n = EX_n$, such that</u>

$$\underline{\frac{\mu_1 + \mu_2 + \dots + \mu_n}{n} \longrightarrow \mu \ , \quad \text{as} \quad n \longrightarrow \infty \ ,} \tag{1.6}$$

<u>where μ is finite and strictly positive.</u>

(T2) <u>For every $\epsilon > 0$</u>

$$\int_{n\epsilon}^{\infty} \underline{\frac{1}{n} \sum_{r=1}^{n} \left\{ 1 - F_r(x) \right\} dx \longrightarrow 0 \ , \quad \text{as} \quad n \longrightarrow \infty} \ . \tag{1.7}$$

(T3) <u>For some $\alpha > 0$, $\gamma \geq 0$, and some non-negative function of slow growth $L(x)$, the sequence of non-negative constants $\{a_n\}$ satisfies either of the equivalent asymptotic relations (1.3) or (1.4); assume further that $\Sigma_{n=1}^{\infty} a_n$ diverges.</u>

<u>Then, if we write $G_n(x) = P\{X_1 + X_2 + \dots + X_n \leq x\}$, in order that</u>

$$(1.8) \qquad \underline{\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{\alpha L(x)}{\Gamma(1+\gamma)} \left(\frac{x}{\mu}\right)^{\gamma}, \quad \text{as} \quad x \to \infty},$$

<u>it is sufficient that one of the following two sets of conditions, (T5) or (T6), hold.</u>

(T5) <u>The $\{X_n\}$ are non-negative, and for some unbounded non-decreasing function $\ell(n)$, no matter how slowly it may increase,</u>

$$(1.9) \qquad \underline{\liminf_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \int_0^{r/\ell(r)} \{1 - F_r(x)\}\, dx > 0} .$$

(T6) <u>(a) If k is the index of $\{a_n\}$, then there is a distribution function $K(x)$ of a negative-valued random variable with a finite moment of order $(k+2)$, such that $K(x) \geq F_n(x)$ for all n and all x; (b) If $-\kappa$ is the first moment of $K(x)$, then for some $v > \kappa$ and every $\epsilon > 0$</u>,

$$(1.10) \qquad \underline{\liminf_{n \to \infty} \int_{-\infty}^{\epsilon n/\log n} \frac{1}{n} \sum_{r=1}^{n} \{U(x) - F_r(x)\}\, dx > 2/(k+1)\, v\, \epsilon} .$$

In condition (T3) above we have required that $\Sigma_{n=1}^{\infty} a_n$ shall diverge; it will be appreciated that this assumption is made only to avoid triviality. A consequence of the divergence of $\Sigma_{n=1}^{\infty} a_n$ is that the index $k \geq -1$; therefore the distribution function $K(x)$ which appears in condition (T6)(a) will always have a finite mean; this justifies the introduction of $-\kappa$ in condition (T6)(b).

The special case of Theorem 1 in which the $\{X_n\}$ are non-negative can be given the following form.

Theorem 2. Suppose that (i) $\{X_n\}$ is a sequence of non-negative random variables such that (T1), (T2), and (T5) of Theorem 1 hold; (ii) $A(n)$ is a non-decreasing function of n for which constants $\alpha > 0$, $\gamma \geq 0$, and a non-negative function of slow growth $L(n)$, can be found such that

$$(1.11) \qquad A(n) \sim \alpha n^{\gamma} L(n), \qquad \text{as } n \to \infty \quad .$$

Then it follows that

$$(1.12) \qquad E\,A(N_x) \sim \alpha \left(\frac{x}{\mu}\right)^{\gamma} L(x), \quad \text{as } x \to \infty .$$

We note that by letting $A(n) = n$ in Theorem 2 we obtain a version of the Elementary Renewal Theorem for independent, non-identically distributed, non-negative random variables. Alternatively, by taking $a_n = 1$ for all n in Theorem 1, we obtain the following version for the case when the random variables may assume negative values.

Theorem 3. If conditions (T1) and (T2) of Theorem 1 hold; and if both parts of (T6) hold for $k = 0$, then

$$(1.13) \qquad \frac{H(x)}{x} \to \frac{1}{\mu} \quad , \quad \text{as } x \to \infty \quad ,$$

where $H(x) = \Sigma_{n=1}^{\infty} G_n(x)$ is the expected number of partial sums $S_n \leq x$.

From (1.13) we can infer that, for any fixed $h > 0$,

$$(1.14) \qquad \frac{1}{t} \int_t^{t+h} H(x)\,dx \to \frac{h}{\mu} \quad , \quad \text{as } t \to \infty .$$

Therefore

$$\frac{1}{t} \int_0^t \left\{ H(x+h) - H(x) \right\} dx \to \frac{h}{\mu} , \quad \text{as } t \to \infty ;$$

or, in other words,

$$(1.15) \qquad \lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{n=1}^{\infty} P\left\{ x < S_n \leq x + h \right\} dx = \frac{h}{\mu} .$$

This last limit (1.15) is the form taken by Kawata's result (1956); we see that it is implied by the simpler statement (1.13). On the other hand it is not difficult to deduce (1.13) from (1.15), so that (1.15) seems an unduly complicated form for the result. For (1.15) is equivalent to (1.14); and from (1.14) and the monotone character of $H(x)$ we can infer that lim sup $H(t)/t \leq \mu^{-1}$ and lim inf $H(t + h)/t \geq \mu^{-1}$.

We close this introduction with some remarks about the conditions of Theorem 1. The easiest proofs of the Elementary Renewal Theorem for the case of identically distributed random variables make use of the weak law of large numbers to show that $S_n/n \to \mu$ in probability, as $n \to \infty$. The present investigation will also depend on establishing such weak convergence of S_n/n, and conditions (T1) and (T2), aided by (T6)(a) when the random variables can take negative values, are concerned with this task.

To understand the raison d'être of the alternative conditions (T5) or (T6)(b) it is necessarily to inquire a little into our mode of proof. We shall, as just noted, begin by establishing that $S_n/n \to \mu$ in probability. If only S_n would not deviate too much from its "average value" $n\mu$ our theorems would then be an easy consequence of this weak law of large numbers. Unfortunately, considerable deviation of S_n from $n\mu$ is possible; the main obstacle we have to overcome is presented by sequences $\{ S_n \}$ which tend to adhere, for long stretches of n, to the same value (or even gradually decrease over such stretches), and then indulge in a rare, but very large, increase in value. This kind of pathological behavior is exemplified by a sequence $\{X_n\}$ for which $P\{X_n = 0\}$ becomes arbitrarily near unity as n increases. Conditions (T5) or (T6)(b) are concerned with controlling this kind of awkwardness.

Condition (T6)(a), which is unnecessary when the random variables are non-negative, is introduced to ensure the finiteness of the quantities with which we deal; it will be understood better in relation to the following theorem, which we shall use later in this paper but prove elsewhere (Smith (1963)).

Theorem A. *If $\{X_n\}$ is a sequence of independent and identically distributed random variables with $0 < E X_n \leq \infty$, and if k is a non-negative integer, then a necessary and sufficient condition for the convergence of the series*

$$(1.16) \qquad \sum_{n=1}^{\infty} n^k P\{X_1 + X_2 + \dots + X_n \leq x\} \quad , \quad -\infty < x < +\infty ,$$

is that $E\{|\min(0,X_n)|^{k+2}\} < \infty$. Furthermore, when this condition is met, if X_0 is any other random variable, independent of the $\{X_n\}$, such that $E\{|\min(0,X_0)|^{k+2}\} < \infty$, then

$$(1.17) \qquad \sum_{n=1}^{\infty} n^k P\{X_0 + X_1 + \dots + X_n \leq x\}, \; -\infty < x < +\infty \quad ,$$

is also convergent.

Thus we see that (T6)(a) must be satisfied when the $\{X_n\}$ are identically distributed, and it therefore seems reasonable to require the satisfaction of some condition like (T6)(a) even in the general case, if we are to ensure the finiteness of the left-hand side of (1.8).

