

Comparison of Bit Error Rates for Binary and Differential Phase  
Shift Keying in a Power Line Communication Environment

by

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## Abstract

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This paper emphasizes the derivation of expressions for the bit error rate (BER) when (binary) phase shift keying (PSK) or differential phase shift keying (DPSK) modulation techniques are used in a power line communication environment. This environment consists of white, gaussian-distributed, background noise plus 60 Hz harmonics of relatively high power.

Two different BER derivations were performed. The first one was for the case of 0,1 or 2 interfering harmonics and the other was for the case of many interfering harmonics ( $\gg 1$ ). The initial derivation was based on considering specific forms for the signal, noise, and harmonics and passing this received signal through the PSK and DPSK receivers one stage at a time. A decision variable for each system was first derived. Then the BER was derived by considering the value of these decision variables in relation to a given threshold level at the sample times. The decision variable for the PSK receiver was simple to derive since that system is linear, while the DPSK system is nonlinear and that derivation was quite complicated.

The second derivation was necessitated by limitations in the original direct derivation approach. These were caused by computational limits for the PSK derivation and sheer analytical complexity

problems for the nonlinear DPSK case if more than 2 harmonics were considered. The second derivation was based on using the central limit theorem to approximately describe the contribution of the harmonic noise to the analysis. With this simplification the BER was easily derived using published procedures and the resulting expressions were simple enough to be useful.

These BER expressions were used to compare PSK and DPSK performance for two specific systems. One is a present-day system with  $R=76.22$  baud and the other is a possible future system with  $R=500$  baud. The direct derivation was used for the first case and the simplified derivation was used for the second system. The results show PSK to be far superior to DPSK in the first application while the performances were quite similar for the second system. Based on this observation one might choose the DPSK technique for the second system since its hardware requirements are less demanding. This second observation is undoubtedly influenced by the gross generalization made about the harmonic interference in that derivation.

## Biography

Kenneth E. Martin, Jr. was born March 19, 1960, in Cleveland, Ohio. He graduated as valedictorian from Olmsted Falls High School in 1978 and was employed by the Cleveland Electric Illuminating Company under their Scholastic Awards Program. This employment was in conjunction with the Cooperative Education Program of the University of Cincinnati which he attended from 1978 to 1983. He left the Illuminating Company in 1980 and completed his co-op assignments with NASA's Lewis Research Center and General Electric in Cleveland, Ohio. Graduating summa cum laude in June, 1983 with a Bachelor of Science degree in electrical engineering, he entered graduate school at North Carolina State University in July, 1983. He received a fellowship from the Center for Communication and Signal Processing (CCSP) at NCSU for one year of graduate study. He was then employed as a research assistant for the CCSP for the duration of his graduate study and will be awarded his Master of Science degree in May, 1985.

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## Table of Contents

	Page
1. Introduction.....	1
2. Binary Phase Shift Keying (BPSK) Transmission for PLC.....	4
2.1 System Description.....	4
2.2 Derivation of BER with Two Harmonic Interferers.....	17
2.3 Computer Calculation of BER.....	22
2.4 Results for BPSK Modulation with Two Harmonic Interferers....	27
3. Differential Phase Shift Keying (DPSK) Transmission for PLC.....	33
3.1 System Description.....	33
3.2 Derivation of BER with Two Harmonic Interferers.....	37
3.3 Computer Calculation of BER.....	42
3.4 Results for DPSK Modulation with Two Harmonic Interferers....	45
4. BER Expression for BPSK with Many Harmonic Interferers.....	48
4.1 Derivation of BER.....	48
4.2 Computer Results for BPSK.....	50
5. BER Expression for DPSK with Many Harmonic Interferers.....	55
5.1 Derivation of BER.....	55
5.2 Computer Results for DPSK.....	57
6. Comparison of BPSK and DPSK Modulation for Two PLC Systems.....	60
6.1 Comparison for Two Interferer Case.....	60
6.2 Comparison for Nineteen Interferer Case.....	65
6.3 Conclusions.....	69

