ON THE CONVERGENCE OF FINITE ELEMENT SCHEMES BASED ON NON-CONFORMING AND SIMPLIFIED HYBRID DISPLACEMENT METHODS

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SUMMARY

Although the convergence of compatible displacement models can be almost automatically guaranteed if the completeness condition of the employed shape functions is satisfied, they have special difficulties when applied to plate bending problems on account of the imposed continuity conditions on assumed lateral deflection fields. Since available shape functions of plate bending in this category are not necessarily simple and accurate enough from the practical point of view, non-conforming shape functions have been frequently used for practical purposes with often reasonable results obtained. However, such an approach is obviously not admissible in the framework of the classical calculus of variation. So the present authors proposed a simplified hybrid displacement method, which may be the simplest modification of non-conforming method based on the Lagrange multiplier technique, and have provided some numerical results to establish the feasibility of the method experimentally.

This paper presents a set of convergence criteria for both of the above-mentioned finite element approaches with their applications made to concrete finite element schemes. The criteria consist of conditions to assure the stability of the schemes and of those to assure the approximation of the actual deflection patterns. If they are satisfied, the rate of convergence can be easily established with the aid of functional analysis. It is particularly to be taken notice of that the establishment of these conditions becomes much more difficult than in usual conforming schemes because of introduction of the boundary terms required in the above approaches.

The examples of finite element schemes to which the present theory may be applicable are:

(1) Rectangular finite element for plane stress problems proposed by R.H. Gallagher: In this scheme, the displacements are approximated by non-conforming quadratic polynomials which satisfy homogeneous equations of equilibrium in each element.
(2) Triangular finite element for plate bending by Fujino and Morley: The lateral deflection is approximated by non-conforming quadratic polynomials. This scheme is equivalent to the mixed one proposed by Herrmann.
(3) Some other elements for plate bending derived by the present authors.

Furthermore, the convergence of the well-known ACM non-conforming scheme may be also established by the combined use of the present theory and Miyoshi's techniques with his error estimations improved. In these examples, it is generally observed that the consistency conditions are not easily verified in non-conforming schemes, while the stability ones are difficult to check in hybrid displacement schemes.

1. Introduction

The convergence of finite element schemes utilizing conforming shape functions can be automatically established if the so-called completeness condition in the energy space, i.e., the constant strain or stress condition is satisfied. The compatible displacement method is, however, not easy enough to apply to plate bending analysis based on Kirchhoff-Love's thory on account of the difficulty in achieving the continuity conditions imposed on lateral deflection field. This is probably the main reason why the avilable schemes in the above category are, in general, much complicated for computational purposes. Thus, non-conforming models have been often employed to avoid such difficulties of the above method, but they are obviously not admissible from the standpoint of orthodox theory of calculus of variation (Zienkiewicz [1], Washizu [2]). On the other hand, an alternative was proposed by the authors (Kikuchi and Ando [3] -[6]) and named simplified hybrid displacement method, in which some boundary-integral terms are introduced to the original non-conforming schemes by the use of the Lagrange multiplier method so that the continuity requirements for shape functions may be much relaxed. Consequently, this method possesses a sound background of variational principle in contrast to the non-conforming method, and the numerical results obtained by this approach appear to be generally reasonable. Unfortunately, no general convergence study based on numerical analysis (as a branch of functional analysis) has been conducted with respect to the above-mentioned hybrid method probably due to some latent difficulties.*)

This paper is to discuss the convergence of the above two methods by presenting some sufficient conditions as criteria for stability and approximation properties of schemes. Then the criteria are applied to a few simple finite element schemes based on these methods with convergence rates evaluated for sufficiently smooth exact solutions. Although the class of shape functions available in these methods becomes much wider than in compatible one thanks to the relaxation of continuity conditions, the check of convergence character becomes more difficult since additional estimations are now necessary for boundary integral terms. Finally it is to be noted that the results to be obtained for non-conforming schemes in this paper may be regarded as a generalization of Miyoshi's ones on the ACM-scheme (Miyoshi [7]).