2. Some preliminary lemmas.

We begin by showing that the conditions of Theorem 1 are sufficient to ensure that $S_n/n \longrightarrow \mu$ in probability as $n \longrightarrow \infty$. This could be done by appeal to classical results; however, it is not difficult to proceed from first principles, and our proof conveniently introduces an argument of a sort which we shall use several times in the course of this paper.

Lemma 2. If conditions (T1) and (T2) hold, then a sufficient condition for ensuring that $S_n/n \longrightarrow \mu$ in probability as $n \longrightarrow \infty$ is either: (a) the random variables $\{X_n\}$ are non-negative; or (b) condition (T6)(a) holds with $k = -1$.

Proof. Consider first the case of a sequence of independent, non-negative, random variables $\{X_n\}$ whose distribution functions satisfy (T2) and whose mean values satisfy the condition that $\mu_1 + \mu_2 + \ldots + \mu_n = O(n)$, a condition less restrictive than (1.6). Write $m_n = n^{-1}(\mu_1 + \mu_2 + \ldots + \mu_n)$ and let η be a small strictly positive number. Then, in virtue of a familiar inequality, for every fixed $t \geq 0$ we have

$$P\left\{ S_n \leq n(m_n - \eta)\right\} \leq e^{n(m_n - \eta)} E\left\{ e^{-tS_n}\right\} .$$

Furthermore, if we make use of the Laplace-Stieltjes transforms

$$\Phi_j(t) = \int_{0-}^{\infty} e^{-tx} \, dF_j(x) ,$$

we may rewrite this last inequality thus:

$$P\left\{ S_n \leq n(m_n - \eta)\right\} \leq e^{n(m_n - \eta)} \prod_{j=1}^{n} \Phi_j(t) .$$

But, as may easily be verified,

$$\Phi_j(t) \leq e^{\Phi_j(t) - 1} ;$$

and so we have that

$$(2.1) \qquad P\left\{ S_n \leq n(m_n - \eta) \right\} \leq e^{W_n(t)} \quad ,$$

where, after some integrations by parts, we see that

$$(2.2) \qquad W_n(t) = n(m_n - \eta)t - t \sum_{j=1}^{n} \int_0^\infty e^{-tx} \left\{ 1 - F_j(x) \right\} dx \quad .$$

Choose a small $\epsilon > 0$ and set

$$\delta_n(\epsilon) = \int_{n\epsilon}^{\infty} \frac{1}{n} \sum_{j=1}^{n} \left\{ 1 - F_j(x) \right\} dx \quad ;$$

by (T2), $\delta_n(\epsilon) \longrightarrow 0$ as $n \longrightarrow \infty$. From (2.2) we then deduce that

$$(2.3) \quad W_n(t) \leq n(m_n - \eta)t - nte^{-nt\epsilon} \left[\delta_n(0) - \delta_n(\epsilon) \right] .$$

If we observe that $\delta_n(0) = m_n$ and put $t = (n\sqrt{\epsilon})^{-1}$ in (2.3), we find that

$$(2.4) \qquad W_n\left(\frac{1}{n\sqrt{\epsilon}}\right) \leq \frac{m_n - \eta}{\sqrt{\epsilon}} - \frac{e^{-\sqrt{\epsilon}}}{\sqrt{\epsilon}} \left[m_n - \delta_n(\epsilon) \right]$$

$$= \frac{1}{\sqrt{\epsilon}} \left\{ m_n(1 - e^{-\sqrt{\epsilon}}) + e^{-\sqrt{\epsilon}} \delta_n(\epsilon) - \eta \right\} \quad .$$

Recall that m_n is positive and bounded; thus $m_n(1-e^{-\sqrt{\epsilon}})$ can be made arbitrarily small for all n, by choosing ϵ sufficiently small. Hence the expression in braces in the last inequality can be made $< -\frac{1}{2}\eta$, for all sufficiently large n, by choice of ϵ. Therefore, from (2.1), we see that

$$P\left\{ S_n \leq n(m_n - \eta) \right\} \leq \exp\left\{ - \frac{\eta}{2\sqrt{\epsilon}} \right\} \quad ,$$

for all sufficiently large n. Since ϵ can be chosen arbitrarily small we are led to the conclusion that

$$(2.5) \qquad P\left\{S_n \leq n(m_n - \eta)\right\} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty$$

for every $\eta > 0$.

Let $\rho > 0$ be arbitrarily small. Since $ES_n = n\, m_n$ we have that

$$n\, m_n \geq n(m_n + \rho) \; P\left\{S_n \geq n(m_n + \rho)\right\}$$

$$+\, n(m_n - \eta) P\left\{n(m_n - \eta) < S_n < n(m_n + \rho)\right\} ,$$

and from this inequality it follows that

$$(2.6) \qquad (\rho + \eta) P\left\{S_n \geq n(m_n + \rho)\right\} < \eta + m_n\, P\left\{S_n \leq n(m_n - \eta)\right\} .$$

If we choose η arbitrarily small, and observe once again that m_n is bounded, we can infer from (2.5) and (2.6) that

$$(2.7) \qquad P\left\{S_n \geq n\,(m_n + \rho)\right\} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

for every $\rho > 0$. The coupling of (2.5) and (2.7) produce the desired conclusion that for every $\eta > 0$

$$(2.8) \qquad P\left\{\left|\frac{S_n}{n} - m_n\right| \geq \eta\right\} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty \quad .$$

We remark that (2.8) proves the lemma for the case when the $\{X_n\}$ are non-negative (see the result due to A. A. Bobrov quoted on page 141 of Gnedenko and Kolmogorov (1954)). We turn now to the general case, and begin by defining

$$X_n^+ = X_n \quad , \quad \text{if} \quad X_n \geq 0 \; ,$$

$$= 0 \quad , \quad \text{otherwise} \quad ,$$

and $X_n^- = X_n^+ - X_n$. Thus both X_n^+ and X_n^- are non-negative random variables and we shall write $\mu_n^+ = E\, X_n^+$, $\mu_n^- = E\, X_n^-$, $m_n^+ = n^{-1}(\mu_1^+ + \mu_2^+ + \ldots + \mu_n^+)$, $m_n^- = n^{-1}(\mu_1^- + \mu_2^- + \ldots + \mu_n^+)$, and, of course, $m_n = n^{-1}(\mu_1 + \mu_2 + \ldots + \mu_n)$.

Since (T6)(a) holds for $k = -1$, the mean of $K(x)$ is finite; if we call this mean $-\kappa$ then it also follows from (T6)(a) that $\mu_n^- \leq \kappa$ for all n, and hence $m_n^- \leq \kappa$. Moreover, the fact that $K(x) \geq F_n(x)$, for all n and all x, ensures that (1.7) will hold when the $\{F_n(x)\}$ in that condition are replaced by the corresponding distribution functions of the variables $\{X_n^-\}$. We can now infer from the result established for non-negative random variables at the start of this proof that, if we write $S_n^- = X_1^- + X_2^- + \ldots + X_n^-$,

$$(2.9) \qquad P\left\{ \left| \frac{S_n^-}{n} - m_n^- \right| \geq \eta \right\} \longrightarrow 0 \text{ as } n \longrightarrow \infty \; ,$$

for every $\eta > 0$.

Let us turn now to a consideration of $S_n^+ = X_1^+ + X_2^+ + \ldots + X_n^+$. We note first that, since $\mu_n^+ - \mu_n^- = \mu_n$, we have $m_n^+ = m_n^- + m_n$. But we have just shown that m_n^- is bounded by κ ; and we may now suppose $m_n \longrightarrow \mu$ as $n \longrightarrow \infty$, by (1.6). Thus m_n^+ is a bounded function of the integer n, and we can appeal to our preliminary result to deduce that

$$(2.10) \qquad P\left\{ \left| \frac{S_n^+}{n} - m_n^+ \right| \geq \eta \right\} \longrightarrow 0 \text{ as } n \longrightarrow \infty \; ,$$

for every $\eta > 0$. The lemma follows from (2.9) and (2.10) .