7. List of References.....	72
8. Appendices.....	75
8.1 60 Hz Harmonic Noise.....	75
8.2 Derivation of Decision Variable U for BPSK with Two Harmonic Interferers.....	80
8.2.1 Input $s(t)$ .....	82
8.2.2 Input $h_i(t)$ .....	83
8.2.3 Input $n(t)$ .....	83
8.3 Derivation of Decision Variable V for DPSK with Two Harmonic Interferers.....	86
8.3.1 Derivation of Multiplier Output, $W(t)$ .....	86
8.3.2 Derivation of Low Pass Filter (LPF) Output, $V(t)$ .....	91
8.3.3 Derivation of Sample/Hold Output, $V$ .....	105
8.4 Derivation of Decision Variable U Using Central Limit Theorem.....	107
8.5 Q Function Approximation.....	113

## 1. Introduction

Power line communications (PLC) is a topic of growing interest to the electric utility industry. Rising energy costs make it increasingly important for this industry to economize wherever possible, and automation of its system is at the heart of this process. This automation requires a two-way communications link between the utility's operations control center and the customer sites. With this link in place, the utility can monitor power usage and flow and can initiate changes in the system to allow it to operate as economically as possible.

Of particular interest is the utility's control of residential appliances. This comes under the heading of "load control" and represents one of the utility's greatest opportunities for cost savings. This opportunity exists because the utility must be sure to have adequate generating capacity to handle any peak load which may occur, and this is very expensive. Supplying this peak load may mean buying power from neighboring utilities, using oil-fired "peaking units", or building more generating plants than are necessary for normal operations. All of these are expensive and create an incentive to find a better way to operate. The ideal solution for the utility is to be able to selectively control the load which it must supply during peak times. To be effective, the utility must be able to turn on and off a large number of appliances. Therefore, the utility must have access to a large number of customers. This requires a far-reaching communications network. At this time the



utilities are making use of radio-controlled relays to turn appliances on and off. However, this system is limited by the one-way nature of the communication. To be truly useful the communication method should allow two-way transmission between the residential site and the utility's control center. Two-way transmission would allow the utility not only the ability to control its load, but also the opportunity to perform tasks such as remote reading of electric meters and monitoring devices or surveying of its load for planning purposes.

An obvious solution is for the utility to make use of its own power transmission network as a communications medium. This network connects the control center to all of the customers and has the capability of two-way transmission. However, the problems associated with trying to communicate over such a channel are significant. This channel represents a very harsh communications environment. The channel changes as loads are added and dropped, there are standing waves on the line at communication frequencies due to wave reflections, and 60 Hz harmonic interference is prevalent throughout the system [12]. All of these are significant problems, but this research will be concerned with only the last one. How does the presence of 60 Hz harmonics in addition to background (additive, white, gaussian-distributed) noise affect the integrity of transmitted data?

In order to answer that question, it is necessary to consider a particular transmission scheme. For this work the performance of two

modulation methods will be compared, specifically binary phase shift keying (BPSK) and differential phase shift keying (DPSK). Both methods are widely used in the transmission of binary digital data. Generally BPSK will give higher performance while DPSK requires simpler and less expensive hardware. To properly assess this trade-off (for use in a PLC system) it is necessary to measure the performance of each method under the conditions described. This requires the selection of an appropriate performance measure, which for digital data is typically the bit error rate (BER). This gives the average probability of a bit error and will be the measure used in this study.

The objective of this research is to derive BER expressions for BPSK and DPSK and use these to compare their performance in the context of two typical PLC systems. The first system represents a commercial product in operation today which transmits at 76.22 baud with a carrier frequency of 12.5 kHz. The second system represents a possible system of the future. This system transmits data at a much higher rate (500 baud) and will be assumed to use a carrier frequency of 6 kHz. Based on this information it should be possible to select the more appropriate modulation scheme for each PLC system.

## 2. Binary Phase Shift Keying (BPSK) Transmission for PLC

### 2.1. System Description

Phase shift keying (PSK) is a widely used bandpass transmission method for digital communications [4]. In a PSK system the information is transmitted via the phase of the signal. For binary data the signal only requires two different phases (eg. 0 or  $\pi$  radians) to transmit the data. This type of transmission is called binary phase shift keying (BPSK). See figure 2.1 for typical binary data and the associated BPSK signal. It is of course possible to use more data levels or send more bits per symbol (by encoding) if more choices of phase are allowed. However, this would make the system more complicated than required for this application. More phase choices also makes the detection problem more difficult and this is already a primary concern in the PLC system. The block diagram for the proposed BPSK transmission system is shown in figure 2.2. Note that this figure also shows the DPSK system. See section 3.1 for a description of that system. Before continuing with the discussion of this system it is helpful to provide some background on more general data transmission systems.

