An integral equation method to boundary value problems in elastostatics

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1 INTRODUCTION

The boundary element method (BEM) is already a well established numerical technique for solving some boundary value problems in elastostatics - Brebbia and Walker (1980). The main feature of this approach is the use of fundamental solutions which reduces the dimension of the problem by one and results in finding some unknown functions on the boundary only. So if we want to use the BEM we need: first - the fundamental solutions, and second - the boundary integral equations which are usually constructed by means of Betti's law or Green's second identity. In many cases of practical importance however, the fundamental solutions are not known, or they are so complicated that the effective implementation of the BEM is under question. On the other hand, if the thickness of the domain in the two dimensional case is not constant, or the material is orthotropic the solution with boundary element method is complicated in a similar way.

A method for solving the integral equations of the thin shells, derived by means of Betti's reciprocal theorem is proposed by Kilchevskii (1963). The bending and membrane solutions of the corresponding plate are obtained as a first step. After that a few ways of including the curvatures and other parameters of the shell are given. This idea is further developed by Vainberg and Siniavskii (1961) and Forbes and Robinson (1969). A few other papers, concerning this subject, have appeared recently - Bezine,G. (1980), Tanaka,M. & K. Miyazaki (1986).

A variant of this approach is applied recently by the present author (1985). It is based on the integral equations, derived by means of Green's second identity and the use of fundamental solutions, which do not satisfy the governing differential equations of the problem. This results in a set of simultaneous algebraic equations for nodal unknowns in the whole domain. The number of unknowns is larger than in the boundary element method, but the proposed numerical technique is very general. It can be successfully applied in wide range of static and dynamic problems of structural mechanics.

2 A BRIEF THEORY OF THE METHOD

Consider a system of partial differential equations (1) and the associate homogeneous boundary conditions (2):

(1) $Lu(x,y) = q(x,y)$, (2) $Nu(x,y)\big|_S = 0$,

where L and N are differential operator matrices, $u(x,y)$ is a vector of unknown functions, $q(x,y)$ is a vector of external loads.
We are trying to find the solution $u(x,y)$ in the two dimensional region Ω , bounded by a curve S . The region Ω may be simply or multiply connected, while its boundary S may be piecewise smooth, i.e., it may have corners (Figure 1).

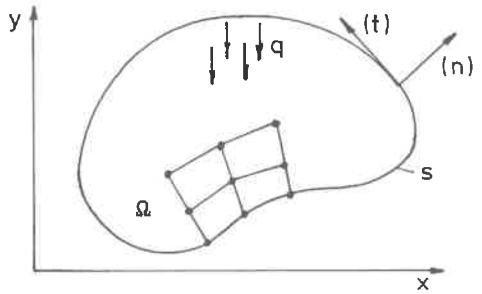

Figure 1

Let us presume that the set of differential equations (1) has not a matrix of fundamental solutions $u(x,y)$. We can make the following decomposition:

(3) $Lu(x,y) = L_1 u(x,y) + L_2 u(x,y)$.

In relation (3) L_1 is a matrix of differential operators, for which the matrix of fundamental solutions u_1^* and the Green's second identity are known i.e.,:

(4) $L_1 u_1^*(x,y) = \delta_i$,

(5) $\int_\Omega (uL_1 u_1^* - u_1^* L_1 u)\, d\Omega = \int_S (uN_1 u_1^* - u_1^* N_1 u)\, ds$.

Here δ_i is the Dirac delta function, N_1 is a known matrix of differential operators. As a rule we choose the diagonal coefficients of L_1 among the so-called poliharmonic operators: ∇^2 , ∇^4, ∇^6..., because they have fundamental solutions in a simple closed form.
In fact many differential equations of mathematical physics possess these operators or might be reconstructed properly as to have this quality.

On introducing equation (3) into (5) and bearing in mind (4) we have:

(6) $cu_i = \int_\Omega q u_1^*\, d\Omega - \int_\Omega u_1^* L_2 u\, d\Omega + \int_S (uN_1 u_1^* - u_1^* N_1 u)\, ds$,

in which c is a numerical matrix, which coefficients depend on dislocation of point i .
Now we have to solve the system of integral equations (6), associated with the given boundary conditions (2). It is important to point out that the domain integrals in (6) include the unknown functions $u(x,y)$ and their derivatives.
Integrating by parts, if necessary, we obtain a new set of integral equations which include the unknown functions and their derivatives on the boundary and the same functions in the domain. If the fundamental solutions of equations (1) are available, there is no need of decomposition (3) and integral equations (6) will become a system of boundary integral equations. The next important step is to discretizy the domain and the boundary in order to get the system of algebraic equations (Figure 1). This might be done by using the usual boundary (on the boundary) and finite (in the domain) element's procedures.
The stresses may be obtained by a proper differentiation of equations(6).

