ABSTRACT

YANG, SHUOWEI. Nonlinear Control Design Techniques Via Higher Order Lyapunov Functions. (Under the direction of Dr. Fen Wu.)

Control of systems having inherent nonlinear characteristics is among the most challenging problems in systems and control theory. In this dissertation, for a class of nonlinear systems with polynomial vector fields, we propose a novel synthesis method via higher order Lyapunov functions by using power transformation based system augmentation.

In solving nonlinear control problems, because of their simple structures, quadratic Lyapunov functions are most often used. However, it has been generally accepted that quadratic Lyapunov functions can be very conservative when dealing with complicated nonlinearities. This research explores the usage of higher order Lyapunov functions to achieve performance improvement during optimal control process. The uniqueness of this research is taking advantage of power transformation to augment the state vector and then derive higher order Lyapunov functions in quadratic form of augmented state variables, which has not been addressed in literature until current date.

The first part of this dissertation studies the state feedback $\mathcal{H}_\infty$ control for polynomial nonlinear systems. By combining power transformation with Sum-of-Squares (SOS) techniques, we can augment the systems with more state variables representing higher order combinations of the original ones. Then, higher order Lyapunov functions can be synthesized in convex condition and can provide better $\mathcal{H}_\infty$ performance comparing to quadratic ones. Besides, taking advantage of the flexible form of the augmented system matrix, an iterative process can also been adopted to continually improve the $\mathcal{H}_\infty$ performance.

After the discussion of standard nonlinear systems, we further utilize the proposed method to improve controlled performance of polynomial nonlinear systems subjected to actuator saturations. Two type of control problems are considered: region of attraction (ROA) enlargement and disturbance attenuation. Since the higher order Lyapunov level set has flexible shape, we

can enlarge the ROA of a saturated polynomial nonlinear system from the quadratic ellipsoid shape. Additionally, we solve $\mathcal{H}_\infty$ control problems for saturated polynomial nonlinear systems. The solvability condition for both problems have been derived in convex conditions with higher order Lyapunov functions. The entire process and comparison result has been demonstrated by a few examples, including an animal population control problem using Lotka-Volterra equation.

Finally, we investigate the usage of higher order Lyapunov function to improve the performance of output feedback nonlinear $\mathcal{H}_\infty$ controller synthesis. The output feedback nonlinear $\mathcal{H}_\infty$ control problem has been formulated as appropriate Hamilton-Jacobi-Isaacs (HJI) equations (inequalities). With system augmentation, the proposed approach will present the higher order terms of the conditions as convex LMIs, which can be computed efficiently through SOS programming. This technique has been applied to a spacecraft with two control torques to improve the command tracking performance.

Nonlinear Control Design Techniques Via Higher Order Lyapunov Functions

by
Shuowei Yang

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APPROVED BY:

Dr. Gregory D. Buckner	Dr. Scott M. Ferguson
Dr. Paul I. Ro	Dr. Fen Wu Chair of Advisory Committee

DEDICATION

To my parents *Jiucheng Yang and Gengchun Tang.*

BIOGRAPHY

Shuowei Yang was born in Beijing, China. He received his B.S. degree in Mechanical Engineering and Automation from North China Electric Power University in 2006. After that, he moved to Bethlehem, Pennsylvania and attended Dept. of Mechanical Engineering and Mechanics at Lehigh University. In May 2009, he received his M.S. degree and started to pursue the Ph.D. degree in Dept. of Mechanical and Aerospace Engineering at NCSU in August of the same year.

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$0_{n\times m}$	the zero element of $\mathbf{R}^{n\times m}$
$\mathrm{diag}(a_1, a_2, \ldots, a_n)$	the n by n diagonal matrix with elements a_1, a_2, …, a_n on the diagonal line
$\mathcal{F}_l(\cdot,\cdot)$	linear fractional transformation
I_n	the n-dimensional identity matrix
$\mathcal{L}_2$	space of square integrable functions
M^T	the transpose of the matrix M
M^{-1}	the inverse of the invertible matrix M
$M > 0$ ($M \geq 0$)	the matrix M is positive definite (positive semi-definite)
$M < 0$ ($M \leq 0$)	the matrix M is negative definite (negative semi-definite)
$\mathbf{N}^n$	set of n-dimensional non-negative natural vectors
$\mathbf{R}$	set of real numbers
$\mathbf{R}_+$	set of non-negative real numbers
$\mathbf{R}^n$	set of n-dimensional real vectors
$\mathbf{R}^{m\times n}$	set of real $m \times n$ matrices
$\mathbf{S}^{n\times n}$	set of symmetric matrices in $\mathbf{R}^{n\times n}$
$\mathbf{S}_+^{n\times n}$	set of positive definite matrices in $\mathbf{R}^{n\times n}$
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Kronecker sum of matrices A and B
Z^n	$:=\{\alpha \in \mathbf{N}^n : \alpha_i \in \{0,1\}\}$
$\|u\|_2$	$= \left[\int_0^\infty u^T(t)u(t)dt\right]^{\frac{1}{2}}$ for $u \in \mathcal{L}_2$
$\|x\|$	$= (x^T x)^{\frac{1}{2}}$ for vector x
$\|\cdot\|_\infty$	for $u \in \mathbf{R}^n$, $\|u\|_\infty := \max_i \|u_i\|$

$\frac{\partial V}{\partial x}$	$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}$
$M^{[r]}(x)$	a monomial vector of x with all components having degree no more than r
Φ_{SOS}	the set of SOS polynomials
$\|z\|_2^2$	$= \int_{-\infty}^{\infty} trace[z^*(t)z(t)]dt$
$\max_{x(0),w} \frac{\|z\|_2}{\|w\|_2}$	the induced $\mathcal{L}_2/\mathcal{H}_\infty$ norm from w to z
$\mathbf{I}[k_1, k_2]$	for two integers k_1, k_2, $k_1 < k_2$, $\mathbf{I}[k_1, k_2] := \{k_1, k_1+1, \ldots, k_2\}$
$\text{sat}(\cdot)$	the symmetric saturation function with implicit saturation level given by $\bar{u} \in \mathbf{R}^n$
$\bar{U}$	$\text{diag}(\bar{u}_1, \ldots, \bar{u}_m)$ where $\bar{u}_i > 0$ is the saturation level for the ith component of $sat(\cdot)$
$\text{dz}(u)$	the deadzone function, $\text{dz}(u) := u - \text{sat}(u)$
HeX	for a square matrix X, He$X := X + X^T$
$\varepsilon(P)$	for $P \in \mathbf{R}^{n\times n}$, $P = P^T \geq 0$, $\varepsilon(P) := \{x \in \mathbf{R}^n : \ x^T P x \leq 1\}$
$\mathcal{L}(H)$	for $H \in \mathbf{R}^{m\times n}$, $\mathcal{L}(H)$:=$\{x \in \mathbf{R}^n : \ \|Hx\|_\infty < 1\}$

Chapter 1

Introduction

1.1 Motivations and Objectives

The dynamics of physical systems studied in engineering and science always contain different kinds of nonlinearities. Many of these nonlinear effects can be neither ignored nor linearized without loss of generality. The increasing demand for good control performance of practical systems with natural nonlinear complexities encourages the exploration to improve the solvability and applicability of existing nonlinear control design techniques.

In the area of nonlinear control research, Lyapunov stability theory has became one of the most important tools to conduct nonlinear analysis as well as design problems. Among the tremendous amount of related researches, especially in the past, quadratic Lyapunov function has been the priority of researcher's choice comparing with Lyapunov candidates with other forms for a long time. This dominance is mainly due to its simplest structure. Taking advantage of Lyapunov functions with quadratic form, a nonlinear analysis or synthesis condition can be formulated as a convex state-dependent linear matrix inequality (LMI), which can be recast as a positive or negative semidefinite programming condition. Then, it can be efficiently computed using existing commercial softwares such as the MATLAB robust control toolbox, or Sum of Square (SOS) programming based toolboxes, like SOSTOOLS or YALMIP, which can provide

more tractable solutions. However, for more complicated system such as nonlinear systems or linear systems with structured uncertainties, it is now broadly accepted that, quadratic Lyapunov function is not adequate and often leads to conservative results. As a result, non-quadratic Lyapunov functions have been proposed in the literature.

A polyhedral Lyapunov function can be used to stabilize a linear system with structured time-varying parameter uncertainties and input disturbances to a bounded neighborhood of the origin by a continuous or piecewise linear controller [10]. But the computational cost is very high and it remains an issue for large dimensional systems. Maximum and convex hull Lyapunov functions have been proposed in [29] to study stability of linear differential inclusions (LDIs). Works along this line can be found in the area of saturation control and switching control of linear systems.

Another powerful class of non-quadratic Lyapunov functions in analysis and control of nonlinear and linear uncertain systems is higher order polynomial Lyapunov functions. Peet [65] presented a proof that the existence of a polynomial Lyapunov function is necessary and sufficient for exponential stability of a sufficiently smooth nonlinear vector field on a bounded set. In another word, the local exponential stability of a smooth nonlinear system is equivalent to the existence of a polynomial Lyapunov function with at least degree of two. Moreover, the solvability of this kind of Lyapunov functions is no longer a issue, at least for polynomial nonlinear systems, because of the proved result of Peet and Papachristodoulou [66], which states that exponential stability of a polynomial vector field on a bounded set implies the existence of a polynomial SOS Lyapunov function. This important statement implies that semidefinite programming can be used to analyze stability and synthesize stabilizing control laws of a polynomial nonlinear system.

From the two references quoted above, it is not hard to conclude that, by using system augmentation based on power transformation [13], when a polynomial nonlinear system is quadratically stabilizable, it is always feasible to find a higher order Lyapunov function to verify its stability for both the original and augmented systems. If a polynomial nonlinear system is stabi-

lizable, its augmented system is stabilizable as well, because they represent the same dynamics as long as the coupling relationship between new states and original states is guaranteed. Therefore, based on the result of [65], if a quadratic Lyapunov function exist to stabilize the original system, another quadratic counterpart stabilizing the augmented system exists as well, and it leads to a higher order Lyapunov function for the original system. Moreover, [66] ensures that we are able to find this higher order one using SOS programming techniques.

Motivated by the above conclusion and the limitation of quadratic Lyapunov functions, **in this dissertation**, we will propose a systematic approach to use power transformation based system augmentation to derive higher order Lyapunov functions in order to improve control performance, comparing with the quadratic based results. Following the system augmentation, the LMI conditions used for original systems will be implemented based on the augmented ones and solved by SOS programming together with coupling constraints to maintain the relationship between new states and original states. Not like non-convex ones, this method is mor computationally more efficient and has the capacity of involving more higher order terms to search for better results.

Although higher order Lyapunov functions has been successfully used on stability domains analysis of nonlinear systems [64], most synthesis problems of polynomial control systems pursuing higher order Lyapunov functions are not reducible to convex problems. Some previous researches are able to compute for higher order Lyapunov functions. An iterative algorithm was proposed in [95] to solve the nonlinear $\mathcal{H}_\infty$ control problem by constructing a higher order Lyapunov function and its associated nonlinear controllers starting from quadratic Lyapunov functions. However, the iterative algorithm may not converge to its global minimum and the result could be conservative. [67] proposed a dual approach to state feedback synthesis based on density function. Nevertheless, it is currently not clear how a performance objective can be incorporated in the density-based approach. On the other hand, a series of related work, [68, 24, 25, 42, 43] represented the open-loop nonlinear systems in a pre-selected state dependent linear-like form to solve the convex sufficient LMI conditions for nonlinear control

problems to achieve higher-order Lyapunov functions. Although these results have demonstrated to work well for some examples, this pre-selection process requires knowledge of the system and no method is available to select the most suitable state vector $Z(x)$ and the associated system representations. Moreover, bad $Z(x)$ selections might lead to infeasibility. It is well known that, the representation of state space system matrices is not unique. Without pre-knowledge and experience, the decision to choose a specific form of a nonlinear system to derive optimal control performance is always not easy. No effective approach in literature has been discussed, although the existence of this issue has been pointed out [68].

Thanks to system augmentation, the dynamics of original states will have more flexible representations. **In this dissertation**, we will treat these flexibilities as decision variables and solve them together with regular decision variables associated with Lyapunov functions iteratively. By doing this, we can expect a control performance at least not worse, at the most of times better, than the results derived by traditional quadratic methods. The proposed method will firstly be used to solve a state feedback $\mathcal{H}_\infty$ control design problem of a polynomial nonlinear system and compare the results with quadratic Lyapunov functions.

Additionally, as one of the most common nonlinearities in engineering control systems, saturation is a widely encountered phenomenon that has received extensively increasing attention by researchers. However, unlike the linear system oriented researches, such as [56, 74, 22, 23, 79] etc., there are a very limited number of papers that have been witnessed discussing about analysis and synthesis problems of saturated nonlinear systems. Since the techniques developed for linear systems can not be applied directly to the nonlinear systems, nonlinear saturation control research has been obviously restricted. Although a few works, like [55, 34, 15], have been observed, most of these efforts paid attention to the construction of quadratic Lyapunov functions. However, with more monomials, higher order Lyapunov functions can involve more capabilities into the optimization process to improve the performance.

In this dissertation, we will try to take advantage of the flexible shape of the higher order Lyapunov level set to enlarge the region of attraction (ROA) of a saturated polynomial

nonlinear system from the quadratic ellipsoid shape. Moreover, a comparison between $\mathcal{H}_\infty$ control performance will be addressed as well for the system subjected to actuator saturations. Both of these two synthesis problems will be state feedback based.

Beyond state feedback control problems, we are also interested in improving performance of output feedback nonlinear $\mathcal{H}_\infty$ controller synthesis. In the past decade, a theoretical framework for exploring output feedback $\mathcal{H}_\infty$ control of nonlinear systems based on appropriate Hamilton-Jacobi-Isaacs (HJI) equations (inequalities) has been proposed in [85, 44, 86] to interpreting nonlinear $\mathcal{H}_\infty$ control in terms of dissipativity and differential game [7]. For hyperbolic nonlinear systems whose linearized plant are stabilizable, [85, 86] characterized the solution of HJI equation by an invariant manifold of Hamiltonian vector fields using differential geometric theory. Later on, the result has been generalized to non-hyperbolic nonlinear systems via output feedback control [44]. In order to derive nonlinear output feedback $\mathcal{H}_\infty$ controllers, it is necessary and sufficient to have solutions of a pair of coupled HJI equations with separate structures, to synthesize a state feedback and an output estimation controller, respectively. Parallel to linear $\mathcal{H}_\infty$ control theory, a separation principle was also established under a detectability hypothesis [45].

The HJI equation or inequality is the nonlinear version of the Riccati equation considered in the $\mathcal{H}_\infty$ control problem for linear systems. Unlike the Riccati equation for linear systems, which can be solved easily by efficient numerical algorithms, the solution of HJI equation PDE in the nonlinear context is very difficult to solve for. Therefore, one of the key challenges of nonlinear output feedback $\mathcal{H}_\infty$ control theory is the solvability of HJI equations. Beginning from the polynomial approximation approach proposed in [60] and [30], Taylor series expansion were considered to solve HJI equation term by term in an iterative fashion. Along this line, many authors have articles with similar approaches to the solution of the problem, such as [86, 49, 46, 33, 17, 83, 92]. But these approaches cannot provide solutions sharing a closed form. And also, these solutions may not converge to an analytic solution. Furthermore, Galerkin successive approximation was proposed in [8, 9] to reduce HJI equations to a sequence of linear

partial differential equations. Then some works applied the Galerkin successive method to find an approximated solution, for example, in [27]. As a result, the computation cost increases exponentially as the number of states increase.

On the other hand, [59] provided a convex parametrization of nonlinear $\mathcal{H}_\infty$ control problem based on a pair of positive definite matrix functions $P(x), Q(x)$. Unfortunately, it is difficult, if not impossible, to specify the form of $P(x)$ such that $\partial V(x)/\partial x = 2x^T P(x)$ except for the trivial case when $P(x)$ is a constant matrix. Taking advantage of the tractable SOS programming [69, 35] methods, in [70], the $\mathcal{L}_2$ gain analysis problem for polynomial nonlinear systems was formulated as a convex state-dependent linear matrix inequality (LMI). An iterative synthesis method based on SOS programming to solve a special state feedback $\mathcal{H}_\infty$ problem was proposed in [88].

Because of the approach proposed in [96], we are able to efficiently solve the nonlinear output feedback $\mathcal{H}_\infty$ control problem for a more generalized class of system. Moreover, this method also revealed that, when dealing with higher order polynomial systems, it is very difficult, if not impossible, for quadratic Lyapunov functions to render the output feedback estimation HJI condition as an SOS. Therefore, **in this dissertation**, we will use higher order Lyapunov functions to address this solvability issue and improve the $\mathcal{H}_\infty$ performance as well. With system augmentation, the proposed approach will present the higher order terms of the conditions as convex LMIs such that, the computational efficiency can be achieved.

1.2 Dissertation Outline

The detailed outline of this dissertation is as follows:

Chapter 1 reviews the development of research work for polynomial nonlinear systems and emphasizes related literature that motivates our work. The objectives of the dissertation are also described.

Chapter 2 contains definitions, concepts and information about the mathematical background and related tools. They are useful throughout the work and provide foundations for the

derivation of theoretical results in the later chapters.

Chapter 3 establishes a state feedback $\mathcal{H}_\infty$ strategy for polynomial nonlinear systems using higher order Lyapunov functions. Using power transformation, a nonlinear polynomial system is augmented systematically. Then, a higher order Lyapunov function of the original system as well as the associated nonlinear controller can be derived by quadratically stabilizing the augmented system. Coupling constraints are embedded in the LMI condition to guarantee the relationship between the augmented and the original states. Optional regional constraints can be added as well to provide a local solution. An iteration process is also available to solve for the system matrix annihilators and regular decision variables respectively at each step. This could provide a powerful tool to not only use higher order Lyapunov functions, but also find the suitable representation of system matrix to improve the $\mathcal{H}_\infty$ performance.

In Chapter 4, the application of the proposed system augmentation strategy is expanded to the area of saturation control. By using norm-bounded differential inclusion (NDI), a nonlinear system subjected to actuator saturation can be included by a NDI system representation. By applying the iterative approach proposed in Chapter 3 for unsaturated $\mathcal{H}_\infty$ problem to NDI nonlinear systems, saturation $\mathcal{H}_\infty$ problem can be solved in terms of state-dependent LMI conditions. Moreover, this chapter also expands the region of attraction (ROA) of a saturated nonlinear polynomial system to expand the usage of the proposed method. By solving for system matrix representations and higher order Lyapunov functions iteratively, the Lyapunov level set can be enlarged in pre-selected directions gradually to a non-ellipsoid shape.

Chapter 5 continually explores the applicability of our study with higher order Lyapunov functions from state feedback to output feedback synthesis issues. For a class of polynomial nonlinear systems, the output feedback $\mathcal{H}_\infty$ control problem can be related to the solution of a pair of HJI inequality conditions. A computationally efficient $\mathcal{H}_\infty$ control design procedure is proposed to convexly solve the HJI conditions based on system augmentation to derive higher order Lyapunov solutions. By applying this procedure, attitude command tracking problem of an axi-symmetric spacecraft with two control torques are solved in a local region. In this

process, higher order Lyapunov function is used to improve the solvability issue of output feedback estimation that are very difficult, if not impossible, solved by quadratic Lyapunov functions. Besides, with the help of system augmentation, the augmented system has more flexible forms that have the chance to provide a $\mathcal{H}_\infty$ performance improvement, than using quadratic based nonlinear control laws.

Chapter 6 concludes the research contribution in this dissertation as well as provides a remark on some future research directions.

Chapter 2

Preliminaries

In this chapter, we provide some background knowledge which forms the basis for the later study. Power transformation is discussed in Sections 2.1 , which plays a very important role in this research. Since Sum-of-Squares (SOS) and semi-definite programming is extensively applied throughout this work, some background material will be provided in Section 2.2.

2.1 Power Transformation

For the vector $x = [x_1 \quad \cdots \quad x_n]^T \in \mathbf{R}^n$, its power transformation of degree p is a nonlinear change of coordinates that forms a new homogeneous vector $x^{[p]}$ of all integer powered monomials of degree p that can be made from the original x vector

$$x_\ell^{[p]} := x_1^{k_{\ell 1}} x_2^{k_{\ell 2}} \dots x_n^{k_{\ell n}}, \ \ell \in \mathbf{I}[1, m]$$

where $k_{\ell j} \in \{0, 1, \dots, p\}$ such that $\sum_{j=1}^{n} k_{\ell j} = p, \forall \ell$, and the ordering of $x_\ell^{[p]}$s is lexicographical in the $k_{\ell j}$s. The maximal number of independent monomials of degree p in n variables is determined by

$$N_n^p = \begin{pmatrix} n+p-1 \\ p \end{pmatrix} = \frac{(n+p-1)!}{p!(n-1)!}.$$

For example, when $n = 2, p = 3, N_2^3 = 4$, the four monomials are $x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$.

The power transformation preserves certain linear relation which makes the transformation attractive for studying linear systems. For $x, y \in \mathbf{R}^n$, the matrix $A = [a_{ij}] \in \mathbf{R}^{n\times n}$ is a linear map from x to y defined by $y_i = \sum_{j=1}^n a_{ij}x_j, i \in \mathbf{I}[1, n]$. We denote by matrix $A^{[p]}$ the map from $\mathbf{R}^{N_n^p}$ to $\mathbf{R}^{N_n^p}$, which transforms the linear relation $y = Ax$ to

$$y^{[p]} = A^{[p]}x^{[p]}.$$

The above equation implies that, the set of all linearly independent p-degree terms $y_iy_j \dots y_k$ depend linearly on the set of all linearly independent p-degree terms $x_ix_j \dots x_k$ [13]. The principal properties of $A^{[p]}$ are covered by the following theorem. Its proof is omitted here for space reason.

Theorem 1 ([13]) *Suppose matrices $A, B \in \mathbf{R}^{n\times n}$, then $A^{[p]}$ and $B^{[p]}$ satisfy*

1. $I_n^{[p]} = I_{N_n^p}$,
2. $(AB)^{[p]} = A^{[p]}B^{[p]}$,
3. $(A^q)^{[p]} = (A^{[p]})^q$, *where p, q are integers,*
4. $(A^T)^{[p]} = (A^{[p]})^T$.

Power transformation is also a useful tool to transform states of a dynamic system to any arbitrary high degree [13]. For a linear ODE $\dot{x}(t) = A(t)x(t)$, we use $A_{[p]}$ to denote the coefficient matrix in the augmented differential equation as

$$\frac{d}{dt}x^{[p]}(t) = A_{[p]}(t)x^{[p]}(t), \qquad p \in \mathbf{Z}_+.$$

Thus, the set of all p-degrees monomials in $x_1, x_2, \dots, x_n$ satisfies a linear differential equation with a coefficient matrix which is derived from A. It is not difficult to see that $A_{[p]}$ is an infinitesimal version of $A^{[p]}$ which depends linearly on the elements of A. From the result of [13],

if A is Hurwitz, then $A_{[p]}$ is Hurwitz as well. Therefore, the stability of a linear ODE is preserved under the power transformation. In sequel, we will try to extend the power transformation idea to polynomial nonlinear systems in facilitating nonlinear control designs.

2.2 SOS Programming

Given the vector $x = [x_1 \quad \cdots \quad x_n]^T \in \mathbf{R}^n$ and the multi-index vector $k = [k_1, \cdots, k_n]^T \in \mathbf{Z}^n$ with $p = \sum_{i=1}^n k_i$, we define for integers $p, r \in \mathbf{Z}$

1. $m_p(x)$ is a monomial of x with degree p

$$m_p(x) = x^p = x_1^{k_1} x_2^{k_2} \dots x_n^{k_n},$$

2. $M(x)$ is a monomial vector whose elements $M_i(x)$ are monomials of x

$$M^{[p]}(x) = \left\{ \left[M_1^{[p]}(x) \quad \dots \quad M_t^{[p]}(x) \right]^T : \ M_i^{[p]}(x) = m_{pi}(x), \ i \in \mathbf{I}[1,t] \right\},$$

3. $M^{(r)}(x)$ is a monomial vector of x with all components $M_i^{(r)}(x)$ having degree no more than r

$$M^{(r)}(x) = \left\{ \left[M_1^{(r)}(x) \quad \dots \quad M_q^{(r)}(x) \right]^T : \ M_i^{(r)}(x) = m_{pi}(x), \ p \le r, \ i \in \mathbf{I}[1,q] \right\},$$

4. $f(x)$ is a polynomial of x composed of combination of monomials.

If a polynomial $f(x)$ is SOS decomposable, then it implies $f(x) \ge 0$ for any $x \in \mathbf{R}^n$. Given the polynomial of degree $2r$, the SOS decomposition condition is equivalent to the existence of a positive semi-definite matrix G such that

$$p(x) = [M^{(r)}(x)]^T G M^{(r)}(x),$$

where G is called Gramian matrix [19]. Therefore, by solving a semi-definite programming problem [12, 69, 84], we can search for a non-negative Gramian matrix associated with a polynomial to determine if it is an SOS [64]. SOS programming has also been generalized to polynomial matrix forms in [35].

The main advantages of SOS programming are the resulting computational tractability and the algorithmic character of its solution procedure [64]. This could help to provide coherent methodology of synthesizing Lyapunov functions for nonlinear systems. Besides, for many difficult optimization problems, SOS technique can provide tractable relaxations. In the field of control theory, many optimization problems (such as dissipativity and passivity) are in the form of indefinite quadratic problems [84]. The maximization of these functionals subject to indefinite quadratic constraints is often computationally complicated, sometimes even NP-hard. Consider the following feasibility problem :

$$f(x) = x^T Q x < 0 \ \forall x \text{ with constrains } g_i(x) = x^T P_i x \geq 0, \ i \in [1, n]. \tag{2.1}$$

A lot of Lyapunov based control problems can be formulated as LMIs in the above form. Using the S-procedure [12], a sufficient condition to (2.1) can be written as

$$\exists \ \alpha_1 > 0, \cdots, \alpha_n > 0 \text{ such that } Q + \sum_{i=1}^{n} \alpha_i P_i < 0. \tag{2.2}$$

By lifting the degree restriction of multipliers [64], a more powerful conditions of the original statement will be

$$f(x) + \sum_{i=1}^{n} A_i(x) g_i(x) + \sum_{0 \leq i < j \leq n} \beta_{ij} g_i(x) g_j(x) < 0 \quad \forall x, \tag{2.3}$$

in which $A_i(x) = x^T A_i x$, $A_i > 0$ and $\beta_{ij} \geq 0$. The new relaxation is always at least as powerful as the standard one: this can be easily verified by taking $A_i = \alpha_i I$ and $\beta_{ij} = 0$.

Because of the above advantages, SOS programming has become a popular technique for nonlinear systems research. [63, 78] used it on stability analysis problems and [67, 68, 94] used it on stabilization problems.

2.3 Kronecker Product

The Kronecker product, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix. It is a generalization of the outer product from vectors to matrices. Many works in literature introduce the concepts and applications of Kronecker product, such as [31]. In this section, basic concepts and properties of Kronecker product will be listed for further references.

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

It can be verified that the Kronecker product has the usual properties of a product.

1. Bilinearity and associativity:

$(A + B) \otimes C = A \otimes C + B \otimes C$

$A \otimes (B + C) = A \otimes B + A \otimes C$

$A \otimes (\alpha B) = \alpha(A \otimes B)$

$A \otimes (B \otimes C) = (A \otimes B) \otimes C$

in which A, B, C are matrices and α is a scalar.

2. The mixed-product property: If A, B, C and D are matrices of compatible size such that one can form the matrix products AC and BD, then

$(A \otimes B)(C \otimes D) = AC \otimes BD$.

3. $A \otimes B$ is invertible if and only if A and B are invertible, in which case the inverse is given by $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

4. Transpose: The operation of transposition is distributive over the Kronecker product:

$(A \otimes B)^T = A^T \otimes B^T$.

5. Kronecker sum: If the dimension of A is $n \times n$, and the dimension of B is $m \times m$ and I_k denotes the $k \times k$ identity matrix then we can define

$A \oplus B = A \otimes I_m + I_n \otimes B$.

6. Determinant: Let A be an $n \times n$ matrix and let B be a $p \times p$ matrix. Then

$|A \otimes B| = |A|^p |B|^n$.

Chapter 3

Nonlinear State Feedback $\mathcal{H}_\infty$ Control for Polynomial Nonlinear Systems

In this chapter, we will address $\mathcal{H}_\infty$ control problem for state-dependent polynomial nonlinear systems using state feedback control laws. The control design objective is to construct a higher order Lyapunov function, rather than a quadratic one, as well as its associated polynomial state feedback controller to stabilize the closed-loop nonlinear system and optimize its performances. Different performances such as disturbance rejection ability and region of attraction (ROA) expansion can be specified by different optimization conditions. In this chapter, we focus on improving the disturbance attenuation performance.

To derive higher order Lyapunov functions and polynomial nonlinear controllers effectively, the original nonlinear systems are augmented under the rule of power transformation. The augmented systems have more state variables and the additional variables represent higher order combinations of the original ones. As a result, the design of higher order Lyapunov functions for original nonlinear systems can be recast to the search of quadratic Lyapunov functions for augmented systems. Moreover, taking the advantage of the flexible form of the augmented

system matrix, a polynomial annihilator [43] can be introduced to optimize the system matrices for performance improvement. The annihilator and the combination of Lyapunov function and controller gain can be treated as known and unknown variables iteratively to improve the performance. The Sum-of-Squares (SOS) programming [64, 69] is then used to solve the quadratic Lyapunov functions of augmented variables and synthesize the corresponding controllers through computationally efficient convex optimizations.

Section 3.1 covers a review of the nonlinear control design techniques that are capable of computing higher order Lyapunov functions. In Section 3.2, system augmentation based on power transformation will be introduced. Section 3.3 proves the existence of a higher order Lyapunov function for an asymptotically stable polynomial nonlinear system. Then, using the augmented system, we will employ higher order Lyapunov functions to solve nonlinear state feedback $\mathcal{H}_\infty$ problems for polynomial nonlinear systems in Section 3.4. An example will be given in Section 3.5 to demonstrate the proposed nonlinear control approach and compare the results derived by using quadratic and higher order Lyapunuv functions. Finally, we will conclude the chapter in Section 3.6.

3.1 Introduction

Although higher order Lyapunov functions has been successfully used on stability domains analysis of nonlinear systems [64], most synthesis problems of polynomial control systems pursuing higher order Lyapunov functions are not reducible to convex problems. The reason for their non-convexity is the difficulty of searching for a polynomial Lyapunov function and polynomial controller variables simultaneously. There are currently several synthesis approaches that are capable of solving nonlinear control problems for polynomial nonlinear systems. A convex parametrization of nonlinear output feedback $\mathcal{H}_\infty$ control condition was derived in [59] based on a pair of positive definite matrix functions $P(x), Q(x)$. Unfortunately, it is difficult, if not impossible, to specify the functional form of $P(x)$ such that $\partial V(x)/\partial x = 2x^T P(x)$ except for the trivial case when $P(x)$ is a constant matrix. Therefore, the proposed synthesis condition does

not naturally lead to computationally tractable solution algorithms for nonlinear $\mathcal{H}_\infty$ control. As an improvement to [59], an iterative algorithm was proposed in [95] to solve the nonlinear $\mathcal{H}_\infty$ control problem by constructing a higher order Lyapunov function and its associated nonlinear controllers starting from quadratic Lyapunov functions. However, the iterative algorithm may not converge to its global minimum and the result could be conservative. Since the set of polynomial Lyapunov functions $V(x)$ and stabilizing control laws $K(x)$ are not jointly convex, a dual approach to the state feedback synthesis based on density function has been proposed [67]. Nevertheless, it is currently not clear how a performance objective can be incorporated in the density-based approach.

