ANALYSIS OF THERMAL-PLASTIC RESPONSE OF SHELLS OF REVOLUTION BY NUMERICAL INTEGRATION

J.W. LEONARD

Department of Civil Engineering, Illinois Institute of Technology Chicago, Illinois 60616, U.S.A.

H.U. AHMED

Argonne National Laboratory, Argonne, Illinois 60439, U.S.A.

SUMMARY

An economic technique for the numerical analysis of the elasto-plastic behavior of shells of revolution would be of considerable value in the nuclear reactor industry. When the elastic analysis rules of the ASME nuclear vessel code (Section III) and of the associated high temperature code case 1592 cannot be met, the obvious alternative is to perform an inelastic analysis to evaluate structural integrity. In the design of reactor fuel containments, and of reactor fuel pins, the extremely high temperature environments and complicated loading systems require an accurate as well as economic numerical analysis method. Currently available techniques based on simultaneous equation solution, e.g. the finite element method, require considerable computer storage and computational time for the repetitive calculations necessary for inelastic analyses.

A numerical method based instead on the numerical integration of the governing shell equations has been shown, for elastic cases, to be more efficient than the finite element method when applied to shells of revolution. In the numerical integration method, the governing differential equations of motion are converted into a set of initial-value problems. Each initial-value problem is integrated numerically between meridional boundary points and recombined so as to satisfy boundary conditions. For large-deflection elasto-plastic behavior, the equations are nonlinear and, hence, are recombined in an iterative manner using the Newton-Raphson procedure. Suppression techniques are incorporated in order to eliminate extraneous solutions within the numerical integration procedure.

The Reissner-Meissner shell theory for shells of revolution is adopted to account for large deflection and higher-order rotation effects. The computer modelling of the equations is quite general in that specific shell segment geometries, e.g. cyclindrical, spherical, toroidal, conical segments, and any combinations thereof can be handled easily.

The elasto-plastic constitutive relations adopted are in accordance with currently recommended constitutive equations for inelastic design analysis of FFTF Components. The Von Mises yield criteria and associated flow rule is used and the kinematic hardening law is followed. Examples are considered in which stainless steels common to LMFBR application are used.

1. Introduction

In the nuclear industry there is a dire need to develop accurate and economic methods for the analysis of structures subjected to complicated thermal loadings. It is, therefore, necessary to account for the large deflection elastic-plastic behavior of structures.

The use of an appropriate non-linear shell theory is of considerable importance in elastic-plastic analysis of shell structures. Although, numerous shell theories can not always be made due to limited experimental data. Sander's non-linear shell theory (9) which invokes Kirchoff's hypothesis (normals to the undeformed middle surface of the shell remain normal to the deformed middle surface) is used in the following work. Sander's equations of equilibrium and the strain displacement relations that are presented in the following are in the reduced component form (Ref. 9). The generalized equations have been reduced to a form applicable to axisymmetric shells subjected to symmetric loadings and limited to small rotations and small rotation about the normal; otherwise the large deflection considerations have been retained.

A numerical integration scheme, which follows a Runge-Kutta-Gill procedure with a suppression technique of Refs. (6,7), has been extend herein to large deflection elastic-plastic problems. In Ref. (5) is introduced an integration scheme applicable to the multisegment method of numerical integration for the analysis of nonlinear elastic shells. Later, Gerdeen (3) applied this procedure to non-linear elastic-plastic analysis of shells of revolution. Both in the suppression technique that is presented here for the large deflection elastic-plastic analysis and in the multisegment method of integration, the governing differential equations of motion are coverted into a set of initial value problems. Each initial value problem is integrated numerically between meridional boundary points and recombined so as to satisfy the boundary conditions. Unlike the multisegment method, the suppression technique does not require the solution to a set of simultaneous equations but eliminates the extraneous solution in a marching process until the boundary has been reached. At that point the entire solution is corrected by satisfying the boundary conditions.

For the non-linear problem a quazilinearization technique, which was initially developed by Bellman (10) and later used in Refs. (11,12) has been incorporated here with the suppression technique. In the quazilinearization technique a boundary value problem is converted to an initial value problem by assuming arbitrary initial conditions so that it becomes directly applicable in conjunction with the suppression technique. The quazilinearized form of the shell equations derived in the following have not been presented before. For the elastic-plastic problem the constitutive relatins adopted follow the Von-Mises yield criteria and associated flow rule. The solution process for elastic-plastic problem is generally followed from Refs. (1), (2) and (3).