3 AN APPLICATION TO A SHALLOW SHELL OF DOUBLE CURVATURE

As an example we consider a shallow shell with constant curvatures $k_1 = 1/R_1$, $k_2 = 1/R_2$, $k_{12} = 0$, and with constant thickness h , subjected to an external load with components $X(x,y), Y(x,y)$ and $Z(x,y)$ (Figure 2).

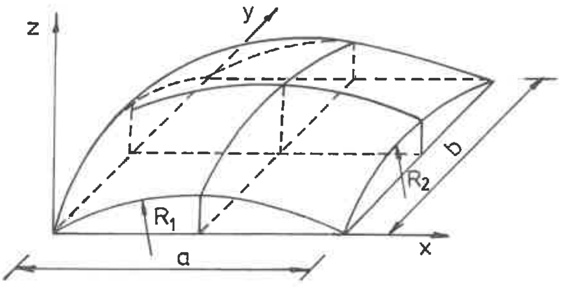

Figure 2

The exact fundamental solutions are obtained by Matsui and Matsuoka (1978) in an extremely complicated form. That is why since now they are not used in conventional BEM. The method described in section 2 gives a very simple alternative to get a solution of this problem.
As is well known, the equilibrium conditions of a shell element give the following set of differential equations (Tepavitcharov et al. 1980):

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} + k_{12}\frac{\partial w}{\partial x} + \frac{1-\nu^2}{Eh}X = 0,$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} + k_{21}\frac{\partial w}{\partial y} + \frac{1-\nu^2}{Eh}Y = 0, \tag{7}$$

$$D\nabla^4 w + \frac{Eh}{1-\nu^2}\left[(k_1 k_{12} + k_2 k_{21})w + k_{12}\frac{\partial u}{\partial x} + k_{21}\frac{\partial v}{\partial y}\right] - Z = 0,$$

where $k_{12}=k_1+\nu k_2$, $k_{21}=k_2+\nu k_1$, $D=Eh^3/12(1-\nu^2)$, $k=\frac{Eh}{1-\nu^2}(k_1k_{12}+k_2k_{21})$
E is Young's modulus, ν is Poisson's ratio, u , v and w are the displacements of an arbitrary point on the middle surface of the shell, which correspond to axes x , y and z . The equations (7) may be reconstructed as follows:

$$\nabla^2 u = \bar{X}, \quad \nabla^2 v = \bar{Y}, \quad D\nabla^4 w = \bar{Z}, \tag{8}$$

where ∇^2 and ∇^4 are harmonic and biharmonic operators and

$$\begin{aligned}
\bar{X} &= -\frac{1-\nu^2}{Eh}X + \frac{1+\nu}{2}\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x\partial y}\right) - k_{12}\frac{\partial w}{\partial x}, \\
\bar{Y} &= -\frac{1-\nu^2}{Eh}Y + \frac{1+\nu}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x\partial y}\right) - k_{21}\frac{\partial w}{\partial y}, \\
\bar{Z} &= Z - \frac{Eh}{1-\nu^2}\left(kw + k_{12}\frac{\partial u}{\partial x} + k_{21}\frac{\partial v}{\partial y}\right).
\end{aligned} \tag{9}$$

It should be noted that the Green's second formula and the fundamental solutions for the operators ∇^2 and ∇^4 are well known, which leads to a system of four integral equations. To relate the equations (7), (8) and (9) with the approach described in section 2 we write the matrices L , L_1 and u_1^* and consider the following as a first variant:

1 variant

$$L = \begin{bmatrix}
\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2} & \frac{1+\nu}{2}\frac{\partial^2}{\partial x\partial y} & k_{12}\frac{\partial}{\partial x} \\
\frac{1+\nu}{2}\frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} & k_{21}\frac{\partial}{\partial y} \\
\frac{Eh}{1-\nu^2}k_{12}\frac{\partial}{\partial x} & \frac{Eh}{1-\nu^2}k_{21}\frac{\partial}{\partial y} & D\nabla^4 + k
\end{bmatrix},$$