On the other hand, by representing the open-loop nonlinear systems in a pre-selected state dependent linear-like form, sufficient conditions for the solutions of nonlinear control problems were formulated in terms of state-dependent linear matrix inequalities (LMIs) [68]. Specifically, the nonlinear dynamics will be written in the form of $\dot{x} = A(x)Z(x) + B(x)u$ using a pre-selected polynomial vector $Z(x)$ of the state variables and nonlinear control problems are solved using a higher-order Lyapunov function $V(x) = Z^T(x)PZ(x)$. An alternative approach [24, 25] employed a transformed vector $Z(x) = W(x)x$ of the same dimension as state variables and $W(x)$ as a square, lower triangular polynomial matrix with all diagonal elements equal to 1. Further works along this line have been done in [42, 43] by focusing on optimal control and disturbance rejection problems. A non-iterative two-step design procedure was proposed with the first step finding a feasible solution to the state-dependent LMI, while the second step utilizing polynomial annihilators to reduce the conservatism of the first step. Although these results have demonstrated to work well for some examples, one of the drawbacks is that the method provided in [68] may lead to infeasibility for some selections of $Z(x)$. Moreover, this pre-selection process requires pre-knowledge of the system and no method is available to select the most suitable state vector $Z(x)$ and the associated system representations.

3.2 System Augmentation

Consider the affine-input, nth-order polynomial nonlinear system of the following form

$$\begin{cases} \dot{x} &= A(x)x + B_1(x)d + B_2(x)u \\ e &= C(x)x + D_1(x)d + D_2(x)u \end{cases} \tag{3.1}$$

where the state $x \in \mathbf{R}^n$, the disturbance $d \in \mathbf{R}^{n_d}$, and the control input $u \in \mathbf{R}^{n_u}$. We assume all states are measurable and the state-space entries are polynomial functions of the state x. It is noted that $A(x)$ is not unique because all the expressions of the vector field can be written as $(A(x) + \mathcal{N}_A(x))x$, where $\mathcal{N}_A(x)$ satisfies $\mathcal{N}_A(x)x = 0$ for all $x \in \mathbf{R}^n$. The nonlinear state feedback controller is in a polynomial form of $u = F(x)x$.

Given a positive integer r, $x^{(r)}$ will be used to denote an augmented vector containing all possible monomials of degree $k \leq r$ of the state vector x, ordered lexigraphically. So $x^{(r)} = \begin{bmatrix} x^{[1]} & x^{[2]} & \dots & x^{[r]} \end{bmatrix}^T$. Then $x^{(r)}$ has the dimension of $\tilde{N}_n^r = \sum_{k=1}^r N_n^k$, in which $x^{[r]}$ and N_n^k has been defined in Section 2.1. As an example, the monomial of a two-variable vector of degree no more than 2 is

$$x^{(2)} = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}^T := \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 \end{bmatrix}^T \in \mathbf{R}^5.$$

Note that the augmented variables are not independent from each other. Clearly, we have the following coupling constraints among elements of $x^{(2)}$ vector

$$\phi_1(\tilde{x}) = \tilde{x}_3 - \tilde{x}_1^2 = 0, \quad \phi_2(\tilde{x}) = \tilde{x}_4 - \tilde{x}_1\tilde{x}_2 = 0, \quad \phi_3(\tilde{x}) = \tilde{x}_5 - \tilde{x}_2^2 = 0,$$
$$\phi_4(\tilde{x}) = \tilde{x}_2\tilde{x}_4 - \tilde{x}_1\tilde{x}_5 = 0, \quad \phi_5(\tilde{x}) = \tilde{x}_2\tilde{x}_3 - \tilde{x}_1\tilde{x}_4 = 0, \quad \phi_6(\tilde{x}) = \tilde{x}_4^2 - \tilde{x}_3\tilde{x}_5 = 0.$$

They reflect the equivalence relationship among the multiple combinations of augmented state variables. The number of coupling constraints will depend on the form of monomial basis vector. Table 3.1 lists the maximal number of coupling constraints for n state variables consisting of

all the possible monomials up to degree of r.

Table 3.1: Maximal number of coupling constraints.

states number (n)	highest monomial order (r)	augmented states number ($\tilde{N}_n^r$)	maximal number of coupling constraints
2	2	5	6
2	3	9	27
2	4	14	75
3	2	9	20
3	3	19	126
3	4	34	465
4	2	14	50
4	3	34	420
4	4	69	1990

In order to synthesize a polynomial controller for the nonlinear system (3.1), we augment the state vector so that it becomes $x^{(r)}$. Then, the augmented nonlinear system becomes

$$\begin{cases} \dot{\tilde{x}} &= A_{(r)}(\tilde{x})\tilde{x} + B_{1(r)}(\tilde{x})d + B_{2(r)}(\tilde{x})u \\ e &= C_{(r)}(\tilde{x})\tilde{x} + D_{1(r)}(\tilde{x})d + D_{2(r)}(\tilde{x})u \end{cases} \tag{3.2}$$

where $\tilde{x} = x^{(r)}$, the augmented system matrix $A_{(r)} \in \mathbf{R}^{\tilde{N}_n^r \times \tilde{N}_n^r}$ and the input matrix $B_{2(r)}(\tilde{x}) \in \mathbf{R}^{\tilde{N}_n^r \times n_u}$. $B_{1(r)}, C_{(r)}$ and $D_{2(r)}, D_{1(r)}$ are the relevant matrices after the augmentation, respectively. Consequently, the goal of the controller design becomes computing the control gain $\widetilde{F}(x^{(r)})$ such that $u = \tilde{F}(x^{(r)})x^{(r)}$ can stabilize the closed-loop system by satisfying certain conditions. Note that $A_{(r)}(\tilde{x})$ does not satisfy the definition of $A_{[p]}$ in Section 2.1. As an example,

given

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \qquad B_1(x) = \begin{bmatrix} b_{11}(x) \\ b_{12}(x) \end{bmatrix}, \qquad B_2(x) = \begin{bmatrix} b_{21}(x) \\ b_{22}(x) \end{bmatrix}$$
$$C(x) = \begin{bmatrix} c_1(x) & c_2(x) \end{bmatrix}, \qquad D_1(x) = d_1(x), \qquad D_2(x) = d_2(x)$$

By the nonlinear state transformation, the original system state x can be augmented to its power 2 vector $\tilde{x} = x^{(2)}$. Reformulating the original system (3.1) with respect to the new states $\tilde{x}$, we get one of the state space description of the augmented system

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \\ \dot{\tilde{x}}_5 \end{bmatrix} = \begin{bmatrix} a_{11}(x) & a_{12}(x) & 0 & 0 & 0 \\ a_{21}(x) & a_{22}(x) & 0 & 0 & 0 \\ 0 & 0 & 2a_{11}(x) & 2a_{12}(x) & 0 \\ 0 & 0 & a_{21}(x) & a_{11}(x)+a_{22}(x) & a_{12}(x) \\ 0 & 0 & 0 & 2a_{21}(x) & 2a_{22}(x) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix}$$
$$+ \begin{bmatrix} b_{11}(x) \\ b_{12}(x) \\ 2b_{11}(x)\tilde{x}_1 \\ b_{11}(x)\tilde{x}_2 + b_{12}(x)\tilde{x}_1 \\ 2b_{12}(x)\tilde{x}_2 \end{bmatrix} d + \begin{bmatrix} b_{21}(x) \\ b_{22}(x) \\ 2b_{21}(x)\tilde{x}_1 \\ b_{21}(x)\tilde{x}_2 + b_{22}(x)\tilde{x}_1 \\ 2b_{22}(x)\tilde{x}_2 \end{bmatrix} u$$
$$e = \begin{bmatrix} c_1(x) & c_2(x) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 \end{bmatrix}^T + D_1(x)d + D_2(x)u,$$

which is in the form of (3.2):

$$\dot{\tilde{x}} := A_{(2)}(\tilde{x})\tilde{x} + B_{1(2)}(\tilde{x})d + B_{2(2)}(\tilde{x})u$$
$$e := C_{(2)}(\tilde{x})\tilde{x} + D_{1(2)}(\tilde{x})d + D_{2(2)}(\tilde{x})u.$$

Following the same procedure, the original nonlinear system can be extended to its higher-order

$(r > 2)$ equivalence as well. Since the representation of nonlinear state space system matrix is not unique, $A_{(r)}$ can always be rewritten as $A_{(r)}+\mathcal{N}_A$, in which the nonlinear polynomial annihilator $\mathcal{N}_A$ satisfies $\mathcal{N}_A x^{(r)} = 0$. In this thesis, together with power transformation, the annihilator $\mathcal{N}_A$ will be used to address the issue of SOS feasibility and performance improvement when solving LMI conditions. The application of this idea has not been witnessed in literature to solve LMI based control problems.

3.3 Existence of Higher Order Lyapunov Functions

In [66], using the Picard iteration, Peet proved that exponential stability of a polynomial vector field on a bounded set implies the existence of a Lyapunov function which is a sum-of-squares of polynomials. Nevertheless, it did not specify particular Lyapunov function form. In this section, we will prove the existence of higher order Lyapunov functions for quadratically stabilizable polynomial nonlinear systems. To this end, a useful lemma for subsequent development will be stated and proved. The format of augmented system matrices provided in this lemma can help efficiently construct one representation of the augmented system during the control synthesis programming. This lemma is extended from a special case of Lemma 1 in [18].

Lemma 1 *For a polynomial nonlinear system $\dot{x} = A(x)x + B(x)F(x)x$, we define $A_{[r]}(x) \in \mathbf{R}^{N_n^r \times N_n^r}$, $B_{[r]}(x) \in \mathbf{R}^{N_n^r \times (r \cdot n^{r-1} \cdot n_u)}$ and $F_{[r]}(x) \in \mathbf{R}^{(r \cdot n^{r-1} \cdot n_u) \times N_n^r}$ for any $r \in Z_+$ such that*

$$\frac{d}{dt}x^{[r]} = \frac{\partial x^{[r]}}{\partial x}\left[A(x)x + B(x)F(x)x\right] = A_{[r]}(x)x^{[r]} + B_{[r]}(x)F_{[r]}(x)x^{[r]}. \tag{3.3}$$

Moreover, let $x^{\langle r \rangle} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{r}$ and $T_r \in \mathbf{R}^{n^r \times N_n^r}$ be the matrix satisfying $x^{\langle r \rangle} = T_r x^{[r]}$.

Then $A_{[r]}$ and $B_{[r]}, F_{[r]}$ can be expressed as

$$A_{[r]}(x) = (T_r^T T_r)^{-1} T_r^T \sum_{i=0}^{r-1} (I_{n^{r-1-i}} \otimes A(x) \otimes I_{n^i}) T_r \tag{3.4}$$

$$B_{[r]}(x) = (T_r^T T_r)^{-1} T_r^T \left[(I_{n^{r-1}} \otimes B(x) \otimes I_{n^0}) \cdots (I_{n^0} \otimes B(x) \otimes I_{n^{r-1}}) \right]$$

$$F_{[r]}(x) = \begin{bmatrix} (I_{n^{r-1}} \otimes F(x) \otimes I_{n^0}) \\ \vdots \\ (I_{n^0} \otimes F(x) \otimes I_{n^{r-1}}) \end{bmatrix} T_r.$$

Proof: First, we introduce $A_{\langle r \rangle}(x) \in \mathbf{R}^{n^r \times n^r}$ and $B_{\langle r \rangle}(x) \in \mathbf{R}^{n^r \times (r \cdot n^{r-1} \cdot n_u)}$ and $F_{\langle r \rangle}(x) \in \mathbf{R}^{(r \cdot n^{r-1} \cdot n_u) \times n^r}$ as

$$\frac{\partial x^{\langle r \rangle}}{\partial x} \left[A(x) + B(x) F(x) \right] x = A_{\langle r \rangle}(x) x^{\langle r \rangle} + B_{\langle r \rangle}(x) F_{\langle r \rangle}(x) x^{\langle r \rangle}. \tag{3.5}$$

Using the properties of Kronecker product, it can be shown that

$$\begin{aligned}
\frac{\partial x^{\langle r \rangle}}{\partial x} &= \sum_{i=0}^{r-1} x^{\langle r-1-i \rangle} \otimes I_n \otimes x^{\langle i \rangle} \\
(x^{\langle r-1-i \rangle} \otimes I_n \otimes x^{\langle i \rangle}) Ax &= (x^{\langle r-1-i \rangle} \otimes I_n \otimes x^{\langle i \rangle})(I_{n^0} \otimes Ax \otimes I_{n^0}) \\
&= [(x^{\langle r-1-i \rangle} \otimes I_n)(I_{n^0} \otimes Ax)] \otimes [x^{\langle i \rangle} I_{n^0}] \\
&= [(x^{\langle r-1-i \rangle} I_{n^0}] \otimes [I_n Ax)] \otimes [x^{\langle i \rangle} I_{n^0}] \\
&= (I_{n^{r-1-i}} x^{\langle r-1-i \rangle}) \otimes Ax \otimes (I_{n^i} x^{\langle i \rangle}) \\
&= [(I_{n^{r-1-i}} \otimes A) \cdot (x^{\langle r-1-i \rangle} \otimes x)] \otimes (I_{n^i} x^{\langle i \rangle}) \\
&= (I_{n^{r-1-i}} \otimes A \otimes I_{n^i}) \cdot (x^{\langle r-1-i \rangle} \otimes x \otimes x^{\langle i \rangle}) \\
&= (I_{n^{r-1-i}} \otimes A \otimes I_{n^i}) x^{\langle r \rangle}.
\end{aligned}$$

Then we have

$$A_{\langle r\rangle} = \sum_{i=0}^{r-1} I_{n^{r-1-i}} \otimes A \otimes I_{n^i}. \tag{3.6}$$

Following the same procedure, one can obtain

$$\begin{aligned}
\frac{\partial x^{\langle r\rangle}}{\partial x}\dot{x} &= A_{\langle r\rangle}x^{\langle r\rangle} + \sum_{i=0}^{r-1}(x^{\langle r-1-i\rangle} \otimes I_n \otimes x^{\langle i\rangle})BFx \\
&= A_{\langle r\rangle}x^{\langle r\rangle} + \sum_{i=0}^{r-1}(I_{n^{r-1-i}} \otimes B \otimes I_{n^i}) \cdot (I_{n^{r-1-i}} \otimes F \otimes I_{n^i})x^{\langle r\rangle} \\
&= A_{\langle r\rangle}x^{\langle r\rangle} + B_{\langle r\rangle}F_{\langle r\rangle}x^{\langle r\rangle}
\end{aligned}$$

with

$$B_{\langle r\rangle}(x) = \begin{bmatrix}(I_{n^{r-1}} \otimes B(x) \otimes I_{n^0}) \cdots (I_{n^0} \otimes B(x) \otimes I_{n^{r-1}})\end{bmatrix} \quad F_{\langle r\rangle}(x) = \begin{bmatrix}(I_{n^{r-1}} \otimes F(x) \otimes I_{n^0}) \\ \vdots \\ (I_{n^0} \otimes F(x) \otimes I_{n^{r-1}})\end{bmatrix}. \tag{3.7}$$

From eqn. (3.5), one can derive

$$T_r \frac{\partial x^{[r]}}{\partial x}\left[A(x) + B(x)F(x)\right]x = A_{\langle r\rangle}(x)T_r x^{[r]} + B_{\langle r\rangle}(x)F_{\langle r\rangle}(x)T_r x^{[r]}. \tag{3.8}$$

Note that T_r has full column rank and contains N_n^r linearly independent columns. Multiplying both sides of (3.8) by $(T_r^T T_r)^{-1}T_r^T$ and following eqns. (3.3) and (3.6)-(3.7), we obtain the expressions of $A_{[r]}$ and $B_{[r]}, F_{[r]}$ as desired. *Q.E.D.*

The following theorem confirms the existence of higher order Lyapunov functions for quadratically stabilizing nonlinear systems. These Lyapunov functions will be beneficial for the stabilization and $\mathcal{H}_\infty$ control of polynomial nonlinear systems. For space reason, the dependency of

nonlinear state-space matrices on x is often omitted.

Theorem 2 *For a polynomial nonlinear system (3.1) with $d = 0$, if there exists a quadratic Lyapunov function $V(x) = x^T Px > 0$ and a control $u = F(x)x$ such that $x^T(PA + PB_2F + A^TP + F^TB_2^TP)x < 0$, then for any $r \in \mathbf{Z}_+$, the closed-loop system also admits $2r$th order Lyapunov function $V(x^{(r)}) = x^{(r)T}P_{(r)}x^{(r)} > 0$ and a matrix function $F_{(r)}(x)$ satisfying*

$$x^{(r)T}\left[P_{(r)}A_{(r)} + P_{(r)}B_{2(r)}F_{(r)} + A_{(r)}^TP_{(r)} + F_{(r)}^TB_{2(r)}^TP_{(r)}\right]x^{(r)} < 0. \tag{3.9}$$

Proof: Let us augment the state vector x to $\tilde{x} = x^{(2)} = \begin{bmatrix} x \\ x^{[2]} \end{bmatrix}$. Then the augmented system can be written as

$$\dot{\tilde{x}} = A_{(2)}(\tilde{x})\tilde{x} + B_{2(2)}(\tilde{x})F_{(2)}(\tilde{x})\tilde{x} = \begin{bmatrix} A & 0 \\ 0 & A_{[2]} \end{bmatrix}\begin{bmatrix} x \\ x^{[2]} \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_{2[2]} \end{bmatrix}\begin{bmatrix} F & 0 \\ 0 & F_{[2]} \end{bmatrix}\begin{bmatrix} x \\ x^{[2]} \end{bmatrix}. \tag{3.10}$$

Since $x^TPx > 0$ and $x^T(PA + PB_2F + A^TP + F^TB_2^TP)x < 0$, we have

$$\begin{aligned}
&[x^TPx] \otimes [x^T(PA + A^TP + PB_2F + F^TB_2^TP)x] < 0 \\
\Leftrightarrow &[x^TP] \otimes [x^T(PA + A^TP + PB_2F + F^TB_2^TP)] \cdot [x \otimes x] < 0 \\
\Leftrightarrow &[x^T \otimes x^T] \cdot [P \otimes (PA + A^TP + PB_2F + F^TB_2^TP)] \cdot [x \otimes x] < 0 \\
\Leftrightarrow &x^{\langle 2\rangle T}\left\{(P \otimes P)\left[(I_n \otimes A) + (I_n \otimes B_2F)\right] + \left[(I_n \otimes A)^T + (I_n \otimes B_2F)^T\right](P \otimes P)\right\}x^{\langle 2\rangle} < 0.
\end{aligned} \tag{3.11}$$

Similarly, from $[x^T(PA + A^TP + PB_2F + F^TB_2^TP)x] \cdot [x^TPx] < 0$ one can obtain that

$$x^{\langle 2\rangle T}\left\{(P \otimes P)\left[(A \otimes I_n) + (B_2F \otimes I_n)\right] + \left[(A \otimes I_n)^T + (B_2F \otimes I_n)^T\right](P \otimes P)\right\}x^{\langle 2\rangle} < 0. \tag{3.12}$$

Adding eqns. (3.11) and (3.12), we obtain

$$x^{\langle 2\rangle T}\left[(P\otimes P)(A\otimes I_n + I_n\otimes A) + (A\otimes I_n + I_n\otimes A)^T(P\otimes P)\right]x^{\langle 2\rangle}$$
$$+ x^{\langle 2\rangle T}\left\{(P\otimes P)\left[(B_2F\otimes I_n) + (I_n\otimes B_2F)\right] + \left[(B_2F\otimes I_n) + (I_n\otimes B_2F)\right]^T(P\otimes P)\right\}x^{\langle 2\rangle} < 0,$$

which is equivalent to

$$\begin{aligned} & x^{\langle 2\rangle T}\left[(P\otimes P)\sum_{i=0}^{1}(I_{n^{1-i}}\otimes A\otimes I_{n^i}) + \sum_{i=0}^{1}(I_{n^{1-i}}\otimes A\otimes I_{n^i})^T(P\otimes P)\right]x^{\langle 2\rangle} \\ & + x^{\langle 2\rangle T}(P\otimes P)\sum_{i=0}^{1}(I_{n^{1-i}}\otimes B_2\otimes I_{n^i})\cdot(I_{n^{1-i}}\otimes F\otimes I_{n^i})x^{\langle 2\rangle} \\ & + x^{\langle 2\rangle T}\sum_{i=0}^{1}(I_{n^{1-i}}\otimes F\otimes I_{n^i})^T\cdot(I_{n^{1-i}}\otimes B_2\otimes I_{n^i})^T(P\otimes P)x^{\langle 2\rangle} < 0. \end{aligned} \tag{3.13}$$

Utilizing (3.6) and (3.7) with $r = 2$, condition (3.13) can be rewritten to

$$x^{\langle 2\rangle T}\left[(P\otimes P)A_{\langle 2\rangle} + A_{\langle 2\rangle}^T(P\otimes P) + (P\otimes P)B_{2\langle 2\rangle}F_{\langle 2\rangle} + F_{\langle 2\rangle}B_{2\langle 2\rangle}^T(P\otimes P)\right]x^{\langle 2\rangle} < 0. \tag{3.14}$$

Substituting $x^{\langle r\rangle} = T_r x^{[r]}$ into eqn. (3.14) and invoking Lemma 1, one can derive

$$x^{[2]T}\left(P_{[2]}A_{[2]} + A_{[2]}^TP_{[2]} + P_{[2]}B_{2[2]}F_{[2]} + F_{[2]}^TB_{2[2]}^TP_{[2]}\right)x^{[2]} < 0,$$

with $P_{[2]} = T_2^T(P\otimes P)(T_2T_2^T)^{-1}T_2(T_2^TT_2) = T_2^T(P\otimes P)T_2 > 0$. Therefore, for the nonlinear system (3.10), we are able to construct a Lyapunov function $\tilde{x}^TP_{(2)}\tilde{x} > 0$ with $P_{(2)} = \begin{bmatrix} P & 0 \\ 0 & P_{[2]} \end{bmatrix}$ such that

$$x^{(2)T}\left[P_{(2)}A_{(2)} + P_{(2)}B_{2(2)}F_{(2)} + A_{(2)}^TP_{(2)} + F_{(2)}^TB_{2(2)}^TP_{(2)}\right]x^{(2)} < 0, \tag{3.15}$$

in which $F_{(2)}$ is defined as $F_{(2)} = \begin{bmatrix} F & 0 \\ 0 & F_{[2]} \end{bmatrix}$. For any $r > 2$, the similar procedure can be applied by further augmenting (3.10). The only difference is multiplying additional $(x^T P x)^i, (x^T P x)^j, i+j = r-1$ on the left and right sides of (3.11) and (3.12) respectively. *Q.E.D.*

3.4 State Feedback Control of Polynomial Nonlinear Systems

Applying the S-procedure [12], we obtain the state-feedback stabilization and $\mathcal{H}_\infty$ conditions for polynomial nonlinear systems.

Theorem 3 *For the nonlinear system (3.1),*

1. *it is stabilizable by a polynomial state feedback control if there exist a positive definite matrix $Q \in \mathbf{S}_+^{\tilde{N}_n^r \times \tilde{N}_n^r}$ and a rectangular matrix function $M(\tilde{x}) : \mathbf{R}^{\tilde{N}_n^r} \to \mathbf{R}^{n_u \times \tilde{N}_n^r}$ such that*

$$\tilde{z}^T R(\tilde{x}) \tilde{z} + \sum_i \tilde{z}^T \Lambda_i(\tilde{x}) \tilde{z} \phi_i(\tilde{x}) < 0, \tag{3.16}$$

   where

$$R(\tilde{x}) = He\left\{ A_{(r)}(\tilde{x}) Q + B_{2(r)}(\tilde{x}) M(\tilde{x}) \right\}.$$

   $\tilde{z}$ is an auxiliary vector of the same dimension as $\tilde{x}$. Λ_i are symmetric multipliers of dimension $\tilde{N}_n^r \times \tilde{N}_n^r$ to enforce the coupling constraints $\phi_i(\tilde{x}) = 0$ among augmented state variables.

2. *it has $\mathcal{L}_2$ gain less than a given level $\gamma > 0$ if there exist a positive definite matrix Q and a matrix function $M(\tilde{x})$ such that*

$$\tilde{z}^T T(\tilde{x}) \tilde{z} + \sum_i \tilde{z}^T \Lambda_i(\tilde{x}) \tilde{z} \phi_i(\tilde{x}) < 0, \tag{3.17}$$

where

$$T(\tilde{x}) = \begin{bmatrix} He\{A_{(r)}(\tilde{x})Q + B_{2(r)}(\tilde{x})M(\tilde{x})\} & B_{1(r)}(\tilde{x}) & QC_{(r)}^T(\tilde{x}) + M^T(\tilde{x})D_{2(r)}^T(\tilde{x}) \\ B_{1(r)}^T(\tilde{x}) & -\gamma I & D_{1(r)}^T(\tilde{x}) \\ C_{(r)}(\tilde{x})Q + D_{2(r)}(\tilde{x})M(\tilde{x}) & D_{1(r)}(\tilde{x}) & -\gamma I \end{bmatrix}.$$

Moreover, an admissible polynomial state feedback control law is given by

$$u = \tilde{F}(x^{(r)})x^{(r)} = M(x^{(r)})Q^{-1}x^{(r)}. \tag{3.18}$$

Proof:

1. For a Lyapunov function $V(\tilde{x}) = \tilde{x}^T Q^{-1}\tilde{x} > 0$, its derivative along the augmented system (3.2) is given by

$$\begin{aligned} \dot{V}(\tilde{x}) &= \dot{\tilde{x}}^T Q^{-1}\tilde{x} + \tilde{x}^T Q^{-1}\dot{\tilde{x}} \\ &= \tilde{x}^T \left\{ \left[A_{(r)}(\tilde{x}) + B_{2(r)}(\tilde{x})\widetilde{F}(\tilde{x}) \right]^T Q^{-1} + Q^{-1} \left[A_{(r)}(\tilde{x}) + B_{2(r)}(\tilde{x})\widetilde{F}(\tilde{x}) \right] \right\} \tilde{x} \\ &= \tilde{z}^T R(\tilde{x})\tilde{z}, \end{aligned}$$

   where $\tilde{z} = Q^{-1}\tilde{x}$ and $M(\tilde{x}) = \tilde{F}(\tilde{x})Q$. If $\phi_i(\tilde{x}) = 0$ for all indices i, the augmented system becomes an equivalent description for the original nonlinear dynamics. Therefore condition (3.16) implies $\tilde{z}^T R(\tilde{x})\tilde{z} < 0$, indicating that the Lyapunov stability condition is satisfied.

2. By defining $M(\tilde{x}) = \tilde{F}(\tilde{x})Q$ and multiplying $\text{diag}\{Q^{-1}, I, I\}$ from left-hand side and its

transpose from right-hand side of $T(\tilde{x})$, we obtain

$$
\begin{aligned}
T_m(\tilde{x}) &:= \mathrm{diag}\{Q^{-1}, I, I\}\ T(\tilde{x})\ \mathrm{diag}\{Q^{-1}, I, I\} \\
&= \begin{bmatrix} He\{Q^{-1}[(A_{(r)} + B_{2(r)}\tilde{F}]\} & \star & \star \\ B_{1(r)}^T Q^{-1} & -\gamma I & \star \\ [C_{(r)} + D_{2(r)}\tilde{F}] & D_{1(r)} & -\gamma I \end{bmatrix}.
\end{aligned}
$$

It is clear from eqn. (3.17), $T_m(\tilde{x}) < 0$ if the coupling constraint $\phi_i(\tilde{x}) = 0$ holds for all i. Then for the Lyapunov function $V(\tilde{x}) = \tilde{x}^T Q^{-1} \tilde{x}$, we have

$$
\begin{aligned}
&\dot{V}(\tilde{x}) + \frac{1}{\gamma} e^T e - \gamma d^T d \\
&= \tilde{x}^T \left\{ \left[A_{(r)}(\tilde{x}) + B_{2(r)}(\tilde{x})\tilde{F}(\tilde{x}) \right]^T Q^{-1} + Q^{-1} \left[A_{(r)}(\tilde{x}) + B_{2(r)}(\tilde{x})\tilde{F}(\tilde{x}) \right] \right\} \tilde{x} \\
&\quad + d^T B_{1(r)}(\tilde{x})^T Q^{-1} \tilde{x} + \tilde{x}^T Q^{-1} B_{1(r)}(\tilde{x}) d - \gamma d^T d \\
&\quad + \frac{1}{\gamma} \left\{ \left[C_{(r)}(\tilde{x}) + D_{2(r)}(\tilde{x})\tilde{F}(\tilde{x}) \right] \tilde{x} + D_{1(r)}(\tilde{x}) d \right\}^T \left\{ [C_{(r)}(\tilde{x}) + D_{2(r)}(\tilde{x})\tilde{F}(\tilde{x})]\tilde{x} + D_{1(r)}(\tilde{x}) d \right\} \\
&= \begin{bmatrix} \tilde{x}^T & d^T \end{bmatrix} \begin{bmatrix} \left\{ \begin{array}{c} He\{Q^{-1}[(A_{(r)} + B_{2(r)}\tilde{F}]\} \\ + \frac{1}{\gamma}[C_{(r)} + D_{2(r)}\tilde{F}]^T [C_{(r)} + D_{2(r)}\tilde{F}] \end{array} \right\} & \star \\ B_{1(r)}^T Q^{-1} + \frac{1}{\gamma} D_{1(r)}^T [C_{(r)} + D_{2(r)}\tilde{F}] & -\gamma I + \frac{1}{\gamma} D_{1(r)}^T D_{1(r)} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ d \end{bmatrix} < 0.
\end{aligned}
$$

The last inequality is by taking Schur complement of $T_m(\tilde{x}) < 0$. Integrating on both sides of the above inequality, it confirms that the $\mathcal{L}_2$ gain of the original nonlinear system is bounded by γ.

Q.E.D.

Theorem 3 provides the conditions to solve the nonlinear state feedback control problem by constructing higher order Lyapunov functions as well as polynomial controller gains. The synthesis condition (3.17) determines a quadratic Lyapunov function for augmented systems in a convex form. Consequently, it provides a higher-order Lyapunov function for the original non-

linear systems. The proposed approach can be thought as a generalization of existing methods in [68, 24, 43] when $Z(x) = x^{(r)}$. For some nonlinear systems, it is not necessary to include all of the monomials to guarantee the solvability of synthesis conditions. The number of coupling constraints will thus depend on the form of monomial basis vector. The proposed method is closely related to the approach in [68]. By introducing a new state vector $\tilde{x}$ to lexicographically represent the r-th degree power transformation of the original state vector x. As a result, one can derive a more general representation of $2r$-th order Lyapunov function including the representations in [68, 24, 25, 42, 43] for Lyapunov functions as special cases.

Moreover, the original system will be augmented from the original states to the newly defined states, as $\dot{\tilde{x}} = \tilde{A}(\tilde{x})\tilde{x} + \tilde{B}(\tilde{x})u$. This augmentation helps us to replace $M(x)A(x)$ and $M(x)B(x)$ in [68] of the LMI conditions by single matrices $\tilde{A}(\tilde{x})$ and $\tilde{B}(\tilde{x})$, respectively to overcome limited representations. Therefore the feasibility and performance of the synthesis problems can be improved. Finally, the synthesis condition is different from previous results by including coupling constraint terms. The roles of coupling constraints are two folds: One is to enforce the augmented system to be equivalent to the original nonlinear dynamics. If the condition hold true for the augmented system, it holds true for the original system as well. However, this is just a sufficient condition. The inverse statement is not true. On the other hand, it helps to find a more appropriate nonlinear representation in solving the synthesis condition.