2. General Shell Equations

The Sander's shell equations (9) simplified to axisymmetric shells subjected to symmetrical loadings and limited to the assumptions described earlier are as follows:

Equations of Equilibrium

$$N_{\phi,s} + \frac{\cos\phi}{r}(N_\phi - N_\theta) + \frac{1}{R_\phi}(Q_\phi - \gamma_\phi N_\phi) + P_\phi = 0 \qquad (1)$$

$$Q_{\phi,s} + \frac{\cos\phi}{r} Q_\phi - (R_\Theta + \frac{\sin\phi}{r}) N_\Theta + \frac{1}{R_\phi} Q_\phi \gamma_\phi$$

$$- [\gamma_{\phi,s} + \frac{1}{R_\phi} (1 + \gamma_\phi^2)] N_\phi + p = 0 \tag{2}$$

$$M_{\phi,s} + \frac{\cos\phi}{r} (M_\phi - M_\Theta) - Q_\phi = 0 \tag{3}$$

Strain Displacement Relations

$$e_\Theta = \frac{1}{r} (u_\phi \cos_\phi + w \sin\phi) \tag{4}$$

$$e_\phi = u_{\phi,s} + \frac{w}{R_\phi} + \frac{1}{2} \gamma_\phi^2 \tag{5}$$

$$\gamma_\phi = \frac{1}{R_\phi} u_\phi - w_{,s} \tag{6}$$

$$R_\Theta = \frac{1}{r} \gamma_\phi \cos\phi \tag{7}$$

$$R_\phi = \gamma_{\phi,s} \tag{8}$$

In the above, use has been made of the expression

$$\frac{\partial}{\partial\phi}(\cdot) = R_\phi \frac{\partial}{\partial s} (\quad) \tag{9}$$

a comma denotes partial differentiation, and the following definitions are assumed:

ϕ - meridional coordinate

Θ - circumferential coordinate

s - independent variable

R_ϕ - meridional radius of curvature

r - circumferential radius of curvature

p - force applied normal to shell

p_ϕ - force applied along ϕ directio..

e_Θ, e_ϕ - circumferential and meridional strains

k_Θ, k_ϕ - circumferential and meridional rotations

Fundamental Variables:

w - deflection normal to the middle surface of the shell

u_ϕ - deflection along ϕ direction

N_ϕ - stress resultant in ϕ direction

γ_ϕ - rotation about ϕ and as defined above.

M_ϕ - moment resultant about ϕ

Q_ϕ - shear stress resultant normal to ϕ

Rearranging the governing shell equations, one obtains the required set of first order nonlinear differential equations in terms of auxiliary and fundamental variables.

$$W_{,s} = \frac{u_\phi}{R_\phi} - \gamma_\phi \tag{10}$$

$$\gamma_{\phi,s} = R_\phi \tag{11}$$

$$M_{\phi,s} = \frac{\cos\phi}{r}(M_\Theta - M_\phi) + Q_\phi \tag{12}$$

$$U_{\phi,s} = \epsilon_\phi - \frac{w}{R_\phi} - \frac{1}{2}\gamma_\phi^2 \tag{13}$$

$$Q_{\phi,s} = N_\Theta \left(\frac{\sin\phi}{r} + K_\Theta\right) + N_\phi \left[\frac{1}{R_\phi}(1+\gamma_\phi^2) + \gamma_\phi\right]$$

$$- Q_\phi \left(\frac{\cos\phi}{r} + \frac{1}{R_\phi}\gamma_\phi\right) - P \tag{14}$$

$$N_{\phi,s} = \frac{\cos\phi}{r}(N_\Theta - N_\phi) + \frac{1}{R_\phi}(\gamma_\phi N_\phi - Q_\phi) - P_\phi \tag{15}$$

Quazilinearization of Shell Equations

In the quazilinearization technique the non-linear differential equation is first represented by a set of simultaneous first order differential equations. Each of the first order differential equations is linearized using Taylor series expansions with second and higher order terms omitted. Iterative solutions of the resulting linearized differential equations usually converges quadratically to the solution of the original equations such that each iteration approximately doubles the accuracy.

