Linear Mechanical Modelling Using Dynamical Condensation

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Abstract.

This paper provides a new technique aided to reduce degrees of freedom (DOF) in linear mechanical systems when dynamic behaviour is studied.DOF are chosen to be nodal displacements in sense of Finite Element Method (FEM) and are derived into 'master' and 'slave'.'Slave' displacements are eliminated but their contribution in matrix transfer function is included in implicit form.Since integral transform is used to perform elimination the accuracy of mathematical description is not violated.As a product of reduction the order of derivatives in govering equations rises.A modified Gauss algorithm to achieve 'condensed' description is employed.Some computational advantages over other techniques are discussed.

1.Introduction.

FEM is currently in use in engineering applications nowadays.In particular, one may use FEM to determine the dynamic response of idealized structures of nuclear power plant.In order to give more detail description of the structure a large number of nodal displacements are required.The system of govering equations become large and the computer storage may be exeeded.On the other hand, computational time rises significantly.

Widespread manner to reduce DOF is so called 'static condensation' technique. Nodal displacements are divided into 'master' and 'slave'.'Master' DOF are predominant and the 'slave' DOF are not so typical and important for analysis.They are eliminated from govering equations of the motion under the assumption that the inertial effects of the masses lumped in 'slave' DOF are neglected according to ZIENKIEWICZ[1].That leads to changes in the natural spectrum and they are more significant in high frequencies.High frequencies modes are truncated and response contains only contribution of low tones.

Nuclear power plant equipments are subjected to high frequency excitations and keeping natural spectrum of the structure without changes is important for dynamic perturbation analysis.

Following the basic assumption of the 'static condensation' technique is should be noted that this technique is not recommended to be applied when 'slave' DOF contain large masses lumped.For instance, if the motion of the reactor basement should be described by 'slave' DOF neglecting of the mass will lead to incredible results.

The objective of this paper is to develop the 'dynamical condensation' technique which has the following features: at first - the natural spectrum of the structure is kept unchanged after elimination so a wide range of perturbed frequencies are adopted and second - inertial effects in 'slave' DOF are not neglected and limitation of the masses is avoided.As a first step of the algorithm proposed is transforming govering equations of motion which have been obtained using time derivatives by the integral transform.As a product of this transform an algebraic system of equations may be obtained and used as a starting one.Elimination of the unknown displacements is attended by rising of the order of time derivatives.This strategy may be performed over all structure or over all separate parts of the structure - substructures.Since no inertial effects have been neglected the elimination will not cause the loss of accuracy and natural spectrum of the structure will remain without changes.Comparing the spectrum of the starting structure and the spectrum of the 'condensed' structure both will coinside completely.

To perform the elimination of the 'slave' DOF mentioned above a modified Gauss algorithm is employed.It is quite similar to the well known Gauss equation solver but coefficients are polinimials.

2.Dynamic Condensation Technique.

Let's consider a system of 'n' linear differential equations with constant coefficients, which are used to describe dynamical behaviour of a linear mechanical system with 'n' DOF:

$$M.\ddot{\underset{\sim}{Y}} + C.\dot{\underset{\sim}{Y}} + K.\underset{\sim}{Y} = \underset{\sim}{F} \qquad (1)$$

When the initial conditions are chosen to be $Y(0)=\dot{Y}(0)=0$ after Laplace transform over left and right sides of (1) equations of motion become:

$$A(P).\underset{\sim}{Y}(P) = \underset{\sim}{F}(P) \qquad (2)$$

where $A(p)$ may be realized as a dynamical stiffness matrix, which is mass-,damping- and static stiffness dependent:

$$A(P) = P^2.M + P.C + K \qquad (3)$$

Suppose we have to eliminate '1' components of the vector $\underset{\sim}{Y}(p)$.Let's denote by $\underset{\sim}{Y}(p)$, $A(p)$ and $\underset{\sim}{F}(p)$ vector of unknown transformed displacements,dynamical stiffness matrix and vector of external loads of order 'n' respectivly.We denote by $\bar{\underset{\sim}{Y}}(p)$ $\bar{A}(p)$ and $\bar{\underset{\sim}{F}}(p)$ their corresponding matrices order (n-1) in 'condensed' structure after elimination:

$$\bar{A}(P).\bar{\underset{\sim}{Y}}(P) = \bar{\underset{\sim}{F}}(P) \qquad (4)$$

where the new dynamical stiffness matrix $\bar{A}(p)$ has an order N higher than 2 with respect to 'p' and may be written in the form:

$$\bar{A}(P) = \bar{A}_0 + P.\bar{A}_1 + P^2.\bar{A}_2 + P^3.\bar{A}_3 + \ldots + P^N.\bar{A}_N = \sum_{i=0}^{N} P^i.\bar{A}_i \qquad (5)$$

For our further consideration we shall accept substructure approach wellknown from FEM.We assume that the structure is composed of substructures and 'master' DOF are chosen only to connect two or more adjacent substructures.Then we denote by $\underset{\sim}{V}^f(P)$ the forces acting on substructure 'f' in it's 'master' DOF and which represent the interaction between adjaccent substructures according to the First Newton's low.It is clear that the assemblage will not contain $\underset{\sim}{V}^f(p)$ and after summation $\underset{\sim}{V}^f(p)$ have to vanish $\sum_f \underset{\sim}{V}^f(p) = 0$.Following (2) and (4) equations for dynamical equilibrim of 'condensed' substructure 'f' may be written in the form: $\bar{A}^f(p)\bar{\underset{\sim}{Y}}^f(p) = \bar{\underset{\sim}{F}}^f(p) + \bar{\underset{\sim}{V}}^f(p)$. Forces $\bar{\underset{\sim}{V}}^f(p)$ are, in general unknown, but they are acting only on 'master' DOF and after assemblying they would similary to $\underset{\sim}{V}^f(p)$ be balanced at any 'master' DOF.The global system of govering equations of the 'condensed' structure is obtained after summation over substructures:

$$\sum_f \bar{A}^f(P).\underset{\sim}{\bar{Y}}^f(P) = \sum_f \underset{\sim}{\bar{F}}^f(P) + \sum_f \underset{\sim}{\bar{V}}^f(P) \tag{6}$$

where:

$$\sum_f \bar{A}^f(P) = \sum_f \Big(\sum_{i=0}^{N^f} P^i . \bar{A}^f_i \Big) = \sum_{i=0}^{N^f} \Big(\sum_f \bar{A}^f_i \Big) = \sum_{i=0}^{N} P^i . A_i = \bar{A}(P) \tag{7}$$

and $N = \max(N^f)$, $f=1,2,\ldots$

3.Numerical example.

The following example is invoked to ilustrate elimination algorithm.Let's consider a simple cantilever composed of four pure sear elements as shown in Fig.1. For the sake of simplicity $C = 0$, and matrices M , K , $\underset{\sim}{F}$ are respectively:

$$M = \begin{bmatrix} 1&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \quad ; \quad K = \begin{bmatrix} 1&-1&0&0 \\ -1&2&-1&0 \\ 0&-1&2&-1 \\ 0&0&-1&2 \end{bmatrix} \quad ; \quad \underset{\sim}{F} = -\begin{Bmatrix} 1\\1\\1\\1 \end{Bmatrix} \ddot{X}_G \tag{8}$$

The transformed system has the form:

$$\begin{bmatrix} P^2+1 & -1 & 0 & 0 \\ -1 & P^2+2 & -1 & 0 \\ 0 & -1 & P^2+2 & -1 \\ 0 & 0 & -1 & P^2+2 \end{bmatrix} \begin{Bmatrix} Y_1(P) \\ Y_2(P) \\ Y_3(P) \\ Y_4(P) \end{Bmatrix} = -\begin{Bmatrix} 1\\1\\1\\1 \end{Bmatrix} \ddot{X}_G(P) \tag{9}$$