Lemma 3. Under the conditions of Lemma 2 ,

$$(2.11) \qquad \int_{\mu}^{\infty} \left\{ 1 - G_n(nx) \right\} dx \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty .$$

Proof. It is easy to verify that

$$m_n = \int_{-\infty}^{+\infty} \left\{ U(x) - G_n(nx) \right\} dx ,$$

$$= \int_{\mu}^{\infty} \left\{ 1 - G_n(nx) \right\} dx + \int_{0}^{\mu} \left\{ 1 - G_n(nx) \right\} dx$$

$$- \int_{-\kappa}^{0} G_n(nx)\, dx - \int_{-\infty}^{-\kappa} G_n(nx)\, dx ,$$

$$(2.12) \qquad = A_n + B_n - C_n - D_n , \quad \text{say} .$$

By Lemma 2, $G_n(nx) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $x < \mu$. Thus $B_n \longrightarrow \mu$ and $C_n \longrightarrow 0$ as $n \longrightarrow \infty$, by bounded convergence. But $m_n \longrightarrow \mu$, by (1.6). Thus we can see from (2.12) that in order to establish the required result, $A_n \longrightarrow 0$, we need only prove that $C_n \longrightarrow 0$. In the case when the $\{X_n\}$ are non-negative there is, of course, no need for further argument.

Write $K_n(x)$ for the familiar n-fold Stieltjes convolution of $K(x)$ with itself. Then plainly, since $K(x) \geq F_n(x)$ for all n and all x, $K_n(x) \geq G_n(x)$ for all n and all x. Thus it will be enough if we can prove that

$$(2.13) \qquad \int_{-\infty}^{-\kappa} K_n(nx)dx \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty .$$

However,

$$\int_{-\infty}^{0} K_n(nx)dx = \kappa \quad , \qquad \text{for all } n \; ,$$

so that (2.13) would follow if we proved that

$$(2.14) \qquad \int_{-\kappa}^{0} K_n(nx)dx \longrightarrow \kappa \qquad \text{as } n \longrightarrow \infty \; .$$

To prove (2.14) we need only remark that, by the weak law of large numbers for identically distributed random variables, $K_n(nx) \longrightarrow 1$ as $n \longrightarrow \infty$ for all $x > -\kappa$; thus (2.14) follows from the theorem on bounded convergence. This proves the lemma.

Lemma 4. If the non-negative constants $\{a_n\}$ satisfy (1.3) then, as $s \longrightarrow 0+$,

$$\sum_{n=1}^{\infty} a_n e^{-\mu s n} \sim \frac{\alpha}{\mu^{\gamma} s^{\gamma}} L(\tfrac{1}{s}) \quad .$$

Proof. Plainly, $e^{-\mu s} \longrightarrow 1 - 0$ as $s \longrightarrow 0+$. Therefore, as $s \longrightarrow 0+$,

$$\sum_{n=1}^{\infty} a_n e^{-\mu s n} \sim \frac{\alpha}{(1 - e^{-\mu s})^{\gamma}} L(\frac{1}{1 - e^{-\mu s}})$$

$$\sim \frac{\alpha}{\mu^{\gamma} s^{\gamma}} L\left(\frac{1}{1 - e^{-\mu s}}\right) \quad .$$

But, as we have remarked in our introduction, $L(rx)/L(x) \longrightarrow 1$ as $x \longrightarrow \infty$ uniformly for r in any interval not containing 0. Thus it transpires that

$$L\left(\frac{1}{1-e^{-\mu s}}\right) \sim L\left(\frac{1}{s}\right)$$

as $s \longrightarrow 0+$, since $1 - e^{-\mu s} \sim \mu s$ for small s. Thus the lemma is proved.

Lemma 5. Under the same conditions as Lemma 4, as $s \longrightarrow 0+$,

$$\sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \sim \frac{\gamma\, \alpha}{(\mu s)^{\gamma+1}} L\left(\frac{1}{s}\right) .$$

Proof. Choose η, $0 < \eta < 1$. Then

$$\mu\, n\, e^{-\mu n s} < \frac{e^{-\mu s n \eta} - e^{-\mu s n}}{(1-\eta)s} .$$

Thus

$$\mu \cdot \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu n s} < \frac{\sum_{n=1}^{\infty} a_n\, e^{-\mu s n \eta} - \sum_{n=1}^{\infty} a_n\, e^{-\mu s n}}{(1-\eta)s}$$

and so

$$\frac{\mu\, s^{\gamma+1}}{L\left(\frac{1}{s}\right)} \sum_{n=1}^{\infty} n\, a_b\, e^{-\mu s n} < \frac{s^{\gamma}}{(1-\eta)L\left(\frac{1}{s}\right)} \left\{ \sum_{n=1}^{\infty} a_n\, e^{-\mu s n \eta} - \sum_{n=1}^{\infty} a_n\, e^{-\mu s n} \right\} .$$

It follows therefore, from Lemma 4, that

$$(2.15) \qquad \limsup_{s \to 0+} \frac{\mu\, s^{\gamma+1}}{L\left(\frac{1}{s}\right)} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \leq \frac{\alpha}{\mu^{\gamma}} \left\{ \frac{\eta^{-\gamma} - 1}{1-\eta} \right\} .$$

If we let $\eta \longrightarrow 1 - 0$ in (2.15) we obtain

$$(2.16) \qquad \limsup_{s \to 0+} \frac{\mu\, s^{\gamma+1}}{L\left(\frac{1}{s}\right)} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \leq \frac{\alpha\, \gamma}{\mu^{\gamma}} .$$

Similarly, by taking $\eta > 1$ and using the fact that

$$\mu\, n\, e^{-\mu ns} > \frac{e^{-\mu sn} - e^{-\mu sn\eta}}{\eta - 1}$$

we can show

$$(2.17) \qquad \liminf_{s \to 0+} \frac{\mu\, s^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu sn} \geq \frac{\alpha\, \gamma}{\mu^{\gamma}} .$$

The lemma follows from (2.16) and (2.17) .

3. Proof of Theorem 1.

We shall write β for an upper bound to the numbers $\{ a_n/n^k \}$, where k is the index of the non-negative coefficient sequence $\{ a_n \}$; we shall also write $\eta > 0$ for an arbitrary small number; it is supposed that $\eta < \mu$.

Consider, to begin with,

$$(3.1) \qquad K_n = \int_{n\eta}^{n\mu} e^{-sx}\, G_n(x)\, dx ,$$

$$= n \int_{\eta}^{\mu} e^{-nsx}\, G_n(nx)\, dx .$$

Evidently,

$$0 \leq K_n \leq n\, e^{-n\eta s} \int_{\eta}^{\mu} G_n(nx)\, dx .$$

But $G_n(nx) \to 0$ as $n \to \infty$, for all $x < \mu$, by Lemma 2. Hence we can appeal to bounded convergence and write

$$(3.2) \qquad K_n = n\, e^{-n\eta s}\, \delta_n' \qquad ,$$

where $\delta_n' \to 0$ as $n \to \infty$, uniformly in $s \geq 0$.

Next consider

$$(3.3) \qquad L_n = \int_{n\mu}^{\infty} e^{-sx} \{1 - G_n(x)\} \, dx$$

$$= n \int_{\mu}^{\infty} e^{-nsx} \{ 1 - G_n(nx) \} \, dx \quad .$$

In view of Lemma 3 and the assumption that $\eta < \mu$ we may thus conclude that

$$(3.4) \qquad L_n = n \, e^{-n\eta s} \, \delta_n'' \quad ,$$

where $\delta_n'' \longrightarrow 0$ as $n \longrightarrow \infty$, uniformly in $s \geq 0$.