For example let us consider the following non-linear first order differential equation

$$\frac{dy}{dx} = f(x,y) \tag{16}$$

with boundary conditions $y(0) = \alpha$, where $f(x,y)$ is a nonlinear function which can be linearized around $y = y^*$ as follows

$$\delta f = \frac{\partial f}{\partial y}\bigg|_{y=y^*} \delta y$$

$$\text{or } f(x,y) = f(x,y^*) + \frac{\partial f(x,y)}{\partial y}\bigg|_{y=y^*} \tag{17}$$

In other words, the differential equation can be represented around $y=y^*$ as

$$\frac{dy}{dx} = f(x,y^*) + f_y(x,y)(y-y^*) \tag{18}$$

where $f_y(x,y) = \frac{\partial f}{\partial y}\Big|_{y=y^*}$

If y is represented in the vector form $(y_1, y_2, \ldots, y_n)$ and f represented in the vector form $(f_1, f_2, \ldots, f_n)$ the above differential equation can be written as

$$\frac{dy}{dx} = f(x,y_1) + J(y)(y-y_1) \tag{19}$$

where $$J(y) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots\cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots\cdots & \frac{\partial f_2}{\partial y_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots\cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \tag{20}$$

Thus, one obtains a recurrence relation in the following form:

$$\frac{dy_i}{dx} = f_i\ (y_1^*,\ y_2^* \cdots\cdots y_n^*) + \sum_i \frac{\partial f_i\ (x, y_i)}{\partial y_i} \tag{21}$$

$$i = 1, 2, \ldots\ldots, n$$

The general quasilinearized form of the first order non-linear differential equation, derived from the procedure outlined above, are to be used both for elastic and elastic-plastic cases. They are,

$$w,_s = \frac{u_\phi}{R_\phi} - B_\phi \tag{22}$$

$$\gamma_{\phi,s} = k_\phi \tag{23}$$

$$M_{\phi,s} = \frac{\cos\phi}{r}\ (M_\Theta - M_\phi) + Q_\phi \tag{24}$$

$$\begin{aligned} Q_{\phi,s} = {} & \frac{1}{R_\phi}\ (2\ \gamma_{\phi 0}\ N_{\phi 0})\ \gamma_\phi + N_{\phi 0}\ R_\phi + N_{\Theta 0}\ k_\Theta \\ & + \left(\frac{\sin\phi}{r} + k_{\Theta 0}\right) N_\Theta - \left(\frac{\gamma_{\phi 0}}{R_\phi} + \frac{\cos\phi}{r}\right) Q_\phi \\ & + \left(\frac{1}{R_\phi} + \frac{\gamma_{\phi 0}{}^2}{R_\phi} + R_{\phi 0}\right) N_\phi \\ & + \left(k_{\phi 0}\ N_{\phi 0} - k_{\phi 0}\ N_{\Theta 0} - \frac{2}{R_\phi}\ \gamma_{\phi 0}^2\ N_{\phi 0} + \frac{\gamma_{\phi 0}\ Q_{\phi 0}}{R_\phi} - p\right) \end{aligned} \tag{25}$$

$$U_{\phi,s} = e_\phi - \frac{w}{R_\phi} - \gamma_{\phi 0}\ \gamma_\phi + \frac{1}{2}\ \gamma_{\phi 0}{}^2 \tag{26}$$

$$\begin{aligned} N_{\phi,s} = {} & \frac{\cos\phi}{r}\ N_\Theta - \frac{\cos\phi}{r}\ N_\phi - \frac{Q_\phi}{R_\phi} - \frac{\gamma_{\phi 0}\ N_{\phi 0}}{R_\phi} \\ & + \frac{1}{R_\phi}\ (\gamma_{\phi 0}\ N_\phi + N_{\phi 0}\ \gamma_\phi) - P_\phi \end{aligned} \tag{27}$$

where $N_{\phi 0}$, $M_{\phi 0}$, $Q_{\phi 0}$, $Y_{\phi 0}$, $R_{\phi 0}$, K_{Θ}, $N_{\Theta 0}$ etc. are residual values of the fundamental and auxiliary variables. There use will be specified in section 5 of this paper.