(10)

$$L_1 = \begin{bmatrix}
\nabla^2 & 0 & 0 \\
0 & \nabla^2 & 0 \\
0 & 0 & D\nabla^4
\end{bmatrix}, \quad u_1^* = \begin{bmatrix}
\frac{1}{2\pi}\ln r & 0 & 0 \\
0 & \frac{1}{2\pi}\ln r & 0 \\
0 & 0 & \frac{1}{8\pi D}r^2\ln r
\end{bmatrix},$$

$$u^T = \{u, v, w\}, \quad q^T = \left\{-\frac{1-\nu^2}{Eh}X, \ -\frac{1-\nu^2}{Eh}Y, \ Z\right\}.$$

Because of limited space we do not write the matrices L_2, N_1 and the derived system of integral equations. Here we shall not discuss the way the boundary conditions are satisfied and integration by parts (it concerns the domain integrals: $\int_\Omega \bar{X}d\Omega$, $\int_\Omega \bar{Y}d\Omega$ and $\int_\Omega \bar{Z}d\Omega$), which leads to the final system of integral equations.

2 variant

$$L = \begin{bmatrix} \nabla^2 + \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial x^2} & \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial x\,\partial y} & \frac{2k_{12}}{1-\nu}\frac{\partial}{\partial x} \\ \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial x\,\partial y} & \nabla^2 + \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial y^2} & \frac{2k_{21}}{1-\nu}\frac{\partial}{\partial y} \\ \frac{Eh}{1-\nu^2}k_{12}\frac{\partial}{\partial x} & \frac{Eh}{1-\nu^2}k_{21}\frac{\partial}{\partial y} & D\nabla^4 + k \end{bmatrix},$$

(11)

$$L_1 = \begin{bmatrix} \nabla^2 + \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial x^2} & \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial x\,\partial y} & 0 \\ \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial x\,\partial y} & \nabla^2 + \frac{1+\nu}{1-\nu}\frac{\partial^2}{\partial y^2} & 0 \\ 0 & 0 & D\nabla^4 \end{bmatrix}, \quad u_1^* = \begin{bmatrix} u_x^* & v_x^* & 0 \\ u_y^* & v_y^* & 0 \\ 0 & 0 & \frac{1}{8\pi D}r^2\ln r \end{bmatrix}.$$

$$u^T = \{u, v, w\}, \qquad q^T = \left\{\frac{-2(1+\nu)}{Eh}X, -\frac{2(1+\nu)}{Eh}Y, Z\right\}.$$

In the second variant we use a slightly modificated form of the differential operator matrix L. As it can be seen the matrix L_1 is separated into two parts: the upper part is related to the membrane stress state of the shell, since the term $D\nabla^4$ is related to the bending state. In this respect the matrix u_1^* consists of two well known systems of fundamental solutions: the Kelvin's plane-stress solutions and the fundamental solutions of the thin plate in bending. It is important to note that the boundary conditions are easy to be satisfied for this variant. The integral equations can be derived similarly.

4 NUMERICAL EXAMPLE. CONCLUSIONS

As an illustration we consider a square shallow shell of constant thickness and with different curvatures - see figure 2. Having the coordinate system situated at the centre of the square we choose the displacement functions as follows:

(12) $w(x,y)=(x + y + 2)/2, \quad u(x,y)=x, \quad v(x,y)=y$.

Introducing these functions into the integral equations, obtained from the first of the given variants, we discretizy the contour into n boundary elements and the domain into m finite elements. A numerical integration scheme is applied for both - the line and domain integrals. In this way we have the so called invers problem: the displacement functions and their derivatives are given and employing the discretized integral identities we find the approximate values of the coefficient of matrix c - see formula (6). We know the exact values of these coefficients for any point on the boundary

and in the domain.
It was observed that the results are quite good for n=32 and m=64.

Pertinent to the method proposed is its generality with respect to the contour geometry, the boundary conditions and the mechanical and thermal loads. The main problem with BEM - the lack of fundamental solutions for the high-order differential operators is avoided here by using new simple kernel functions. Therefore the main advantages of the BEM (the high accuracy) and the FEM (its generality) are successfully combined. On the other hand this procedure leads to non symmetrical and fully populated matrix.

Some improvements of the computer programs in order to get a real test solution of a shallow shell with double Gaussian curvature, are still subject to further research and development.

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