The block diagram of a general binary baseband transmission system is shown in figure 2.3. For this discussion assume that the channel does not attenuate or distort the phase of the signal, ie.  $G_c(f) = 1$ . If channel effects must be included then they can be incorporated into the transmitter and/or receiver filters. The main concern in data transmission is the minimization of the probability

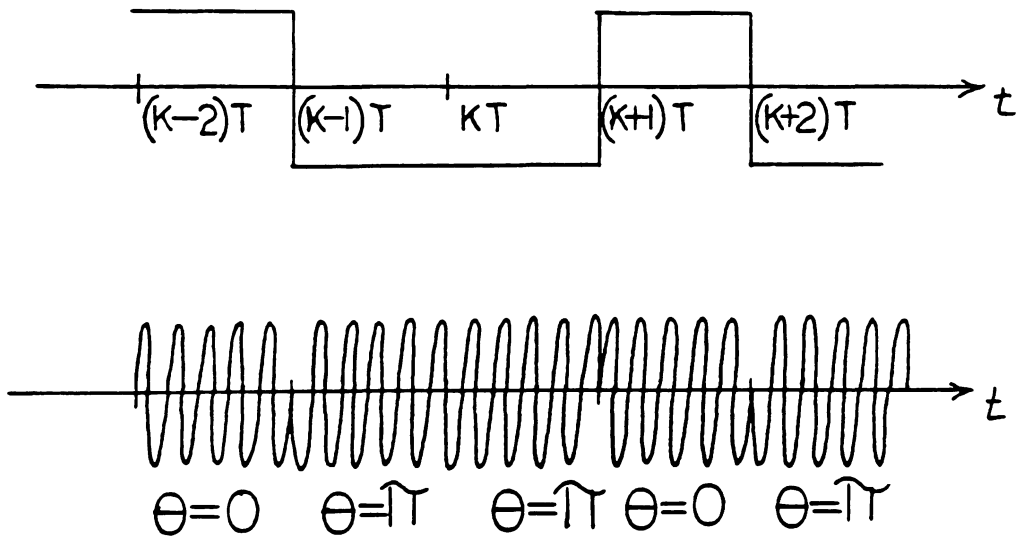


Figure 2.1 Binary Data and Resulting BPSK Waveform

of a detection error. This is accomplished by minimizing the effect of intersymbol interference (ISI) and maximizing the signal to noise ratio at the detector input at the sampling times,  $t=kT$ .

Minimizing the effect of ISI requires that the overall system transfer function,  $G(f)$  [ $= G_T G_R(f)$ ], meet the Nyquist criterion [14]. In the time domain this means that the system impulse response,  $g(t)$ , must equal 0 at  $t=kT$  for  $k \neq 0$ . Then the shape of  $g(t)$  can be arbitrary for the remainder of the time since it cannot affect the value of future or past data pulses at their corresponding sample times. These  $g(t)$  are called Nyquist pulses. See figure 2.4 for an example. It can be seen that such pulses are noncausal and hence are not physically realizable. However, by making the "tails" of the pulse decay more rapidly and introducing an appropriate time delay, an approximate Nyquist pulse can be physically realized. A typically used set of Nyquist pulses are those of the raised cosine family [5]:

$$P(f, \beta) = \frac{\pi}{4\beta} \cos\left(\frac{\pi f}{2\beta}\right) \text{Rect}\left(\frac{f}{2\beta}\right) * \frac{1}{R} \text{Rect}\left(\frac{f}{R}\right) \quad 0 < \beta \leq \frac{R}{2} \quad (2.1.1)$$

where  $R$  is the bit rate,  $\beta$  is a parameter and

$$\begin{aligned} \text{Rect}\left(\frac{f}{T}\right) &\equiv 1 & |f| \leq \frac{T}{2} \\ &\equiv 0 & |f| > \frac{T}{2}. \end{aligned} \quad (2.1.2)$$