4. Constitutive Relations

4.1 Elastic Case

$$N_\Theta = K\,(e_\Theta + \upsilon\, e_\phi) - \frac{Eh\ \alpha T_0}{(1-\upsilon)} \tag{28}$$

$$N_\phi = K\,(e_\phi + \upsilon\, e_\Theta) - \frac{Eh\ \alpha T_0}{(1-\upsilon)} \tag{29}$$

$$M_\Theta = D\,(\kappa_\Theta + \upsilon\,\kappa_\phi) - D\,\alpha\,(1+\upsilon)\,T_1 \tag{30}$$

$$M_\phi = D\,(\kappa_\phi + \upsilon\,\kappa_\Theta) - D\,\alpha\,(1+\upsilon)\,T_1 \tag{31}$$

where $K = \frac{Eh}{1-\upsilon^2}$ and $D = \frac{Eh^3}{12\,(1-\upsilon^2)}$, E = Elastic modulus, υ = Poisson"s ratio e_Θ, e_ϕ, κ_Θ, κ_ϕ are the circumferential and the meridional strain and rotation terms. N_Θ is the stress resultant along the circumference, M_Θ is the circumferential moment, h is the thickness of the shell, α the coefficient of thermal expansion, and

$$T_0\,(\phi) = \frac{1}{h}\int_{-h/2}^{h/2} T\,(\phi, z)\,dz \tag{32}$$

$$T_1\,(\phi) = \frac{12}{h^3}\int_{-h/2}^{h/2} z\,T\,(\phi, z)\,dz \tag{33}$$

T being the temperature distribution in the shell, z the coordinate through the thickness

4.2 Elastic-Plastic Case

For elastic-plastic analysis consideration, the following incremental constitutive relations can be formulated for thermal stress-strain behaviors with the aid of Ref. (1) and Ref. (2). The appropriate temperature terms included herein are such that the elastic and elastic-plastic behaviors are identified accordingly.

The incremental stress-strain relations

$$\begin{bmatrix} \Delta N_\phi \\ \Delta N_\Theta \\ \Delta M_\phi \\ \Delta M_\Theta \end{bmatrix} = \begin{bmatrix} \frac{\partial N_\phi}{\partial e_\phi} & \frac{\partial N_\phi}{\partial e_\Theta} & \frac{\partial M_\phi}{\partial R_\phi} & \frac{\partial N_\phi}{\partial R_\Theta} \\ \frac{\partial N_\Theta}{\partial e_\Theta} & \frac{\partial N_\Theta}{\partial e_\Theta} & \frac{\partial N_\Theta}{\partial R_\phi} & \frac{\partial N_\Theta}{\partial R_\Theta} \\ \frac{\partial M_\phi}{\partial e_\phi} & \frac{\partial M_\phi}{\partial e_\Theta} & \frac{\partial M_\phi}{\partial R_\phi} & \frac{\partial M_\phi}{\partial R_\Theta} \\ \frac{\partial M_\Theta}{\partial e_\phi} & \frac{\partial M_\Theta}{\partial e_\Theta} & \frac{\partial M_\Theta}{\partial R_\phi} & \frac{\partial M_\Theta}{\partial R_\Theta} \end{bmatrix} \begin{bmatrix} \Delta e_\phi \\ \Delta e_\Theta \\ \Delta R_\phi \\ \Delta R_\Theta \end{bmatrix} +$$

$$\begin{bmatrix} \left(\frac{\partial N_\phi}{\partial e_\phi} + \frac{\partial N_\phi}{\partial e_\theta}\right) \alpha T_0 \\ \left(\frac{\partial N_\theta}{\partial e_\theta} + \frac{\partial N_\theta}{\partial e_\theta}\right) \alpha T_0 \\ \left(\frac{\partial M_\phi}{\partial e_\phi} + \frac{\partial M_\phi}{\partial e_\theta}\right) \alpha T_1 \\ \left(\frac{\partial M_\theta}{\partial e_\phi} + \frac{\partial M_\theta}{\partial e_\theta}\right) \alpha T_1 \end{bmatrix} \tag{34}$$

where Δ identifies incremental value of the quantity and the generalized stiffness coefficients are related to partial stiffness coefficients (Ref. 1) by

$$\frac{\partial N_i}{\partial e_j} = \int_{-h/2}^{h/2} \frac{\partial \sigma_i}{\partial e_j}\, dz \quad , \quad \frac{\partial N_i}{\partial R_i} = \int_{-h/2}^{h/2} \frac{\partial \sigma_i}{\partial e_j}\, zdz \tag{35}$$