Remark 1 *Suppose that the nonlinear matrix inequality $\tilde{z}^T R(\tilde{x})\tilde{z} < 0$ ensures a specific stabilization/performance control conditions for the original system (3.1). After generating the augmented system (3.2), the corresponding matrix inequality to guarantee the same stabilization/performance control condition within an restricted area can be written as*

$$\tilde{z}^T R(\tilde{x})\tilde{z} + \sum_i \tilde{z}^T \Lambda_i(\tilde{x})\tilde{z}\phi_i(\tilde{x}) + \sum_j \tilde{z}^T \Theta_j(\tilde{x})\tilde{z}\psi_j(\tilde{x}) < 0, \tag{3.19}$$

where $\psi_j(\tilde{x}) > 0$ are regional constraints. $\tilde{z}^T\Theta_j(\tilde{x})\tilde{z} > 0$ represents SOS multipliers to enforce the regional constraints.

Theoretically, the regional constraint terms are optional. However, the richness of these terms are helpful to render the synthesis condition feasible. Therefore, in order to render the polynomial conditions SOS decomposable, it is often needed to amend the conditions with regional constraints. The number and the form of the regional constraints could vary from case to case. If the augmented system has m states, the most commonly used regional constraints are in the form of

$$\psi_j(\tilde{x}) = c_j - \tilde{x}_j^q > 0, \quad j \in \mathbf{I}[1, m]$$

in which c_j and q are pre-defined positive constants. Consequently, the local stabilization region is specified by the intersection of constraints $\psi_j(\tilde{x}) > 0, j \in \mathbf{I}[1, m]$.

Remark 2 *Rewriting $A_{(r)}(\tilde{x})$ as $A_{(r)}(\tilde{x}) + \mathcal{N}_A(\tilde{x})$, with a polynomial annihilator $\mathcal{N}_A(\tilde{x})$ satisfying $\mathcal{N}_A(\tilde{x})\tilde{x} = 0$, a better optimization result with certain performance requirements might be obtained by solving for a suitable $\mathcal{N}_A(\tilde{x})$ iteratively. In the first part of each iteration, the optimal result is obtained using fixed $\mathcal{N}_A(\tilde{x})$ from the previous cycle. Then, the results are substituted into the conditions to solve the same optimization problem for a new $\mathcal{N}_A(\tilde{x})$. Using $f(\mathcal{N}_A(\tilde{x}), Q)$ to represent a condition, this iterative process can be shown in the flowchart in Fig. 3.1.*

Remark 3 *Let n be the total number of decision variables and m denote the number of rows of the LMIs. Using different Semi-Definite Programming (SDP) solver, the computational complexity will be different. The algorithm in MATLAB LMI Control Toolbox has a complexity of $O(n^3m)$, while the computational complexity of the solver SEDUMI used by the MATLAB toolbox SOSTOOLS is in $O(n^2m^{2.5} + m^{3.5})$[76].*

It is well known that the state space representation of a dynamic system is not unique. Since the given or pre-selected form of a known system matrix $A_{(r)}(\tilde{x})$ is just one among many possible representations, it is likely that there exists a $\mathcal{N}_A(\tilde{x})$ such that, $A_{(r)}(\tilde{x}) + \mathcal{N}_A(\tilde{x})$ is a proper one to achieve better optimization result when involving certain performance requirements. By involving entries of $\mathcal{N}_A(\tilde{x})$ as new variables, however, it will lead to bilinear matrix inequalities

multiplying two undetermined matrices, like $[A_{(r)}(\tilde{x}) + \mathcal{N}_A(\tilde{x})]Q + Q[A_{(r)}(\tilde{x}) + \mathcal{N}_A(\tilde{x})]^T$. The iteration process described in Remark 2 can be utilized to resolve this issue. Take $\mathcal{L}_2$ gain optimization problems for example, during each iteration, we are trying to minimize a γ value, with one of $\mathcal{N}_A(\tilde{x})$ and Q as fixed, and the other as unknown, alternatively. The iteration process will provide monotonic decreasing γ values until it converges.

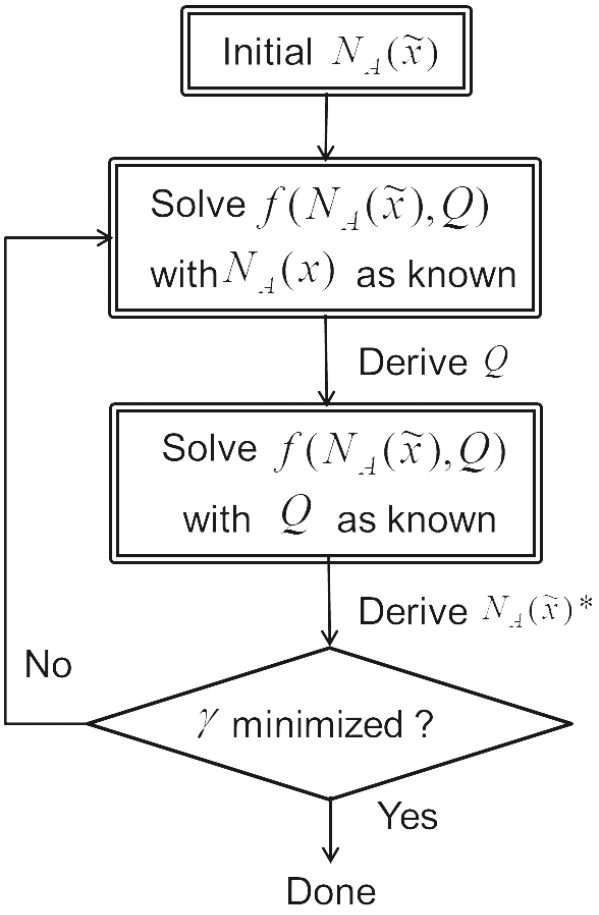


Figure 3.1: Flowchart of the iterative process to derive the system matrices and Lyapunov functions.

3.5 Example

In this example, we will design a nonlinear controller to optimize the $\mathcal{L}_2$ gain of a polynomial nonlinear system

$$\dot{x} = \begin{bmatrix} x_1 & 1+\frac{1}{5}x_2 \\ x_1^2 - 1 & -x_2 - x_1 x_2 \end{bmatrix} x + \begin{bmatrix} 0.2 & 0.25 \\ 0.25 & 0.2 \end{bmatrix} d + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{3.20}$$

$$e = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix} u. \tag{3.21}$$

We are going to compute and compare the minimal γ value derived by using quadratic and quartic Lyapunov functions. The augmented system can be derived using the form of (3.2) and $\tilde{x} = x^{(2)}$. As described in Remark 2, $A_{(2)}(\tilde{x})$ can be rewritten as $A_{(2)}(\tilde{x}) + \mathcal{N}_A(\tilde{x})$ with

$$\mathcal{N}_A(\tilde{x}) = \begin{bmatrix} \sum_{i=1}^{5} \alpha_i \tilde{x}_i^2 & 0 & \begin{pmatrix} \alpha_1 \tilde{x}_1 + \\ \alpha_3 \tilde{x}_1 \tilde{x}_3 \end{pmatrix} & \alpha_4 \tilde{x}_1 \tilde{x}_4 & \begin{pmatrix} \alpha_2 \tilde{x}_1 + \\ \alpha_5 \tilde{x}_1 \tilde{x}_5 \end{pmatrix} \\ 0 & -\sum_{i=1}^{5} \alpha_{i+5} \tilde{x}_i^2 & \begin{pmatrix} \alpha_6 \tilde{x}_2 + \\ \alpha_8 \tilde{x}_2 \tilde{x}_3 \end{pmatrix} & \alpha_9 \tilde{x}_2 \tilde{x}_4 & \begin{pmatrix} \alpha_7 \tilde{x}_2 + \\ \alpha_{10} \tilde{x}_2 \tilde{x}_5 \end{pmatrix} \\ \begin{pmatrix} -\alpha_{11} \tilde{x}_1 - \\ \alpha_{12} \tilde{x}_1 \tilde{x}_3 \end{pmatrix} & 0 & \begin{pmatrix} \alpha_{11} + \alpha_{12} \tilde{x}_1^2 \\ +\alpha_{13} \tilde{x}_2^2 \end{pmatrix} & 0 & -\alpha_{13} \tilde{x}_1^2 \\ -\alpha_{16} \tilde{x}_2 \tilde{x}_5 & -\alpha_{14} \tilde{x}_1 & -\alpha_{15} \tilde{x}_1 \tilde{x}_2 & \begin{pmatrix} \alpha_{14} + \alpha_{15} \tilde{x}_1^2 \\ +\alpha_{16} \tilde{x}_2^2 \end{pmatrix} & 0 \\ -\alpha_{18} \tilde{x}_1 \tilde{x}_5 & \begin{pmatrix} -\alpha_{17} \tilde{x}_2 - \\ \alpha_{19} \tilde{x}_2 \tilde{x}_5 \end{pmatrix} & 0 & 0 & \begin{pmatrix} \alpha_{17} + \alpha_{18} \tilde{x}_1^2 \\ +\alpha_{19} \tilde{x}_2^2 \end{pmatrix} \end{bmatrix}$$

such that $\mathcal{N}_A(\tilde{x})\tilde{x} = 0$. Applying Theorem 3, a $\mathcal{H}_\infty$ controller can be designed by condition (3.17) to minimize the $\mathcal{H}_\infty$ index $\gamma > 0$. By solving the problem iteratively as described in Remark 2,

the results of each optimization loop is listed in Table 3.2. During each odd step, $\mathcal{N}_A(\tilde{x})$ is fixed. Alternatively, Q is fixed in even steps. From the table we can clearly see that, the optimized γ value in each iteration decreases and the improvement is becoming smaller. If we proceed enough iterations, the γ values will converge to the minimum value. For comparison, $\gamma_2 = 2.1362$ with 2nd-order Lyapunov function; while using 4th-order Lyapunov function, $\gamma_4 = 1.7173$. The corresponding control laws are

$$
\begin{aligned}
u_2 &= -42.4962x_1 + 21.2374x_2 + 0.0018x_1^2 + 0.0052x_1x_2 - 0.0071x_2^2 - 0.0043x_2^3, \\
u_4 &= -107.3641x_1 - 65.0999x_1^2 - 61.4079x_1x_2 - 33.4939x_1^3 - 8.6229x_1x_2^2 + 52.5632x_2 \\
&\quad + 32.3301x_2^2 - 12.9297x_1^2x_2 - 23.8567x_2^3.
\end{aligned}
$$

With the disturbance as shown in Fig. 3.2, we can see the trajectories of the control inputs using 4^{th} and 2^{th} order Lyapunov functions in Fig. 3.3. The simulation results of the nonlinear system using the above control laws are provided in Fig. 3.4. The comparison clearly shows the advantage of u_4 for disturbance attenuation. On the other hand, using the method in [68], we failed to find feasible solutions based on $Z(x) = [x_1 \;\; x_2 \;\; x_1^2]^T$, $[x_1 \;\; x_2 \;\; x_1x_2]^T$, $[x_1 \;\; x_2 \;\; x_2^2]^T$ or $[x_1 \;\; x_2 \;\; x_1^2 \;\; x_1x_2 \;\; x_2^2]^T$. By multiplying the derivative matrices of these $Z(x)$'s with any form of $A(x)$ and $B(x)$, it can be verified that some of $\tilde{z}_i^2, i > 2$ are missing in condition (3.17). Lacking these critical even power terms, the SOS condition will be infeasible. These issues also arise when the approach of [42, 43] is considered. Moreover, references [25, 24] introduced $Z(x)$ of the same dimension with x such that $Z(x) = W(x)x$, in which $W(x)$ is a pre-specified square polynomial matrix as

$$
W(x) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ w_{11}(x) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1}(x) & w_{n2}(x) & \dots & 1 \end{bmatrix}.
$$

Table 3.2: Optimization results.

Iteration #	1	2	3	4	5	6	7	8
γ value	2.4918	2.0566	1.8686	1.8261	1.7741	1.7607	1.7252	1.7173

It is not difficult to verify that this lower triangular $W(x)$ is invertible and $\det(W(x)) = 1$. Regardless of the pre-selection process of $W(x)$, one obvious drawback of this method is that, it cannot provide all monomials of augmented state variables. For this system with two states, the $W(x)$ simply cannot generate the x_2^2 term, which is crucial to render the synthesis condition feasible.

It is worthy to mention that, for some nonlinear systems, the γ_4 value derived in the first iteration might be even larger than γ_2. However, the iterative process can decrease γ dramatically. This example clearly indicate that a suitable representation of a system matrix may lead to a better synthesis result.

3.6 Summary

In this chapter, we developed a novel nonlinear control design technique by combining power transformation and SOS programming to solve disturbance attenuation synthesis problems for polynomial nonlinear systems using higher order Lyapunov functions. Power transformation provides a systematic procedure to augment polynomial nonlinear systems to their higher order equivalence and facilitate the formulation of nonlinear control synthesis conditions in a convex form. Consequently, the resulting state-dependent matrix inequalities can be solved effectively by SOS programming. Furthermore, taking the advantage of the flexible form of the augmented system matrix $A_{(r)}(\tilde{x}) + \mathcal{N}_A(\tilde{x})$, The $\mathcal{H}_\infty$ performance can be improved iteratively.

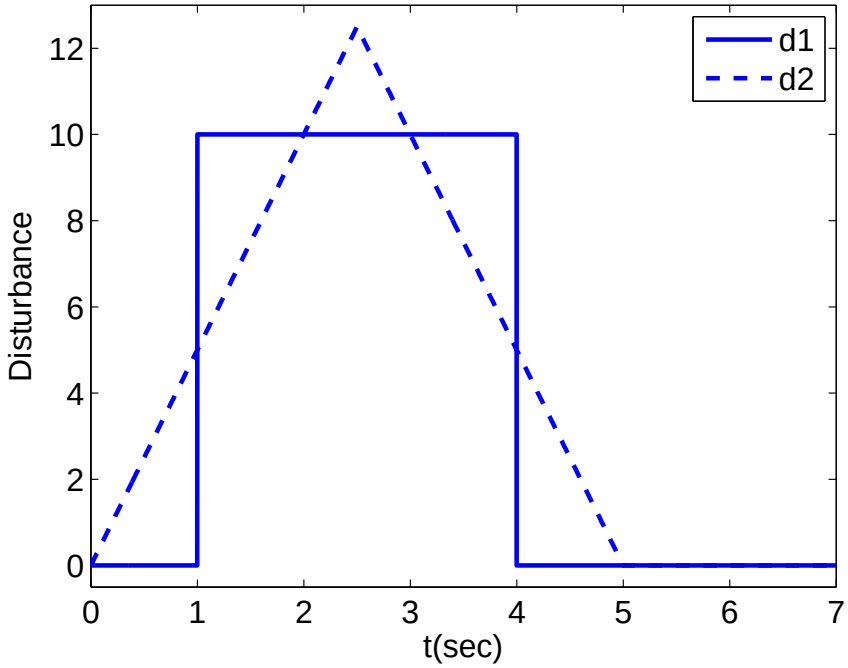


Figure 3.2: Disturbance.

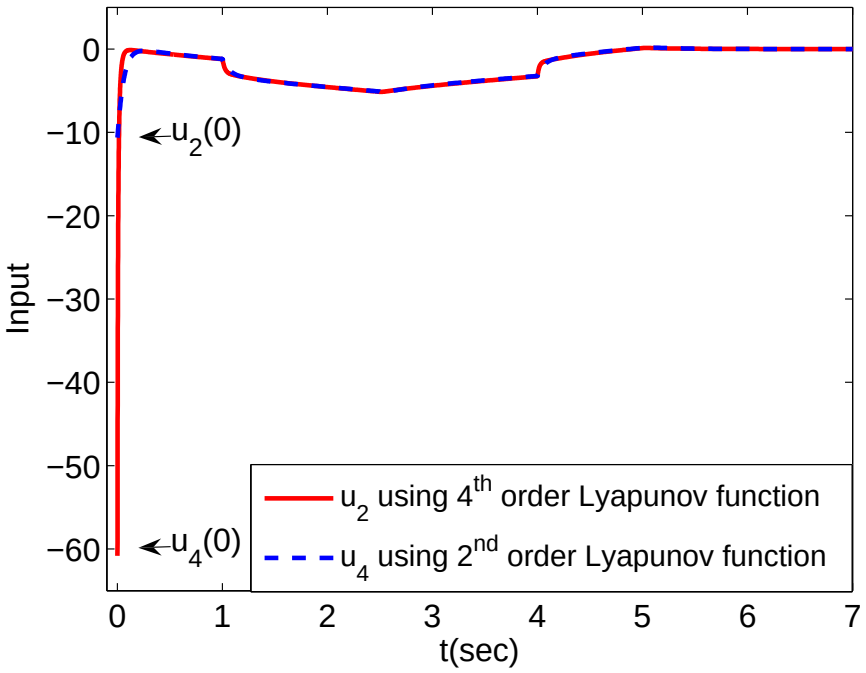
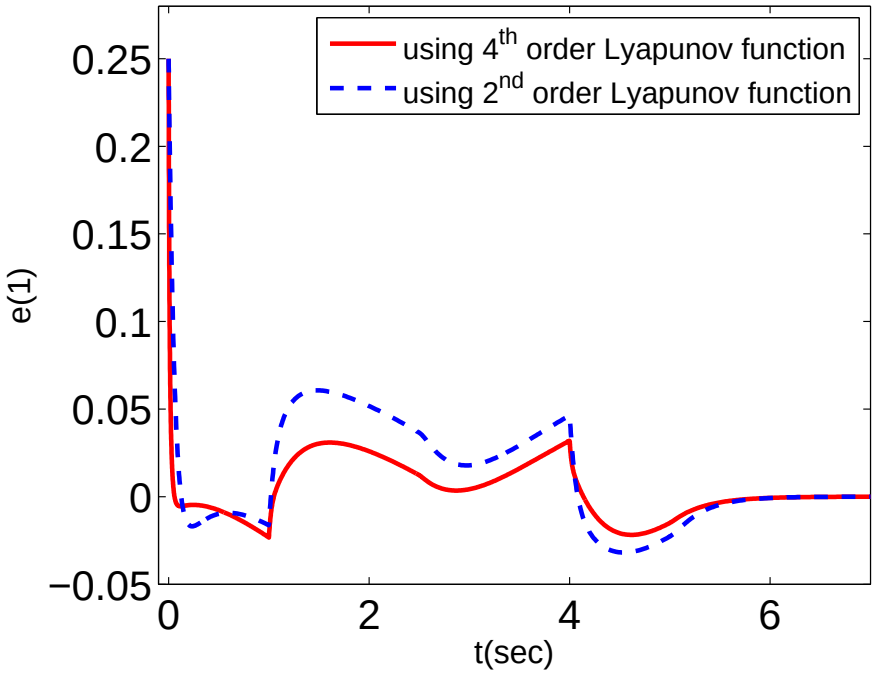
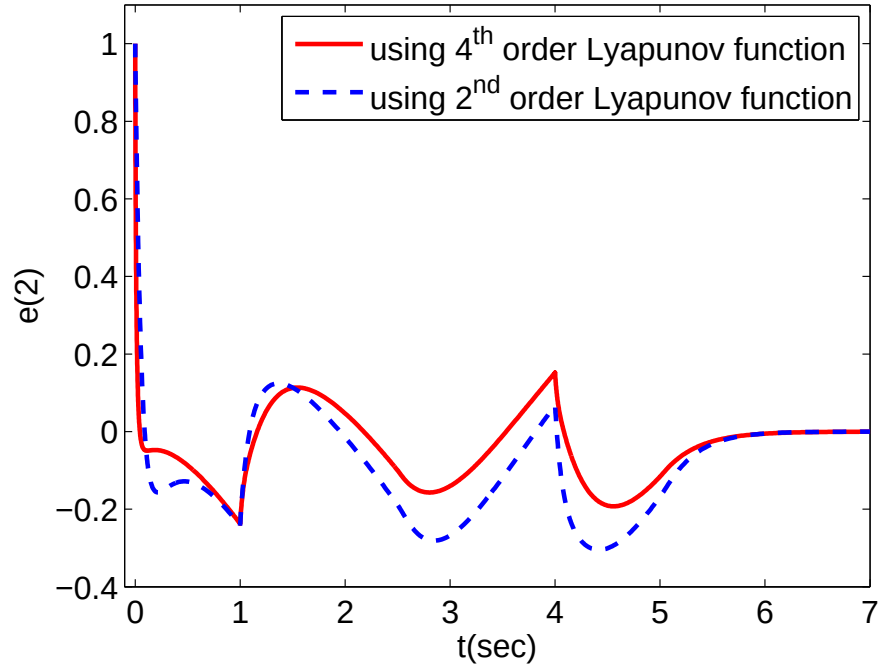


Figure 3.3: Comparison of inputs.



(a) Output e_1



(b) Output e_1

Figure 3.4: Comparison of output trajectories.

Chapter 4

Nonlinear State Feedback Control for Saturated Polynomial Nonlinear Systems

The objective of this chapter is to use higher order Lyapunov functions to design state feedback controllers for saturated polynomial nonlinear systems to achieve region of attraction (ROA) enlargement and performance improvements. By combining power transformation with Sum-of-Squares (SOS) techniques, Chapter 3 augmented unsaturated systems with more state variables representing higher order combinations of the original ones. Then, the search of higher order Lyapunov functions for original systems became the design of quadratic Lyapunov functions for augmented systems. In this chapter, the same idea will be applied to saturated systems. By computing for higher order Lyapunov functions using norm-bounded differential inclusion (NDI) LMI conditions, the flexible representations of augmented systems can help us to achieve better performance than quadratic based method. Two performance oriented optimization problems will be addressed on saturated nonlinear polynomial systems, the ROA expansion problem and the $\mathcal{H}_\infty$ disturbance attenuation problem.

This chapter is organized as follows: In Section 4.1, we will summarize the existing methods

that can be used to solve control design problems for nonlinear systems subjected to saturation nonlinearity. Concepts of dead-zone functions and NDI will be introduced in Section 4.2. Section 4.3 proves the existence of a higher order Lyapunov function for an asymptotically stable polynomial nonlinear system with saturation. Based on the NDI technique, stabilization conditions for saturated nonlinear polynomial systems and the following ROA expansion will be discussed in Section 4.4. Section 4.5 is devoted to the $\mathcal{H}_\infty$ saturation control aspect of saturated polynomial nonlinear systems. The advantage of the proposed control methods will be illustrated in the examples in Section 4.6. Section 4.7 will conclude this entire chapter.

4.1 Introduction

As one of the most common nonlinearities in engineering control systems, saturation is a widely encountered phenomenon that has received extensively increasing attention by researchers. In this area, a wide range of topics have been discussed in great depth, including stabilization (e.g., [51, 56, 74]), output regulation (e.g., [22, 91, 23]) and disturbance rejection (e.g., [37, 39, 91, 23]). Recently, a focus to exponentially unstable linear open-loop systems has been witnessed in literature (see, e.g. [79, 36, 90]).

Comparing with the tremendous number of these works dealing with linear systems, there are a very limited number of papers that have been witnessed discussing about analysis and synthesis problems of nonlinear systems subject to saturation nonlinearity. The fact that tools developed for linear systems can not be applied directly to the nonlinear systems seriously restricts the research on saturation control of nonlinear systems. Reference [55] obtained a 'universal' construction of Artstein's theorem, proposing a control formula for smooth nonlinear systems when the control input is bounded. Nevertheless, the construction of such a stabilizing control law relies on a priori information of control Lyapunov functions, which may not always available. In [34], an optimality-based nonlinear control was proposed for nonlinear systems with sector-bounded input nonlinearities by solving the HJB equation for unconstrained nonlinear systems. Then, a family of globally stabilizing controllers parameterized by the cost

functional minimization is constructed. Recently, a special class of nonlinear systems involving linear saturated systems affected by sector-bounded nonlinearities has been studied in [15]. The saturated nonlinear systems were stabilized regionally using nonlinearity-dependent saturation control laws. Nevertheless, most of research effort has been focused on the development and construction of quadratic Lyapunov functions in nonlinear saturation control problems. It is conceivable that higher order Lyapunov functions will be helpful in improving control design of polynomial nonlinear systems.

4.2 Norm-Bounded Differential Inclusion (NDI)

4.2.1 NDI of Linear Systems

Consider the open-loop linear system subject to saturation nonlinearity and disturbance:

$$\begin{cases} \dot{x} &= A_p x + B_{pu} sat(u) + B_{pd} d \\ u &= C_{pu} x + D_{p,uu} sat(u) + D_{p,ud} d \\ e &= C_{pe} x + D_{p,eu} sat(u) + D_{p,ed} d \end{cases} \tag{4.1}$$

where $x \in \mathbf{R}^n$ is the plant state, $u \in \mathbf{R}^{n_u}$ is the unsaturated control input and $d \in \mathbf{R}^{n_d}$ is the disturbance. In the above system, $\mathrm{sat}(\cdot)$ is a vector saturation function with the saturation levels given by a vector $\bar{u} \in \mathbf{R}^{n_u}$, and $\mathrm{sat}(u) = [\ \mathrm{sat}(u_1)\ \ \mathrm{sat}(u_2)\ \ \ldots\ \ \mathrm{sat}(u_{n_u})\ ]^T$, where

$$\mathrm{sat}(u_i) = \begin{cases} \bar{u}_i & u_i \geq \bar{u}_i \\ u_i & u_i \in [-\bar{u}_i, \bar{u}_i] \\ -\bar{u}_i & u_i \leq -\bar{u}_i. \end{cases} \quad , \quad \bar{u}_i > 0, i \in \mathbf{I}[1, n_u].$$

In this thesis we consider symmetric saturation functions. In order to derive tractable synthesis conditions, we need to introduce the deadzone functions and rewrite the system based on deadzone functions rather than saturated inputs. The deadzone nonlinearity is closely related to the saturation function and is given by $\mathrm{dz}(u) := u - \mathrm{sat}(u)$ for all $u \in \mathbf{R}^m$. Then, the

saturated linear systems can be represented as the following form using deadzone functions [39]

$$\begin{cases} \dot{x} &= A_0 x + B_0 q + B_1 d \\ u &= C_0 x + D_{00} q + D_{01} d \\ e &= C_1 x + D_{10} q + D_{11} d \\ q &= \mathrm{dz}(u) \end{cases}. \tag{4.2}$$

This general form has been used to study antiwindup systems in [52, 61, 93, 32]. Also, let $\mathcal{V}$ be the set of $n_u \times n_u$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^{n_u} elements in $\mathcal{V}$. Suppose these elements of $\mathcal{V}$ are labeled as $\Delta_j,\ j \in \mathbf{I}[1, 2^{n_u}]$, where for $j = z_1 2^{n_u-1} + z_2 2^{n_u-2} + \cdots + z_{n_u} + 1$ with $z_i \in \{0, 1\}$, the diagonal elements of Δ_j are $\{1-z_1, 1-z_2, \cdots, 1-z_{n_u}\}$. Denote $\Delta_j^- = I - \Delta_j$. Clearly, $\Delta_j^- \in \mathcal{V}$ if $\Delta_j \in \mathcal{V}$. In global analysis of saturated linear systems, we often use

$$\mathrm{dz}(u) \in \mathrm{sect}[0, I]u,$$

where sect$(\cdot)$ means the nonlinear sector. This relation holds for all $u \in \mathbf{R}^{n_u}$ but could be conservative over a local region where the system operates. On the other hand, the deadzone function could be sharpened for regional analysis. To this end, a norm-bounded differential inclusion (NDI) can be used to cover the original system.

Lemma 2 ([39]) *Let M be a positive diagonal matrix. Suppose*

$$2I - M^{-1} D_{00} M - M D_{00}^T M^{-1} = S^2$$

where S is nonsingular and symmetric. Then,

$$\begin{aligned} &Co\{(I - \Delta_j D_{00})^{-1} \Delta_j, j \in \mathbf{I}[1, 2^{n_u}]\} \\ &\subseteq \{M(S^{-2} + S^{-1}\Omega S^{-1}) M^{-1} : \|\Omega\| \le 1\} \end{aligned}$$

where $\|\Omega\|$ *is the largest singular value of* Ω*. Furthermore, each vertex of the left-hand side is on the boundary of the righthand side.*

The proof of above lemma can be found in [40]. By applying Lemma 2, it has been shown in [39] that the linear system (4.1) can be covered by a NDI with a matrix $H \in \mathbf{R}^{n_u \times n}$ such that

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} \in \mathrm{Co} \left\{ \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} : \|\Omega\| \leq 1 \right\}, \tag{4.3}$$

in which $\Omega \in \mathbf{R}^{n_u \times n_u}$ and

$$\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} := \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_0 \\ D_{10} \end{bmatrix} \times M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} \begin{bmatrix} C_0 - H & D_{01} \end{bmatrix}.$$

We are mainly concerned about the existence of a matrix H.

4.2.2 NDI of Nonlinear Systems

After introducing NDI of linear systems subjected saturation nonlinearity and disturbance, it is natural to expand the application of NDI techniques to represent saturated polynomial nonlinear systems.

Consider the affine-input, nth-order polynomial nonlinear system subject to actuator saturation

$$\begin{cases} \dot{x} &= A(x)x + B_1(x)d + B_2(x)sat(u) \\ u &= C_u(x)x + D_{01}(x)d + D_u sat(u) \\ e &= C_e(x)x + D_{11}(x)d + D_e sat(u) \end{cases} \tag{4.4}$$

where the state $x \in \mathbf{R}^n$, the disturbance $d \in \mathbf{R}^{n_d}$, and the control input $u \in \mathbf{R}^{n_u}$. We assume that all states are measurable and the state-space entries are polynomial functions of the state x. Then, (4.4) can be represented as the following form similarly to (4.2) using deadzone functions

[39]

$$\begin{cases} \dot{x} &= A_0(x)x + B_0(x)q + B_1(x)d \\ u &= C_0(x)x + D_{00}(x)q + D_{01}(x)d \\ e &= C_1(x)x + D_{10}(x)q + D_{11}(x)d \\ q &= \mathrm{dz}(u) \end{cases}. \tag{4.5}$$

Applying Lemma 2, the NDI description of (4.4) can be written as

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} \in \left\{ \begin{bmatrix} A_\Omega(x) & B_\Omega(x) \\ C_\Omega(x) & D_\Omega(x) \end{bmatrix} \begin{bmatrix} \dot{x} \\ d \end{bmatrix} : \|\Omega\| \le 1 \right\}, \tag{4.6}$$

$$\text{where} \quad \begin{bmatrix} A_\Omega(x) & B_\Omega(x) \\ C_\Omega(x) & D_\Omega(x) \end{bmatrix} := \begin{bmatrix} A_0(x) & B_1(x) \\ C_1(x) & D_{11}(x) \end{bmatrix} + \begin{bmatrix} B_0(x) \\ D_{10}(x) \end{bmatrix} M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} \begin{bmatrix} C_1(x) - H & 0 \end{bmatrix}.$$

By doing this, the saturation function or the deadzone function can be included in a sector so that the system (4.5) can be cast into the general framework of absolute stability.