2. Formulation

2.1 Preliminaries

We will consider an elastic body occupying a bounded open region Q in R^n (n = positive integer) with smooth boundary ∂Q. For mathematical treatment of the finite element method, the use of thoery of Hilbert space is very convenient and usually regarded as inevitable (Strang [8] , Aubin [9]). A Hilbert space H of displacements and the corresponding applied forces defined on Q will be employed with its inner product and norm denoted as

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*) A paper dealing speciffically with convergence of this approach applied to plate bending will be published soon (Kikuchi and Ando [10]).

(,) and $\| \ \|$. The concept of strains and stresses is also fundamental in the theory of elasticity, and another Hilbert space H^* is also to be used for strain (or stress) fields on Q with inner product $\langle\ ,\ \rangle$ provided. It will be assumed that the boundary conditions associated with the pressent discussions are limited to homogeneous ones so as to simplify the theoretical treatment.

The strain-displacement relation is to be written in the operational form as

$$Tu = v \qquad (T : D(T) \subset H \longrightarrow H^*) \quad , \tag{1}$$

where $u \in D(T)$ is a displacement field in Q, $v \in H^*$ the corresponding strain field and D(T) the domain of definition of linear operator T. As usual, the fact that $u \in D(T)$ implies not only that the strain field calculated from u is well defined as an element of H^* but also that u satisfies imposed homogeneous geometric boundary conditions.

Then, it is well known that the equilibrium equation can be expressed as

$$T^* v = f \qquad (T^* : D(T^*) \subset H^* \longrightarrow H) \quad , \tag{2}$$

where T^* is the dual operator of T, which may be regarded as an extension of the concept of the transposed matrix in the theory of matrix algebra, $v \in H^*$ a strain field and $f \in H$ applied external force. Any element of $D(T^*)$ satisfies some continuity or smoothness conditions in Q as well as kinematic homogeneous boundary conditions (if any). It is here to be noted that T and T^* possess the property

$$\langle Tu, v \rangle = (u, T^* v) \qquad \text{for } \forall u \in D(T) \text{ and } \forall v \in D(T^*) \quad . \tag{3}$$

Some examples of such dual relations in mathematical physics can be found in Ref. [11] (Fujita).

As seen from the preceding discussions, the typical problem in the theory of elasticity is to solve a system of eqs. (1) and (2), or to find the solution u of the single equation*)

$$T^* T u = f \qquad \text{with } u \in D(T) \text{ and } Tu \in D(T^*). \tag{4}$$

We decompose Q into finite elements $\{Q_{hi}\}_{i=1}^{m(h)}$, in which h is the representative mesh size which will reduce to 0 as the discretization proceeds, and m(h) is number of finite elements. We usually regard Q_{hi} as open and simply-connected. Furthermore, it is required that each finite element has piece-wise sufficiently smooth boundary so that it permits the use of the divergence theorem of the type

$$\langle v, T_{hi} u \rangle_{Q_{hi}} = (T^*_{hi} v, u)_{Q_{hi}} - G(v,u)_{\partial Q_{hi}} \qquad \text{for } \forall u \in D(T) \text{ and } \forall v \in D(T^*) \tag{5}$$

*) The word "single" is to be interpretted in operational sense. As well known, the n-dimension theory of elasticity usually requires n equations with respect to n-component displacement field.

where T_{hi} and T^*_{hi} are the restrictions of T and T^* to Q_{hi}, respectively, and G is a bilinear form defined on ∂Q_{hi}, which is an integral of product between displacements and their corresponding surface tractions. In the above expressions, all the quantities with subscript Q_{hi} are considered in Q_{hi}.