(i,j denotes d, θ)

$$\frac{\partial M_i}{\partial e_j} \int_{-h/2}^{h/2} \frac{\partial \sigma_i}{\partial e_j}\, zdz \quad , \quad \frac{\partial M_i}{\partial K_j} = \int_{-h/2}^{h/2} \frac{\partial \sigma_i}{\partial e_j}\, z^2 dz \tag{36}$$

such that the incremental form of the constitutive relations are

$$\begin{bmatrix} d\sigma_\phi \\ d\sigma_\theta \\ d\bar{e}_p \end{bmatrix} = \begin{bmatrix} \frac{\partial \sigma_\phi}{\partial e_\phi} & \frac{\partial \sigma_\phi}{\partial e_\theta} & \frac{\partial \sigma_\phi}{\partial e_p} \\ \frac{\partial \sigma_\theta}{\partial e_\phi} & \frac{\partial \sigma_\theta}{\partial e_\theta} & \frac{\partial \sigma_\theta}{\partial e_p} \\ \frac{\partial \bar{e}_p}{\partial e_\phi} & \frac{\partial \bar{e}_p}{\partial e_\theta} & \frac{\partial \bar{e}_p}{\partial \bar{e}_p} \end{bmatrix} \begin{bmatrix} de_\phi \\ de_\theta \\ 0 \end{bmatrix} \tag{37}$$

where $\bar{e}_p$ = effective plastic strain.

The last of these is obvious replaceable by Prandt-Reuss relations. From Ref. (11)

$$\begin{bmatrix} de_\phi \\ de_\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 1/_E & -\nu/_E & 3/_2 \frac{\sigma'_\phi}{\sigma} \\ -\nu/_E & 1/_E & 3/_2 \frac{\sigma'_\theta}{\sigma} \\ \frac{3}{2}\frac{\sigma'_\theta}{\sigma} & \frac{3}{2}\frac{\sigma'_\theta}{\sigma} & -H' \end{bmatrix} \begin{bmatrix} d\sigma_\phi \\ d\sigma_\theta \\ d\bar{e}_p \end{bmatrix} \tag{38}$$

where σ = effective stress, $H' = \partial\sigma/\partial\bar{e}_p$ = hardening parameter, and σ^1_ϕ, σ^1_θ are the deviatoric stresses

$$\sigma^1_\phi = \frac{1}{3}(2\sigma_\phi - \sigma_\theta) \tag{39}$$

$$\sigma^1_\theta = \frac{1}{3}(2\sigma_\theta - \sigma_\phi) \tag{40}$$

The inversion of this matrix gives partial stiffness coefficients as

$$\frac{\partial\sigma_\phi}{\partial e_\phi} = \frac{1}{D}\left\{H' + \frac{9}{4}\left(\frac{\sigma_\theta'}{\sigma}\right)^2 E\right\} \tag{41}$$

$$\frac{\partial\sigma_\phi}{\partial e_\theta} = \frac{1}{D}\left\{\nu H' - \frac{9}{4} E \left(\frac{\sigma_\phi'}{\sigma}\right)\left(\frac{\sigma_\theta'}{\sigma}\right)\right\} \tag{42}$$

$$\frac{\partial\sigma_\phi}{\partial e_p} = \frac{3}{2D}\left\{\left(\frac{\sigma_\phi'}{\sigma}\right) + \nu\left(\frac{\sigma_\theta'}{\sigma}\right)\right\} \tag{43}$$

$$\frac{\partial\sigma_\theta}{\partial e_\phi} = \frac{1}{D}\left\{\nu H' - \frac{9}{4} E \left(\frac{\sigma_\phi'}{\sigma}\right)\left(\frac{\sigma_\theta'}{\sigma}\right)\right\} \tag{44}$$

$$\frac{\partial\sigma_\theta}{\partial e_\theta} = \frac{1}{D}\left\{H' + \frac{9}{4} E \left(\frac{\sigma_\phi'}{\sigma}\right)^2\right\} \tag{45}$$

$$\frac{\partial\sigma_\theta}{\partial e_p} = \frac{3}{2D}\left\{\left(\frac{\sigma_\theta'}{\sigma}\right) + \nu\left(\frac{\sigma_\phi'}{\sigma}\right)\right\} \tag{46}$$

$$\frac{\partial e_p}{\partial e_\phi} = \frac{3}{2D}\left\{\left(\frac{\sigma_\theta'}{\sigma}\right) + \nu\left(\frac{\sigma_\theta'}{\sigma}\right)\right\} \tag{47}$$

$$\frac{\partial e_p}{\partial e_\phi} = \frac{3}{2D}\left\{\left(\frac{\sigma_\theta'}{\sigma}\right) + \nu\left(\frac{\sigma_\phi'}{\sigma}\right)\right\} \tag{48}$$

$$\frac{\partial e_p}{\partial e_p} = \frac{1}{D}\left\{- \frac{(1-\nu^2)}{E}\right\} \tag{49}$$

where

$$D = (1-\nu^2)\frac{H'}{E} + \frac{9}{4}\left(\frac{\sigma_\phi'}{\sigma}\right)^2 + \frac{9\nu}{2}\left(\frac{\sigma_\theta'}{\sigma}\right) + \frac{9}{4}\left(\frac{\sigma_\theta'}{\sigma}\right)^2 \tag{50}$$