In this research, we would like to synthesize a higher order Lyapunov function and a nonlinear state feedback controller to stabilize polynomial nonlinear systems and minimize its disturbance effect. To this end, the NDI-based saturation analysis conditions in [39] will be applied to nonlinear control synthesis. For disturbance attenuation, we are mainly concerned with a class of energy-bounded disturbances

$$\mathcal{W}_s = \left\{ d : \ \mathbf{R}_+ \to \mathbf{R}^{n_d}, \int_{\tau=0}^{\infty} d^T(\tau)d(\tau)d\tau \le s^2, \ d \in \mathcal{L}_2 \right\}.$$

The level of disturbance attenuation will be measured by the performance index

$$\sup_{d \in \mathcal{W}_s, x(0)=0} \frac{\|e\|_2}{\|d\|_2},$$

which specifies a regional $\mathcal{L}_2$ gain.

4.3 Existence of Higher Order Lyapunov Functions

In Section 3.3, we have proved that, for an asymptotically stable polynomial nonlinear system without saturation, there always exists higher order Lyapunov functions. In this section, we will extend that argument to saturated systems.

Lemma 3 *For a saturated polynomial nonlinear system expressed using deadzone function,*

$$\begin{cases} \dot{x} &= A(x)x + B_0(x)q \\ u &= C_1(x)x + D_{10}(x)q \ , \\ q &= dz(u) \end{cases} \tag{4.7}$$

$x \in \mathbf{R}^n$, *let us define* $A_{[r]}(x) \in \mathbf{R}^{N_n^r \times N_n^r}$ *and* $B_{0[r]}(x) \in \mathbf{R}^{N_n^r \times n_u}$ *as*

$$\frac{d}{dt}x^{[r]} = \frac{\partial x^{[r]}}{\partial x}(Ax + B_0 q) = A_{[r]}x^{[r]} + B_{0[r]}q, \tag{4.8}$$

where n_u *denotes the number of inputs.* $x^{\langle r \rangle} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{r}$ *and* $T_r \in \mathbf{R}^{n^r \times N_n^r}$ *be the matrix satisfying*

$$x^{\langle r \rangle} = T_r x^{[r]}. \tag{4.9}$$

Then $A_{[r]}$ *and* $B_{0[r]}$ *can be expressed using* A *and* B_0 *respectively as:*

$$A_{[r]} = (T_r^T T_r)^{-1} T_r^T \sum_{i=0}^{r-1} (I_{n^{r-1-i}} \otimes A \otimes I_{n^i}) T_r, \tag{4.10}$$

$$B_{0[r]} = (T_r^T T_r)^{-1} T_r^T \sum_{i=0}^{r-1} (x^{\langle r-1-i \rangle} \otimes B_0 \otimes x^{\langle i \rangle}). \tag{4.11}$$

Proof: Since (4.10) and (3.4) are identical, so the proof of (4.10) can be found in the proof of Lemma 1.

Now we define $B_{0\langle r\rangle} \in \mathbf{R}^{n^r \times n_u}$ as

$$\frac{\partial x^{\langle r\rangle}}{\partial x} B_0 = B_{0\langle r\rangle}. \tag{4.12}$$

Using the properties of Knonecker product, it can be shown that

$$\frac{\partial x^{\langle r\rangle}}{\partial x} = \sum_{i=0}^{r-1} x^{\langle r-1-i\rangle} \otimes I_n \otimes x^{\langle i\rangle} \quad \text{and}$$
$$(x^{\langle r-1-i\rangle} \otimes I_n \otimes x^{\langle i\rangle}) B_0 = (x^{\langle r-1-i\rangle} \otimes B_0 \otimes x^{\langle i\rangle}).$$

Then we have

$$B_{0\langle r\rangle} = \sum_{i=0}^{r-1} x^{\langle r-1-i\rangle} \otimes B_0 \otimes x^{<i>}. \tag{4.13}$$

From (4.9) and (4.12), we can derive

$$T_r \frac{\partial x^{[r]}}{\partial x} (B_0 q) = B_{0\langle r\rangle} q. \tag{4.14}$$

We can finally derive (4.11) by following (4.8) and (4.13), and multiply $(T_r^T T_r)^{-1} T_r^T$ on both sides of (4.14). *Q.E.D.*

Theorem 4 *For a saturated polynomial nonlinear system (4.6) without disturbance, if there exists a quadratic Lyapunov function $x^T P x > 0$, $x^T (P A_\Omega + A_\Omega^T P) x < 0$, $\|\Omega\| \le 1$ to render the asymptotical stability, the system also has a higher order Lyapunov function $x^{(r)T} P_{(r)} x^{(r)} > 0$, such that for any $\mathcal{N}_A$ satisfying $\mathcal{N}_A x^{(r)} = 0$ with compatible size,*

$$x^{(r)T} \left(P_{(r)} (A_{\Omega(r)} + \mathcal{N}_A) + (A_{\Omega(r)} + \mathcal{N}_A)^T P_{(r)} \right) x^{(r)} < 0, \ r \ge 2, \tag{4.15}$$

Proof: By augmenting the state vector x as $\tilde{x} = x^{(2)} = \begin{bmatrix} x \\ x^{[2]} \end{bmatrix}$, we can write the augmented system without disturbance as

$$\dot{\tilde{x}} = A_{(2)}(\tilde{x})\tilde{x} + B_{0(2)}(\tilde{x})q = \begin{bmatrix} A(x) & \mathbf{0} \\ \mathbf{0} & A_{[2]}(x) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} + \begin{bmatrix} B_0(\tilde{x}) \\ B_{0[2]}(\tilde{x}) \end{bmatrix} q. \tag{4.16}$$

The existence of quadratic Lyapunov function $x^T P x > 0$ indicates that,

$$\begin{aligned} x^T \left[A_\Omega(\tilde{x})^T P + P A_\Omega(\tilde{x}) \right] x = x^T \Big[\big(A(\tilde{x}) + B_0 M (S^{-2} + S^{-1}\Omega S^{-1}) M^{-1} (C_1(\tilde{x}) - H) \big)^T P \\ + P \big(A(\tilde{x}) + B_0 M (S^{-2} + S^{-1}\Omega S^{-1}) M^{-1} (C_1(\tilde{x}) - H) \big) \Big] x < 0. \end{aligned} \tag{4.17}$$

Pre-multiplying (4.17) by $x^T P x$ and proceeding Kronecker product operations, one can derive

$$\begin{aligned} & [x^T P x] \cdot [x^T (P A_\Omega + A_\Omega^T P) x] = [x^T P x] \otimes [x^T (P A_\Omega + A_\Omega^T P) x] < 0 \\ \Rightarrow & (x \otimes x)^T \left[(P \otimes P)(I_n \otimes A) + (I_n \otimes A)^T (P \otimes P) \right] (x \otimes x) + \\ & (x \otimes x)^T \left[(P \otimes P)(x \otimes B_0) \left(M \bar{S} M^{-1} (C_1 - H) \right) \right] x + \\ & x^T \Big[\big(M \bar{S} M^{-1} (C_1 - H) \big)^T (x \otimes B_0)^T (P \otimes P) \Big] (x \otimes x) < 0, \end{aligned} \tag{4.18}$$

in which $\bar{S} = S^{-2} + S^{-1}\Omega S^{-1}$. Since there always exists $C_{1\langle 2 \rangle}$ such that, $C_{1\langle 2 \rangle}(x \otimes x) = C_1 x$, by selecting $H_{\langle 2 \rangle}$ with suitable dimension, (4.18) can be written as

$$\begin{aligned} & (x \otimes x)^T \left[(P \otimes P)(I_n \otimes A) + (I_n \otimes A)^T (P \otimes P) \right] (x \otimes x) + \\ & (x \otimes x)^T \left[(P \otimes P)(x \otimes B_0) \left(M \bar{S} M^{-1} \right) \left(C_{1\langle 2 \rangle} - H_{\langle 2 \rangle} \right) \right] (x \otimes x) + \\ & (x \otimes x)^T \Big[\big(C_{1\langle 2 \rangle} - H_{\langle 2 \rangle} \big)^T \left(M \bar{S} M^{-1} \right)^T (x \otimes B_0)^T (P \otimes P) \Big] (x \otimes x) < 0. \end{aligned} \tag{4.19}$$

$[x^T(PA_\Omega + A_\Omega^T P)x] \cdot [x^T P x] < 0$, can be proceeded in the same way and derive the following expression:

$$\begin{aligned}
&(x \otimes x)^T \left[(P \otimes P)(A \otimes I_n) + (A \otimes I_n)^T (P \otimes P)\right](x \otimes x) + \\
&(x \otimes x)^T \left[(P \otimes P)(B_0 \otimes x)\left(M\bar{S}M^{-1}\right)\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right)\right](x \otimes x) + \\
&(x \otimes x)^T \left[\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right)^T \left(M\bar{S}M^{-1}\right)^T (B_0 \otimes x)^T (P \otimes P)\right](x \otimes x) < 0. \qquad (4.20)
\end{aligned}$$

Then, the summation of (4.19) and (4.20) can be obtained:

$$\begin{aligned}
&x^{\langle 2\rangle T}\left[(P \otimes P)\sum_{i=0}^{1}(I_{n^{1-i}} \otimes A \otimes I_{n^i}) + \sum_{i=0}^{1}(I_{n^{1-i}} \otimes A \otimes I_{n^i})^T (P \otimes P)\right] x^{\langle 2\rangle} + \\
&x^{\langle 2\rangle T}\left[(P \otimes P)\sum_{i=0}^{1}(x^{\langle 1-i\rangle} \otimes B_0 \otimes x^{\langle i\rangle})\left(M\bar{S}M^{-1}\right)\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right) + \right. \\
&\left.\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right)^T \left(M\bar{S}M^{-1}\right)^T \sum_{i=0}^{1}(x^{\langle 1-i\rangle} \otimes B_0 \otimes x^{\langle i\rangle})^T (P \otimes P)\right] x^{\langle 2\rangle} < 0. \qquad (4.21)
\end{aligned}$$

Involving $A_{\langle r\rangle}$ and $B_{0\langle r\rangle}$ with $r = 2$ by following (3.6) and (4.13) respectively, (4.21) can be written as

$$\begin{aligned}
&x^{\langle 2\rangle T}\left[(P \otimes P)\left(A_{\langle 2\rangle} + B_{0\langle 2\rangle}\left(M\bar{S}M^{-1}\right)\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right) + \right.\right. \\
&\left.\left. + \left(A_{\langle 2\rangle} + B_{0\langle 2\rangle}\left(M\bar{S}M^{-1}\right)\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right)\right)^T (P \otimes P)\right] x^{\langle 2\rangle} < 0 \\
\Rightarrow &x^{\langle 2\rangle T}\left[(P \otimes P)A_{\Omega\langle 2\rangle} + A_{\Omega\langle 2\rangle}^T (P \otimes P)\right] x^{\langle 2\rangle} < 0, \qquad (4.22)
\end{aligned}$$

in which $A_{\Omega\langle 2\rangle} = A_{\langle 2\rangle} + B_{0\langle 2\rangle}\left(M\bar{S}M^{-1}\right)\left(C_{1\langle 2\rangle} - H_{\langle 2\rangle}\right)$.

Moreover, substituting (4.9), (4.10) and (4.11) into (4.22) with $r = 2$, a homogenous Lyapunov function based on augmented state vector $x^{[2]}$ following power transformation can be achieved

as

$$x^{[2]T}\left[P_{[2]}\left(A_{[2]}+B_{0[2]}\left(M\bar{S}M^{-1}\right)\left(C_{1[2]}-H_{[2]}\right)\right)+\right.$$
$$\left.\left(A_{[2]}+B_{0[2]}\left(M\bar{S}M^{-1}\right)\left(C_{1[2]}-H_{[2]}\right)\right)^{T}P_{[2]}\right]x^{[2]}<0,$$

in which
$$P_{[2]}=T_2^T(P\otimes P)(T_2T_2^T)^{-1}T_2(T_2^TT_2)=T_2^T(P\otimes P)T_2>0,$$
$$A_{[2]}=(T_2^TT_2)^{-1}T_2^TA_{\langle 2\rangle}T_2=(T_2^TT_2)^{-1}T_2^T\sum_{i=0}^{1}(I_{n^{1-i}}\otimes A\otimes I_{n^i})T_2,$$
$$B_{0[2]}=(T_2^TT_2)^{-1}T_2^TB_{0\langle 2\rangle}=(T_2^TT_2)^{-1}T_2^T\sum_{i=0}^{1}(x^{\langle 1-i\rangle}\otimes B_0\otimes x^{\langle i\rangle}),$$
$$C_{1[2]}=C_{1\langle 2\rangle}T_2,\text{ and }H_{[2]}=H_{\langle 2\rangle}T_2.$$

Therefore, for system (4.16) involving both original and augmented states, a Lyapunov function $\tilde{x}^TP_{(2)}\tilde{x}>0$ can be constructed as following

$$x^{(2)T}[P_{(2)}A_{\Omega(2)}(x^{(2)})+A_{\Omega(2)}^T(x^{(2)})P_{(2)}]x^{(2)}<0,\text{ in which}\tag{4.23}$$
$$P_{(2)}=\begin{bmatrix}P & \mathbf{0}\\ \mathbf{0} & P_{[2]}\end{bmatrix}\text{ and }A_{\Omega(2)}=\tilde{A}_{(2)}(\tilde{x})+\tilde{B}_{0(2)}M\left(S^{-2}+S^{-1}\Omega S^{-1}\right)M^{-1}(C_{1(2)}(\tilde{x})-H_{(2)}),$$
$$\text{where }C_{1(2)}(\tilde{x})=\begin{bmatrix}C_1 & C_{1[2]}\end{bmatrix}\text{ and }H_{(2)}=\begin{bmatrix}H & H_{[2]}\end{bmatrix}.$$

(4.15) can be derived by adding $x^{(2)T}\left(P_{(2)}\mathcal{N}_A+\mathcal{N}_A^TP_{(2)}\right)x^{(2)}=0$ and (4.23) for the case of $r=2$. For $r>2$, similarly to the proof of Theorem 2, additional $(x^TPx)^i,(x^TPx)^j,i+j=r-1$ can be multiplied on the left and right sides of (4.20) and (4.20) respectively. *Q.E.D.*

4.4 Region of Attraction (ROA) Expansion of Saturated Polynomial Nonlinear Systems

Consider the affine-input, nth-order polynomial nonlinear system subject to actuator saturation

$$\begin{cases} \dot{x} &= A(x)x + B_1(x)d + B_2(x)\text{sat}(u) \\ e &= C(x)x + D_1(x)d + D_2(x)\text{sat}(u) \end{cases} \tag{4.24}$$

where the state $x \in \mathbf{R}^n$, the disturbance $d \in \mathbf{R}^{n_d}$, and the control input $u \in \mathbf{R}^{n_u}$. We assume that all states are measurable and the state-space entries are polynomial functions of the state x. The ROA of a saturated system can be estimated by the largest level set of its Lyapunov functions V to guarantee $V \leq 1$. Pursuing the largest ROA of a system is a very important optimization issue for a system subjected to actuator saturations to evaluate the performance of the closed-loop system. Since the level set of higher order Lyapunov functions may have different shape from ellipsoid, this advantage can help us to achieve ROA larger than those derived by using quadratic Lyapunov functions. Further more, the iterative process to derive the system matrices and Lyapunov functions explained by Remark 2 in Section 3.4 can help us to expand the level set more efficiently in pre-selected desired directions.

Like what has been done in Chapter 3 of augmenting system 3.1 to 3.2, the original saturated system 4.24 can be augmented as following:

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}(\tilde{x})\tilde{x} + \tilde{B}_1(\tilde{x})d + \tilde{B}_2(\tilde{x})\text{sat}(u) \\ e &= \tilde{C}(\tilde{x})\tilde{x} + \tilde{D}_1(\tilde{x})d + \tilde{D}_2(\tilde{x})\text{sat}(u) \end{cases} \tag{4.25}$$

with suitable dimensions. From now on, all of the theorems and conditions will be stated based on this augmented system. Similar to its linear counterpart, the augmented polynomial

nonlinear system can be written as

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}(\tilde{x})\tilde{x} - \tilde{B}_2(\tilde{x})q + \tilde{B}_1(\tilde{x})d + \tilde{B}_2(\tilde{x})u \\ u &= F(\tilde{x})\tilde{x} \\ e &= \tilde{C}(\tilde{x})\tilde{x} - \tilde{D}_2(\tilde{x})q + \tilde{D}_1(\tilde{x})d + \tilde{D}_2(\tilde{x})u \\ q &= \mathrm{dz}(u). \end{cases} \tag{4.26}$$

For space reason, the "$(\tilde{x})$" parts of the nonlinear matrices in (4.26) are often omitted. By substituting $u = F(\tilde{x})\tilde{x}$ into the $\dot{\tilde{x}}$ and e equations in the same system, the system (4.26) can be written as

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}_0(\tilde{x})\tilde{x} + \tilde{B}_0(\tilde{x})q + \tilde{B}_1(\tilde{x})d \\ u &= F(\tilde{x})\tilde{x} + \tilde{D}_{00}(\tilde{x})q + \tilde{D}_{01}(\tilde{x})d \\ e &= \tilde{C}_1(\tilde{x})\tilde{x} + \tilde{D}_{10}(\tilde{x})q + \tilde{D}_{11}(\tilde{x})d \\ q &= \mathrm{dz}(u) \end{cases} \tag{4.27}$$

with $\tilde{A}_0(\tilde{x}) = \tilde{A}(\tilde{x}) + \tilde{B}_2(\tilde{x})F(\tilde{x})$, $\tilde{B}_0(\tilde{x}) = -\tilde{B}_2(\tilde{x})$, $\tilde{C}_1(\tilde{x}) = \tilde{C}(\tilde{x}) + \tilde{D}_2(\tilde{x})F(\tilde{x})$, $\tilde{D}_{10}(\tilde{x}) = -\tilde{D}_2(\tilde{x})$, $\tilde{D}_{11}(\tilde{x}) = \tilde{D}_1(\tilde{x})$, and both $\tilde{D}_{00}(\tilde{x})$, $\tilde{D}_{01}(\tilde{x})$ equal to zero.

Given $Q \in \mathbf{S}_+^{\tilde{N}_n^r \times \tilde{N}_n^r}$, we define a Lyapunov function $V(\tilde{x}) = \tilde{x}^T Q^{-1} \tilde{x}$ and $\varepsilon(Q^{-1}) = \{\tilde{x} \in \mathbf{R}^{\tilde{N}_n^r} : \tilde{x}^T Q^{-1} \tilde{x} \leq 1\}$. The Lyapunov matrix Q and controller gain $F(\tilde{x})$ in (4.26) are to be determined but all other nonlinear matrices in (4.26) as well as the saturation level $\bar{u}$ of u are known.

Lemma 4 *Given X, Y, Z, S, of compatible dimensions, where S is symmetric and nonsingular. If*

$$He \begin{bmatrix} Z & X \\ Y & -S^2/2 \end{bmatrix} \leq 0, \tag{4.28}$$

then $He(Z + XS^{-1}(I + \Omega)S^{-1}Y) \leq 0 \ \ \forall \|\Omega\| \leq 1$.

This lemma follows directly using Schur complements and from $M\Omega N + N^T\Omega^T M^T \leq MM^T +$

$NN^T \leq 0 \ \ \forall \|\Omega\| \leq 1.$

Theorem 5 *For the augmented system (4.26) with actuator saturation, $\phi_i(\tilde{x}) = 0$ denotes the coupling constraints of augmented state variables. Given $s > 0$, if there exist $Y(\tilde{x}), K(\tilde{x}) \in \mathbf{R}^{n_u \times \tilde{N}_n^r}$ and a diagonal matrix $U(\tilde{x}) \in \mathbf{S}_+^{n_u \times n_u}$ satisfying*

$$\tilde{z}_1^T \left\{ \begin{bmatrix} He\left\{\tilde{A}Q + \tilde{B}_2 K\right\} & \star \\ K - Y - U\tilde{B}_2^T & -2U \end{bmatrix} \right\} \tilde{z}_1 + \sum_i \tilde{z}_1^T \Lambda_{1i}(\tilde{x}) \tilde{z}_1 \phi_i(\tilde{x}) < 0 \tag{4.29}$$

$$\tilde{z}_2^T \begin{bmatrix} \frac{\bar{u}_l^2}{s^2} & Y_l \\ Y_l^T & Q \end{bmatrix} \tilde{z}_2 + \sum_i \tilde{z}_2^T \Lambda_{2i}(\tilde{x}) \tilde{z}_2 \phi_i(\tilde{x}) > 0 \quad \forall l > 0, \tag{4.30}$$

where $\tilde{z}_1$ and $\tilde{z}_2$ are free variables with compatible dimensions. Then the controller gain $F(\tilde{x}) = K(\tilde{x})Q^{-1}$ ensures that $s\varepsilon(Q^{-1})$ is a contractively invariant ellipsoid in the augmented state space.

In Theorem 5, $\tilde{z}_j$ are auxiliary vectors and Λ_{ji} represent multipliers without any restrictions to enforce the coupling constraints $\phi_i(\tilde{x}) = 0$. For some nonlinear systems, it is not necessary to include all of the monomials to guarantee the solvability of synthesis conditions. The number of coupling constraints will thus depend on the form of monomial basis vector. The Remarks 1-3 wrote for Theorem 3 in Chapter 3 are also applicable to Theorem 5.

Proof: Defining $U = M^2$ and $\tilde{z}_1 = \text{diag}(Q^{-1}, M^{-1}) \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}$, in which $\tilde{v}$ is a free vector with compatible size, (4.29) can be written as

$$\begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}^T \text{diag}(Q^{-1}, M^{-1}) \left\{ \begin{bmatrix} \text{He}\left\{\tilde{A}Q + \tilde{B}_2 K\right\} & \star \\ K - Y - U\tilde{B}_2^T & -2M^2 \end{bmatrix} \right\} \text{diag}(Q^{-1}, M^{-1}) \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}$$
$$+ \sum_i \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}^T \bar{\Lambda}_{1i}(\tilde{x}) \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} \phi_i(\tilde{x}) < 0, \text{ in which} \tag{4.31}$$
$$\bar{\Lambda}_{1i}(\tilde{x}) = \text{diag}(Q^{-1}, M^{-1}) \cdot \Lambda_{1i}(\tilde{x}) \cdot \text{diag}(Q^{-1}, M^{-1}).$$

Noticing that $\mathrm{He}(-S^2/2) = \mathrm{He}(-I + M^{-1}\tilde{D}_{00}(\tilde{x})M)$, in which $\tilde{D}_{00}(\tilde{x})M = 0$, and $Y = HQ$, (4.31) is equivalent to

$$\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \mathrm{He}\begin{bmatrix} Q^{-1}\tilde{A}_0(\tilde{x}) & Q^{-1}\tilde{B}_0(\tilde{x})M \\ M^{-1}(F(\tilde{x})-H) & -S^2/2\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix} + \sum_i \begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \bar{\Lambda}_{1i}(\tilde{x})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}\phi_i(\tilde{x}) < 0 \qquad (4.32)$$

Utilizing Lemma 4, (4.32) can be following expressed as

$$\begin{aligned}
&\tilde{x}^T\Big[\Big(\tilde{A}_0(\tilde{x}) + \tilde{B}_0 M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1}(F(\tilde{x}) - H)\Big)^T Q^{-1} \\
&+Q^{-1}\Big(\tilde{A}_0(\tilde{x}) + \tilde{B}_0 M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1}(F(\tilde{x}) - H)\Big)\Big]\tilde{x} + \sum_i \begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \bar{\Lambda}_{1i}(\tilde{x})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}\phi_i(\tilde{x}) < 0 \\
\Rightarrow\ &\tilde{x}^T\Big(\tilde{A}_\Omega(\tilde{x})^T Q^{-1} + Q^{-1}\tilde{A}_\Omega(\tilde{x})\Big)\tilde{x} + \sum_i \begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \bar{\Lambda}_{1i}(\tilde{x})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}\phi_i(\tilde{x}) < 0 \qquad (4.33)
\end{aligned}$$

(4.33) makes a sufficient condition for $\tilde{x}^T\Big(\tilde{A}_\Omega(\tilde{x})^T Q^{-1} + Q^{-1}\tilde{A}_\Omega(\tilde{x})\Big)\tilde{x} < 0$. Defining $\tilde{z}_2 = \mathrm{diag}(1, Q^{-1})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}$, (4.30) can be expressed as

$$\begin{aligned}
&\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \mathrm{diag}(1, Q^{-1})\begin{bmatrix}\frac{\bar{u}_l^2}{s^2} & Y_l \\ Y_l^T & Q\end{bmatrix}\mathrm{diag}(1, Q^{-1})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix} + \sum_i \begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \bar{\Lambda}_{2i}(\tilde{x})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}\phi_i(\tilde{x}) > 0 \quad \forall l > 0, \\
\Rightarrow\ &\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \begin{bmatrix}\frac{\bar{u}_l^2}{s^2} & H_l \\ H_l^T & Q^{-1}\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix} + \sum_i \begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}^T \bar{\Lambda}_{2i}(\tilde{x})\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}\phi_i(\tilde{x}) > 0 \quad \forall l > 0, \qquad (4.34)
\end{aligned}$$

in which $\bar{\Lambda}_{2i}(\tilde{x}) = \mathrm{diag}(1, Q^{-1}) \cdot \Lambda_{2i}(\tilde{x}) \cdot \mathrm{diag}(1, Q^{-1})$, H_l is the l-th row of H and $\bar{u}_l$ is the l-th diagonal element of $\bar{U}$[37].

Since for a given $s > 0$, we have $s\varepsilon(P) \subseteq \mathcal{L}(\bar{U}^{-1}H) \Longleftrightarrow \begin{bmatrix}\frac{\bar{u}_l^2}{s^2} & H_l \\ H_l^T & P\end{bmatrix} \geq 0 \ \ \forall\ l$, (4.34) becomes

a sufficient condition to guarantee that, $|\bar{U}^{-1}Hx|_\infty \leq 1$ (i.e., $|h_l(x)| \leq \bar{u}_l$ for all l) for all $x \in s\varepsilon(P)$. *Q.E.D.*

Theorem 5 has been proved to be able to stabilize a saturated nonlinear system. In order to derive a Lyapunov function with a level set that can be maximized in pre-selected directions, the method in [38] is adopted to shape the Lyapunov functions. Combining with Theorem 5, this optimization problem can be formulated as

$$\begin{aligned} &\inf_{Q>0} \quad \gamma \\ &\text{s.t.} \begin{bmatrix} \gamma & v_i^T \\ v_i & Q \end{bmatrix} \geq 0, i = 1, 2, \dots \\ &\qquad \text{and } (4.29) - (4.30) \text{ are satisfied,} \end{aligned} \tag{4.35}$$

where $v_i \in \mathbf{R}^n$ are those vectors defining the desired directions. The detailed procedure of derivation can be found in [38].

4.5 State Feedback $\mathcal{H}_\infty$ Control of Saturated Polynomial Nonlinear Systems

The state feedback $\mathcal{H}_\infty$ control problem of polynomial nonlinear systems has been discussed in Chapter 3. In this section, the parallel study focusing on saturated systems will be addressed. For ease of reference, the deadzone version of the augmented system (4.25) is repeated as follows:

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}(\tilde{x})\tilde{x} - \tilde{B}_2(\tilde{x})q + \tilde{B}_1(\tilde{x})d + \tilde{B}_2(\tilde{x})u \\ u &= F(\tilde{x})\tilde{x} \\ e &= \tilde{C}(\tilde{x})\tilde{x} - \tilde{D}_2(\tilde{x})q + \tilde{D}_1(\tilde{x})d + \tilde{D}_2(\tilde{x})u \\ q &= \mathrm{dz}(u). \end{cases} \tag{4.36}$$

Similar to Theorem 5 for stabilization problems, we have the following theorem for $\mathcal{H}_\infty$

control problems:

Theorem 6 *For the augmented system (4.36) with actuator saturation, $\phi_i(\tilde{x}) = 0$ denotes the coupling constraints of augmented state variables. Given $\gamma, s > 0$, if there exist $Y(\tilde{x}), K(\tilde{x}) \in \mathbf{R}^{n_u \times \tilde{N}_n^r}$ and a diagonal matrix $U(\tilde{x}) \in \mathbf{S}_+^{n_u \times n_u}$ satisfying*

$$\tilde{z}_1 \left\{ \begin{bmatrix} He\left\{\tilde{A}(\tilde{x})Q + \tilde{B}_2(\tilde{x})K\right\} & \star & \star & \star \\ \tilde{B}_1(\tilde{x})^T & -I & \star & \star \\ \tilde{C}(\tilde{x})Q + \tilde{D}_2(\tilde{x})K & \tilde{D}_1(\tilde{x}) & -\gamma^2 I & \star \\ K - Y - U\tilde{B}_2(\tilde{x})^T & \boldsymbol{0} & -U\tilde{D}_2(\tilde{x})^T & -2U \end{bmatrix} \right\} \tilde{z}_1$$
$$+ \sum_i \tilde{z}_1^T \Lambda_{1i}(\tilde{x}) \tilde{z}_1 \phi_i(\tilde{x}) < 0, \tag{4.37}$$

$$\tilde{z}_2^T \begin{bmatrix} \frac{\bar{u}_l^2}{s^2} & Y_l \\ Y_l^T & Q \end{bmatrix} \tilde{z}_2 + \sum_i \tilde{z}_2^T \Lambda_{2i}(\tilde{x}) \tilde{z}_2 \phi_i(\tilde{x}) > 0 \quad \forall l \in I[1, m], \tag{4.38}$$

where $\tilde{z}_1$ and $\tilde{z}_2$ are free variables with compatible dimensions. Then the controller gain $F(\tilde{x}) = K(\tilde{x})Q^{-1}$ guarantees that $\dot{V}(\tilde{x}) + \frac{1}{\gamma^2} e^T e \leq d^T d$ for all $\tilde{x} \in s\varepsilon(Q^{-1}), d \in \mathbf{R}^r$. If $\tilde{x}(0) = 0$ and $\|d\|_2 \leq s$, then $\|e\|_2 \leq \gamma \|d\|_2$.

In Theorem 6, $\tilde{z}_j$ are auxiliary vectors and Λ_{ji} represent multipliers without any restrictions to enforce the coupling constraints $\phi_i(\tilde{x}) = 0$. Again, the Remarks 1-3 wrote for Theorem 3 in Chapter 3 are also applicable for Theorem 6. Since the condition (4.38) is exactly the same with (4.30), only the proof of (4.37) is needed. Similarly with the proof of Theorem 5, a new lemma is necessary to work together with Lemma 4 to help us with the proof.

Proof: Defining $U = M^2$, $\mathbf{D} = \text{diag}(Q^{-1}, I, I, M^{-1})$, $\bar{\mathbf{x}} = \begin{bmatrix} \tilde{x} & d & e & \tilde{v} \end{bmatrix}^T$ and $\tilde{z}_1 = \mathbf{D}\bar{\mathbf{x}}$, in

which $\tilde{v}$ is a free vector with compatible size, (4.37) can be written as

$$\bar{\mathbf{x}}^T\mathbf{D}\begin{bmatrix}\mathrm{He}\left\{\tilde{A}Q+\tilde{B}_2K\right\} & \star & \star & \star\\ \tilde{B}_1^T & -I & \star & \star\\ \tilde{C}Q+\tilde{D}_2K & \tilde{D}_1 & -\gamma^2 I & \star\\ K-Y-U\tilde{B}_2^T & \mathbf{0} & -U\tilde{D}_1^T & -2U\end{bmatrix}\mathbf{D}\bar{\mathbf{x}}+\sum_i \bar{\mathbf{x}}^T\bar{\Lambda}_{1i}(\tilde{x})\bar{\mathbf{x}}\phi_i(\tilde{x})<0, \tag{4.39}$$

in which $\bar{\Lambda}_{1i}(\tilde{x})=\mathbf{D}\cdot\Lambda_{1i}(\tilde{x})\cdot\mathbf{D}$.