Thus, if we write $\delta_n = \delta_n' - \delta_n''$,

$$(3.5) \qquad \sum_{n=1}^{\infty} a_n(L_n - K_n) = \sum_{n=1}^{\infty} n \, a_n \, \delta_n \, e^{-n\eta s} \quad .$$

Given an arbitrary $\epsilon > 0$, we can find $n_o(\epsilon)$ such that $|\delta_n| < \epsilon$ for all $n > n_o$. Moreover we can assume that $s^{-\gamma} L(s^{-1}) \longrightarrow \infty$ as $s \longrightarrow 0+$, since we suppose $\Sigma \, a_n$ to be divergent. Thus

$$(3.6) \qquad \left| \sum_{n=1}^{\infty} n \, a_n \, \delta_n \, e^{-n\eta s} \right|$$

$$< \sum_{n=1}^{n_o} n \, a_n \, | \, \delta_n \, | \, e^{-n\eta s} + \epsilon \sum_{n=1}^{\infty} n \, a_n \, e^{-n\eta s} \quad .$$

Therefore, by Lemma 5 ,

$$(3.7) \qquad \limsup_{s \to 0+} \frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \left| \sum_{n=1}^{\infty} n \, a_n \, \delta_n \, e^{-n\eta s} \right| \leq \epsilon \, \gamma \, \alpha \quad .$$

But ϵ is arbitrary, and we can therefore deduce from (3.7) and (3.5) that, as $s \longrightarrow 0+$,

$$(3.8)\qquad \frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} a_n \, (L_n - K_n) \longrightarrow 0 .$$

Now consider the function

$$(3.9)\qquad H_\eta(x) = \sum_{n=1}^{\infty} a_n \, G_n(x) \, U(x - n\eta) .$$

Evidently $H_\eta(x)$ is non-decreasing, since each term in the summation is non-decreasing. We also note that

$$(3.10)\qquad H_\eta(x) = \sum_{n=1}^{\infty} a_n \, U(x - n\mu)$$

$$- \sum_{n=1}^{\infty} a_n \left\{ U(x - n\mu) - G_n(x) \right\} \quad U(x - n\eta) .$$

Let us denote the Laplace transform of a function $A(x)$, say, thus:-

$$A^o(s) = \int_0^\infty e^{-sx} \, A(x) \, dx .$$

Then, from (3.10), we have

$$(3.11)\qquad H_\eta^o(s) = \frac{1}{s} \sum_{n=1}^{\infty} a_n \, e^{-n\mu s} + \sum_{n=1}^{\infty} a_n \, (L_n - K_n) ;$$

the term-by-term integration being justified by monotone convergence.

From (3.11), (3.8), and Lemma 4, it now appears that

$$(3.12)\qquad \frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \, H_\eta^o(s) \longrightarrow \alpha\,\mu , \qquad \text{as } s \longrightarrow 0+ .$$

An appeal to Doetsch (1950, p. 511) then allows the inference

$$(3.13)\qquad \frac{\mu^\gamma \, \Gamma(1+\gamma)}{t^\gamma \, L(t)} \, H_\eta(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty .$$

But, by (3.9),

$$(3.14) \qquad \sum_{n=1}^{\infty} a_n \; G_n(x) = H_\eta(x) + \Psi_\eta(x) \quad , \text{ say },$$

where

$$(3.15) \qquad \Psi_\eta(x) = \sum_{n=1}^{\infty} a_n \, G_n(x) \; \left\{ 1 - U(x - n\eta) \right\} \quad .$$

If we were to prove that

$$(3.16) \qquad \lim_{\eta \to 0} \; \limsup_{x \to \infty} \; \frac{\Psi_\eta(x)}{x^\gamma \, L(x)} = 0$$

then the theorem would follow from (3.14) and (3.13). The proof of (3.16) under fairly weak hypotheses is quite involved, however, and we therefore present it in the following two separate sections.

4 Completion of proof under (T6).

If $\{X_n\}$ is the renewal sequence under study let us write $X_n^- = -X_n$ when $X_n < 0$, $X_n^- = 0$ when $X_n \geq 0$. When (T6) holds we can introduce the distribution function $K(x)$ which, as has already been explained in § 1, may be assumed to have a finite first moment $-\kappa$. Therefore, if we write $\nu_n = E\,X_n^-$, we have $0 \leq \nu_n \leq \kappa$ for all n.

Let us also write $X_n^+ = X_n + X_n^-$, $S_n^+ = X_1^+ + X_2^+ + \dots + X_n^+$, and $S_n^- = X_1^- + X_2^- + \dots + X_n^-$.

<u>Lemma 6.</u> <u>When (T6) holds we can find $\eta > 0$, $\delta > 0$, such that</u>

$$(4.1) \qquad \underline{P\left\{ S_n^+ \leq \nu_1 + \nu_2 + \dots + \nu_n + n\eta \right\} = O\left(\frac{1}{n^{k+\delta+1}} \right)} \quad ,$$

<u>where k is the index of the coefficient sequence $\{a_n\}$.</u>

Proof. Let us write $\bar{\nu}_n = n^{-1}(\nu_1 + \nu_2 + \dots + \nu_n)$. Then, for any $t \geq 0$, $\eta > 0$, it is plain that

$$P\left\{ S_n^+ \leq n(\bar{\nu}_n + \eta) \right\} \leq e^{n(\eta + \bar{\nu}_n)t} \, E \, e^{-t\, S_n^+}$$

$$(4.2) \qquad = e^{n(\eta + \bar{\nu}_n)t} \prod_{j=1}^{n} \Phi_j^+(t) \quad ,$$

where

$$\Phi_j^+(t) = F_j(0+) + \int_{0+}^{\infty} e^{-tx} \, dF_j(x) \quad .$$

If we now use the familiar inequality already employed in § 2 we can deduce from (4.2) that

$$(4.3) \qquad P\left\{ S_n^+ \leq n\,(\bar{\nu}_n + \eta) \right\} \leq e^{W_n(t)} \quad ,$$

where, after some integrations by parts, we now have

$$(4.4) \qquad W_n(t) = n(\eta + \bar{\nu}_n)t - t \sum_{j=1}^{n} \int_0^{\infty} e^{-tx} \left\{ 1 - F_j(x) \right\} dx \; .$$

In this section we are assuming (1.10) to hold, and so, given any fixed $\epsilon > 0$, we have for all sufficiently large n that

$$\sum_{j=1}^{n} \int_{-\infty}^{\epsilon n/\log n} \left\{ U(x) - F_j(x) \right\} dx \quad > \quad 2n\sqrt{(k+1)\, v\, \epsilon} \quad ,$$

where $v > \kappa$ is independent of ϵ .

We can rewrite this last inequality as follows.

$$(4.5) \qquad \sum_{j=1}^{n} \int_{0}^{\epsilon n/\log n} \left\{ 1 - F_j(x) \right\} dx \quad > \quad n\, \bar{\nu}_n + 2n\sqrt{(k+1)\, \nu\, \epsilon} \quad .$$

But, from (4.4), it is plain that

$$W_n(t) \leq n(\eta + \bar{\nu}_n)t - nt\, e^{-t\epsilon n/\log n}\,(\bar{\nu}_n + 2\sqrt{(k+1)\, v\, \epsilon}\,) .$$

If, in the latter inequality, we make the substitutions

$$t = t_n = \frac{\lambda \log n}{\epsilon n} ,$$

$$\lambda = \sqrt{\frac{\epsilon(k+1)}{\kappa}} ,$$

and if we note incidentally that $\kappa \geq \bar{\nu}_n$ for all n, and $1 - e^{-x} < x$ for all $x > 0$, then we find that

$$(4.6) \qquad \frac{W_n(t_n)}{\log n} < \frac{\lambda\,\eta}{\epsilon} + (k + 1) - \lambda\, e^{-\lambda}(k+1)\sqrt{\frac{v}{\kappa}} .$$

By taking ϵ sufficiently small we can make $e^{-\lambda}$ arbitrarily near unity and thus make

$$(k+1) - \lambda\, e^{-\lambda}(k+1)\sqrt{\frac{v}{\kappa}} < -(k + 2\delta + 1)$$

for some small $\delta > 0$ (recall that $v > \kappa$). Next choose η so small that $\lambda\eta < \epsilon\,\delta$ and it follows from (4.6) that

$$(4.7) \qquad W_n(t_n) < -(k + \delta + 1) \log n .$$

Lemma 6 follows from (4.7) and (4.3) .