Note that the parameter  $\beta$  determines the necessary channel bandwidth and hence the rate of decay of the tails. Some tradeoff is necessary between bandwidth and tail decay since a small bandwidth means a slower decay and hence a less easily realized pulse. Selection of an appropriate  $\beta$  depends on the particular system requirements. So if

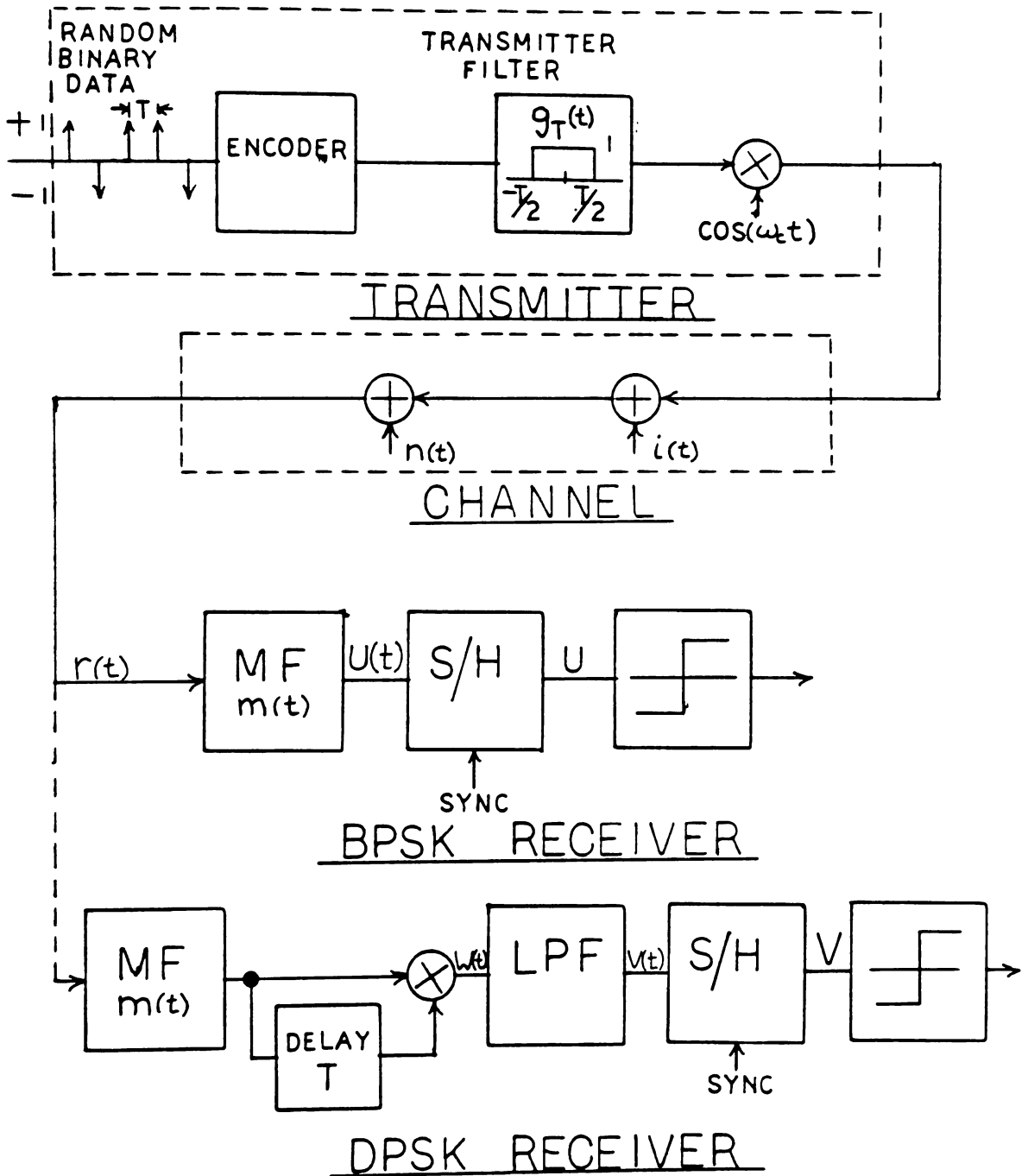


Figure 2.2 BPSK and DPSK System Block Diagram

raised cosine pulses are used then the following must be true to get no ISI:

$$G(f) = G_T G_R(f) = P(f, \beta). \quad (2.1.3)$$

To get the best system performance in the presence of white noise the receiver uses a matched filter [17]. Assuming that the transmitter filter impulse response has even symmetry it can be shown that the matched filter and transmitter filter have identical impulse responses. That means  $g_R(t) = g_T(t)$  and so  $G_R(f) = G_T(f)$ . Under this assumption,

$$G_T(f) = G_R(f) = P^{1/2}(f, \beta). \quad (2.1.4)$$

In that case, selection of a  $\beta$  value completely determines the necessary transmitter and receiver filters.

It is certainly possible to use this type of system in this study. However, in the interest of analytical simplicity the transmission system here utilizes rectangular pulses in the baseband portion of the system rather than Nyquist pulses. Rectangular pulses generally require an infinite channel bandwidth in order to have no ISI. However, the use of a "correlation processor" type matched filter will make the ISI negligible. That is because this filter is "dumped" at the end of each bit period so there is no contribution by previous pulses to the present output and hence no ISI. With these assumptions it is time to continue with the description of the actual system model used for this work.

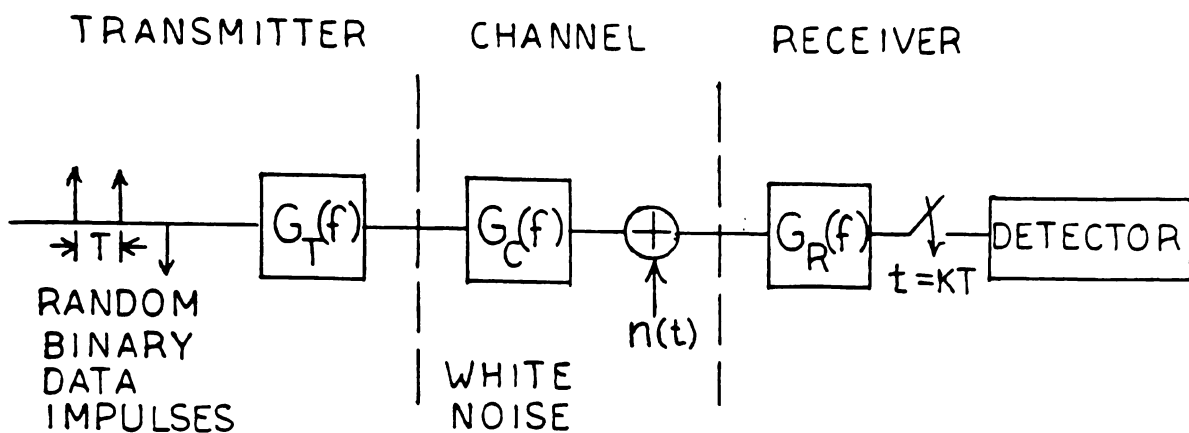


Figure 2.3 General Binary Baseband Data Transmission System

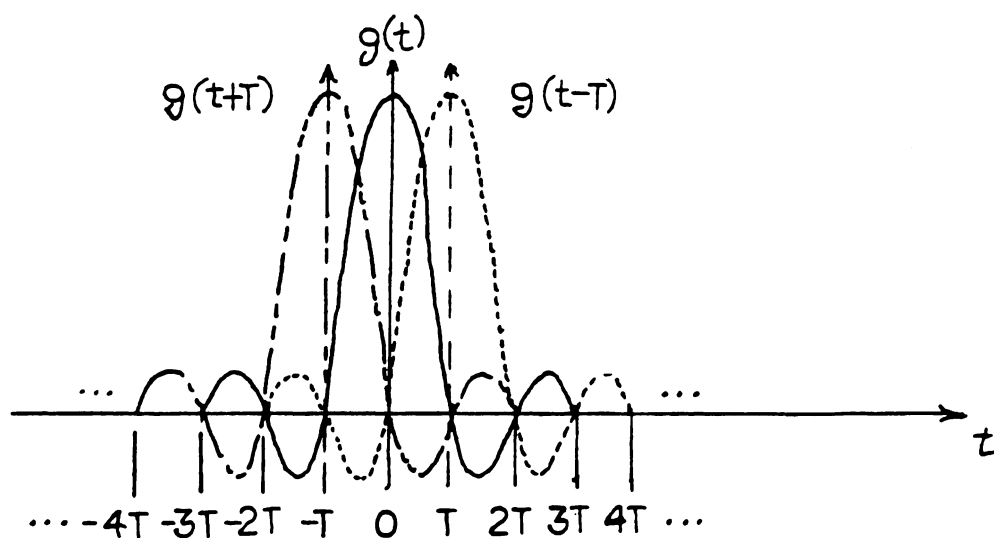


Figure 2.4 Example of Nyquist Pulses



The transmitter is the first stage of the system. It is very simple and generates the output signal by directly modulating the carrier with the output from the transmitter filter. This filter has a rectangular impulse response,  $g_T(t)$ , so a random binary rectangular pulse train is produced by the random binary impulses which represent the data. Therefore, over each  $T$  second bit period the resulting transmitted signal has the form  $\cos(\omega_c t + \theta)$  (where the magnitude has been scaled to unity for convenience) with carrier angular frequency  $\omega_c$  and  $\theta = 0$  or  $\pi$  radians. The next stage is the transmission channel.

The transmission channel shown in the figure consists of the power line together with its noise sources. The main noise sources are residential and industrial loads as well as the ever-present background random white noise. As discussed in section 8.1, the industrial and residential loads are sources of 60 Hz harmonic interference, which is the main concern of this study. In particular, voltage control devices (eg. light dimmers and SCRs) and