We will now construct a finite element space S^h ($\subset H$) spanned by a non-conforming basis $\{\phi_{hj}\}_{j=1}^{n(h)}$, in which $n(h)$ indicates the dimension of S^h. An element of S^h will be denoted such as U_h, that is

$$U_h = \sum_{j=1}^{n(h)} a_{hj}\,\phi_{hj} \quad , \tag{6}$$

where the coefficients a_{hj}'s can be usually interpretted as nodal displacements in the finite element method. We further assume that each $U_h \in S^h$ (strictly speaking, the restriction of U_h to Q_{hi}) belongs to $D(T_{hi})$ for every i so that $T_{hi}U_h$ may be well defined. We also introduce $\widetilde{U}_h$ for every $U_h \in S^h$, which is defined uniquely by a_{hj}'s on both boundary ∂Q and inter-element boundaries of interconnecting elements and coincides with U_h at nodal points where displacements can be specified as nodal variables.*) As for boundary conditions, it is required that $\widetilde{U}_h$ satisfies the imposed geometric boundary conditions on ∂Q. If S^h is a conforming finite element space, it is understood that $\widetilde{U}_h = U_h$ on ∂Q_{hi}.

Then the following semi-norm and bilinear form can be defined for $\forall U_h \in S^h$ and $\forall V_h \in S^h$ (and the corresponding $\widetilde{U}_h$ and $\widetilde{V}_h$) :

$$|||\, U_h \,|||_h = \sqrt{\sum_{i=1}^{m(h)} \langle T_{hi}U_h, T_{hi}U_h \rangle_{Q_{hi}}} \quad . \tag{7-1}$$

$$B_h(U_h, V_h) = \sum_{i=1}^{m(h)} \left\{ G(T_{hi}U_h, V_h - \widetilde{V}_h)_{\partial Q_{hi}} + G(T_{hi}V_h, U_h - \widetilde{U}_h)_{\partial Q_{hi}} \right\} . \tag{7-2}$$

The semi-norm defined by eq. (7-1) will be called non-conforming energy norm, which reduces to the usual energy norm (Mikhlin [12]) if S^h is a space of conforming functions.

2.2 Finite element schemes based on non-conforming shape functions

Now that a non-conforming space S^h is introduced in section 2-1 together with the concept of boundary and interelement displacement $\widetilde{U}_h$, we can consider two types of finite element schemes belonging to the generalized displacement method.

(1) Non-conforming scheme : The finite element solution $U_h \in S^h$ for the exact solution of eq. (4) is determined by the relation

$$\sum_{i=1}^{m(h)} \langle T_{hi}U_h, T_{hi}V_h \rangle_{Q_{hi}} = (f, V_h) \qquad \text{for } \forall V_h \in S^h . \tag{8}$$

*) It is to be noted that U_h and $\widetilde{U}_h$ are uniquely defined by common a_{hj}'s.

The use of the divergence theorem (eq.(5)) to the right hand side of eq.(8) yields

$$\sum_{i=1}^{m(h)} \langle T_{hi}U_h, T_{hi}V_h \rangle_{Q_{hi}} = \sum_{i=1}^{m(h)} \{ \langle T_{hi}u, T_{hi}V_h \rangle_{Q_{hi}} + G(Tu, V_h)_{\partial Q_{hi}} \} , \quad (9)$$

where the exact solution u is assumed to be sufficiently smooth so as to permit the above transformation. It is to be noted that the second term in the right hand side of the above equation does not in general vanish except when S^h is a conforming finite element space. This fact is just what brings the main difficulty to the convergence study of non-conforming finite element schemes. It is also clear that eq.(8) may be rewritten as

$$\sum_{i=1}^{m(h)} \langle T_{hi}U_h, T_{hi}V_h \rangle_{Q_{hi}} = \sum_{i=1}^{m(h)} \langle T_{hi}u, T_{hi}V_h \rangle_{Q_{hi}} + B_h(u, V_h) , \quad (10)$$

if we introduce $\tilde{u} = u$ on ∂Q_{hi}.