We apply elimination strategy over all structure to eliminate $Y_2(p)$ and $Y_4(p)$.After that we obtain:

$$\begin{bmatrix} P^4+3P^2+1 & -1 \\ -1 & P^4+4P^2+2 \end{bmatrix} \begin{Bmatrix} Y_1(P) \\ Y_3(P) \end{Bmatrix} = -\begin{Bmatrix} P^2+3 \\ P^2+4 \end{Bmatrix} \ddot{X}_G(P) \tag{10}$$

Using substructure description we have to eliminate midle DOF $Y_2(p)$ and $Y_4(p)$.Initial description of the both substructures is:

$$\begin{bmatrix} P^2+1 & -1 & 0 \\ -1 & P^2+2 & -1 \\ 0 & -1 & \frac{1}{2}P^2+1 \end{bmatrix} \begin{Bmatrix} Y_1(P) \\ Y_2(P) \\ Y_3(P) \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{Bmatrix} \ddot{X}_G(P) + \begin{Bmatrix} 0 \\ 0 \\ V_3(P) \end{Bmatrix} ; \begin{bmatrix} \frac{1}{2}P^2+1 & -1 \\ -1 & P^2+2 \end{bmatrix} \begin{Bmatrix} Y_3(P) \\ Y_4(P) \end{Bmatrix} = -\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \ddot{X}_G(P) + \begin{Bmatrix} -V_3(P) \\ 0 \end{Bmatrix} \tag{11}$$

Elimination algorithm over (11) leads to:

$$\begin{bmatrix} P^4+3P^2+1 & -1 \\ -1 & \frac{1}{2}P^4+2P^2+1 \end{bmatrix} \begin{Bmatrix} Y_1(P) \\ Y_3(P) \end{Bmatrix} = -\begin{Bmatrix} P^2+3 \\ \frac{1}{2}P^2+2 \end{Bmatrix} \ddot{X}_G(P) + \begin{Bmatrix} 0 \\ (P^2+2).V_3(P) \end{Bmatrix} ; \left[\tfrac{1}{2}P^4+2P^2+1\right]\{Y_3(P)\} = -\{\tfrac{1}{2}P^2+2\}\ddot{X}_G(P) + \{-(P^2+2).V_3(P)\} \tag{12}$$

We note that after assemblying the govering equations for all structure we obtain again (10).From viewpoint of the core used substructure technique is more efficient. It is easy to determine transfer function for $Y_1(p)$ from equation (10) if $Y_3(p)$ would be eliminated:

$$W_1(P) = \frac{Y_1(P)}{\ddot{X}_G(P)} = \frac{P^6 + 7P^4 + 15P^2 + 10}{P^8 + 7P^6 + 15P^4 + 10P^2 + 1} \tag{13}$$

4.Elimination of 'slave' DOF.

Some different approaches to achieve 'condensed' structure are available.Basing on equation (2) a flow - chart has been prepared to provide the relations between Laplaçe transforms of $\underset{\sim}{Y}$ using graph theory approach.Elimination strategy may be performed by structural transformation ISHTEV[2].Meisson's formulae and Fadeev's algorithm KWAKERNAAK, SIVAN[3] are useful for computer aided elimination technique proposed here.

An algorithm has been prepared by the autors is based on the Gauss linear equations solver.Coefficient operations are changed by polinomial operations.In order to avoid remainders polinomial ratious are not used.The highes order of 'p' has been controlled not to induce high order of derivatives in the time domain.

5.Results and discussions.

The technique mentioned above is a convinient one when mathematical description should be done witout losses of accuracy.Consider a system with 'n' DOF.When 'n-1' DOF should be eliminated, the total number of the coefficients in the corresponding single differential equation does not exeed 4n .When a classical form of FEM is used by the description (1) the number of coefficients will be of order 'n^3' To emphasize this advantage of the 'dynamical condensation' more unknown displacements should be involved.

The computational time required to determine the response along the 'master' DOF can essentially be reduced.That can be ilustrated by frequency domain integration.Note, that the computional time is dependent stronger on the number of equations than on the order of derivatives.This approach has been used in reference ISHTEV, PHILIPOV, BOJILOV[4], and it includes spectrum determination of the input signal $X_g(j\omega)$, solving of the system (4) for every values of ω and back Fourier transform to obtain 'master' displacements.