Since $\mathrm{He}(-S^2/2)=\mathrm{He}(-I+M^{-1}\tilde{D}_{00}(\tilde{x})M)$ and $\tilde{D}_{00}(\tilde{x})M=0$, $Y=HQ$, (4.31) is equivalent to

$$\bar{\mathbf{x}}^T\mathrm{He}\begin{bmatrix}Q^{-1}\tilde{A}_0 & Q^{-1}\tilde{B}_1 & \mathbf{0} & Q^{-1}\tilde{B}_0M\\ \mathbf{0} & -I/2 & \mathbf{0} & \mathbf{0}\\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma^2I/2 & \tilde{D}_{10}M\\ M^{-1}(F-H) & M^{-1}\tilde{D}_{01} & \mathbf{0} & -S^2/2\end{bmatrix}\bar{\mathbf{x}}+\sum_i \bar{\mathbf{x}}^T\bar{\Lambda}_{1i}(\tilde{x})\bar{\mathbf{x}}\phi_i(\tilde{x})<0. \tag{4.40}$$

Because of (4.6), we have

$$\begin{bmatrix}Q^{-1}\tilde{A}_\Omega & Q^{-1}\tilde{B}_\Omega & \mathbf{0}\\ \mathbf{0} & -I/2 & \mathbf{0}\\ \tilde{C}_\Omega & \tilde{D}_\Omega & -\gamma^2I/2\end{bmatrix}=\begin{bmatrix}Q^{-1}\tilde{A}_0 & Q^{-1}\tilde{B}_1 & \mathbf{0}\\ \mathbf{0} & -I/2 & \mathbf{0}\\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma^2I/2\end{bmatrix}$$
$$+\begin{bmatrix}Q^{-1}\tilde{B}_0\\ \mathbf{0}\\ \tilde{D}_{10}\end{bmatrix}MS^{-1}(I+\Omega)S^{-1}M^{-1}\begin{bmatrix}F-H & \tilde{D}_{01} & \mathbf{0}\end{bmatrix}\cdot$$

Therefore, by Lemma 4, (4.40) becomes a sufficient condition for

$$\mathbf{x}^T \mathrm{He} \begin{bmatrix} Q^{-1}\tilde{A}_\Omega(\tilde{x}) & Q^{-1}\tilde{B}_\Omega(\tilde{x}) & \mathbf{0} \\ \mathbf{0} & -I/2 & \mathbf{0} \\ \tilde{C}_\Omega(\tilde{x}) & \tilde{D}_\Omega(\tilde{x}) & -\gamma^2 I/2 \end{bmatrix} \mathbf{x} \leq 0 \;\; \forall \|\Omega\| \leq 1, \tag{4.41}$$

in which, $\mathbf{x} = \begin{bmatrix} \tilde{x} & d & e \end{bmatrix}^T$. Finally, by using Schur complements, one can derive

$$\dot{V}(\tilde{x}) + \frac{1}{\gamma^2} e^T e \leq d^T d$$

The proof to guarantee that (4.38) suffices $|\bar{U}^{-1}Hx|_\infty \leq 1$ can be found in the proof of Theorem 5. *Q.E.D.*

4.6 Examples

As the first example, we consider a saturation synthesis problem for the following unstable nonlinear system, trying to maximize the Region of Attraction (ROA).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1.5x_1 - 0.5x_1^2 & x_2 \\ x_2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} \mathrm{sat}(u), \tag{4.42}$$

where the saturation level $\bar{u}$ of u is 0.25. For the given system, we are trying to derive and compare the largest ROAs by using quadratic and quartic Lyapunov functions. The state and

input matrices can be augmented as the form of (4.25) based on (4.16):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix} = \begin{bmatrix} 1.5x_1 - 0.5x_1^2 & x_2 & 0 & 0 & 0\\ x_2 & -1 & 0 & 0 & 0\\ 0 & 0 & 3x_1 - x_1^2 & 2x_2 & 0\\ 0 & 0 & x_2 & \begin{pmatrix} 1.5x_1 - \\ 0.5x_1^2 - 1 \end{pmatrix} & x_2\\ 0 & 0 & 0 & 2x_2 & -2 \end{bmatrix} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix} + \begin{bmatrix} -1\\ -0.5\\ -2x_1\\ -0.5x_1 - x_2\\ -x_2 \end{bmatrix} \text{sat}(u)$$

with the annihilator

$$\mathcal{N}_A(\tilde{x}) = \begin{bmatrix} \sum_{i=1}^{5} \alpha_i \tilde{x}_i^2 & 0 & \begin{pmatrix} \alpha_1\tilde{x}_1 + \\ \alpha_3\tilde{x}_1\tilde{x}_3 \end{pmatrix} & \alpha_4\tilde{x}_1\tilde{x}_4 & \begin{pmatrix} \alpha_2\tilde{x}_1 + \\ \alpha_5\tilde{x}_1\tilde{x}_5 \end{pmatrix}\\ 0 & -\sum_{i=1}^{5} \alpha_{i+5}\tilde{x}_i^2 & \begin{pmatrix} \alpha_6\tilde{x}_2 + \\ \alpha_8\tilde{x}_2\tilde{x}_3 \end{pmatrix} & \alpha_9\tilde{x}_2\tilde{x}_4 & \begin{pmatrix} \alpha_7\tilde{x}_2 + \\ \alpha_{10}\tilde{x}_2\tilde{x}_5 \end{pmatrix}\\ \begin{pmatrix} -\alpha_{11}\tilde{x}_1 - \\ \alpha_{12}\tilde{x}_1\tilde{x}_3 \end{pmatrix} & 0 & \begin{pmatrix} \alpha_{11} + \alpha_{12}\tilde{x}_1^2 \\ +\alpha_{13}\tilde{x}_2^2 \end{pmatrix} & 0 & -\alpha_{13}\tilde{x}_1^2\\ -\alpha_{16}\tilde{x}_2\tilde{x}_5 & -\alpha_{14}\tilde{x}_1 & -\alpha_{15}\tilde{x}_1\tilde{x}_2 & \begin{pmatrix} \alpha_{14} + \alpha_{15}\tilde{x}_1^2 \\ +\alpha_{16}\tilde{x}_2^2 \end{pmatrix} & 0\\ -\alpha_{18}\tilde{x}_1\tilde{x}_5 & \begin{pmatrix} -\alpha_{17}\tilde{x}_2 - \\ \alpha_{19}\tilde{x}_2\tilde{x}_5 \end{pmatrix} & 0 & 0 & \begin{pmatrix} \alpha_{17} + \alpha_{18}\tilde{x}_1^2 \\ +\alpha_{19}\tilde{x}_2^2 \end{pmatrix} \end{bmatrix}$$

in which all α_i's are decision variables. By solving the optimization problem (4.35) for the original system (4.42) and its augmented system respectively, the resulting state feedback controllers are

$$u_2 = 100.40x_1 + 14.32x_2 - 3.16x_1^2 - 4.26x_1x_2 - 1.44x_2^2,$$
$$u_4 = 2.17x_1 - 0.95x_2 + 2.22x_1^2 - 1.48x_1x_2 + 1.39x_2^2,$$

where u_2 and u_4 represent the inputs for 2nd and 4th order Lyapunov function cases, respectively. The associated Lyapunov functions are

$$V_2 = x^T \begin{bmatrix} 15.5543 & -0.1268 \\ -0.1268 & 4.3271 \end{bmatrix} x, \text{ and}$$

$$V_4 = x^{(2)T} \begin{bmatrix} 0.0074 & -0.0098 & 0.0130 & -0.0204 & 0.0077 \\ -0.0098 & 0.0139 & -0.0200 & 0.0339 & -0.0138 \\ 0.0130 & -0.0200 & 0.0683 & -0.0816 & 0.0343 \\ -0.0204 & 0.0339 & -0.0816 & 0.1936 & -0.0651 \\ 0.0077 & -0.0138 & 0.0343 & -0.0651 & 0.0510 \end{bmatrix} x^{(2)},$$

respectively. The optimized level by using quadratic and quartic Lyapunov functions can be seen in Fig. 4.1. The arrows represent predefined v_i directional vectors. From the comparison it can been seen clearly that, by taking advantage of the flexible shape of the higher order Lyapunov function, the ellipsoid of the quadratic function is covered entirely and the ROA of the given system is obviously expanded.

In the second example, we deal with the disturbance attenuation problem for the same nonlinear system with disturbance,

$$\dot{x} = \begin{bmatrix} 1.5x_1 - 0.5x_1^2 & x_2 \\ x_2 & -1 \end{bmatrix} x - \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} sat(u) + \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix} d \tag{4.43}$$

$$e = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} sat(u), \tag{4.44}$$

with $\bar{u} = 0.25$, $s = 0.5$, $\|x_1\| \leq 0.5$ and $\|x_2\| \leq 0.7$. One can augment the system using the form of (4.25) with the same $\mathcal{N}_A(\tilde{x})$ as described in the previous example. Based on the conditions

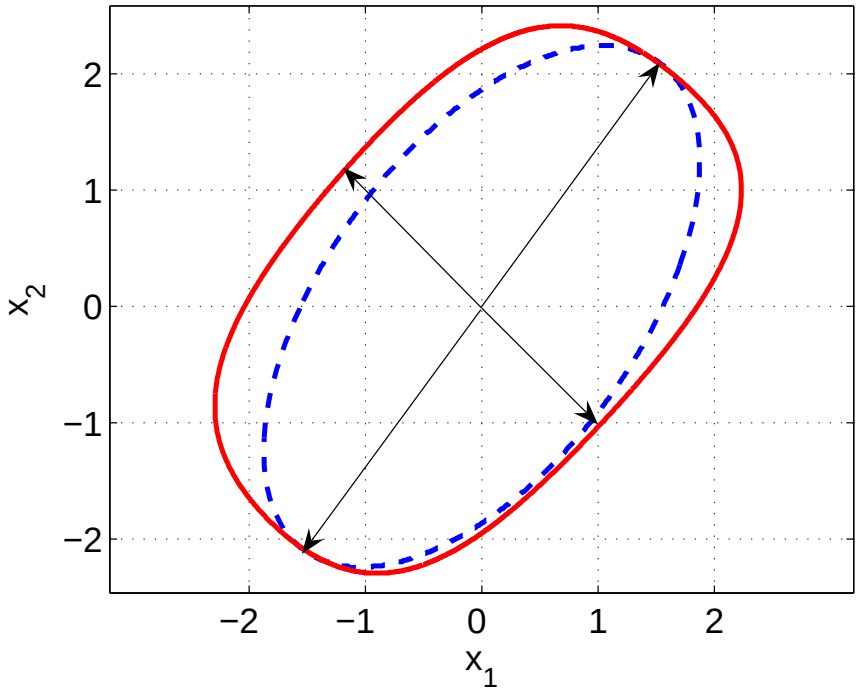


Figure 4.1: DOA as level sets of $V \leq 1$: 2^{nd}-order Lyapunov function (dash), 4^{th}-order Lyapunov function (solid).

of Theorem 6, both quadratic and quartic Lyapunov functions are sought to minimize γ. The smallest γ obtained using quartic Lyapunov function is 1.6507 by using a input

$$u_4 = 39.77x_1 - 15.48x_2 - 4.38x_1^2 + 2.13x_1x_2 - 3.59x_2^2 + 2.71x_1^3 + 0.79x_1^2x_2 + 2x_1x_2^2 + 0.18x_2^3$$ with the Lyapunov function

$$V_4 = x^{(2)T} \begin{bmatrix} 57.5740 & -39.5900 & -3.4955 & 2.8712 & -3.1147 \\ -39.5900 & 48.5966 & 1.9908 & -2.5745 & 1.3567 \\ -3.4955 & 1.9908 & 4.6429 & -3.5327 & 1.9007 \\ 2.8712 & -2.5745 & -3.5327 & 5.8825 & -3.0248 \\ -3.1147 & 1.3567 & 1.9007 & -3.0248 & 4.0026 \end{bmatrix} x^{(2)}.$$

On the other hand, using quadratic Lyapunov function, the optimal γ achieved is 2.1051. The associated Lyapunov function and controller input are

$$V_2 = 99.3201x_1^2 - 175.3614x_1x_2 + 103.3144x_2^2,$$
$$u_2 = 64.5717x_1 - 42.0144x_2.$$

The nonlinear terms of u_2 has been omitted because the coefficients are very small ($< 10^{-3}$). The nonlinear simulation is conducted using a disturbance input

$$d(t) = \begin{cases} 2(t-1.5) & 1.5sec \leq t < 2sec \\ 2(2.5-t) & 2sec \leq t \leq 2.5sec \end{cases}$$

as shown in Fig. 4.2 (a). Fig. 4.2-4.3 illustrate the simulation results by using the controllers synthesized with these two kinds of Lyapunov functions. These comparisons show that, with a decreased γ value, a higher order Lyapunov function could improve the disturbance rejection capability without increasing the control input energy.

Next, we will apply the proposed saturation $\mathcal{H}_\infty$ control method to address the animal population control issue using the famous Lotka-Volterra equation. Wildlife damage is always listed as one of the major challenges of the agriculture industry. The production of farms and ranches can be seriously affected by the attack and resource competition brought by wild animals. For example, in 1953, 47% of agricultural holdings in England were heavily infested with Oryctolagus cuniculus, or European rabbits [77]. Unfortunately, the randomness of these disasters limits the immediate damage control capability (such as labor, money, time, etc.).

In this example, we will try to provide a better control strategy under input limits by considering an animal population dynamics in one area with two species, wild rabbit (damaging plants) and fox (threatening livestock). Although virus spreading was proved to be devastatingly effective for rabbit population, such as Myxomatosis since 1950s and rabbit calicivirus since

1990s, the genetic resistance increased obviously after years of using. More importantly, the diseases had wider consequences on the food chain apart from the death of rabbits. In our simplified biology system, the rapid death of rabbits may force the starving foxes to frequently attack livestock. Therefore, conventional methods including shooting and destruction of warrens and dens are considered in this model to keep the decrease of both animal populations in the similar steady speed. The disadvantage of these methods is also obvious, limited efficiency. Therefore, the saturation control method is used to address the input limitation.

Lotka-Volterra equations are a class of mathematic models that frequently used to describe the predator-prey relationship of biological systems in which two species interact, one a predator and one its prey. The dynamics can be modeled as following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\theta & \delta x_1 \\ -\beta x_2 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} sat(u) + \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{4.45}$$
$$e = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} sat(u),$$

in which x_1 is the number of predator, foxes, and x_2 is the number of prey, rabbits. The derivatives represent the growth rates of the two populations over time, and d_1, d_2 represent the unexpected growth exploration of the species, respectively. Moreover, α, β, θ and δ are parameters describing the interaction of the two species, in which α represents the unrestrained growth rate of the prey in the absence of predators, while $-\theta$ represents the rate that the predators die off in the absence of their prey. The nonlinear terms model the interaction of the two species: the rate of increase in the predators is proportional to the number of available prey, while the rate of decrease in the prey is proportional to the number of predators. The input u indicates the number of animal being killed in unit time, and $sat(u)$ represents the input with maximum killing capability of u.

This model assumes that in a local territory, foxes only consume wild rabbits and the

rabbits have foxes as the unique enemy. However, the real world is obviously more complicated. Therefore, disturbances d_1 and d_2 are introduced to include the perturbation of growth rates caused by extraneous factors for foxes and rabbits, respectively. Such factors include, but not limited to, weather fluctuation, inflow and outflow of population, sudden change of food supplies and enemies that not consider in the model.

The performance output e explains our concern: minimizing the number of rabbits, the total number of foxes and rabbits, and the control input simultaneously under the disturbance d_1 and d_2. All of the variables are assumed to be dimensionless.

Set $\alpha = 1.5$, $\beta = 1$, $\theta = 1$ and $\delta = 1$, one can solve the $\mathcal{H}_\infty$ control problem of this saturated system based on Theorem 6 by minimizing γ. Adopting Lyapunov function with quadratic form, the resultant Lyapunov function and controller input are

$$\begin{aligned} V_2 &= 18.3480x_1^2 - 112.8618x_1x_2 + 231.6973x_2^2, \\ u_2 &= -11.8517x_1 + 51.0891x_2, \end{aligned}$$

and the minimized $\gamma_2 = 1.0016$. By augmenting the system as described in previous examples, conditions of Theorem 6 can be solved for the augmented system. The optimized $\gamma_4 = 0.7236$ and the results are

$$\begin{aligned} u_4 &= -9.6828x_1 + 87.8992x_2 - 0.1284x_1^2 + 0.2545x_1x_2 + 3.3021x_2^2 \\ &\quad + 0.1220x_1^3 + 0.2074x_1^2x_2 + 0.0420x_1x_2^2 + 1.8131x_2^3 \quad \text{with the Lyapunov function} \\ V_4 &= x^{(2)T} \begin{bmatrix} 14.4985 & -76.2593 & -0.2318 & -0.2242 & -0.8869 \\ -76.2593 & 660.9388 & -0.0122 & 1.5352 & 6.1216 \\ -0.2318 & -0.0122 & 0.6510 & 0.0232 & 0.2538 \\ -0.2242 & 1.5352 & 0.0232 & 0.6039 & -0.1075 \\ -0.8869 & 6.1216 & 0.2538 & -0.1075 & 5.0380 \end{bmatrix} x^{(2)}. \end{aligned}$$

Fig. 4.4 plots the progression of the two species over time, in which states x_1 and x_2 are

scaled numbers instead of actual number of animals and the time is dimensionless. The animal population growth progresses with both of the above control strategies can be simulated under the following population perturbations:

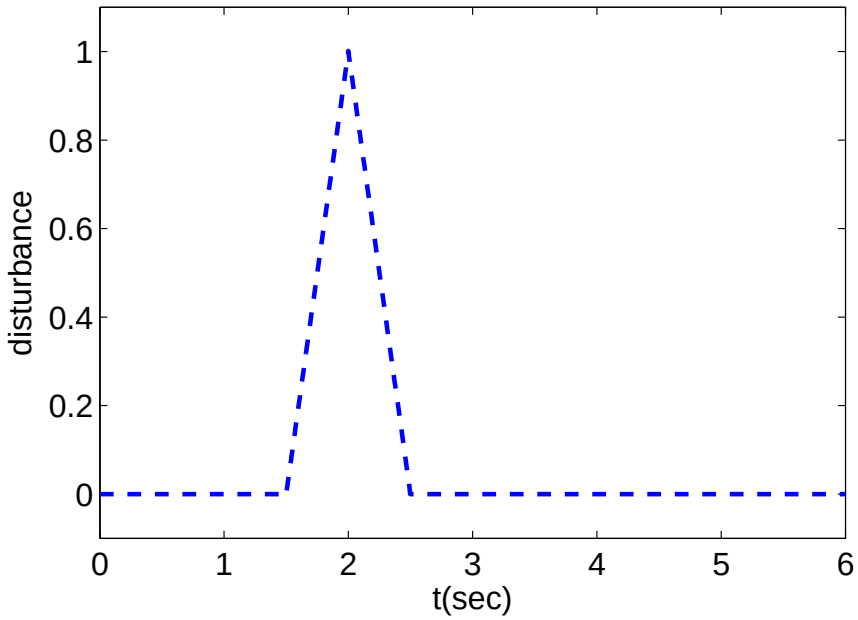
$$d_1(t) = \frac{\sigma_1}{\zeta_1\sqrt{2\pi}} e^{-\frac{(t-\mu_1)^2}{2\zeta_1^2}},\ \sigma_1 = 1\ \ \zeta_1 = 0.8,\ \ \mu_1 = 1.5,$$
$$d_2(t) = \frac{\sigma_2}{\zeta_2\sqrt{2\pi}} e^{-\frac{(t-\mu_2)^2}{2\zeta_2^2}},\ \sigma_2 = 0.5\ \ \zeta_2 = 1,\ \ \mu_2 = 2.7.$$

These disturbance equations indicate that, because of certain extraneous factors, the increase of fox's and rabbit's growth rate satisfies a normal distribution with the peak value at time 1.5 and 2.7, respectively.

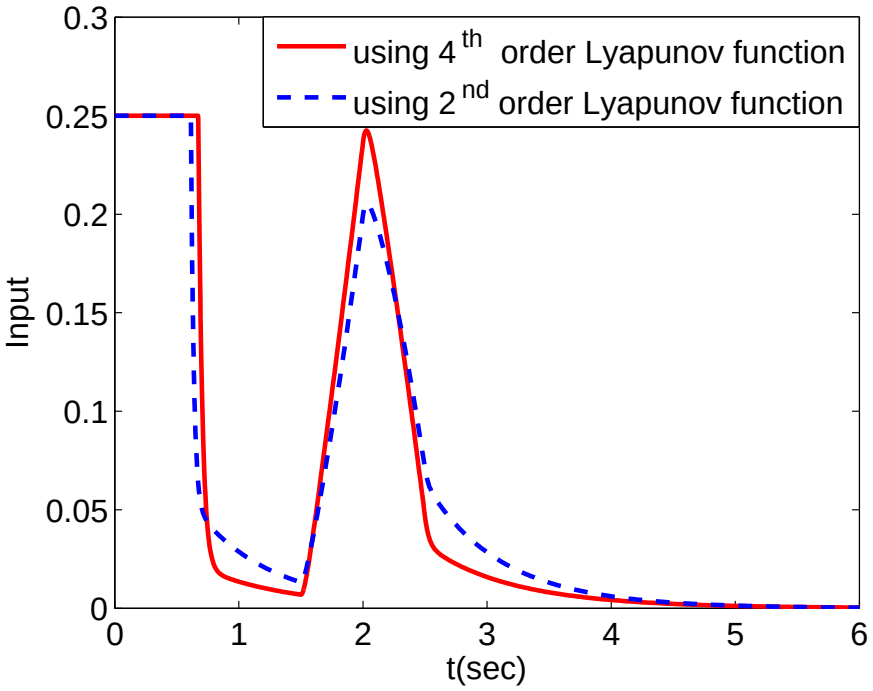
With an initial value of $x_{10} = 0.5$ ($\times 1000$), $x_{20} = 0.2$ ($\times 1000$), and a saturation level $\bar{u} = 0.35$, the simulation results of the nonlinear dynamics can be seen in Fig. 4.5 and Fig. 4.6. From the figures we can clearly conclude that, the control input computed using higher order Lyapunov functions can provide better animal population control performance but with less control effort.

4.7 Summary

In this chapter, we proposed a method to utilize higher order Lyapunov functions to solve state feedback synthesis problems for saturated polynomial nonlinear systems. By taking advantage of the flexible representations of the augmented system, the higher order Lyapunov functions have the ability to improve the closed-loop performance under different performance requirements. Three examples illustrated the effectiveness of the proposed design approach for saturated polynomial nonlinear systems, one for ROA enlargement and another two for $\mathcal{H}_\infty$ performance improvement.

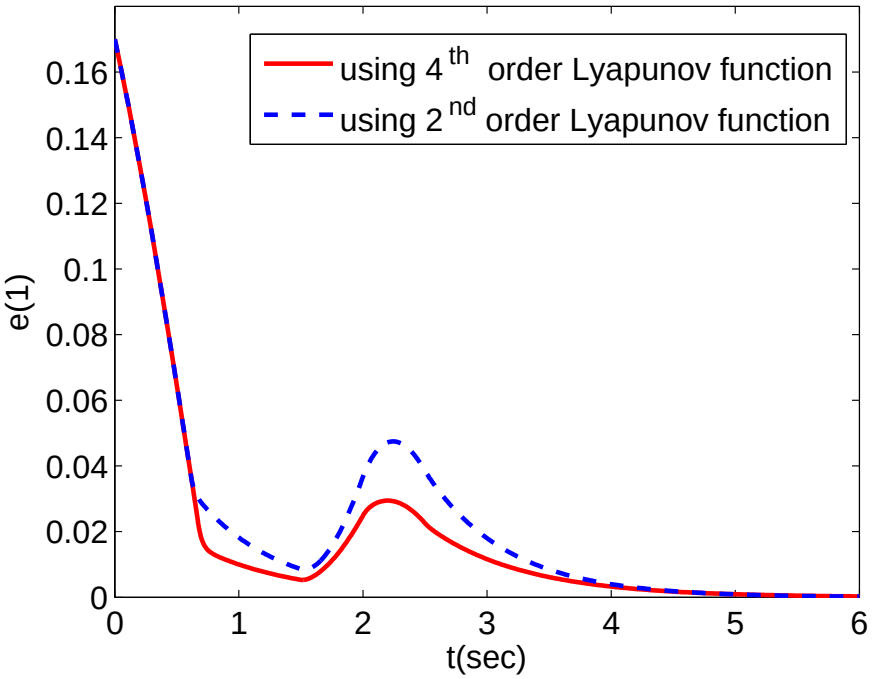


(a) Disturbance

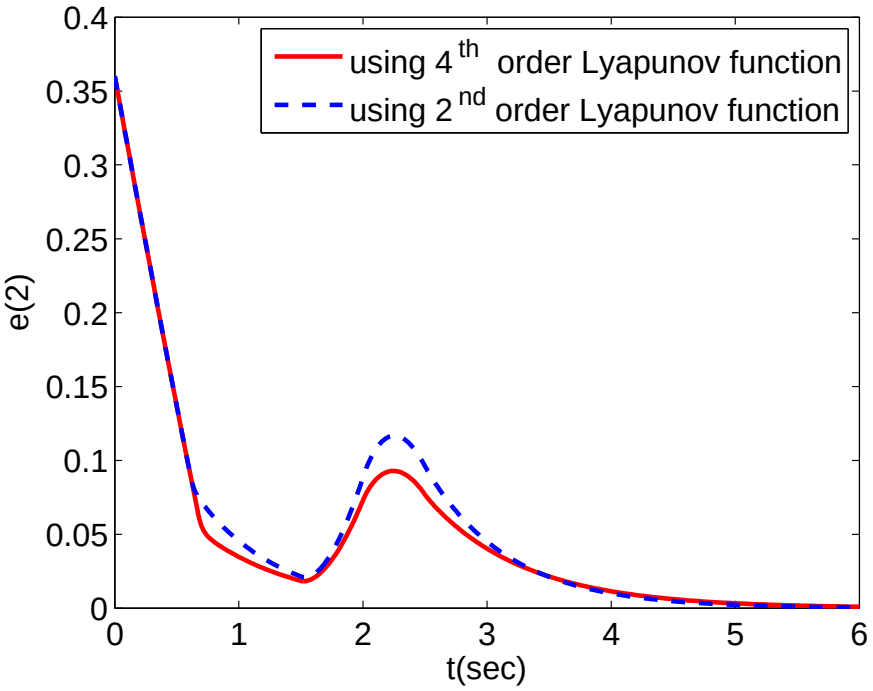


(b) Saturated control input

Figure 4.2: Input trajectories.



(a) Output e_1



(b) Output e_2

Figure 4.3: Comparison of output trajectories.

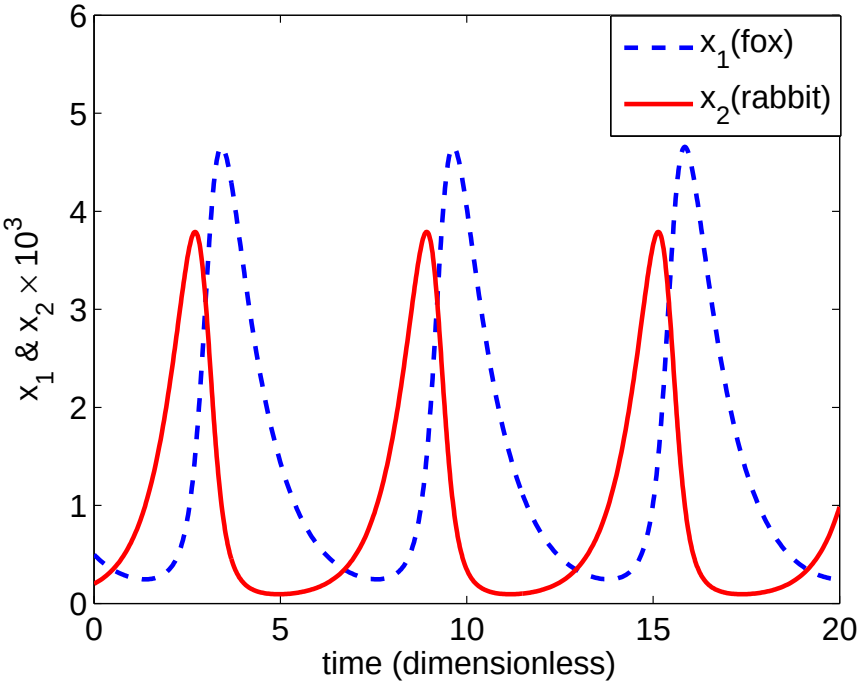
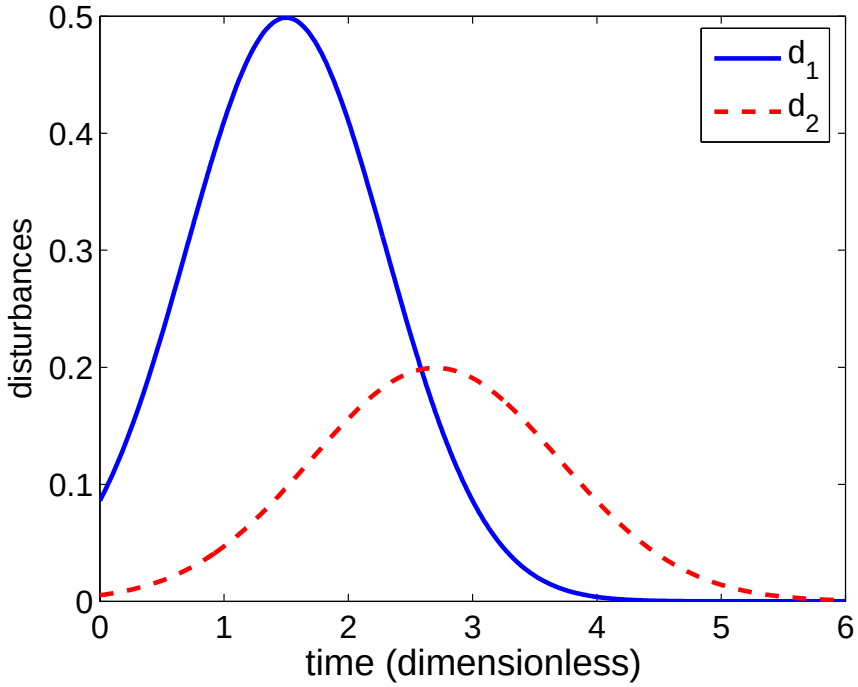
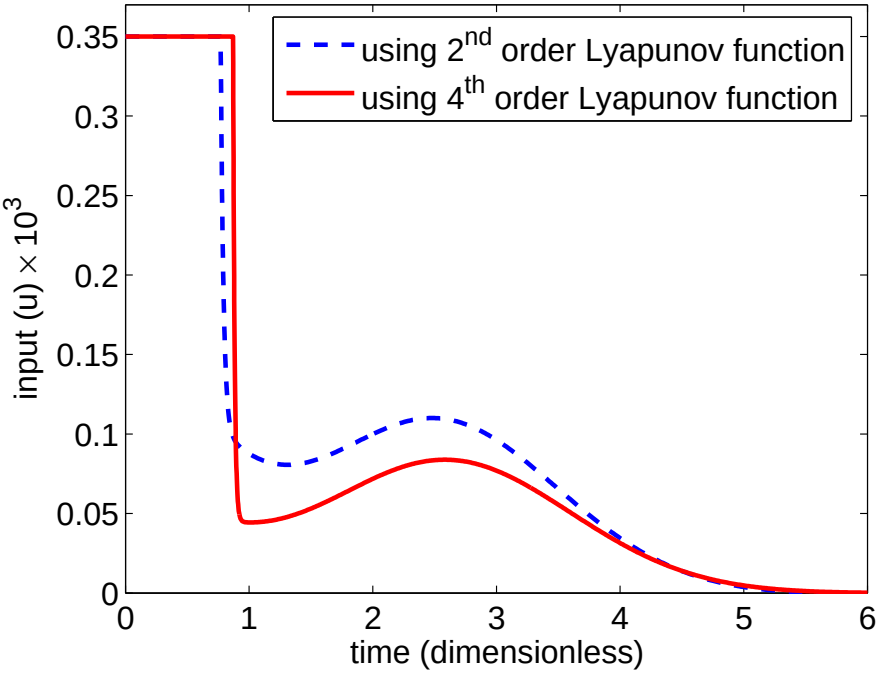


Figure 4.4: Progression of the two species over time without population control and disturbance.