(2) Simplified hybrid displacement scheme : In this scheme, a modification term is added to the corresponding non-conforming scheme utilizing the Lagrange multiplier method (Washizu[2], Kikuchi and Ando [3]-[6]). The scheme thus obtained is

$$\sum_{i=1}^{m(h)} \langle T_{hi}U_h, T_{hi}V_h \rangle_{Q_{hi}} + B_h(U_h, V_h) = (f, V_h) \quad \text{for } \forall V_h \in S^h . \quad (11)$$

If we introduce $\tilde{u} = u$ on ∂Q_{hi} and assume u to be sufficiently smooth, eq.(11) may be transformed into

$$\sum_{i=1}^{m(h)} \langle T_{hi}U_h, T_{hi}V_h \rangle_{Q_{hi}} + B_h(U_h, V_h) = \sum_{i=1}^{m(h)} \langle T_{hi}u, T_{hi}V_h \rangle_{Q_{hi}} + B_h(u, V_h) . \quad (12)$$

Comparing the above equation with eq.(9), we can easily understand that the present scheme has symmetric property in the form of both sides: the right hand side can be obtained by simply replacing U_h by u in the left hand side, and vice versa. This property will play an important role in the error estimation of this type of schemes.

It should be also to be noted that the present method is a modification of hybrid displacement method-II proposed by Tong ([13]). This is why the present scheme is called "simplified" hybrid displacement method.

3. Convergence criteria

The convergence criteria presented in the following are generalized ones of those proposed in the plate bending analysis by the present authors (Kikuchi and Ando[10]).

(1) Stability requirements

We first require that the non-conforming energy norm defined by eq.(7-1) is positive and uniformly bounded from below by $L_2(Q)$ norm in S^h.

(C1) $\quad |||U_h|||_h \geq C_1 \|U_h\| \quad$ for $\forall U_h \in S^h$, (13)

where C_1 is a positive constant independent of U_h and h. This condition is supplied to guarantee the stability of the finite element solution since it may not be automatically realized unless S^h is a conforming finite element space.

In the case of simplified hybrid displacement method, we also require the equivalence of non-conforming energy and pseudo-energy (Kikuchi and Ando [10]) in the following sense :

(C2) $\quad C_2^2 |||U_h|||_h^2 \geq |||U_h|||_h^2 + B_h(U_h, U_h) \geq C_3^2 |||U_h|||_h \quad$ for $\forall U_h \in S^h$, (14)

in which the positive constants C_2 and C_3 are independent of U_h and h. This condition is added because the pseudo-energy is not necessarily positive in actual finite element schemes.

(2) Approximation requirements

If we want the finite element solution U_h to converge to u in both $L_2(Q)$ space and non-conforming energy space as h goes to 0, we should require the following conditions.

(C3) There exists $\hat{U}_h \in S^h$ such that

$$\lim_{h\to 0} \|\hat{U}_h - u\| = 0 \quad , \quad \lim_{h\to 0} |||\hat{U}_h - u|||_h = 0 \quad . \tag{15}$$

The above conditions are rather popular as seen from the experience of the conforming displacement method. However, we should further require some approximation properties with respect to boundary terms in the case of the present schemes.

(C4) In the case of non-conforming schemes, we require that

$$\lim_{h\to 0} \sup_{V_h \in S^h} \frac{|B_h(u, V_h)|}{|||V_h|||_h} = 0 \quad , \tag{16}$$

or, equivalently, that

$$\lim_{h\to 0} \sup_{V_h \in S^h} \frac{\left|\sum_i G(T_{hi}u, V_h)_{\partial Q_{hi}}\right|}{|||V_h|||_h} = 0 \quad . \tag{17}$$

We can use any of the above expressions that is more convenient.

(C5) In the case of simplified hybrid displacement schmes, the condition is given by

$$\lim_{h\to 0} \sup_{V_h \in S^h} \frac{|B_h(\hat{U}_h - u, V_h)|}{|||V_h|||_h} = 0 \quad , \tag{18}$$

where $\hat{U}_h$ is the same as introduced in (C3).

It is apparent that the condition (C4) is rather difficult to check because u itself appears in the expression (16). On the other hand, (C5) may be generally satisfied if S^h is suitably chosen since only the

$\hat{U}_h - u$ needs to be evaluated.

4. Some theorems for existence and error estimations of finite element solutions

From the criteria introduced in section 3, we can derive some general results concerning the existence, uniqueness and convergence of the finite element solutions. We only present the results as theorems since the details are essentially the same as those given in Ref [10].