In what follows we denote the familiar Stieltjes convolution of two distribution functions, say $A(x)$ and $B(x)$, by $A * B(x)$.. We denote $A*A(x)$ by $A^{*2}(x)$, and, generally, $A * A^{*n}(x)$ by $A^{*(n+1)}(x)$, for $n = 1, 2, 3, \ldots$.

Lemma 7. When (T6) holds

$$\sum_{n=1}^{\infty} n^k P\left\{ S_n^- > n(\bar{\nu}_n + \delta)\right\}$$

is convergent for every $\delta > 0$.

Proof. Define $Z_n = \frac{1}{2}\delta + \nu_n - X_n^-$ and write $L_n(x) = P\left\{ Z_n \leq x \right\}$. If we recall that $\nu_n \leq \kappa$ for all n, then we easily see that

$$U(x - \tfrac{1}{2}\delta - \kappa) \leq L_n(x) \leq K(x) \quad .$$

This proves that Z_n is a stochastically stable sequence as defined by Smith (1962), whose Theorem 7 allows us to draw the following conclusion.

For every integer p there is a distribution function $K_p(x)$ such that

$$P\left\{ Z_n + Z_{n+1} + \cdots + Z_{n+p-1} \leq px \right\} \leq K_p(x) \tag{4.8}$$

for all n and all x, where

$$I_p \equiv \int_{-\infty}^{0} K_p(x)\, dx \tag{4.9}$$

is finite for all p, and $I_p \longrightarrow 0$ as $p \longrightarrow \infty$.

Thus we can find $p_o(\frac{1}{2}\delta)$ such that $I_{p_o} < \frac{1}{2}\delta$. If Y is a random variable with distribution function K_{p_o} then it follows from (4.9) that $EY > -\frac{1}{2}\delta$. Moreover, it is clear that $E\left\{ |\min(0,Y)|^{k+2} \right\} < \infty$, since we can certainly suppose $K_p(x) \leq K^{*p}(px)$.

Write $M(x)$ for the supremum of $P\left\{ Z_1 + Z_2 + \cdots + Z_r \leq p_o x \right\}$ for $r = 1, 2, \ldots, p_o - 1$. Then if Y_o is a random variable with distribution function $M(x)$ it is also apparent that $E\left\{ |\min(0,Y_o)|^{k+2} \right\} < \infty$.

Now choose and fix $r = 0$, or 1, or $2, \ldots,$ or $p_o - 1$. It follows from what we have established so far that

(4.10) $$P\left\{ \sum_{j=1}^{n\,p_o+r} Z_j \leq p_o x \right\} \leq M * K_{p_o}^{*n}(x) \quad .$$

Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables, identically distributed, with distribution function $K_{p_o}(x)$; let $Y_1, Y_2, \ldots$ be independent of Y_o. Then $E(Y_j + \frac{1}{2}\delta) > 0$, for $j = 1, 2, 3, \ldots$ and $E\left\{ |\min(0, Y_j + \frac{1}{2}\delta)|^{k+2} \right\} < \infty$ for $j = 0, 1, 2, \ldots$. Thus it follows from Theorem A, quoted in § 1, that

$$\sum_{n=1}^{\infty} n^k \, P\left\{ Y_o + \sum_{j=1}^{n} (Y_j + \tfrac{1}{2}\delta) \leq 0 \right\} < \infty \quad ,$$

that is,

(4.11) $$\sum_{j=1}^{\infty} n^k \; M * K_{p_o}^{*n} \left(-\tfrac{1}{2} n\,\delta \right) < \infty \quad .$$

From (4.10) and (4.11) we conclude that

$$\sum_{n=1}^{\infty} n^k \, P\left\{ (n\,p_o + r)\left(\tfrac{1}{2}\delta + \bar{\nu}_{np_o+r} \right) \leq S^-_{np_o+r} - \tfrac{1}{2} np_o\delta \right\} < \infty \,,$$

whence,

(4.12) $$\sum_{n=1}^{\infty} n^k \, P\left\{ S^-_{np_o+r} \geq (np_o + r)(\delta + \bar{\nu}_{np_o+r}) \right\} < \infty \quad .$$

The lemma follows from (4.12) by letting $r = 0, 1, 2, \ldots, p_o - 1$ in turn.

Lemma 8. When (T6) holds we can find $\eta > 0$ such that

$$\sum_{n=1}^{\infty} n^k \; G_n(n\eta) < \infty \,.$$

Proof. We observe that

$$P\{S_n \leq n\eta\} = P\{S_n^+ \leq n\eta + S_n^-\}$$

$$\leq P\{S_n^+ \leq n\eta + S_n^-, S_n^- < n(\delta + \bar{\nu}_n)\}$$

$$+ P\{S_n^- \geq n(\delta + \bar{\nu}_n)\}$$

for every $\delta > 0$. Hence

$$P\{S_n \leq n\eta\} \leq P\{S_n^+ \leq n(\eta + \delta + \bar{\nu}_n)\}$$

$$+ P\{S_n^- \geq n(\delta + \bar{\nu}_n)\} \quad .$$

The lemma now follows from Lemmas 6 and 7 if we make $\eta + \delta$ sufficiently small.

The proof of (3.16) is now straightforward. We see from (3.15) that

$$\Psi_\eta(x) \leq \sum_{n=1}^{\infty} a_n G_n(n\eta) \quad ,$$

for all x. Therefore, since k is the index of the $\{a_n\}$ sequence, it follows from Lemma 8 that $\Psi_\eta(x)$ is bounded above. Since $x^\gamma L(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, the truth of (3.16) is established.

5 Completion of proof under (T5).

We begin by showing that we can assume, with no loss of generality, certain convenient properties for the function $\psi(n)$ which appears in the condition (T5). All that actually matters are the values taken by $\psi(n)$ for integer values of n; but we may clearly assume $\psi(x)$ to be a continuous function defined for all $x \geq 1$. More to the point, we observe that if (1.9) holds for the function $\psi(x)$ then it also holds for any function $\psi_1(x) \leq \psi(x)$. In this connection we prove the following:

Lemma 9. *If, for $x \geq 1$, $\ell(x)$ is an unbounded, continuous, and non-decreasing function of x then we can find another such function $\ell_1(x) \leq \ell(x)$, defined on the same domain, with the additional property that $\ell_1(x)/x$ is non-increasing for all sufficiently large x, and tends to zero as $x \longrightarrow \infty$.*

Proof. For $x \geq 1$ define

$$(5.1) \qquad \ell_1(x) = \log x + \inf_{1 \leq y \leq x} \left\{ \ell(y) - \log (y) \right\} .$$

We shall show that $\ell_1(x)$ has all the requisite properties.

To begin with, since $\ell(y) - \log y$ is a continuous function in $[1, x]$ it attains its lower bound; we shall write $y(x) \leq x$ for the largest y-value at which this lower bound is attained. Then

$$(5.2) \qquad \ell_1(x) = \log x + \ell(y(x)) - \log y(x) .$$

Evidently $y(x)$ is a non-decreasing function of x. If $y(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ then the fact that $\ell(x)$ is unbounded shows, in (5.2), that $\ell_1(x)$ is also unbounded; if $y(x)$ tends to a finite limit as $x \longrightarrow \infty$, then the fact that $\log x$ is unbounded shows, also in (5.2), that $\ell_1(x)$ is unbounded. Incidentally, it is an easy deduction from (5.1) that $\ell_1(x) \leq \ell(x)$.

Next choose an arbitrary value of x, x_1 say. Our argument will be given in two cases.

Case: $y(x_1) < x_1$. The continuity of $\ell(y) - \log y$ ensures the existence of some open interval G, containing x_1, within which $y(x) = y(x_1)$. Hence, in G,

$$\ell_1(x) = \log x - \log x_1 + \ell_1(x_1) ;$$

from this equation it is clear that $\ell_1(x)$ is increasing in G and, by simple differentiation, $\ell_1(x)/x$ is decreasing in G.