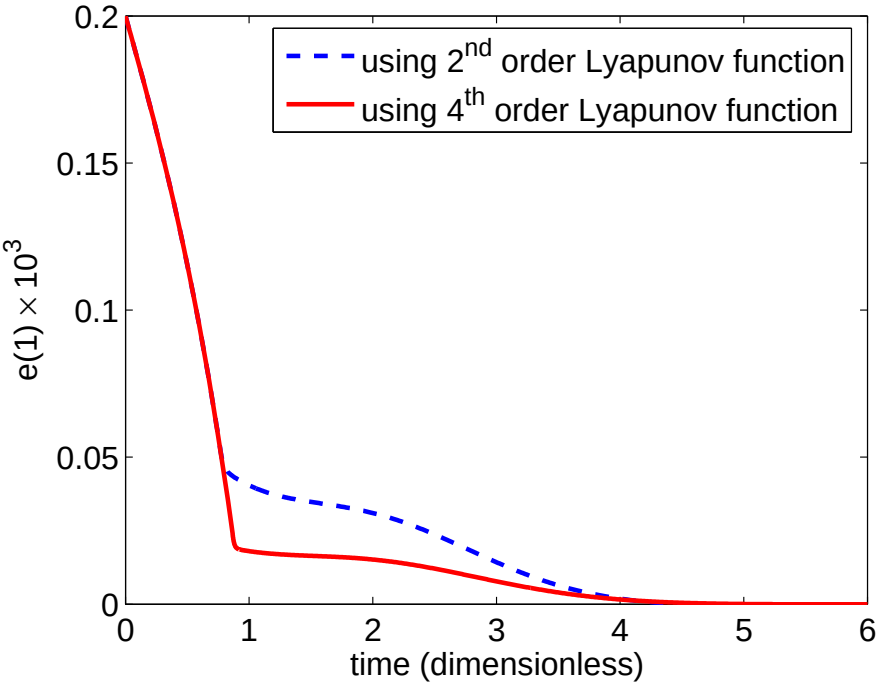


(a) Disturbances

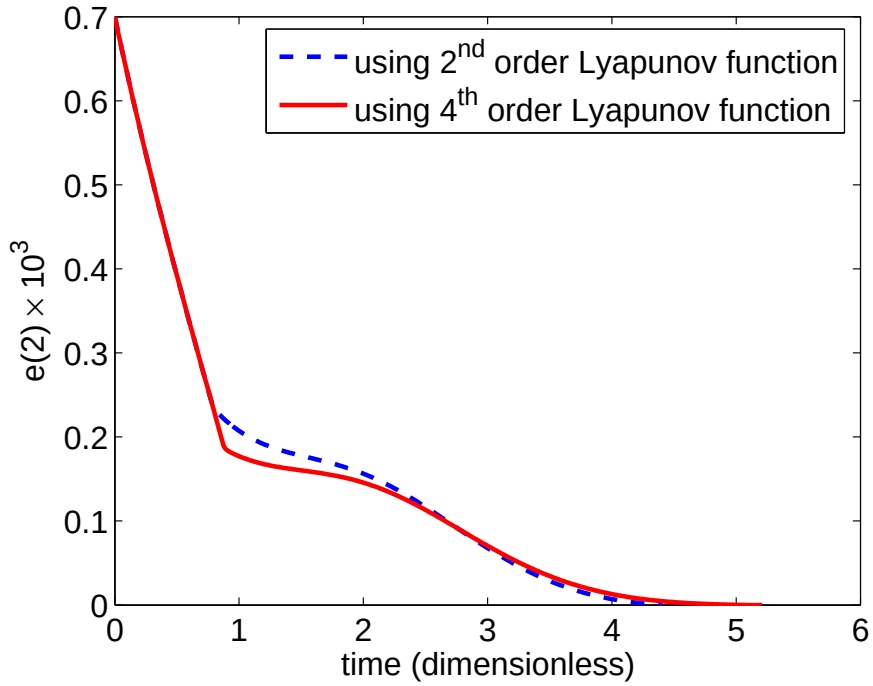


(b) Saturated control input

Figure 4.5: Input trajectories.



(a) Output e_1 (number of rabbits)



(b) Output e_2 (number of foxes and rabbits)

Figure 4.6: Comparison of output trajectories.

Chapter 5

Nonlinear Output Feedback $\mathcal{H}_\infty$ Control for Polynomial Nonlinear Systems

For performance required nonlinear state feedback control problems based on polynomial systems with and without input saturations, Chapter 3 and Chapter 4 demonstrated the advantage of using power transformation to derive higher order Lyapunov functions and associated controller gains. Furthermore, the existence of such advantage in solving output feedback $\mathcal{H}_\infty$ control problems with more complicated conditions will be discussed in this chapter. Firstly, we will provide $\mathcal{H}_\infty$ synthesis conditions in terms of Hamilton-Jacobi-Isaacs (HJI) inequalities for a class of polynomial nonlinear systems without orthonormal and decoupling assumptions on the plant structure. The design procedure of nonlinear $\mathcal{H}_\infty$ control based on an augmented system will then be presented. After that, we will apply this output feedback $\mathcal{H}_\infty$ control design approach to address the command tracking problem for an axi-symmetric rigid body spacecraft with two controls. By using higher order Lyapunov functions, the proposed method will enable the spacecraft to track a given rotational command with and without external disturbances. These simulation results will be compared with the counterparts derived by using quadratic

Lyapunov functions.

This chapter is organized as follows: In Section 5.1, we will summarize the existing methods dedicated for the solution of the HJI equations and the relationship between this work and previously finished research. Section 5.2 generalize output feedback nonlinear $\mathcal{H}_\infty$ synthesis conditions through system augmentation and provide the solution procedure in solving nonlinear $\mathcal{H}_\infty$ analysis and synthesis conditions using SOS programming approach. In Section 5.3, after reviewing the research development in the field of high performance spacecraft control, the nonlinear model of an axi-symmetric spacecraft will be set up. The rest part of Section 5.3 is devoted to compare the results of using higher order Lyapunov functions and quadratic Lyapunov functions for the axis-symmetric spacecraft to achieve command tracking objectives. Finally, the chapter concludes in Section 5.4.

5.1 Introduction

The HJI equation or inequality is the nonlinear version of the Riccati equation considered in the $\mathcal{H}_\infty$ control problem for linear systems. [86] showed that state feedback problems for nonlinear systems are solvable if there exist smooth solutions to some HJI partial differential equations (inequalities). For the output feedback case, additional conditions are needed. In order to derive nonlinear output feedback $\mathcal{H}_\infty$ controllers, it is necessary and sufficient to have solutions of a pair of coupled HJI equations with separate structures, including a state feedback and an output estimation condition, respectively.

The HJI equation is a first-order, nonlinear partial differential equation. In the general case, it cannot be solved analytically. For specific nonlinear systems, it is usually very difficult to be solved for [27]. Therefore, although the formulation of the nonlinear theory of $\mathcal{H}_\infty$ control has been well developed, solving the HJI equations in an efficient systematical numerical approach has been a challenging task for the practical application of the theory ([9, 6, 1]) from at least early 1990s. Many efforts have been attempted to numerically solve the HJI equation. Beginning from the work of [60] and [30] who proposed a polynomial approximation approach, Taylor series

expansion were considered to solve HJI equation term by term in an iterative fashion. Along this line, many authors have articles with similar approaches to the solution of the problem, such as [86, 46, 33, 17, 83, 92]. But these approaches cannot provide solutions sharing a closed form. In addition, these solutions may not converge to an analytic solution. Furthermore, Galerkin successive approximation was proposed in [8, 9] to reduce HJI equations to a sequence of linear partial differential equations. Then some works applied the Galerkin successive method to find an approximated solution, for example, in [27]. Nevertheless, the computation cost increases exponentially as the number of states increase.

On the other hand, while the SOS programming technique [64, 69] has been applied to solve nonlinear analysis [63, 78] and stabilization [67, 68] problems, for polynomial nonlinear systems, [70] formulated the $\mathcal{L}_2$ gain analysis problem as a convex state-dependent linear matrix inequality (LMI). By recasting the $\mathcal{L}_2$ gain analysis problem as an SOS optimization problem, this method was shown promising to overcome the numerical difficulty in solving HJI inequality and provides an analytic solution simultaneously. According to this idea, for a class of polynomial nonlinear systems without orthonormal and decoupling assumptions on the plant structure, [96] proposed an SOS based nonlinear output feedback $\mathcal{H}_\infty$ control design scheme to solve the generalized HJI inequalities. This approach used a non-convex method to synthesize higher order Lyapunov functions as well as the associated nonlinear control laws. As a continuation of the research in [96], this chapter utilize the proposed system augmentation method to synthesis higher order Lyapunov functions as a computationally efficient convex problem such that, more higher order monomials can be involved during the search of Lyapunov functions to improve the output feedback $\mathcal{H}_\infty$ performance.

Considering a polynomial nonlinear system with affine exogenous disturbance and control input

$$\begin{cases} \dot{x} &= A(x)x + B_1(x)d + B_2(x)u \\ e &= C_1(x)x + D_{12}(x)u \\ y &= C_2(x)x + D_{21}(x)d \end{cases} \tag{5.1}$$

where the system state $x \in \mathbf{R}^n$, control input $u \in \mathbf{R}^{n_u}$, exogenous disturbance $d \in \mathbf{R}^{n_d}$, controlled output $e \in \mathbf{R}^{n_e}$, and measured output $y \in \mathbf{R}^{n_y}$. All of the state space entries are polynomial functions of the state x with compatible dimensions. Comparing with past researche in literature dealing with nonlinear output feedback $\mathcal{H}_\infty$ synthesis, such as [44, 86, 58], the assumption of this research is much less restrictive. We only assume that $D_{12}(x)^T D_{12}(x)$ and $D_{21}(x)^T D_{21}(x)$ are invertible and the inverse is either in a constant or polynomial form. Without the decoupling enforcement like in many other researches, the proposed design approach would be widely applicable since a large class of physical systems will be covered and solvable within this framework.

As introduced previously, nonlinear output feedback $\mathcal{H}_\infty$ synthesis problems have two coupled partial differential HJI inequalities with separate structures, one of which is a state feedback condition and the other is an output estimation condition. The aim is to synthesize an output feedback controller to stabilize the closed-loop system asymptotically and guarantee that its $\mathcal{H}_\infty$ norm is less than γ, i.e.

$$\|e\|_2 \leq \gamma \|d\|_2 \quad \text{when} \quad x(0) = 0.$$

For ease of reference, we will provide the nonlinear $\mathcal{H}_\infty$ synthesis condition in the following theorem.

Theorem 7 *Given a scalar $\gamma > 0$, the nonlinear output feedback $\mathcal{H}_\infty$ control problem is (locally) solvable for the nonlinear system (5.1) if there exist two positive definite matrix functions $U(x)$*

and $V(x)$, satisfying $U(0) = V(0) = 0$ such that

$$\mathcal{H}_{FI}(U, \gamma, x) < 0 \tag{5.2}$$

$$\mathcal{H}_{FC}(V, \gamma, x) - \gamma^{-2}\mathcal{H}_{FI}(U, \gamma, x) < 0 \tag{5.3}$$

$$\frac{\partial^2}{\partial x^2}(\mathcal{H}_{FC} - \gamma^{-2}\mathcal{H}_{FI})\bigg|_{x=0} < 0 \tag{5.4}$$

$$W(x) := V(x) - \gamma^{-2}U(x) \geq 0, \tag{5.5}$$

where

$$\begin{aligned}
\mathcal{H}_{FI}(U, \gamma, x) := {} & \frac{\partial U}{\partial x}A(x)x + x^T C_1^T(x)\left[I - D_{12}(D_{12}^T D_{12})^{-1}D_{12}^T\right]C_1(x)x \\
& - \frac{1}{2}\frac{\partial U}{\partial x}B_2(x)(D_{12}^T D_{12})^{-1}D_{12}^T C_1(x)x - \frac{1}{2}x^T C_1^T(x)D_{12}(D_{12}^T D_{12})^{-1}B_2^T(x)\frac{\partial U^T}{\partial x} \\
& + \frac{1}{4}\frac{\partial U}{\partial x}\left[\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)(D_{12}^T D_{12})^{-1}B_2^T(x)\right]\frac{\partial U^T}{\partial x} \\
\mathcal{H}_{FC}(V, \gamma, x) := {} & \frac{\partial V}{\partial x}A(x)x + \frac{1}{4}\frac{\partial V}{\partial x}B_1(x)\left[I - D_{21}^T(D_{21}D_{21}^T)^{-1}D_{21}\right]B_1^T(x)\frac{\partial V^T}{\partial x} \\
& - \frac{1}{2}x^T C_2^T(x)(D_{21}D_{21}^T)^{-1}D_{21}B_1^T(x)\frac{\partial V^T}{\partial x} - \frac{1}{2}\frac{\partial V}{\partial x}B_1(x)D_{21}^T(D_{21}D_{21}^T)^{-1}C_2(x)x \\
& + x^T\left[\frac{1}{\gamma^2}C_1^T(x)C_1(x) - C_2^T(x)(D_{21}D_{21}^T)^{-1}C_2(x)\right]x.
\end{aligned}$$

Moreover, one of the nth-order output feedback controllers is given by

$$\begin{cases} \dot{x}_c &= A(x_c)x_c + B_1(x_c)F_1(x_c) + B_2(x_c)F_0(x_c) + L_0(x_c)\left[C_2(x_c)x_c + D_{21}F_1(x_c) - y\right] \\ u &= F_0(x_c) \end{cases} \tag{5.6}$$

where the matrix functions $F_0(x), F_1(x)$ and $L_0(x)$ are defined as

$$F_0(x) := -(D_{12}^T D_{12})^{-1}\left[\frac{1}{2}B_2^T(x)\frac{\partial U^T}{\partial x} + D_{12}^T C_1(x)x\right] \tag{5.7}$$

$$F_1(x) := \frac{1}{2\gamma^2}B_1^T(x)\frac{\partial U^T}{\partial x} \tag{5.8}$$

$$\frac{\partial W(x)}{\partial x}L_0(x) := -2\left[C_2(x)x + D_{21}F_1(x)\right]^T(D_{21}D_{21}^T)^{-1}. \tag{5.9}$$

Note that solvability conditions (5.2)-(5.3) are also given in the form of generalized HJI inequalities, which remain difficult to solve. Moreover, the bilinear term in condition (5.2), $-\frac{1}{4}\frac{\partial U}{\partial x}B_2(D_{12}^T D_{12})^{-1}B_2^T\frac{\partial U^T}{\partial x}$, and the similar term in (5.3) will render a non-convex problem that can only be processed one by one in the non-convex fashion. The treatment of coupling terms, for example, $x_1^2x_2x_3$, will become more difficult and inconvenient. This difficulty blocked the way to improve the solvability and performance of the output feedback synthesis problem by considering higher order Lyapunov functions with complicated structures.

5.2 Nonlinear Output Feedback $\mathcal{H}_\infty$ Synthesis Problem

Thanks to system augmentation, the non-convexity issue can be avoid and the entire conditions can be write as state-dependent LMIs. Like what we have done to all of the considered systems in Chapter 3 and Chapter 4, before introducing the LMI conditions, the augmented version of the system (5.1) based on power transformation needs to be defined:

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}(\tilde{x})\tilde{x} + \tilde{B}_1(\tilde{x})d + \tilde{B}_2(\tilde{x})u \\ e &= \tilde{C}_1(\tilde{x})\tilde{x} + \tilde{D}_{12}(\tilde{x})u \\ y &= \tilde{C}_2(\tilde{x})\tilde{x} + \tilde{D}_{21}(\tilde{x})d \end{cases} \tag{5.10}$$

in which the system state $\tilde{x} \in \mathbf{R}^{\tilde{n}}$ and other state space entries have suitable dimensions. The conditions in this Chapter will be demonstrated based on this augmented system.

With an augmented state vector $\tilde{x}$, now we can consider $U(\tilde{x})$ and $V(\tilde{x})$ as higher order

Lyapunov functions with quadratic forms, such as $U(\tilde{x}) = \tilde{x}^T R^{-1} \tilde{x}$ and $V(\tilde{x}) = \tilde{x}^T S \tilde{x}$. After this, the HJI conditions (5.2)-(5.3) become a set of state-dependent LMIs even for general polynomial Lyapunov functions with order higher than two. The non-convex bilinear terms can be rewritten by using Schur complement, which is a standard matrix theory technique used in LMI methods. We will redefine the conditions in Theorem 7. Meanwhile, based on these redefined conditions, we will develop a systematic design procedure for nonlinear output feedback $\mathcal{H}_\infty$ control. To this end, let us define

$$\mathcal{M}_{FI}(R,\gamma) := \begin{bmatrix} \left\{ \begin{array}{c} [\tilde{A}(\tilde{x}) - \tilde{B}_2(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{D}_{12}^T(\tilde{x})\tilde{C}_1(\tilde{x})]R \\ +R[\tilde{A}^T(\tilde{x}) - \tilde{C}_1^T(\tilde{x})\tilde{D}_{12}(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{B}_2^T(\tilde{x}) \\ -\tilde{B}_2(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{B}_2^T(\tilde{x})] \end{array} \right\} & \star & \star \\ [I - \tilde{D}_{12}(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{D}_{12}^T(\tilde{x})]^{\frac{1}{2}}\tilde{C}_1(\tilde{x})R & -I & \star \\ \tilde{B}_1^T(\tilde{x}) & 0 & -\gamma^2 I \end{bmatrix}$$

$$\begin{aligned} \mathcal{R}_{FI}(R^{-1},\gamma) :=& R^{-1}[\tilde{A}(\tilde{x}) - \tilde{B}_2(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{D}_{12}^T(\tilde{x})\tilde{C}_1(\tilde{x})] \\ &+ [\tilde{A}^T(\tilde{x}) - \tilde{C}_1^T(\tilde{x})\tilde{D}_{12}(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{B}_2^T(\tilde{x})R^{-1} \\ &+ \tilde{C}_1^T(\tilde{x})[I - \tilde{D}_{12}(\tilde{x})(\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{D}_{12}^T(\tilde{x})]\tilde{C}_1(\tilde{x}) \\ &+ R^{-1}[\frac{1}{\gamma^2}\tilde{B}_1(\tilde{x})\tilde{B}_1(\tilde{x})^T - \tilde{B}_2(\tilde{x})(\tilde{D}_{12}(\tilde{x})^T\tilde{D}_{12}(\tilde{x}))^{-1}\tilde{B}_2^T(\tilde{x})]R^{-1} \end{aligned}$$

$$\mathcal{M}_{FC}(S,\hat{\gamma},R^{-1}) := \begin{bmatrix} \left\{ \begin{array}{c} S[\tilde{A}(\tilde{x}) - \tilde{B}_1(\tilde{x})\tilde{D}_{21}^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{C}_2(\tilde{x})] \\ +[\tilde{A}^T(\tilde{x}) - \tilde{C}_2^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{D}_{21}(\tilde{x})\tilde{B}_1^T(\tilde{x})]S \\ -\tilde{C}_2^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{C}_2(\tilde{x}) - \gamma^{-2}\mathcal{R}_{FI}(R^{-1},\gamma)] \end{array} \right\} & \star & \star \\ [I - \tilde{D}_{21}^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{D}_{21}(\tilde{x})]^{\frac{1}{2}}\tilde{B}_1^T(\tilde{x})S & -I & \star \\ \tilde{C}_1(\tilde{x}) & 0 & -\hat{\gamma}^2 I \end{bmatrix}$$

$$\begin{aligned} \mathcal{R}_{FC}(S,\hat{\gamma},R^{-1}) :=& S[\tilde{A}(\tilde{x}) - \tilde{B}_1(\tilde{x})\tilde{D}_{21}^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{C}_2(\tilde{x})] \\ &+ [\tilde{A}^T(\tilde{x}) - \tilde{C}_2^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}(\tilde{x}))^{-1}\tilde{D}_{21}(\tilde{x})\tilde{B}_1(\tilde{x})]S \\ &+ S\tilde{B}_1(\tilde{x})[I - \tilde{D}_{21}^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{D}_{21}(\tilde{x})]\tilde{B}_1^T(\tilde{x})S \\ &+ \frac{1}{\hat{\gamma}^2}\tilde{C}_1^T(\tilde{x})\tilde{C}_1(\tilde{x}) - \tilde{C}_2^T(\tilde{x})(\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x}))^{-1}\tilde{C}_2(\tilde{x}) - \gamma^{-2}\mathcal{R}_{FI}(R^{-1},\gamma) \end{aligned}$$

$$\mathcal{N}_{FC}(S,\hat{\gamma},R^{-1}) := \mathrm{diag}(\tilde{x}^T, I, I)\mathcal{M}_{FC}(S,\hat{\gamma},R^{-1})\mathrm{diag}(\tilde{x}, I, I).$$

It is not difficult to verify that, for an augmented system (5.10),

$$\mathcal{H}_{FI}(U(\tilde{x}),\gamma) = \tilde{x}^T\mathcal{R}_{FI}(R^{-1},\gamma)\tilde{x}$$

$$\mathcal{H}_{FC}(V(\tilde{x}),\hat{\gamma}) - \gamma^{-2}\mathcal{H}_{FI}(U(\tilde{x}),\gamma) = \tilde{x}^T\mathcal{R}_{FC}(S,\hat{\gamma},R^{-1})\tilde{x}.$$

The matrices $\mathcal{M}_{FI}$ and $\mathcal{M}_{FC}$ are linearly dependent on the Lyapunov matrices R and S, respectively. By multiplying R from both sides of $\mathcal{R}_{FI}$ and followed by a Schur complement, $\mathcal{M}_{FI}$ can be derived. The similar process can be used to generate $\mathcal{M}_{FC}$ from $\mathcal{R}_{FC}$ as well.

For the augmented system (5.10), Theorem 7 can be modified as following:

Theorem 8 *Given a scalar $\gamma > 0$, the nonlinear output feedback $\mathcal{H}_\infty$ control problem is (locally) solvable for the nonlinear system (5.10) if there exists two positive definite matrix functions $U(\tilde{x})$*

and $V(\tilde{x})$, satisfying $U(0) = V(0) = 0$ such that

$$\tilde{z}_1^T \mathcal{M}_{FI}(R,\gamma)\tilde{z}_1 + \sum_i \tilde{z}_1^T \Lambda_{1i}(\tilde{x})\tilde{z}_1\phi_i(\tilde{x}) < 0 \tag{5.11}$$

$$\tilde{z}_2^T \mathcal{M}_{FC}(S,\hat{\gamma},R^{-1})\tilde{z}_2 + \sum_j \tilde{z}_2^T \Lambda_{2j}(\tilde{x})\tilde{z}_2\phi_j(\tilde{x}) < 0 \tag{5.12}$$

$$\frac{\partial^2}{\partial x^2}(\mathcal{H}_{FC} - \gamma^{-2}\mathcal{H}_{FI})\bigg|_{x=0} < 0 \tag{5.13}$$

$$W(x) := V(\tilde{x}) - \gamma^{-2}U(\tilde{x}) \geq 0, \tag{5.14}$$

and the associated controller is given by

$$\begin{cases} \dot{\tilde{x}}_c &= \tilde{A}(\tilde{x}_c)\tilde{x}_c + \tilde{B}_1(\tilde{x}_c)F_1(\tilde{x}_c) + \tilde{B}_2(\tilde{x}_c)F_0(\tilde{x}_c) + L_0(\tilde{x}_c)\left[\tilde{C}_2(\tilde{x}_c)\tilde{x}_c + \tilde{D}_{21}F_1(\tilde{x}_c) - y\right] \\ u &= F_0(\tilde{x}_c) \end{cases} \tag{5.15}$$

where the matrix functions $F_0(\tilde{x}), F_1(\tilde{x})$ and $L_0(\tilde{x})$ are defined as

$$F_0(\tilde{x}) := -(\tilde{D}_{12}^T\tilde{D}_{12})^{-1}\left[\frac{1}{2}\tilde{B}_2^T(\tilde{x})\frac{\partial U^T}{\partial \tilde{x}} + \tilde{D}_{12}^T\tilde{C}_1(\tilde{x})\tilde{x}\right] \tag{5.16}$$

$$F_1(\tilde{x}) := \frac{1}{2\gamma^2}\tilde{B}_1^T(\tilde{x})\frac{\partial U^T}{\partial \tilde{x}} \tag{5.17}$$

$$L_0(\tilde{x}) := -(S - \gamma^{-2}R^{-1})^{-1}\left[\tilde{C}_2(\tilde{x})\tilde{x} + \tilde{D}_{21}F_1(\tilde{x})\right]^T(\tilde{D}_{21}\tilde{D}_{21}^T)^{-1}.$$

Sketch of Proof:

The basic idea of proof is to convert the system (5.10) with general form into another one with a simpler structure by variable changes. After that, the $\mathcal{H}_\infty$ synthesis problem of the transformed system can be solved by reducing it into an output estimation problem. The detailed proof has been provided in [58] as the main result of the paper.

Let $r = d - F_1(\tilde{x})$ and $v = [\tilde{D}_{12}^T\tilde{D}_{12}]^{\frac{1}{2}}(u - F_0(\tilde{x})) = \bar{u} - [\tilde{D}_{12}^T\tilde{D}_{12}]^{\frac{1}{2}}F_0(\tilde{x}$, (5.10) can be

transformed into

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}_a(\tilde{x})\tilde{x} + \tilde{B}_1(\tilde{x})d + \tilde{B}_a(\tilde{x})u \\ v &= -[\tilde{D}_{12}^T\tilde{D}_{12}]^{\frac{1}{2}}F_0(\tilde{x}) + u \\ y &= \tilde{C}_a(\tilde{x})\tilde{x} + \tilde{D}_{21}(\tilde{x})r \end{cases}, \tag{5.18}$$

in which

$$\tilde{A}_a(\tilde{x})\tilde{x} = \tilde{A}(\tilde{x})\tilde{x} + \tilde{B}_1(\tilde{x})F_1(\tilde{x})$$
$$\tilde{B}_a(\tilde{x}) = \tilde{B}_2(\tilde{x})[\tilde{D}_{12}^T\tilde{D}_{12}]^{-\frac{1}{2}}$$
$$\tilde{C}_a(\tilde{x})\tilde{x} = \tilde{C}_2(\tilde{x})\tilde{x} + \tilde{D}_{21}F_1(\tilde{x}).$$

It can be shown that,

$$\begin{aligned} \dot{U}(\tilde{x}) = &-\frac{1}{\gamma^2}e^Te + d^Td - (d - F_1(\tilde{x}))^T(d - F_1(\tilde{x})) \\ &+ \frac{1}{\gamma^2}\left[\bar{u} - [\tilde{D}_{12}^T\tilde{D}_{12}]^{\frac{1}{2}}F_0(\tilde{x})\right]^T\left[\bar{u} - [\tilde{D}_{12}^T\tilde{D}_{12}]^{\frac{1}{2}}F_0(\tilde{x})\right] + \mathcal{H}_{FI}(U(\tilde{x}), \gamma). \end{aligned}$$

Then (5.11) suffices that, $\dot{U}(\tilde{x}) \leq -\frac{1}{\gamma^2}e^Te + d^Td - r^Tr + \frac{1}{\gamma^2}v^Tv$.

Integrating both sides over a finite time interval $[0, T], T > 0$, assuming $x(0) = 0$, we have

$$\int_0^T \left(\frac{1}{\gamma^2}v^Tv - r^Tr\right) \geq \int_0^T \left(\frac{1}{\gamma^2}e^Te - d^Td\right).$$

(5.12) This means, if (5.18) is asymptotically stable and $\|r\|_2 \leq \gamma\|v\|_2$, (5.10) is asymptotically stable too and the $\mathcal{H}_\infty$ norm is less than γ from d to e.