Theorem-1 (Existence and uniqness of finite element solutions)
The non-conforming finite element solution U_h of eq.(8) exists uniquely under (C1) and satisfies the inequalities

$$|||U_h|||_h \leq \frac{1}{C_1} ||f|| \quad , \quad ||U_h|| \leq \frac{1}{C_1^{\,2}} ||f|| \quad . \tag{19}$$

Similarly, the uniqueness and existence of the hybrid displacement solution U_h in eq. (11) can be obtained under (C1) and (C2). In this case the above inequalities are replaced with

$$|||U_h|||_h \leq \frac{1}{C_1 C_2^{\,2}} ||f|| \quad , \quad ||U_h|| \leq \frac{1}{C_1^{\,2} C_2^{\,2}} ||f|| \quad . \tag{20}$$

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The above theorem also implies the stability of the finite element solution with respect to the applied force f.

Let us define the following quantities for each $\hat{U}_h$ in S^h and the exact solution u of eq.(4) :

$$M_1(\hat{U}_h,u) = ||\hat{U}_h - u|| \; , \qquad M_2(\hat{U}_h,u) = |||\hat{U}_h - u|||_h \; ,$$

$$M_3(\hat{U}_h,u) = \sup_{V_h \in S^h} \frac{|B_h(u,V_h)|}{|||V_h|||_h} \; , \tag{21}$$

$$M_4(\hat{U}_h,u) = \sup_{V_h \in S^h} \frac{|B_h(\hat{U}_h - u,V_h)|}{|||V_h|||_h} \; .$$

Then we have the following error estimations for the finite element solutions.

Theorem-2 (Error estimations of non-conforming solution)
The non-conforming solution U_h of eq.(8) satisfies the relations

$$|||U_h - u|||_h \leq \inf_{\hat{U}_h \in S^h} (2M_2(\hat{U}_h,u) + M_3(\hat{U}_h,u)) \quad , \tag{22-1}$$

$$\|U_h - u\| \leqq \inf_{\hat{U}_h \in S^h} \left(M_1(\hat{U}_h,u) + \frac{1}{C_1}(M_2(\hat{U}_h,u) + M_3(\hat{U}_h,u)) \right) . \quad (22\text{-}2)$$

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Similarly, we have the following theorem.

Theorem-3 (Error estimations for simplified hybrid displacement solution) The finite element solution defined by eq.(11) satisfies

$$|||U_h - u|||_h \leqq \inf_{\hat{U}_h \in S^h} \left(M_2(\hat{U}_h,u) + \frac{1}{C_2^{\,2}}(M_2(\hat{U}_h,u) + M_4(\hat{U}_h,u)) \right) , \quad (23\text{-}1)$$

$$\|U_h - u\| \leqq \inf_{\hat{U}_h \in S^h} \left(M_1(\hat{U}_h,u) + \frac{1}{C_1 C_2^{\,2}}(M_2(\hat{U}_h,u) + M_4(\hat{U}_h,u)) \right) . \quad (23\text{-}2)$$

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It follows from the above two theorems that the convergence of the finite element solutions is assured if all of the conditions from (C1) to (C5) are satisfied. The convergence in the present discussions implies the mean convergence of strains and displacements, which amounts to the same results as in the compatible models. It is to be noted that the establishment of the approximation property (C4) is not necessarily easy in non-conforming models, as was already pointed out, while the stability condition (C2) is difficult to hold in hybrid displacement models. Indeed the condition (C2) is violated in the T-2 scheme (Kikuchi and Ando [6]) when the triangular becomes very slender. We have a similar instability phenomenon caused by "kinematic deformation modes" in the hybrid stress method (Pian and Mau [14]), and some special cares should be taken of in the use of hybrid type finite element models. However, once such a difficulty is eliminated, the models will provide with effective tools for structural analysis.

5. Some applications of the theory.

As applications of the present theory, we will consider two types of finite element schmes for plate analysis. They may be regarded as both non-conforming and hybrid displacement schemes since the additional terms in eq. (11) vanish in the final forms of the schemes.