<u>Case: $y(x_1) = x_1$.</u> In this case, for any $h > 0$, $y(x_1 + h) \geq x_1$; thus $\ell(y(x_1 + h)) \geq \ell(x_1) = \ell_1(x_1)$. Hence, by (5.2),

$$\ell_1(x_1 + h) \geq \log(x_1 + h) + \ell_1(x_1) - \log(y(x_1 + h)) . \tag{5.3}$$

Since $x_1 + h \geq y(x_1 + h)$, it follows from (5.3) that $\ell_1(x_1 + h) \geq \ell_1(x_1)$, i.e. $\ell_1(x)$ is increasing at x_1. But, from (5.1),

$$\ell_1(x_1 + h) - \log(x_1 + h) \leq \ell_1(x_1) - \log x_1$$

from which we can infer that

$$\frac{\ell_1(x_1 + h)}{x_1 + h} - \frac{\ell_1(x_1)}{x_1} \leq - \frac{h\,\ell_1(x_1)}{x_1(x_1 + h)} + \frac{h}{x_1(x_1 + h)} . \tag{5.4}$$

The right hand side of (5.4) is negative for all sufficiently large x_1, because $\ell_1(x)$ is unbounded and non-decreasing. Thus $\ell_1(x)/x$ is non-decreasing at x_1.

Finally, we remark that

$$\ell_1(x) \leq \log x + \ell(1) ,$$

from which it is obvious that $\ell_1(x)/x \longrightarrow 0$ as $x \longrightarrow \infty$.

We turn now to the proof of (3.16) under (T5); for this argument we may, by the immediately preceding discussion, assume that $x/\ell(x)$ is unbounded and non-decreasing. We can then define $r^* = r^*(x)$ as the greatest integer such that $r^*/\ell(r^*) \leq x$. We also write $s^* = s^*(x) = r^*((1 + e)x)$; thus $s^*(x)$ is the greatest integer such that $s^*/\ell(s^*) \leq (1 + e)\, x$.

Choose a large positive C and consider the following three cases, in all of which x is assumed to be large.

(i) $\underline{Cx \leq n \leq r^*(x)}$. By considerations similar to those in the proof of Lemma 2 we have

$$G_n(x) \leq e^{W_n(t)} , \tag{5.5}$$

where

$$W_n(t) = tx - t \sum_{j=1}^{n} \int_0^\infty e^{-tu} \left\{ 1 - F_j(u) \right\} du . \tag{5.6}$$

If we substitute $t = 1/(2x)$ and truncate the integrals at x in (5.6) then we find

$$W_n(1/(2x)) \leq \frac{1}{2} - \frac{e^{-\frac{1}{2}}}{2x} \sum_{j=1}^{n} \int_0^x \left\{ 1 - F_j(u) \right\} du . \tag{5.7}$$

Since (T5) is assumed to hold we can, by (1.9), find $\delta > 0$ such that

$$\sum_{j=1}^{n} \int_0^{j/\ell(j)} \left\{ 1 - F_j(u) \right\} du > n\,\delta \tag{5.8}$$

for all sufficiently large n. But $n \leq r^*$ so that (since $x/\ell(x)$ is non-decreasing) $j/\ell(j) \leq x$ for $j = 1, 2, \ldots, n$. (Actually we have only shown that $x/\ell(x)$ is non-decreasing for all sufficiently large x; but this is adequate for our purpose if we note that $x/\ell(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ and assume n large). Hence we can infer from (5.8) that

$$\sum_{j=1}^{n} \int_0^x \left\{ 1 - F_j(u) \right\} du > n\,\delta \tag{5.9}$$

and so, from (5.7), that

$$W_n(1/(2x)) \leq \frac{1}{2} - \frac{e^{-1/2}\, n\,\delta}{2x} .$$

If we use this last inequality in (5.5) we deduce that

$$(5.10) \qquad G_n(x) \leq \exp\left\{ \frac{1}{2} - \frac{e^{-1/2}\, n\, \delta}{2x} \right\}, \qquad n \leq r^* .$$

(ii) $r^*(x) < n \leq s^*(x)$. For this case we modify the kind of inequality we have been using on $G_n(nx)$. Plainly

$$P\left\{ S_n \leq x \right\} \leq P\left\{ X_j \leq x \text{ for all } j \right\}$$

$$= \prod_{j=1}^{n} F_j(x)$$

$$\leq \exp\left\{ -\sum_{j=1}^{n} \left[1 - F_j(x) \right] \right\} .$$

By forming the geometric mean of this last inequality and (5.5) we obtain a new inequality:

$$(5.11) \qquad G_n(x) \leq e^{R_n(t)} ,$$

where

$$(5.12) \qquad R_n(t) = \frac{1}{2}\, tx - \frac{1}{2}\, t \sum_{j=1}^{n} \int_0^{\infty} e^{-tn} \left\{ 1 - F_j(u) \right\} du$$

$$- \frac{1}{2} \sum_{j=1}^{n} \left\{ 1 - F_j(x) \right\} .$$

If we truncate the integrals in (5.12) at x and substitute $t = 1/x$ then we can infer

$$(5.13) \qquad R_n(1/x) \leq \frac{1}{2} - \frac{e^{-1}}{2x} \sum_{j=1}^{n} \int_0^{x} \left\{ 1 - F_j(u) \right\} du$$

$$- \frac{1}{2} \sum_{j=r^*+1}^{n} \left\{ 1 - F_j(x) \right\} .$$

At this point it is convenient to write

$$\lambda_j = \int_0^{j/\ell(j)} \left\{ 1 - F_j(u) \right\} du \quad ;$$

then (5.9) can be rewritten

(5.14) $$(\lambda_1 + \lambda_2 + \ldots + \lambda_n) > n\,\delta\,, \quad n \text{ sufficiently large.}$$

We shall also write

$$\alpha_j(x) = \int_0^x \left\{ 1 - F_j(u) \right\} du \quad .$$

Therefore, for $j > r^*$ (and, consequently, $j/\ell(j) > x$) we have

$$\int_x^{j/\ell(j)} \left\{ 1 - F_j(u) \right\} du = \lambda_j - \alpha_j(x) \quad .$$

A consequence of the last equation is that

(5.15) $$\left(\frac{j}{\ell(j)} - x\right)(1 - F_j(x)) \geq \lambda_j - \alpha_j(x) \quad .$$

However, if $j \leq s^*(x)$, then $j/\ell(j) \leq (1 + e)x$, and from this inequality it follows that

$$\left(\frac{j}{\ell(j)} - x\right) \leq ex \quad .$$

Thus we have, from (5.15), that

$$1 - F_j(x) \geq \frac{\lambda_j - \alpha_j(x)}{ex} \quad , \quad r^* < j \leq s^* \quad .$$

Using the last inequality we can infer from (~~5.12~~ 5.13) that

$$R_n(1/x) < \frac{1}{2} - \frac{1}{2ex}\sum_{1}^{r^*}\lambda_j - \frac{1}{2ex}\sum_{r^*+1}^{n}\alpha_j(x)$$

$$- \frac{1}{2ex}\sum_{r^*+1}^{n}(\lambda_j - \alpha_j(x))$$

$$= \frac{1}{2} - \frac{1}{2ex}\sum_{1}^{n}\lambda_j$$

$$< \frac{1}{2} - \frac{n\,\delta}{2ex} \quad ,$$

by (5.8) .

Therefore, from (5.11), we discover that

$$G_n(x) < \exp\left\{\frac{1}{2} - \frac{n\,\delta}{2ex}\right\} \quad , \qquad r^* < n \leq s^* \quad . \tag{5.16}$$

From (5.10) and (5.16) we can conclude that

$$\sum_{n=Cx}^{s^*(x)} a_n\, G_n(x) < e^{1/2}\sum_{n=Cx}^{\infty} a_n \exp\left\{-\frac{n\,\delta}{2ex}\right\} \quad . \tag{5.17}$$

We quote here a theorem which can be immediately deduced from some results of Karamata (Hardy (1949), pp. 166-169, especially Theorems 110, 111).