Treating (5.12) with Schur complements, one can derive

$$H_a(U, V, \gamma) + \sum_j \tilde{x}^T\bar{\Lambda}_{2j}(\tilde{x})\tilde{x}\phi_j(\tilde{x}) < 0, \tag{5.19}$$

in which $\bar{\Lambda}_{2j}$s are free matrices with compatible dimensions, and

$$\begin{aligned}
H_a(U,V,\gamma) = &\left\{ \frac{\partial V}{\partial \tilde{x}} \tilde{A}(\tilde{x})\tilde{x} - \frac{1}{2}\frac{\partial V}{\partial \tilde{x}} \tilde{B}_1(\tilde{x})\tilde{D}_{21}^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{C}_2(\tilde{x})\tilde{x} \right. \\
&- \frac{1}{2}\tilde{x}^T\tilde{C}_2^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{D}_{21}(\tilde{x})\tilde{B}_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}} \\
&+ \frac{1}{4}\frac{\partial V}{\partial \tilde{x}} \tilde{B}_1(\tilde{x}) \left[I - \tilde{D}_{21}^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{D}_{21}(\tilde{x})\right] \tilde{B}_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}} \\
&\left. +\tilde{x}^T \left[\gamma^{-2}\tilde{C}_1^T(\tilde{x})\tilde{C}_1(\tilde{x}) - \tilde{C}_2^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{C}_2^T(\tilde{x})\right] \tilde{x} \right\} \\
&- \gamma^{-2} \left\{ \frac{\partial U}{\partial \tilde{x}} \tilde{A}(\tilde{x})\tilde{x} - \frac{1}{2}\tilde{x}^T\tilde{C}_1^T(\tilde{x})\tilde{D}_{12}(\tilde{x}) \left[\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x})\right]^{-1} \tilde{B}_2^T(\tilde{x})\frac{\partial U^T}{\partial \tilde{x}} \right. \\
&- \frac{1}{2}\frac{\partial U}{\partial \tilde{x}} \tilde{B}_2(\tilde{x}) \left[\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x})\right]^{-1} \tilde{D}_{12}^T(\tilde{x})\tilde{C}_1(\tilde{x})\tilde{x} \\
&+ \frac{1}{4}\frac{\partial U}{\partial \tilde{x}} \left[\gamma^{-2}\tilde{B}_1(\tilde{x})\tilde{B}_1^T(\tilde{x}) - \tilde{B}_2(\tilde{x}) \left[\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x})\right]^{-1} \tilde{B}_2^T(\tilde{x})\right] \frac{\partial U^T}{\partial \tilde{x}} \\
&\left. \tilde{x}^T\tilde{C}_1^T(\tilde{x}) \left[I - \tilde{D}_{12}(\tilde{x}) \left[\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x})\right]^{-1} \tilde{D}_{12}^T(\tilde{x})\right] \tilde{C}_1(\tilde{x})\tilde{x} \right\}
\end{aligned} \tag{5.20}$$

Choosing $W(\tilde{x}) = V(\tilde{x}) - \gamma^{-2}U(\tilde{x}) \geq 0$ and substituting (5.16) and (5.17) into (5.20), (5.12) becomes a sufficient condition of

$$\begin{aligned}
H_a(W,\gamma) = &\frac{\partial W}{\partial \tilde{x}} \tilde{A}_a(\tilde{x})\tilde{x} - \frac{1}{2}\tilde{x}^T\tilde{C}_a^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{D}_{21}(\tilde{x})\tilde{B}_1^T(\tilde{x})\frac{\partial W^T}{\partial \tilde{x}} \\
&- \frac{1}{2}\frac{\partial W}{\partial \tilde{x}} \tilde{B}_1(\tilde{x})\tilde{D}_{21}^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{C}_a(\tilde{x})\tilde{x} \\
&+ \frac{1}{4}\frac{\partial W}{\partial \tilde{x}} \tilde{B}_1(\tilde{x}) \left[I - \tilde{D}_{21}^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{D}_{21}(\tilde{x})\right] \tilde{B}_1^T(\tilde{x})\frac{\partial W^T}{\partial \tilde{x}} \\
&+ \tilde{x}^T \left[\gamma^{-2}F_0^T(\tilde{x})\tilde{D}_{12}^T(\tilde{x})\tilde{D}_{12}(\tilde{x})F_0(\tilde{x}) - \tilde{C}_a^T(\tilde{x}) \left[\tilde{D}_{21}(\tilde{x})\tilde{D}_{21}^T(\tilde{x})\right]^{-1} \tilde{C}_a(\tilde{x})\right] \tilde{x} < 0,
\end{aligned} \tag{5.21}$$

[58] proved that, (5.18) can be asymptotically stabilized by the controller

$$\begin{aligned}
\dot{\tilde{x}}_c =& \tilde{A}_a(\tilde{x}_c)\tilde{x}_c + \tilde{B}_a(\tilde{x}_c) \left[\tilde{D}_{12}^T\tilde{D}_{12}\right]^{\frac{1}{2}} F_0(\tilde{x}_c) + L_0(\tilde{x}_c)\left[C_a(\tilde{x}_c)\tilde{x}_c - y\right] \\
u =& \left[\tilde{D}_{12}^T\tilde{D}_{12}\right]^{\frac{1}{2}} F_0(\tilde{x}_c)
\end{aligned} \tag{5.22}$$

and has its $\mathcal{H}_\infty$ norm less than γ in a local region, if there exists a positive semi-definite $W(\tilde{x})$ satisfying (5.21) and

$$\frac{\partial^2 W}{\partial \tilde{x}^2}(H_a(W,\gamma))\bigg|_{\tilde{x}=0} < 0. \tag{5.23}$$

It is not difficult to conclude that, control laws (5.15) and (5.22) are identical. *Q.E.D.*

Based on the above new definition of conditions in Theorem 7, we will have the following control design procedure to solve the nonlinear output feedback H_∞ synthesis condition for polynomial nonlinear systems (5.1):

1. Augment the system (5.1) to (5.10).

2. Based on (5.10), obtain a polynomial Lyapunov function $U(\tilde{x}) = \tilde{x}^T R^{-1} \tilde{x}$ by solving the optimization for a performance level $\gamma > 0$

$$\begin{aligned} &\min_{R>0} \ \gamma \\ &\text{s.t.} \quad -\tilde{z}_1^T \mathcal{M}_{FI} \tilde{z}_1 + \sum_i \tilde{z}_1^T \Lambda_i(\tilde{x}) \tilde{z}_1 \phi_i(\tilde{x}) + \sum_j \tilde{z}_1^T \Theta_j(\tilde{x}) \tilde{z}_1 \psi_j(\tilde{x}) \in \Phi_{\text{SOS}} \end{aligned} \tag{5.24}$$

   in which $\tilde{z}_1$ is an auxiliary vector with suitable dimension; $\Lambda_i(\tilde{x})$ represents symmetric multipliers to enforce the coupling constraints $\phi_i(\tilde{x}) = 0$ among augmented state variables; $\psi_j(\tilde{x}) < 0$ denotes optional regional constraints; $\Theta_j(\tilde{x}) > 0$ represents SOS multipliers to enforce the regional constraints.

3. Reformulate conditions (5.3)-(5.5) to obtain a polynomial Lyapunov function $V(\tilde{x}) =$

$\tilde{x}^T S\tilde{x}$ by solving the optimization for a performance level $0 < \hat{\gamma} < \gamma$

$$
\begin{aligned}
&\min_{S>0} \ \hat{\gamma} \qquad (5.25)\\
\text{s.t.} \quad &-\tilde{z}_2^T \mathcal{N}_{FC}\tilde{z}_2 + \sum_i \tilde{z}_2^T \Lambda_i(\tilde{x})\tilde{z}_2\phi_i(\tilde{x}) + \sum_j \tilde{z}_2^T \Theta_j(\tilde{x})\tilde{z}_2\psi_j(\tilde{x}) \in \Phi_{\text{SOS}}\\
&-\tilde{z}_3^T \left[\frac{\partial^2}{\partial\tilde{x}^2}(\mathcal{N}_{FC} - \gamma^{-2}\mathcal{H}_{FI})\bigg|_{\tilde{x}=0}\right]\tilde{z}_3 \in \Phi_{\text{SOS}}\\
&\tilde{x}^T(S - \gamma^{-2}R^{-1})\tilde{x} \in \Phi_{\text{SOS}},
\end{aligned}
$$

where $\tilde{z}_2, \tilde{z}_3$ are free vectors of compatible dimensions and $\Lambda_i(\tilde{x})$ represents symmetric multipliers to enforce the coupling constraints $\phi_i(\tilde{x}) = 0$ among augmented state variables; $\psi_j(\tilde{x}) < 0$ denotes regional constraints and $\Theta_j(\tilde{x}) > 0$ represents SOS multipliers to enforce the regional constraints.

4. After synthesizing higher order Lyapunov functions with quadratic form, $U(\tilde{x})$ and $V(\tilde{x})$ and the optimized performance index γ, an output feedback controller in the following form can be constructed as (5.15). The closed-loop H_∞ norm is guaranteed to be bounded by γ.

In the proposed control design procedure, both problems are solved efficiently using SOS programming techniques with polynomial complexity. The Lyapunov function for the closed-loop system is given by $U(\tilde{x}) + V(\tilde{x} - \tilde{x}_c) - \gamma^{-2}U(\tilde{x} - \tilde{x}_c)$. One thing need to be noticed is that, $L_0(\tilde{x})$ is not unique for a given $W(\tilde{x}) = V(\tilde{x}) - \gamma^{-2}U(\tilde{x})$. The proposed description is just one of the feasible solutions.

Although global stabilization problem was often the priority of most of previous nonlinear control research, its is not always possible to find a global stabilizing controller. Moreover, local controllers will often perform better than global controllers in a restricted region. To solve nonlinear $\mathcal{H}_\infty$ control problem locally, the SOS optimization problems (5.24) and (5.25) need additional region constraints. This will help improve solvability of the SOS decomposition

problems and lead to optimized solutions for nonlinear output feedback $\mathcal{H}_\infty$ control in a local region. These constraints also specify a state space region in which the synthesized nonlinear $\mathcal{H}_\infty$ control is valid.

5.3 Nonlinear $\mathcal{H}_\infty$ Control Designs for Axi-Symmetric Spacecraft

During the control design and simulation of spacecraft, the inherent nonlinear properties, for example, the Coriolis forces and gyroscopic cross coupling between pitch and yaw [28, 50], can hardly be ignored. Being highly sensitive to the unpredictable outer space operating environment, spacecrafts are commanded more and more important and challenging tasks using high performance flight control techniques. These challenging tasks, for example, retrieving a satellite, performing docking, releasing a payload from a shuttle orbiter, or even transporting large loads between spacecrafts, involve large amount of dynamic uncertainties by nature while need high tracking accuracy and maneuver capability spanning large range of angle. On the other hand, the limited storage space on board and high expense of propellant urgently require that, economical control input strategy to be considered during the performance optimization.

Because of the above realistic difficulties and advanced technical expectations, linear control designs based on its linearized dynamics are often inadequate, especially due to the significant cross-coupling effect [26]. The stabilizing feedbacks of a spacecraft with or without disturbance inputs have been studied for a long time, such as [89, 87, 82]. In general, the synthesis of optimal feedback control for a rigid body spacecraft remains a difficult problem and was mainly addressed in the context of time/fuel optimal maneuvers [47]. The main obstruction in obtaining optimal control laws is to efficiently solve the Hamilton-Jacobi-Isaacs (HJI) equation, especially when the cost includes a penalty term on the control effort.

To study rigid body spacecraft trajectory tracking problems, adaptive control [72, 73] and sliding mode control [21] were proposed. However, the full states have to be measured for these

control algorithms and with unmodeled dynamics, the robustness of controlled system cannot be guaranteed. [57] achieved rigid body attitude regulation without velocity measure by using a passivity-based control approach. The effort of using the nonlinear observers to avoid the velocity measurement has been observed in [16]. Unfortunately, these proposed output feedback controllers do not have desired optimality and robustness properties.

5.3.1 Modeling of Axi-Symmetric Spacecraft

In this research, we will focus on the command tracking problem for a spinning axi-symmetric rigid spacecraft with two control torques. No like those works [2, 3, 4, 5, 11, 14, 62, 75] focusing on the stabilization of angular velocities, we adopt the formulation of the kinematic equations developed and studied in [80, 81, 71] for an axi-symmetric rigid body spacecraft model [81] with two control torques acting perpendicular to the symmetry axis and to each other. It is well known that three jet actuators can be used to accomplish arbitrary reorientation maneuvers of a rigid spacecraft using smooth feedback [54]. However, [53] showed that the restricted (nonspinning spacecraft) dynamics cannot be asymptotically stabilized using smooth feedback. The situation considered in this research may arise due to the failure of the actuator on the symmetric axis of the spacecraft. Since we are considering a space operation, the optimization of the command tracking performance of a spacecraft operating in an actuator failure mode, is an important control problem.

Firstly, we need to introduce two reference frames (see Fig. 5.1 and [80, 81, 71]) to describe the spacecraft dynamics. One is the body-fixed reference frame $\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$, and the other one is the inertial reference frame $\mathbf{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$. Secondly, we will use a set of parameters to describe the relative orientation of these two frames. Finally, the kinematic equations will be provided to reflect a relationship between the angular velocity vector and the rates of these parameters.

The body-fixed reference frame $\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ locates at the center of mass and aligned along the principal axes of a rotating rigid body. In the body-fixed reference frame, $\omega_1, \omega_2, \omega_3$

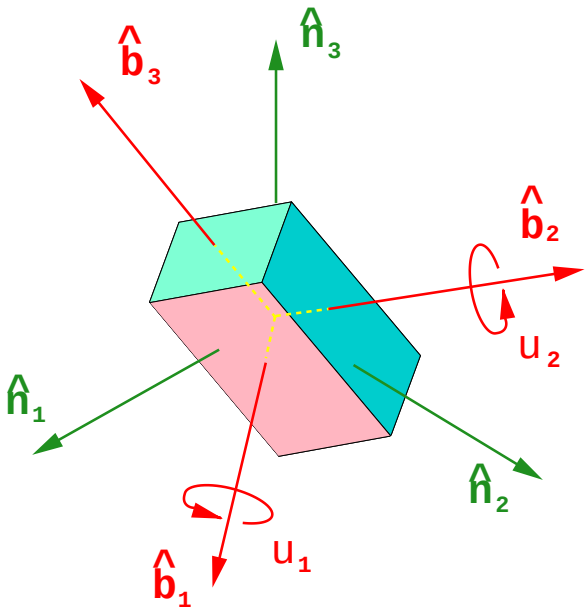


Figure 5.1: Axi-symmetric rigid body with two controls.

and I_1, I_2, I_3 are used to denote the angular velocity vector and moment of inertia to three body axes, respectively. In this model, $\hat{b}_3$ is assumed to point along the symmetry axis and $I_1 = I_2$. As displayed in Fig. 5.1, the control torques T_1 and T_2 active along the $\hat{b}_1$ and $\hat{b}_2$ axis. Clearly they are perpendicular to each other and span the two-dimensional plane orthogonal to the symmetric axis. Furthermore, the rotational motion of this spacecraft can be described by the famous Euler's equations, taking the form of

$$\dot{\omega}_1 = \frac{(I_2 - I_3)}{I_1}\omega_2\omega_3 + \frac{T_1}{I_1} \tag{5.26}$$

$$\dot{\omega}_2 = \frac{(I_3 - I_1)}{I_2}\omega_3\omega_1 + \frac{T_2}{I_2} \tag{5.27}$$

$$\dot{\omega}_3 = \frac{(I_1 - I_2)}{I_3}\omega_1\omega_2 = 0.$$

As a reminder, the spacecraft in static is not controllable using smooth control [53]. So the condition $\omega_3(t) = \omega_{30} \neq 0$ is assumed to render smooth nonlinear control laws, which means the spacecraft rotates constantly about the symmetric axis.

Besides the body-fixed reference frame, we also introduce an inertial reference frame $\mathbf{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$. The two reference frames are related by a rotation matrix R. The three dimensional rotation group, which consists of all matrices sharing the same form with R, is denoted by $SO(3)$. The entries of this set are orthogonal to each other and have determinant of 1.

Because of this, we can use a curve traced by the corresponding rotation vector $R(t) \in SO(3)$ to describe the attitude trajectory of the moving reference frame with respect to the inertial reference frame.

Given by Poisson's system of equations, $R(t)$ satisfies the following differential equation while moving along the trajectory

$$\dot{R} = S(\omega_1, \omega_2, \omega_3)R,$$

where $S(\omega_1, \omega_2, \omega_3)$ is a skew-symmetric matrix and is defined as

$$S(\omega_1, \omega_2, \omega_3) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

There are many ways to parameterize the rotation group, such as Eulerian angles formulation, quaternion, Rodrigues parameters, etc. In this research, we will adopt a parameterization of $SO(3)$ from [81] based on a 3-2-1 Eularian angle sequence [48], using two perpendicular rotations to describe the relative orientation of two reference frames. One rotation provides the description of the location of the designated body axis, which is the spin-axis in this study, in the inertial frame. Another one is an initial rotation about this axis. One of the most important advantages of this rotation matrix's formulation, specifically for our case, is that with the help of a stereographic projection [20], the new parameterization will reduce the number of kinematic variables from three to two.

As a brief introduction, a stereographic projection can be used to describe the location of

inertial axis $\hat{n}_3$ in the body-fixed $\mathbf{b}$ frame by two variables w_1 and w_2

$$w_1 = \frac{\cos\theta_2}{1+\cos\theta_3} \qquad w_2 = \frac{-\cos\theta_1}{1+\cos\theta_3},$$

where θ_1, θ_2 and θ_3 are directional angles of the axis $\hat{n}_3$ with respect to $\mathbf{b}$ frame, i.e., $\hat{n}_3 = \cos\theta_1\hat{b}_1 + \cos\theta_2\hat{b}_2 + \cos\theta_3\hat{b}_3$ with $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$. The stereographic projection is well defined for all orientations except for one, which is $\hat{n}_3$ pointing to the negative direction of $\hat{b}_3$, which means $\cos\theta_1 = 0$, $\cos\theta_2 = 0$ and $\cos\theta_3 = -1$. It is easy to verify that its inverse map is given by

$$\begin{aligned}
\cos\theta_1 &= \frac{-2w_2}{1+w_1^2+w_2^2} \\
\cos\theta_2 &= \frac{2w_1}{1+w_1^2+w_2^2} \\
\cos\theta_3 &= \frac{1-w_1^2-w_2^2}{1+w_1^2+w_2^2}
\end{aligned}$$

and can be used to determine $\cos\theta_1, \cos\theta_2, \cos\theta_3$ for the given w_1, w_2. Under this projection, w_1 and w_2 satisfy the differential equations

$$\dot{w}_1 = \omega_{30}w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2) \tag{5.28}$$

$$\dot{w}_2 = -\omega_{30}w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2) \tag{5.29}$$

Define the states as $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} \omega_1 & \omega_2 & w_1 & w_2 \end{bmatrix}^T$, a state-space model of the axi-symmetric spacecraft will be derived from equation (5.26)-(5.27) and (5.28)-(5.29)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 \\ -am & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{in which,} \tag{5.30}$$

$$a = \frac{I_2 - I_3}{I_1}, \quad m = \omega_{30}, \quad u_1 = \frac{T_1}{I_1}, \quad u_2 = \frac{T_2}{I_2},$$

and $-1 < a < 1$ is assumed for physical consideration. In the control synthesis and simulation processes we will assume $a = 0.5$ and $m = -0.5$ rad/sec. From the model (5.30), we can clearly see that, only the two angular velocities can be controlled and the polynomial nonlinear kinematic equations with degree higher than two are driven by the angular velocities.

5.3.2 Nonlinear $\mathcal{H}_\infty$ Control Design and Simulation of the Command Tracking Problem

We are interested in solving the command tracking problems for the axi-symmetric spacecraft using computationally efficient nonlinear $\mathcal{H}_\infty$ control techniques via SOS programming together with higher order Lyapunov functions. Two facts motivated this exploration. First, spacecraft dynamic (5.30) represents its attitude dynamics by highly nonlinear polynomial vector field, which means the highest degree of the involved monomials is larger than two. Moreover, the structure of LMIs $\mathcal{M}_{FC}$ in (5.24) and $\mathcal{N}_{FC}$ in (5.25) are considerably complicated. Especially, $\mathcal{N}_{FC}$ even increased the degree of each entries in $(1,1)$ block by two with the previous and post multiplication of the state vector. This observation naturally leads to another two related expectations: quadratic Lyapunov functions might meet problems to render the output estimation condition as a feasible SOS decomposition; if this really happened, alternatively, higher order Lyapunov functions may not only be able to overcome this challenge, but also have a chance to rewrite the system matrix as a lower degree one to pursue better $\mathcal{H}_\infty$ performance.

To achieve accurate command tracking with output feedback under the external disturbances such as modeling uncertainty, actuator and sensor inaccuracies, the spacecraft is desired to follow a command trajectory with tracking error as small as possible. The disturbance vector $\begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}^T$ includes disturbances d_1, d_2 along $\hat{b}_1$ and $\hat{b}_2$ axes, output measurement noises d_3, d_4, and the desired x_3 trajectory $x_{3d} = d_5$. The nonlinear control design objective is to minimize the effect of these disturbances on the tracking error $(x_3 - x_{3d})$ with reasonable

control force.

For this command tracking problem, we will specifically introduce a new state variable $x_t = \frac{\tau}{s}(x_3 - x_{3d})$ to eliminate the tracking error asymptotically, in which x_t reflects the accumulated effect of tracking error over the past time and τ is a pre-specified integration coefficient. By incorporating the dynamics of x_t into the original spacecraft model and redefining output variables, the modified system including disturbance d and controlled output variable e becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 & 0 \\ -am & 0 & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m & 0 \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 & 0 \\ 0 & 0 & \tau & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_t \end{bmatrix} + \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.31}$$

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.32}$$

$$y = \begin{bmatrix} 0 & 0 & 325.38 & 0 & 0 \\ 0 & 0 & 0 & 344.38 & 0 \\ 0 & 0 & 0 & 0 & 99.556 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}. \tag{5.33}$$

The new state variable is $\bar{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_t \end{bmatrix}$. The output y include measured attitude w_1, w_2 and accumulated tracking error x_t. Note that this plant for command tracking share the same plant structure of (5.1) and satisfies corresponding assumptions.

Following the design procedure discussed in Section 5.2, and also like all of the problems we handled in previous chapters, the first step is to augment the system (5.31)-(5.33) based on power transformation. However, the state vector with five elements obviously distinguishes this example from those second order ones studied before. if we augment this vector entirely, only with a power of two, the new augmented system will have 20 states. in that case, the LMI conditions along with the increased coupling and regional constraints will create a heavy computational burden that are extremely hard, if not impossible, to be handled by most of the main stream computers as well as the most adopted computational softwares in the time of this thesis. Even if this difficulty can be solved by the efficient utility of large computer memories, the large time consumption is not tolerable. Moreover, not all of the augmented states are necessarily helpful for our optimization. In short, full monomial augmentation for large scaled system with complicated LMI conditions is neither realistic nor necessary at lease for now.

Fortunately, partial augmentation with selected states is good enough to improve the controlled performance with a limited increase on memory use and computing time. Since only the derivatives of x_3 and x_4 are nonlinear, we can add as few as three augmented states, $\tilde{x}_5 = x_3^2$

$\tilde{x}_6 = x_3 x_4$ and $\tilde{x}_7 = x_4^2$. Then, one of the augmented systems can be written as

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \\ \dot{\tilde{x}}_5 \\ \dot{\tilde{x}}_6 \\ \dot{\tilde{x}}_7 \\ \dot{\tilde{x}}_t \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 & 0 & 0 & 0 & 0 \\ -am & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & m & \frac{1}{2}\tilde{x}_1 & \tilde{x}_2 & -\frac{1}{2}\tilde{x}_1 & 0 \\ 0 & \frac{1}{2} & -m & 0 & -\frac{1}{2}\tilde{x}_2 & \tilde{x}_1 & \frac{1}{2}\tilde{x}_2 & 0 \\ 0 & 0 & \tilde{x}_1 & 0 & \tilde{x}_1\tilde{x}_3 & 2m + 2\tilde{x}_2\tilde{x}_3 & -\tilde{x}_1\tilde{x}_3 & 0 \\ 0 & 0 & \frac{1}{2}\tilde{x}_2 & \frac{1}{2}\tilde{x}_1 & \begin{pmatrix} \frac{1}{2}\tilde{x}_1\tilde{x}_4 - \\ \frac{1}{2}\tilde{x}_2\tilde{x}_3 - m \end{pmatrix} & \tilde{x}_2\tilde{x}_4 + \tilde{x}_1\tilde{x}_3 & \begin{pmatrix} \frac{1}{2}\tilde{x}_2\tilde{x}_3 - \\ \frac{1}{2}\tilde{x}_1\tilde{x}_4 + m \end{pmatrix} & 0 \\ 0 & 0 & 0 & \tilde{x}_2 & -\tilde{x}_2\tilde{x}_4 & 2\tilde{x}_1\tilde{x}_4 - 2m & \tilde{x}_2\tilde{x}_4 & 0 \\ 0 & 0 & \tau & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_t \end{bmatrix}$$

$$+ \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.34}$$

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.35}$$

$$y = \begin{bmatrix} 0 & 0 & 325.38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 344.38 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 99.556 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}. \tag{5.36}$$

Note that, only with the help of system augmentation, the power of nonlinearities of $\dot{x}_3$ and $\dot{x}_4$ can be degraded from three to two by using the new states and relocate some of the monomials to new positions.

To solve the command tracking problem in output feedback form, we set $\tau = 1$ and choose both $U(\bar{x})$ and $V(\bar{x})$ as 4^{th} order Lyapunov functions, but with quadratic forms of the augmented state vector $\tilde{x} = [\tilde{x}_1 \;\; \tilde{x}_2 \;\; \tilde{x}_3 \;\; \tilde{x}_4 \;\; \tilde{x}_5 \;\; \tilde{x}_6 \;\; \tilde{x}_7 \;\; \tilde{x}_t]^T$. We will specify the feasible state region in terms of spacecraft angular velocity and attitude as the intersection of $x_3^2 + x_4^2 - 0.5 < 0$ and $x_1^2 + x_2^2 - 0.5 < 0$.

Applying the control design procedure in Section 5.2 with SOS programming, we obtain

$\gamma = 0.9156$

$U(\tilde{x}) = \tilde{x}^T P_4 \tilde{x}$, where P_4 is given by:

$$\begin{bmatrix} 0.2219 & -0.0386 & 1.2253 & -0.2415 & 0 & 0 & 0 & 1.5143 \\ -0.0386 & 0.0857 & -0.2854 & 0.2596 & 0 & 0 & 0 & -0.3525 \\ 1.2253 & -0.2854 & 7.2996 & -1.6083 & 0 & 0 & 0 & 9.2168 \\ -0.2415 & 0.2596 & -1.6083 & 1.0058 & 0 & 0 & 0 & -1.9931 \\ 0 & 0 & 0 & 0 & 0.0259 & -0.0023 & -0.0198 & 0 \\ 0 & 0 & 0 & 0 & -0.0023 & 0.0963 & -0.0042 & 0 \\ 0 & 0 & 0 & 0 & -0.0198 & -0.0042 & 0.0314 & 0 \\ 1.5143 & -0.3525 & 9.2168 & -1.9931 & 0 & 0 & 0 & 12.2192 \end{bmatrix}$$

$\hat{\gamma} = 0.9000$

$V(\tilde{x}) = \tilde{x}^T S_4 \tilde{x}$, where S_4 is given by:

$$10^3 \times \begin{bmatrix} 0.0119 & 0.0000 & -0.1410 & -0.0012 & 0 & 0 & 0 & 0.0002 \\ 0.0000 & 0.0061 & 0.0032 & -0.1082 & 0 & 0 & 0 & 0.0002 \\ -0.1410 & 0.0032 & 2.2330 & -0.0217 & 0 & 0 & 0 & -0.0099 \\ -0.0012 & -0.1082 & -0.0217 & 2.1473 & 0 & 0 & 0 & -0.0076 \\ 0 & 0 & 0 & 0 & 0.0258 & 0.0000 & -0.0012 & 0 \\ 0 & 0 & 0 & 0 & 0.0000 & 0.0416 & 0.0000 & 0 \\ 0 & 0 & 0 & 0 & -0.0012 & 0.0000 & 0.0253 & 0 \\ 0.0002 & 0.0002 & -0.0099 & -0.0076 & 0 & 0 & 0 & 0.0510 \end{bmatrix}$$

With these results, it is straightforward to calculate $F_0(\tilde{x}), F_1(\tilde{x}), L_0(\tilde{x})$ and construct the nonlinear command tracking control law in output feedback form (5.15). The synthesis results are

listed as following:

$$F_0 = \begin{bmatrix} -22.1944\tilde{x}_1 + 3.8570\tilde{x}_2 - 122.5282\tilde{x}_3 + 24.1453\tilde{x}_4 - 151.4306\tilde{x}_t \\ 3.8570\tilde{x}_1 - 8.5651\tilde{x}_2 + 28.5428\tilde{x}_3 - 25.9567\tilde{x}_4 + 35.2549\tilde{x}_t \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0.1324\tilde{x}_1 - 0.0230\tilde{x}_2 + 0.7308\tilde{x}_3 - 0.1440\tilde{x}_4 + 0.9032\tilde{x}_t \\ -0.0460\tilde{x}_1 + 0.1022\tilde{x}_2 - 0.3405\tilde{x}_3 + 0.3096\tilde{x}_4 - 0.4206\tilde{x}_t \\ 0 \\ 0 \\ -1.8064\tilde{x}_1 + 0.4206\tilde{x}_2 - 10.9946\tilde{x}_3 + 2.3776\tilde{x}_4 - 14.5761\tilde{x}_t \end{bmatrix}$$

$$L_0 = \begin{bmatrix} -2.4038 & 0.6707 & -11.6604 \\ 1.0618 & -8.2630 & 8.6528 \\ -0.1939 & 0.0533 & -0.8525 \\ 0.0504 & -0.4570 & 0.3958 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.2260 & 0.1268 & -11.2012 \end{bmatrix}$$

Note that the synthesized controller does not rely on the tracking trajectory explicitly, therefore it is capable to track difference command signals. The usage of augmentation with more states has also been attempted to solve the same problem. A full augmentation of four states (x_1-x_4) plus the tracking error state x_t, which consists of 15 elements can obtain the same γ value as above. However, the computational burden associated for the output estimation process cannot be handled. Two computers have been used: a laptop with Intel i7 2760QM 2.4GHz CPU with four cores, eight threads and a memory of 4GB; and a workstation with Intel E3-1225 3.1GHz CPU with four cores, four threads and a memory of 8GB. Both of them can be considered as

high-end machines at the time of writing. Regardless of the failure of the output design step, the state feedback synthesis using this 15×15 Lyapunov matrix does not provide a better result, after almost six hours of computation, while the partially augmented system uses about one and a half hour on the same machine. This fact indicates that, the use of a partial augmentation with the suitable states are more practical than a full state augmentation.

Moreover, the same $\mathcal{H}_\infty$ design problem has been attempted by using quadratic Lyapunov functions as well. The state feedback synthesis can be finished. Unfortunately, the search for a quadratic Lyapunov function to render the output estimation condition as SOS decomposition failed. In order to finish the comparison between using quadratic and higher order Lyapunov functions, V has to be considered as a higher order one. By choosing $\tilde{x}$ of the output feedback optimization as $[x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_1^2 \;\; x_2^2 \;\; x_t]^T$, we obtain

$$\gamma = 1.1798$$

$U(x) = x^T P_2 x$, where P_2 is given by:

$$\begin{bmatrix} 0.1987 & -0.0226 & 1.2364 & -0.3415 & 1.1936 \\ -0.0226 & 0.1553 & -0.3848 & 0.3074 & -0.3104 \\ 1.2364 & -0.3848 & 9.4647 & -2.8787 & 9.1810 \\ -0.3415 & 0.3074 & -2.8787 & 1.3367 & -2.6534 \\ 1.1936 & -0.3104 & 9.1810 & -2.6534 & 9.5473 \end{bmatrix}$$

$\hat{\gamma} = 0.9000$

$V(\tilde{x}) = \tilde{x}^T S_2 \tilde{x}$, where S_2 is given by:

$$10^3 \times \begin{bmatrix} 0.0152 & 0.0001 & -0.1784 & -0.0035 & 0 & 0 & 0 \\ 0.0001 & 0.0061 & 0.0009 & -0.1157 & 0 & 0 & 0 \\ -0.1784 & 0.0009 & 2.4533 & 0.0220 & 0 & 0 & -0.0025 \\ -0.0035 & -0.1157 & 0.0220 & 2.4164 & 0 & 0 & -0.0004 \\ 0 & 0 & 0 & 0 & 0.0014 & 0.0011 & 0 \\ 0 & 0 & 0 & 0 & 0.0011 & 0.0012 & 0 \\ 0 & 0 & -0.0025 & -0.0004 & 0 & 0 & 0.0430 \end{bmatrix}$$

the matrices $F_0(\tilde{x}), F_1(\tilde{x}), L_0(\tilde{x})$ can be constructed:

$$F_0 = \begin{bmatrix} -19.8702x_1 + 2.2617x_2 - 123.6381x_3 + 34.1515x_4 - 119.3556x_t \\ 2.2617x_1 - 15.5288x_2 + 38.4788x_3 - 30.7421x_4 + 31.0434x_t \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0.0714x_1 - 0.0081x_2 + 0.4441x_3 - 0.1227x_4 + 0.4287x_t \\ -0.0162x_1 + 0.1116x_2 - 0.2764x_3 + 0.2209x_4 - 0.2230x_t \\ 0 \\ 0 \\ -0.0857x_1 + 0.0223x_2 - 0.6596x_3 + 0.1906x_4 - 0.6859x_t \end{bmatrix}$$

$$L_0 = \begin{bmatrix} -3.3512 & 0.8925 & -10.1226 \\ 1.5236 & -9.8095 & 12.3456 \\ -0.2810 & 0.0757 & -0.7965 \\ 0.0715 & -0.5060 & 0.5937 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.1659 & 0.1310 & -11.5454 \end{bmatrix}$$

then the nonlinear command tracking control can constructed in output feedback form (5.6).