(1) Rectangular finite element for plane stress problems (Turner et al.[15], Gallagher [16], Przemieniecki [17]) (Fig. 1)
In this scheme, the in-plane displacements (d_x, d_y) in each finite element are approximated by incomplete quadratic polynomials which are non-conforming and satisfy the homogeneous equations of equilibrium. The displacements along element boundaries are chosen to be linear and have common nodal values to those of the above element displacements.

(2) Triangular finite element for plate bending (Fujino [18], Morley [19]) The lateral deflection w is approximated by non-conforming quadratic polynomials capable of expressing any constant curvature state in each finite element. This scheme yields the same results as the mixed model by Herrmann [20] and Hellan [21]. The nodal arrangement and assumed deflection field are shown in Fig. 2.

Although the details of the convergence proofs of the above two schemes are completely omitted here because of limited number of avilable pages, the establishment of (C3) is rather easy and (C2) is unnecessary in the present schemes. On the other hand, (C1) and (C4) (or (C5)) require some special techniques in evaluating the boundary quantities. As for this point, the results obtained in Ref. [10] will provide with some reference informations. Convergence of some other schemes such as R-4 in Ref [6] may be also discussed in a similar manner, and the convergence rate of ACM scheme obtained by Miyoshi [7] can be improved up to the order of mesh size if the present theory is applied.

In the above two schemes, the convergence rates of strains and deflections are equal to the order of mesh size if the exact solutions are sufficiently smooth and the sequences of mesh divisions satisfy the so-called uniformity conditions in Ref. [8]. Therefore, these schemes can be safely employed to practical purposes due to their simplicity in calculation.

6. Concluding remarks

The convergence of two displacement type finite element methods based on non-conforming shape functions has been discussed. Consequently, it becomes clear that the essential point of such discussions lies in the establishment of some fundamental inequalities for stability and evaluation of boundary terms involved in the schemes. The extension of the present results to dynamic problems may be realized in the same manner as performed in Ref [10]. Further studies should be made for establishing the convergence criteria of more general hybrid displacement methods.

Finally it appears that the present results for non-conforming schemes are closely related to the ones based on the concept of "patch test" recently developed by Strang and Fix [22]. However, it should be emphasized that the present theory for non-conforming schemes has been derived as a natural extension of that for the simplified hybrid displacement method and eq.(16) seems sometimes more convenient than eq.(17) for practical purposes.

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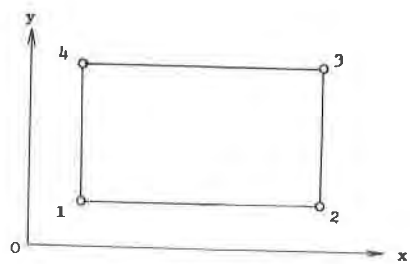


nodal variables : (d_{xi}, d_{yi}) $i = 1,2,3,4.$

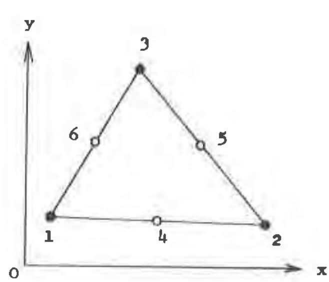
(d_x, d_y) : $d_x = a_1 + a_2 x + a_3 y + a_4 xy - b_4(\nu x^2 + y^2)/2$

$d_y = b_1 + b_2 x + b_3 y + b_4 xy - a_4(\nu y^2 + x^2)/2$

(ν : Poisson's ratio assumed to be const.)

$(\tilde{d}_x, \tilde{d}_y)$: linear along each side

Fig. 1 Rectangular finite element for plane stress problems



nodal variables : w_i $(i = 1,2,3)$, $\partial w/\partial n\big|_i$ $(i = 4,5,6)$

w : complete quadratic polynomials

$\tilde{w}$: linear along each side

$\widetilde{\partial w/\partial n}$: constant along each side

Fig. 2 Triangular finite element for plate bending