<u>Theorem B. Suppose that $\alpha(t)$ is a non-decreasing function of t and that $I(y) = \int_0^\infty e^{-yt}\, d\,\alpha(t)$ is convergent for $y > 0$, and that</u>

$$I(y) \sim \frac{L(y^{-1})}{y^{\gamma}} \quad , \qquad \text{as} \quad y \longrightarrow 0+ \quad , \tag{5.18}$$

<u>where $\gamma \geq 0$ and $L(x)$ is a function of slow growth. Then, if $g(x)$ is a continuous function of bounded variation in $(0,1)$, as $y \longrightarrow 0+$ we have</u>

(a) <u>in case $\gamma > 0$:</u>

$$\underline{\frac{y^{\gamma}}{L(y^{-1})} \int_0^{\infty} e^{-yt} g(e^{-yt})\, d\,\alpha(t) \sim \frac{1}{\Gamma(\gamma)} \int_0^{\infty} e^{-t} t^{\gamma-1} g(e^{-t}) dt} ,$$

(b) <u>in case $\gamma = 0$:</u>

$$\underline{\frac{1}{L(y^{-1})} \int_0^{\infty} e^{-yt} g(e^{-yt})\, d\,\alpha(t) \sim g(1)} .$$

Let us put, in this theorem,

$$\alpha(t) = \sum_{n \leq t} a_n$$

Then Lemma 4 shows that a relation like (5.18) holds .

Define

$$\begin{aligned} g(x) &= 1 , & 0 \leq x \leq a , \\ &= 2 - \frac{x}{a} , & a \leq x \leq 2a , \\ &= 0 , & 2a \leq x \leq 1. \end{aligned}$$

Clearly this $g(x)$ satisfies the conditions of Theorem B. Thus, when $\gamma > 0$ we can deduce that

$$\limsup_{y \to 0+} \frac{y^{\gamma}}{L(y^{-1})} \sum_{n \geq \frac{1}{y} \log \frac{1}{a}} a_n e^{-ny}$$

$$\leq \frac{\alpha}{\mu^{\gamma}\, \Gamma(\gamma)} \int_{\log \frac{1}{2a}}^{\infty} e^{-t}\, t^{\gamma-1}\, dt .$$

In this last result, substitute $y = \delta/(2ex)$ and $\log \frac{1}{a} = C\delta/(2e)$. On being given any prescribed $\epsilon > 0$ we can choose C sufficiently large for us to deduce, via (5.16), that

$$(5.19) \qquad \limsup_{x \to \infty} \frac{1}{x^{\gamma} L(x)} \sum_{n = Cx}^{s^*(x)} a_n G_n(x) < \epsilon .$$

When $\gamma = 0$ a similar result to (5.19) can also be proved by appeal to Theorem B.

<u>(iii) $s^*(x) < n$</u> . For this range of values for n we always have $n/\ell(n) > x$. If we define R_n and λ_n as for case (ii) then by arguments similar to the ones employed in that case we find

$$R_n(x^{-1}) < \frac{1}{2} - \frac{1}{2ex} \sum_{1}^{s^*} \lambda_j - \frac{1}{2ex} \sum_{s^*+1}^{n} \alpha_j(x)$$

$$- \frac{1}{2} \sum_{s^*+1}^{n} \left\{ \frac{\lambda_j - \alpha_j(x)}{\frac{j}{\ell(j)} - x} \right\} .$$

However, when $j > s^*$, $j/\ell(j) > (1 + e)x$ and one can infer that

$$(5.20) \qquad \left(\frac{j}{\ell(j)} - x \right) > e\, x$$

and deduce therefore that

$$(5.21) \quad R_n(x^{-1}) < \frac{1}{2} - \frac{1}{2ex} \sum_{1}^{s^*} \lambda_j - \frac{1}{2} \sum_{s^*+1}^{n} \left\{ \frac{\lambda_j}{\frac{j}{\ell(j)} - x} \right\} .$$

Write

$$T_n = \sum_{s^*+1}^{n} \left\{ \frac{\lambda_j}{\frac{j}{\ell(j)} - x} \right\}$$

and

$$\Lambda_j = \lambda_1 + \lambda_2 + \cdots + \lambda_j \qquad .$$

Then

$$T_n = \frac{\Lambda_n}{\dfrac{n}{\ell(n)} - x} - \frac{\Lambda_{s^*}}{\dfrac{s^*+1}{\ell(s^*+1)} - x}$$

$$+ \sum_{s^*+1}^{n-1} \Lambda_j \left\{ \frac{1}{\dfrac{j}{\ell(j)} - x} - \frac{1}{\dfrac{j+1}{\ell(j+1)} - x} \right\} .$$

We may assume s^* to be large, so that, by (5.14), $\Lambda_j > j\,\delta$ for all j in the range of consideration. Thus

$$T_n > \frac{n\,\delta}{\dfrac{n}{\ell(n)} - x} - \frac{\Lambda_{s^*}}{\dfrac{s^*+1}{\ell(s^*+1)} - x}$$

$$+ \delta \sum_{s^*+1}^{n-1} j \left\{ \frac{1}{\dfrac{j}{\ell(j)} - x} - \frac{1}{\dfrac{j+1}{\ell(j+1)} - x} \right\}$$

$$= \frac{(\delta\, s^* - \Lambda_{s^*})}{\dfrac{s^*+1}{\ell(s^*+1)} - x} + \delta \sum_{s^*+1}^{n} \left\{ \frac{1}{\dfrac{j}{\ell(j)} - x} \right\}$$

If we use this last inequality in (5.21), and also make use of (5.20), then we find

$$2R_n(x^{-1}) < 1 - \frac{s^* \delta}{\frac{s^*+1}{\ell(s^*+1)} - x} - \delta \sum_{s^*+1}^{n} \left\{ \frac{1}{\frac{j}{\ell(j)} - x} \right\}$$

$$< 1 - \frac{s^* \delta}{\frac{s^*+1}{\ell(s^*+1)} - x} - \delta \sum_{s^*+1}^{n} \frac{\ell(j)}{j}$$

$$< 1 - \frac{s^* \delta}{\frac{s^*+1}{\ell(s^*+1)} - x} - \delta\, \ell(s^* + 1) \log\left(\frac{n+1}{s^*+1}\right) .$$

We may therefore conclude, from (5.11), that

$$(5.22) \quad G_n(x) < \left(\frac{s^* + 1}{n + 1}\right)^{\frac{1}{2} \delta\, \ell(s^*+1)} e^{\psi(x)} \quad , \quad n > s^* \quad ,$$

where

$$\psi(x) = \frac{1}{2} - \frac{s^* \delta}{2\left(\frac{s^*+1}{\ell(s^*+1)} - x\right)} .$$

At this point in our argument we need information about the order of magnitude of

$$U_N = \sum_{N}^{\infty} \frac{a_n}{n^\rho} \quad ,$$

for $\rho > \gamma + 2$. To this end, define $\alpha(t) = \Sigma_{n \leq t}\, a_n$, as before. Then there is some constant c such that $\alpha(t) < c\, t^\gamma\, L(t)$ for all large t, by (1.4).