To compare the tracking performance of the controlled spacecraft, we firstly set d_1 to d_4 to be zero and x_{3d} as a trapezoidal profile. Starting from zero initial condition $\bar{x} = 0$, we obtain the simulation results as shown in Fig. 5.2 (a). It is observed that the tracking performance of the controller using 4^{th} order Lyapunov function is obviously better than the performance of the one based on a quadratic Lyapunov function. Moreover, Fig. 5.3 shows the trajectories of the controller inputs for both of the controllers separately. The magnitudes of u_1 and u_2 in Fig. 5.3 (a) are much smaller than those in Fig. 5.3 (b). This comparison shows that, by using $\mathcal{H}_\infty$ control, with higher order Lyapunov functions, instead of the traditional quadratic ones, one can not only improve the solvability of output feedback estimation, but also achieve better

command tracking performance with far less fuel consumption of the jet actuators.

Secondly, we keep d_5 unchanged and set external disturbances $d_1 \sim d_4$ as random white noises with magnitudes of 0.01. Starting from the same initial point, we obtain the simulation results as shown in Fig. 5.2 (b) and the control input trajectories can be shown in Fig. 5.4. It turns out that with small random noises, both of the command tracking performances are not affected significantly. The system behavior is similar to that of no noise. and the 4^{th} order one still shows its advantages.

The trajectories in Fig. 5.2 clearly show a delay gap between the command signal and controlled x_3 during the climbing and the descending period. The responding speed of each system with respect to commands is reflected by the pre-defined integration coefficient τ in the dynamics of the tracking error state x_t in both of the original system (5.31)-(5.33) and the augmented systems (5.34)-(5.36). A larger τ will shrink the delay time when tracking the command, but might increase the difficulty of stabilization. Therefore, it is meaningful to exam the proposed control method with higher order Lyapunov function to deal with a faster system.

Now, we increase τ from 1 to 20 but keep the regional restrictions $x_3^2 + x_4^2 - 0.5 < 0, x_1^2 + x_2^2 - 0.5 < 0$ and try to repeat the design procedure to synthesize a controller that can track the same signal faster. For the solvability consideration, the dynamics of the augmented controlled output e is modified as:

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.014 & 0 \\ 0 & 0.014 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since the proportional relationship does not change, the weighting of tracking performance and input cost remains the same. Following the same design procedure, both of the 4^{th} order based and 2^{nd} order based output feedback controllers can be synthesized and the corresponding $\mathcal{H}_\infty$ indices are $\gamma_4 = 0.9404$ and $\gamma_2 = 1.1727$, respectively. The simulation results to track the same trapozoidal command signal as in Fig. 5.2 without disturbance can be found in Fig. 5.5. From these plots we can easily observe that, the controller based on 4^{th} order Lyapunov functions successfully decreases the delay time as well as finishes the command tracking mission. Unfortunately, on the other side, despite an encouraging kickoff, the quadratic Lyapunov function based controller fails to stabilize the system in the middle of the tracking procedure. This failure might be caused by the shrink of the ROA of the closed-loop system using quadratic Lyapunov functions.

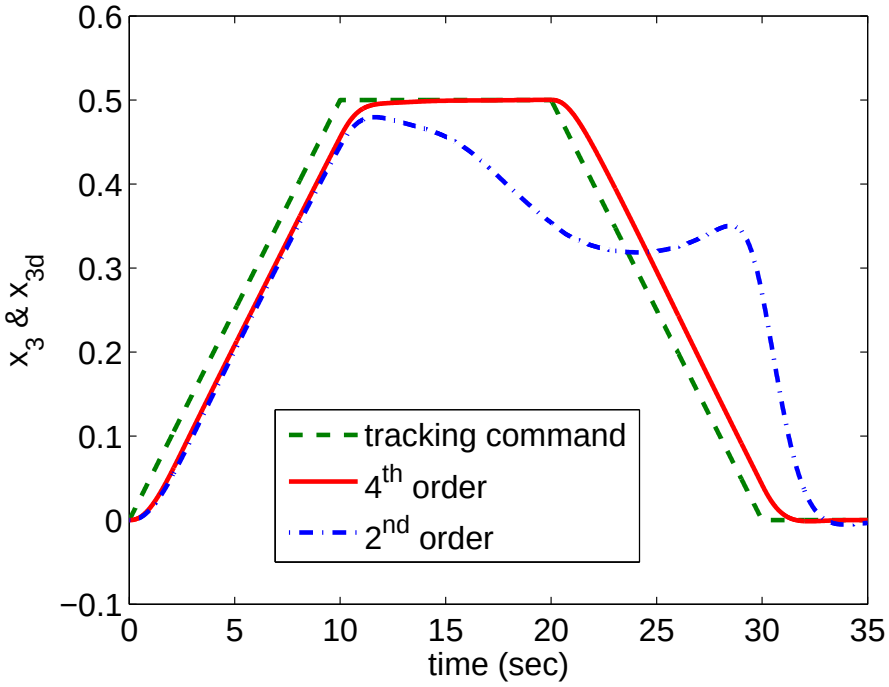
Because of the high dimension of the system dynamic, which is five, the visualization of the closed-loop system's stabilization region would be difficult. The attempts to project it onto coordinates with less dimensions might be conservative for this case by ignoring the evolution of the rest states. Therefore, it is not easy to provide a direct proof to support the speculation of the reason causing the failure of the controller based on quadratic Lyapunov functions.

Fig. 5.6 (a)-(b) list two additional command tracking simulation results with two different maximum command signal levels: 0.48 and 0.52. Comparing Fig. 5.6 (a)-(b) and Fig. 5.5 (a) together, clearly we can see that when the level increases, the starting unstable timing point of the 2^{nd} simulation moves earlier. This may exclude numerical issue as a failure reason.

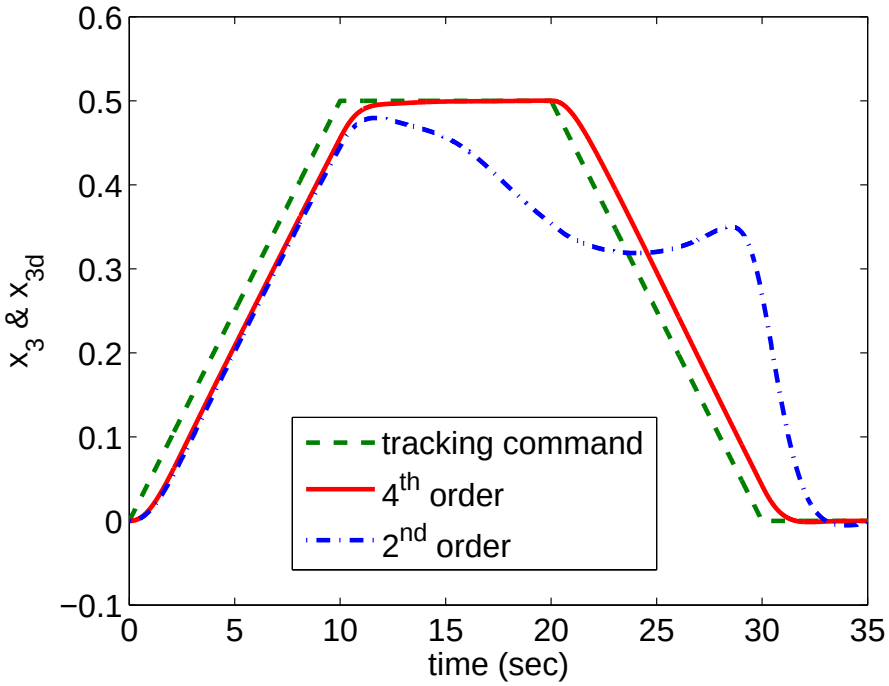
Therefore, it is reasonable to conclude that, at least some points on the tracking trajectory stay out of the ROA of the quadratically controlled system. Fortunately, the ROA of quartically controlled system can embrace the entire trajectory to fulfill the duty to track the specific command.

5.4 Summary

In this chapter, a computational $\mathcal{H}_\infty$ control design procedure has been proposed for a nonlinear output feedback $\mathcal{H}_\infty$ condition with the use of state augmentation to derive higher order Lyapunov functions and the associated output feedback controller. With this procedure, we have developed nonlinear $\mathcal{H}_\infty$ control laws for axi-symmetric spacecraft trajectory tracking problems with two control torques. Since the spacecraft attitude dynamics is in a polynomial nonlinear form, we are able to take advantage of the SOS programming techniques to solve related control design problems. Specifically, the complicated HJI inequalities together with the high order nonlinearities of the spacecraft system model make it difficult, if not impossible, for quadratic Lyapunov functions to render the output feedback estimation condition to be SOS decomposable. The proposed design approach has successfully achieved better controlled performance for the axi-symmetric spacecraft model under different situations.

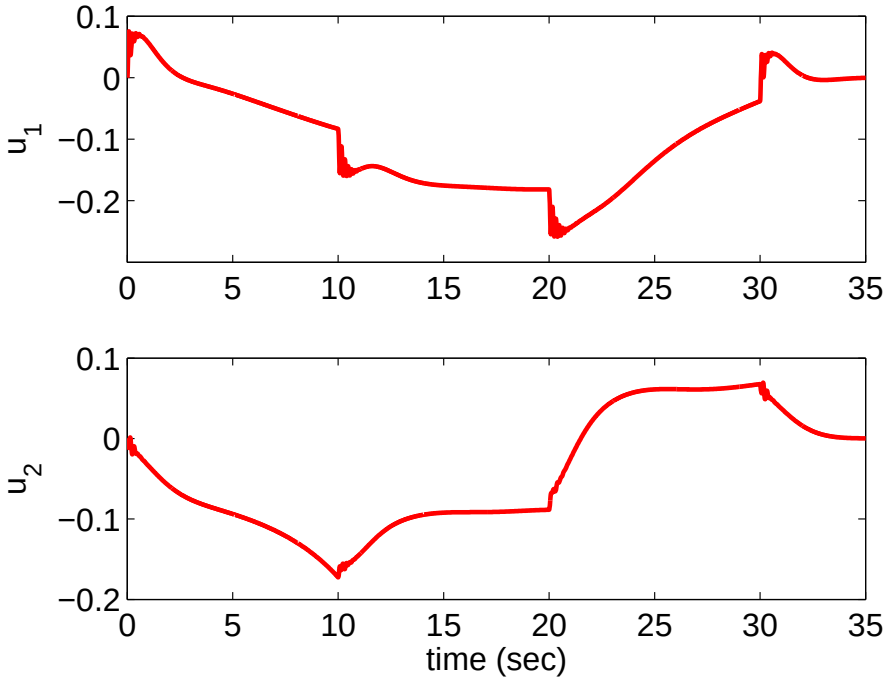


(a) without disturbance

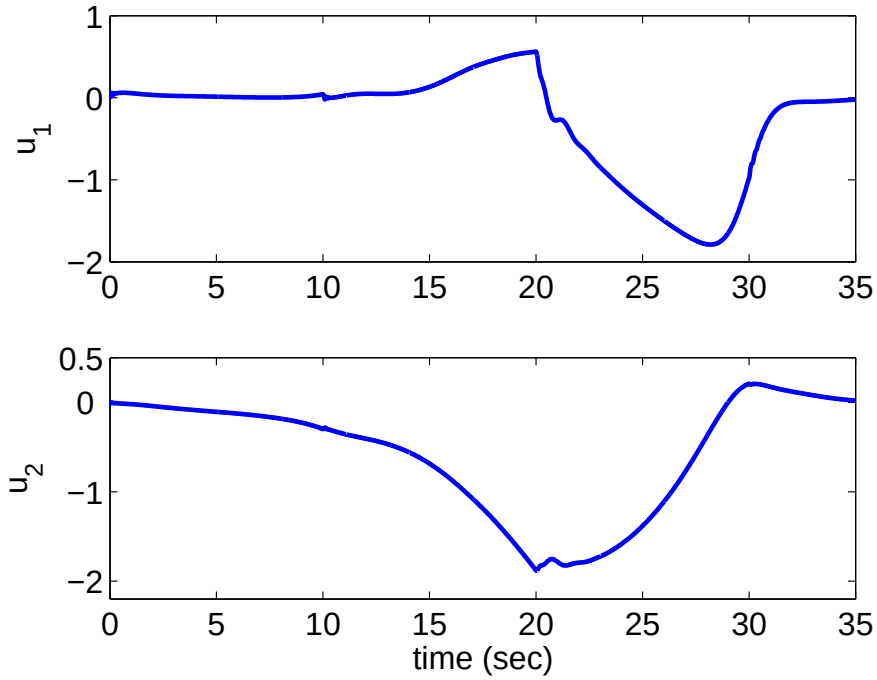


(b) with disturbance

Figure 5.2: Comparison between spacecraft tracking performance using 2^{nd} and 4^{th} order Lyapunov functions, $\tau = 1$.

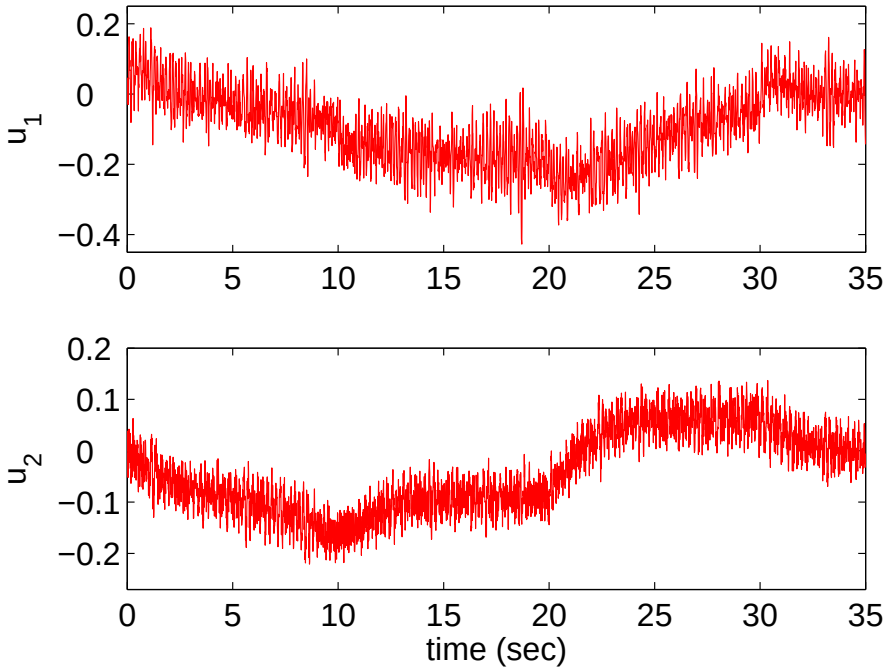


(a) using 4^{th} order Lyapunov function

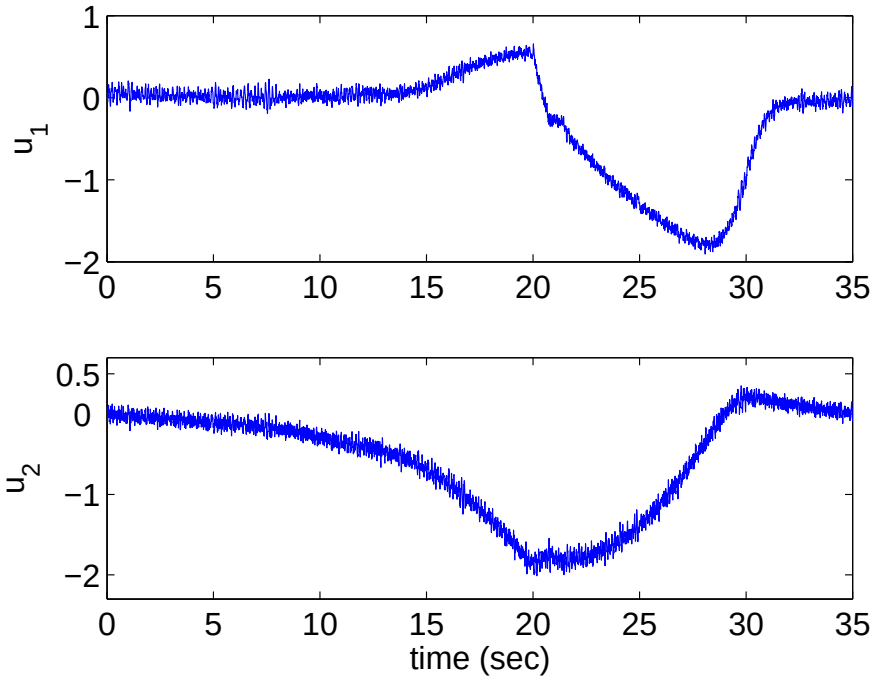


(b) using 2^{nd} order Lyapunov function

Figure 5.3: Control inputs of spacecraft command tracking, $\tau = 1$, without disturbance.

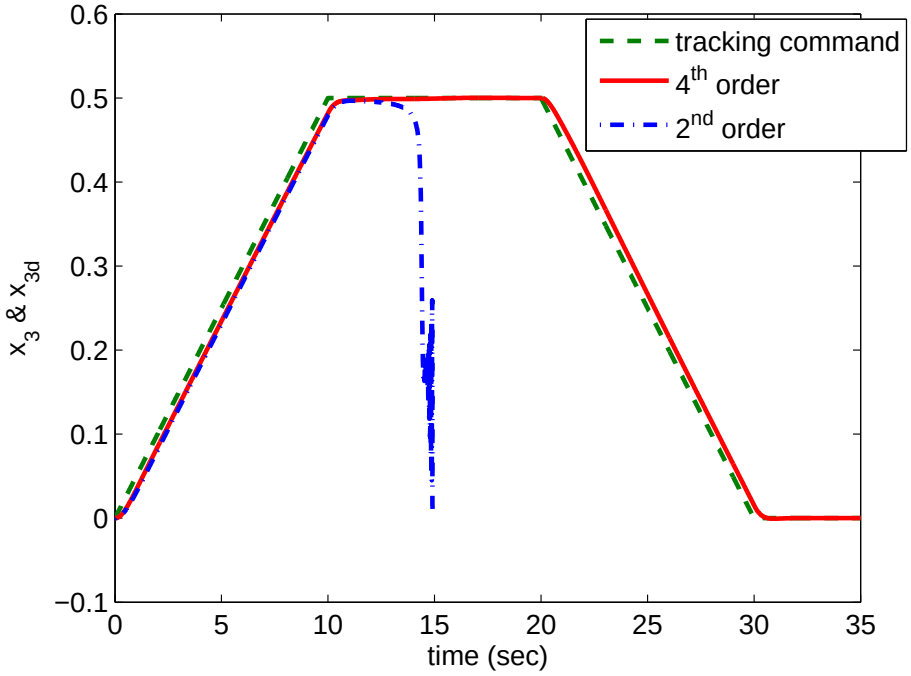


(a) using 4^{th} order Lyapunov function

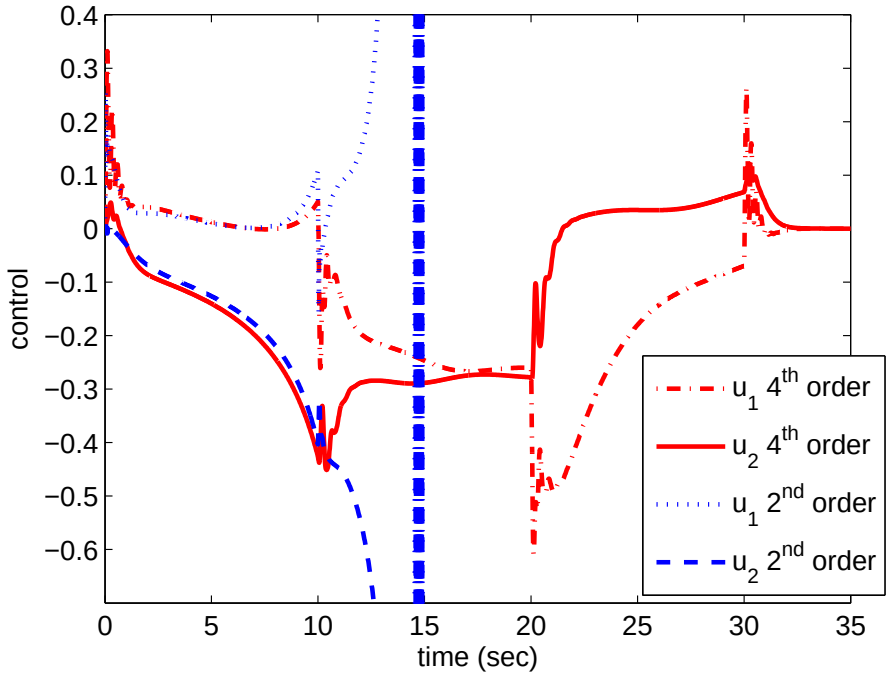


(b) using 2^{nd} order Lyapunov function

Figure 5.4: Control inputs of spacecraft command tracking, $\tau = 1$, with disturbance.

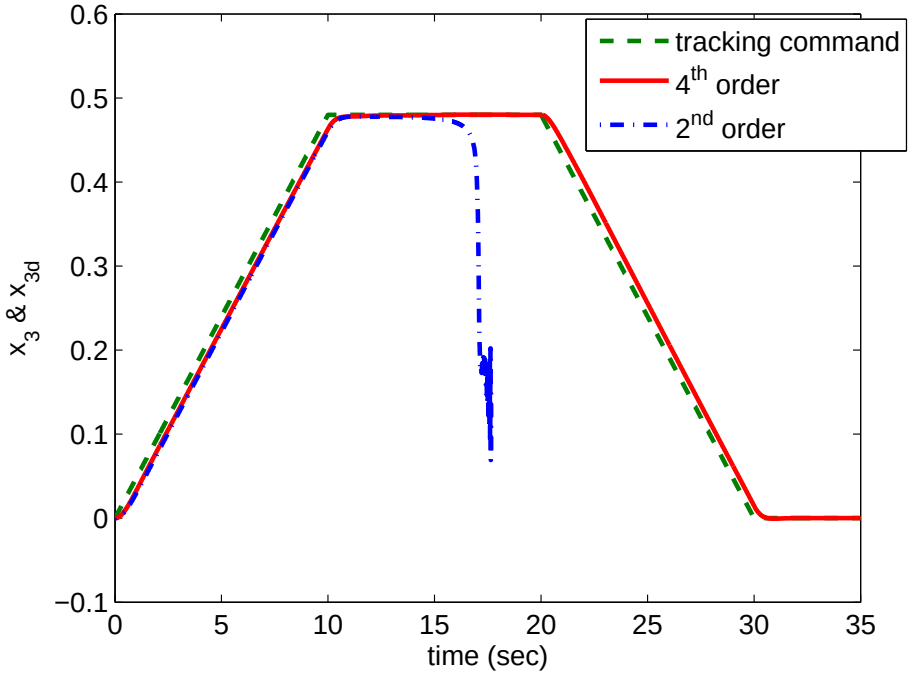


(a) command tracking performance

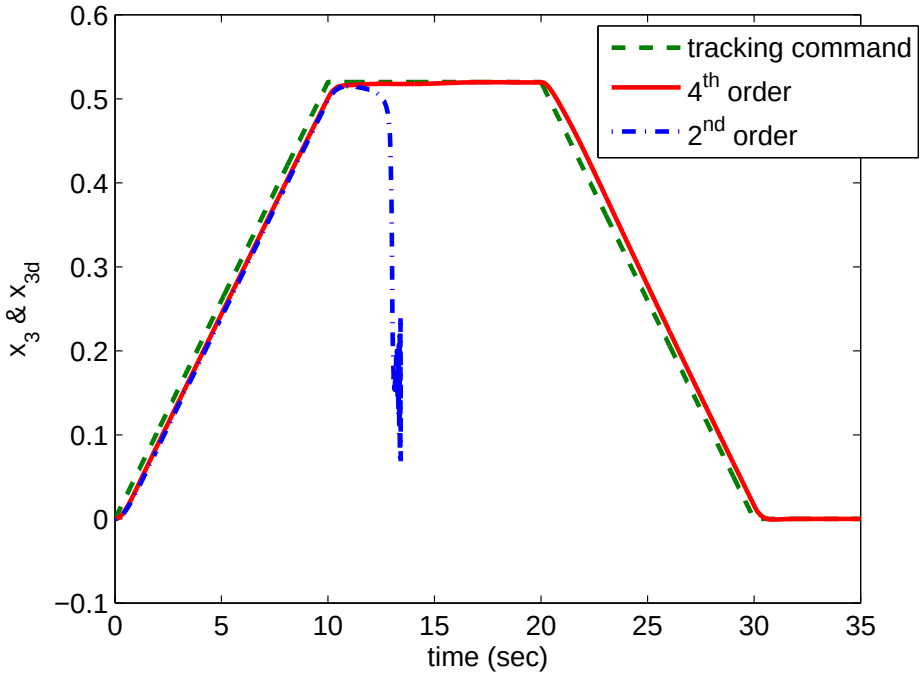


(b) control inputs

Figure 5.5: Comparison between spacecraft command tracking simulation using 2^{nd} and 4^{th} order Lyapunov functions, $\tau = 20$, without disturbance.



(a) maximum of x_{3d} is 0.48



(b) maximum of x_{3d} is 0.52

Figure 5.6: Comparison between spacecraft command tracking simulation using 2^{nd} and 4^{th} order Lyapunov functions, $\tau = 20$, without disturbance.

Chapter 6

Conclusion

6.1 Dissertation Outcome and Contribution

By using Sum Of Squares (SOS) computational tools, this dissertation investigated the optimal control design methods of polynomial nonlinear systems based on system augmentation. Extensive efforts had been devoted and revealed the advantage of higher order Lyapunov functions over the traditionally vastly adopted quadratic ones to improve the solvability and performance of optimal control strategies. The advantage was shown by overcoming control challenges in different aspects, including state and output feedback control problems, systems with and without actuator saturations, along with different optimization objectives, such as $\mathcal{H}_\infty$ control and ROA expansion.

The detailed benefits of the proposed method is summarized as following.

- System augmentation enables us to describe the system nonlinearities in an equivalent form but more flexible ways with increased power degree. It is well known that, the representation of state space system matrices is not unique. Without prior-knowledge and experience, the decision to choose the specific form of a nonlinear system to derive optimal control performance is not always easy. We proposed a novel method to involve annihilators into the augmented system to explore possible choices of the system ma-

trix. By solving them and regular decision variables iteratively, we can achieve a control performance at least not worse, at the most of times better, than the results derived by traditional quadratic methods.

- Moreover, after the system augmentation, the computational procedure can still take advantage of the convex form of the LMI conditions used in quadratic methods. Comparing with previous non-convex methods using higher order Lyapunov functions, this convex formulation is computationally efficient and has the capability of involving more higher order terms to search for better results.

Furthermore, the following highlighted application of the proposed method has been demonstrated to address nonlinear control challenges and make contributions in different aspects.

- Firstly, we achieved better disturbance attenuation performance than quadratic methods on solving state feedback $\mathcal{H}_\infty$ control design problems of polynomial nonlinear systems.

- The similar comparison result was successfully produced for systems subjected to actuator saturations. The effectiveness of the proposed method was verified by solving an animal population control problem using Lotka-Volterra equation. In addition, combining with saturation stabilization conditions, our method expanded the region of attraction (ROA) derived by quadratic Lyapunov function to a larger one with more flexible shape.

- After that, the output feedback $\mathcal{H}_\infty$ control design problem was addressed by solving a pair of HJI inequalities. The proposed method was used to solve this problem for a spacecraft with two torque inputs to fulfill an attitude control command tracking task. Due to the high power degree of the system as well as the complicated condition form, the SOS decomposition of the output estimation condition was computationally expensive, if not impossible, to be achieved by solving for higher order Lyapunov functions directly such that, the attempt in searching one could fail after extensive effort. By using system augmentation, higher order Lyapunov functions not only overcome this barrier smoothly,

but also lead to better tracking performances with less accumulated control input energy comparing with the quadratic Lyapunov function based results.

- In order to avoid the large usage of computer memory caused by full monomial state augmentation of large scaled systems, like the spacecraft model, we adopted partial augmentation with selected nonlinear states to solve the output feedback $\mathcal{H}_\infty$ control problem. The synthesis result showed that, partial augmentation can provide satisfactory results within an acceptable time duration. This alternative system augmentation made the proposed method a practical one to benefit users who desire control performance improvement within a limited time. The users can decide the augmentation levels based on their requirement.

6.2 Future Work

In addition to the results presented in this dissertation, some challenging issues remain to be addressed. Moreover, promising directions of research are worthy to be mentioned for future studies.

- In Chapter 5, we solved the output feedback $\mathcal{H}_\infty$ synthesis conditions for a class of polynomial nonlinear systems with a quite general form. However, the dynamic of controlled output e does not include a term for disturbance, $D_{11}(x)d$. Basically, none-zero D_{11} term will result in HJI conditions with rational function form instead of polynomial form. The research regarding this issue to provide the most complete and generalized results for the output feedback $\mathcal{H}_\infty$ control of polynomial nonlinear systems has been addressed in [97]. The same axi-symmetric spacecraft dynamic model is also studied but only for attitude regulation problem adopting quadratic Lyapunov functions. With the help of the convex condition in deriving higher order Lyapunov function, one might be able to provide good control performance for a more challenging problem (such as command tracking considering controlled output subjected to disturbance) through output feedback $\mathcal{H}_\infty$ control

designs.

- In many practical problems, it is hard or even impossible to find feasible solutions by SOS programming. To deal with this difficulty, as proposed in this dissertation, we add state region constraints to solve their sufficient conditions. Besides, the system augmentation also introduce coupling constraints into the sufficient conditions. Both of these multipliers help enrich the polynomial representation and improve the solvability of the synthesis conditions. Therefore, the composition of these constraints will greatly influence the solutions of the synthesis conditions. In this research, the composition of both of these constraints has been done by checking the interim results, finding desired monomial form for the success of the decomposition, and manually adding them to fill up the necessary ones. For large dimensional systems or systems with higher-order nonlinearities, it will be a time consuming and complicated process. Therefore, it is desirable to continuously explore SOS toolbox and implement this mechanism automatically as well as the capability to select augmented states and augmented system matrices based on inputs from users.

- Additional monomials (regional constraints, coupling constraints and polynomial annihilators) can not only improve the solvability of the synthesis conditions as discussed in the previous paragraph, but also enhance the achievable performance of optimization oriented control methods, such as $\mathcal{H}_\infty$ control. Taking advantage of these monomials to improve control performances is one of the most important features of this research. In the next stage of this research, the tradeoff between performance improvement and computational effort is worth to be further studied. More specifically, a guidance for monomial selection is needed to avoid involving less helpful or even harmful monomials into the computational process. This monomial selection study, together with the augmented state selection study discussed in the above paragraph are responsible to improve the computational efficiency of the proposed method.

- Chapter 5 expanded the applicability of the proposed method into the area of output

feedback control for nonlinear systems without saturation. Similarly, in the future, the application of proposed method on state feedback saturation control discussed in Chapter 4 may be further extended to output feedback control of saturated polynomial nonlinear systems. Comparing the state feedback synthesis conditions in Chapter 4, additional detectability conditions are needed for output feedback saturation controls. Due to the complexity of the LMI based controller constructing procedure, this system augmentation based approach is computationally more efficient than other non-convex methods, to solve for higher order Lyapunov functions.

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