The numerical algorithm for obtaining the solution of the elastic-plastic problem is generally followed from Ref. 11, although some refinements presented in Ref. (13) and here are helpful. The comparisons of the solution for linear-elastic and non-linear elastic problem have been completed and are in general agreement with Refs. (14) and (15).

5. Numerical Scheme

In the numerical integration process, a standard Runge-Kutta-Gill procedure in conjunction with Hamming's predictor-corrector method is used to solved the system of first order differential equations.

In the suppression technique, for a system of order 2j, j quantities are assumed at the initial points and j boundary conditions are satisfied at the terminal point. The correct solution corresponds to some other set of initial values that produce boundary quantities satisfying the terminal boundary conditions. Partial solutions for one [illegible] cular solution and j homogeneous solutions designated by s_0, s_1, ...s_j are carried simultaneously by assuming j initial values for each

$$\{I_0\} = \begin{vmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} \qquad \{I_1\} = \begin{vmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} \quad , \{I_2\} = \begin{vmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} \quad , I_j \begin{vmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_{jj}=1 \end{vmatrix} \tag{51}$$

At a suppression point the values are represented by the vectors $\{t_\ell\}$ with the element $t_{\ell\kappa}$ ($\kappa = 1, \ldots\ldots j$). A convenient selection of the arbitrary boundary values is as follows

$$\{t_0\} = \begin{vmatrix} t_{00} = 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} \qquad \{t_1\} = \begin{vmatrix} t_{11} = 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} \quad , \{t_p\} = \begin{vmatrix} 0 \\ \cdot \\ 0 \\ t_{\ell\ell}=1 \\ \cdot \\ 0 \end{vmatrix} \qquad \{t_j\} = \begin{vmatrix} 0 \\ \cdot \\ \cdot \\ 00 \\ \cdot \\ t_{jj}=1 \end{vmatrix} \tag{52}$$

Let the unsuppressed quantities $\{t_\ell\}$ ($l = 0, 1, \ldots\ldots, j$) represent the displacement quantities at the suppression point, i.e. at a point at which extraneous solution can not be tolerated, so that the following terminal boundary conditions are satisfied

$$\sum_{\ell=1}^{j} B_{\kappa\ell} \{t_\ell\} + \{t_\kappa\} = \{t_\kappa\} \quad (\kappa = 0, 1, \ldots . j) \tag{53}$$

Each solution will yield a set of $B_{\kappa\ell}$ values so that the suppressed solution can be written as

$$\overline{S}_\kappa = S_\kappa + \sum_{\ell=1}^{j} B_{\kappa\ell} S_\ell \quad (\kappa = 0, 1 \ldots\ldots j) \tag{54}$$

$\overline{S}_\kappa$ is the suppressed solution, and S_κ is the unsuppressed solution. At the terminal point only the particular solution need be suppressed to the real terminal boundary values $\{T_0\}$, so that the unsuppressed values $\{T_\kappa\}$ can be combined as follows

$$\{T_0\} = \{T_0\} + \sum_{p=1}^{j} B_{0\ell} \tilde{T}_\ell \tag{55}$$

In the quasilinearization technique a set of residual values must be known as follows

$$R_F = \begin{vmatrix} R_{Fp} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{vmatrix} \quad , R_A = \begin{vmatrix} R_{Aq} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{vmatrix} \tag{56}$$

where p = 1, ...,n and q = 1, ...m and n represents the number of fundamental variables and m represents the number of auxiliary quantities. R_F are the residual values of fundamental variables and R_A are the residual values of auxiliary quantities.

For the elastic case the linear elastic solution can be used as residuals for the nonlinear elastic solution. For the elastic-plastc case, the nonlinear solution corresponding to the yield point can be used as the residual solution. In subsequent iterations, the values from the previous iteration can be used as the residual values for the following iteration until convergence has been obtained. Once residuals have been appropriately defined the suppression process described above is directly applicable owing to the quazi-linearized form of the differential equations.

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