Evidently

$$U_N = \int_{N-0}^{\infty} \frac{1}{u^\rho}\, d\, \alpha(u)$$

$$= \left[\frac{1}{u^{\rho}} \alpha(u) \right]_{N-0}^{\infty} + \rho \int_N^{\infty} \frac{\alpha(u)}{u^{\rho+1}} du$$

$$< c \rho \int_N^{\infty} \frac{L(u)}{u^{\rho-\gamma+1}} du ,$$

if we ignore a negative form. Hence, by an obvious substitution in the integral, we have

$$(5.23) \qquad U_N < \frac{c\rho L(N)}{N^{\rho-\gamma}} \int_1^{\infty} \frac{L(Nv)}{L(N)} \frac{dv}{v^{\rho-\gamma+1}} .$$

But, from (1.2) ,

$$\frac{L(Nv)}{L(N)} = \frac{a(Nv)}{a(N)} \cdot \frac{1}{v} \cdot \exp \left\{ \int_N^{Nv} \frac{a(u)}{u} du \right\} ,$$

where $a(x) \longrightarrow 1$, as $x \longrightarrow \infty$. Therefore, given an arbitrarily small $\epsilon > 0$, we can choose N so large that for all $v \geq 1$

$$\frac{L(Nv)}{L(N)} < \frac{(1+\epsilon)}{v} \exp \left\{ (1+\epsilon) \int_N^{Nv} \frac{du}{u} \right\}$$

$$= (1 + \epsilon) v^{\epsilon} .$$

Hence we can appeal to dominated convergence to infer that

$$\int_1^{\infty} \frac{L(Nv)}{L(N)} \frac{dv}{v^{\rho-\gamma+1}} \longrightarrow \int_1^{\infty} \frac{dv}{v^{\rho-\gamma+1}}$$

as $N \longrightarrow \infty$. It then transpires, from (5.23), that

$$(5.24) \qquad U_N = O \left(\frac{L(N)}{N^{\rho-\gamma}} \right) .$$

For all sufficiently large x we shall have $\frac{1}{2}\delta\, \ell(s^*+1) > \gamma + 2$, and hence may infer from (5.22) and (5.24) that

$$\sum_{n=s^*+1}^{\infty} a_n G_n(x) = 0\left((s^*+1)^{\gamma}\; L(s^*+1)\, e^{\psi(x)}\right)$$

and hence that

$$(5.25)\quad \frac{1}{x^{\gamma}L(x)} \sum_{n=s^*+1}^{\infty} a_n\; G_n(x) = 0\left(\left(\frac{s^*+1}{x}\right)^{\gamma} \frac{L(s^*+1)}{L(x)}\; e^{\psi(x)}\right) .$$

Since

$$\frac{s^*+1}{\ell(s^*+1)} = \left(\frac{s^*+1}{s^*}\right)\left(\frac{\ell(s^*)}{\ell(s^*+1)}\right)\left(\frac{s^*}{\ell(s^*)}\right)$$

$$\leq \left(\frac{s^*+1}{s^*}\right)(1+e)x \quad ,$$

it is clear that for all large x

$$\frac{s^* + 1}{\ell(s^* + 1)} < (1 + 2e)\, x \; .$$

Therefore, from the definition of $\psi(x)$ which follows ~~(2.2)~~ (5.22) we infer that

$$(5.26)\qquad \psi(x) < \frac{1}{2} - \frac{s^*\delta}{4\, ex} \; .$$

In addition, we can deduce from (1.2) that

$$\frac{L(s^* + 1)}{L(x)} = \frac{a(s^* + 1)}{a(x)}\left(\frac{x}{s^* + 1}\right) \exp\left\{\int_x^{s^*+1} \frac{a(u)}{u}\, du\right\}$$

and so,

$$(5.27)\qquad \frac{L(s^* + 1)}{L(x)} = 0\left(\left(\frac{s^* + 1}{x}\right)^{\epsilon}\right)$$

for any $\epsilon > 0$.

If we combine (5.26) and (5.27) with (5.25) we discover that

$$(5.28)\quad \frac{1}{x^{\gamma}\, L(x)} \sum_{n=s^*+1}^{\infty} a_n\, G_n(x) = O\left(\left(\frac{s^* + 1}{x}\right)^{\gamma+\epsilon} e^{-\frac{s^*\delta}{4ex}}\right) .$$

But $(s^*+1) > (1 + e)\, x / (s^*+1)$, so that $(s^*+1)/x \longrightarrow \infty$ as $x \longrightarrow \infty$. We can therefore deduce from (5.28) that

$$\frac{1}{x^{\gamma} L(x)} \sum_{n=s^*+1}^{\infty} a_n\, G_n(x) \longrightarrow 0 , \quad \text{as} \quad x \longrightarrow \infty .$$

On combining this last result with (5.19) we find that given any $\epsilon > 0$ we can choose a sufficiently large $C > 0$ so that

$$\limsup_{x \longrightarrow \infty} \frac{1}{x^{\gamma}\, L(x)} \sum_{n=Cx}^{\infty} a_n\, G_n(x) < \epsilon .$$

This result establishes (3.16) and completes our proof, since

$$\Phi_{\eta}(x) = \sum_{n > \frac{x}{\eta}}^{\infty} a_n\, G_n(x) .$$

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Addendum to the report "ON THE ELEMENTARY RENEWAL THEOREM FOR NON-IDENTICALLY DISTRIBUTED VARIABLES". (Revised version)

It turns out that, when the random variables are non-negative, conditions (T1) and (T2) automatically imply that (1.9) will hold for some function $\ell(n)$. Thus (T5) can be greatly simplified to read merely "The X_n are non-negative". The proof runs as follows:-

(T2), being true for every $\epsilon > 0$, implies the existence of an unbounded increasing function $\lambda(n)$, say, such that

$$(A1) \qquad \frac{1}{n} \int_{n/\lambda(n)}^{\infty} \sum_{r=1}^{n} \left\{ 1 - F_r(x) \right\} dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

Therefore, from (1.6),

$$(A2) \qquad \sum_{r=1}^{n} \frac{1}{n} \int_{0}^{n/\lambda(n)} \left\{ 1 - F_r(x) \right\} dx \longrightarrow \mu \quad \text{as} \quad n \longrightarrow \infty .$$

Let $\ell(n)$ be another unbounded increasing function, but one which increases very much more slowly than $\lambda(n)$. Let $\rho(n)$ be such that $r/\ell(r) > n/\lambda(n)$ for all $r > \rho(n)$. Then the result (A2) implies that

$$(A3) \qquad \frac{1}{n} \sum_{r=1}^{\rho(n)} \int_{0}^{n/\lambda(n)} \left\{ 1 - F_r(x) \right\} dx + \frac{1}{n} \sum_{\rho(n)+1}^{n} \int_{0}^{r/\ell(r)} \left\{ 1 - F_r(x) \right\} dx$$

$$= T_1(n) + T_2(n), \text{ say, } > (\mu - \epsilon), \quad \epsilon > 0,$$

for all sufficiently large n. Now

$$\frac{1}{n} \sum_{r=1}^{\rho(n)} \int_{0}^{n/\lambda(n)} \left\{ 1 - F_r(x) \right\} dx \leq \frac{\mu_1 + \cdots + \mu_{\rho(n)}}{n}$$

and so we may from (1.6) conclude that $T_1(n) \longrightarrow 0$ if we can show $\rho(n)/n \longrightarrow 0$. It would then follow that $T_2(n) > (\mu - 2\epsilon)$ for all large n and from this, in view of the arbitrariness of ϵ, we would have

$$\liminf_{n \to \infty} \ \frac{1}{n} \sum_{r=1}^{n} \int_{0}^{r/\ell(r)} \left\{ 1 - F_r(x) \right\} dx \ \geq \ \mu$$

from which, of course,

$$\frac{1}{n} \sum_{r=1}^{n} \int_{0}^{r/\ell(r)} \left\{ 1 - F_r(x) \right\} dx \ \longrightarrow \ \mu \ ,$$

which is far more than we need.

Now for all large n

$$\lambda(n) \geq \lambda\left(n \ \frac{\log \lambda(n)}{\lambda(n)}\right)$$

and so, if we put $s(n) = (\log \lambda(n))/\lambda(n)$,

$$\frac{ns(n)}{\log \lambda(ns(n))} \geq \frac{n}{\lambda(n)} \quad .$$

We may, by Lemma 9, assume $n/\lambda(n)$ is increasing for all large n; from this it follows easily that $n/(\log \lambda(n))$ is increasing for all large n. Therefore, if we let $\ell(n) = \log \lambda(n)$,, we have that $r/\ell(r) \geq n/\lambda(n)$ for all $r \geq ns(n)$. Hence $\rho(n) \leqslant ns(n)$ and it may be seen that $s(n) \longrightarrow 0$; thus our claims are proved.