Fig.4 shows the relationship between time required to solve the problem and number of DOF.Computer PDP 11 34 was used to demonstrate numerical examples.Line 2 shows the time consumed to compute 1 000 values of response of all displacements without elimination.Line 1 shows the time required to compute 1 000 values of one response when 'n-1' displacements are eliminated.Dotted line 3 represents the time consumed using 'dynamic condensation' technique multiplied by number of DOF.One may observe from Fig.4 that 'dynamic condensation' technique is more efficient when number of DOF is significant.Also even if 'dynamical condensation' is not indispensable it's better to eliminate 'n-1' DOF than to determine all 'n' - responses simultaneously.

When linear modelling in frequency domain is applied, the frequency transfer function is used to evaluate the natural spectrum of the structure.Fig.5 shows the frequency transfer functions in logaritmic scale.It could be seen that line 1 which corresponds to the 'condensed' and initial system also contains all four natural frequencies.To compare with line 1 are shown results after 'static condensation' technique respectively - line 2 (two DOF are eliminated) and line 3 (three DOF are eliminated).Evidently the diference between line 3 and line 1 will grow in general as it would be expected.The shifting between line 3 and 1 is more significant than between lines 1 and 2.It is also seen that after 'dynamical condensation' the transfer function is not violated and it's kept equal to the initial one.

6.References.

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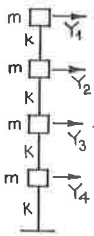


FIG.1

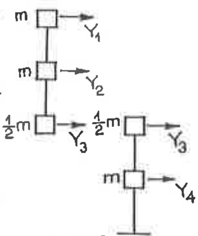


FIG.2

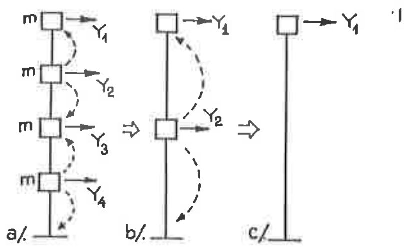
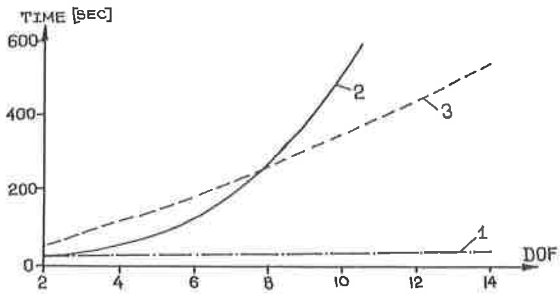


FIG.3

FIG.4



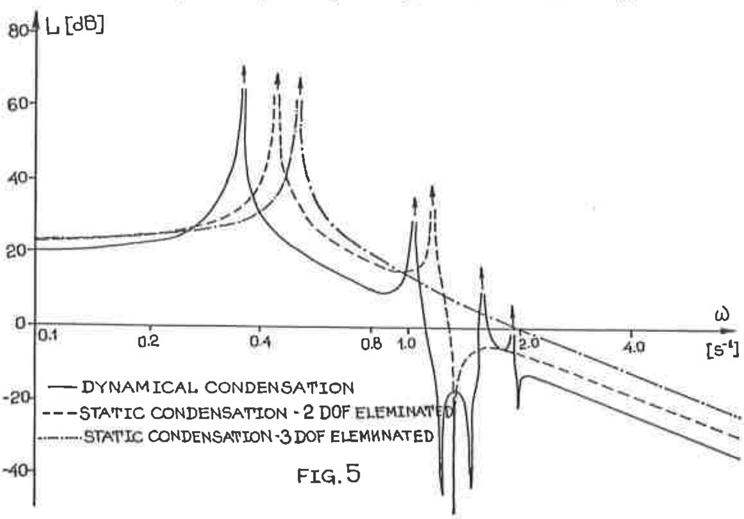


FIG.5