other equipment generate pulses in sync with the 60 Hz power signal. It is assumed that these are additive noise sources. In general, channel attenuation is also present. This attenuation is frequency and position dependent, and affects both the signal and harmonic interference. The harmonic interference of interest is at a frequency close to that of the signal, but the signal and interference have different points of origin. Therefore, they are attenuated by different amounts. The degree of attenuation also changes as the system

changes due to shifting loads. These load shifts are random in nature and so the attenuation is also a random process. This will not be a subject for this research but is an important consideration. This work will simply assume that the ratio of signal level to interference levels (called  $\gamma$ ), and the signal to white noise ratio (called  $\text{snr}$ ) are known at the front end of the receiver.

The received signal will have the form

$$r(t) = s(t) + i(t) + n(t) \quad (2.1.5)$$

where

$$s(t) = A \cos(\omega_c t + \vartheta) \quad \text{for } (k-1)T \leq t < kT \quad (2.1.6)$$

$$i(t) = h_1(t) + h_2(t) + \dots + h_m(t) + \dots + h_N(t) \quad (2.1.7)$$

( $N$  large) with

$$h_j(t) = B_j \cos(\omega_j t + \varphi_j) \quad (2.1.8)$$

and

$$n(t) = \text{white gaussian noise} \quad (2.1.9)$$

with mean = 0 and variance =  $\sigma^2$ .

Here  $\omega_c$  is the carrier frequency,  $\vartheta = 0$  or  $\pi$  radians,  $\omega_j = n_j$  ( $2\pi \cdot 60$ ) for the  $n_j$ th harmonic of the 60 Hz power signal, the  $\varphi_j$  are independent uniformly distributed random variables over  $(-\pi, \pi)$ , and  $B_j$  is a random variable with an unknown probability distribution. For this work the  $B_j$  are assumed to be known parameters.

The matched filter at the front end of the receiver transforms  $r(t)$  into the output signal,

$$U(t) = s_M(t) + i_M(t) + n_M(t). \quad (2.1.10)$$

The matched filter is designed to pass most of  $s(t)$  while limiting how much of  $i(t)$  and  $n(t)$  can reach the detection portion of the receiver. In fact, if only white noise is present then use of the matched filter (matched to the signal) ensures optimum detection of the data. Since harmonic interference is also present, however, the matched filter will have to be matched to the signal plus interference to have optimum detection. This is probably impossible, therefore, some degradation is allowed in detection so the system will be relatively simple. In addition, recall that the goal of this study is to compare BPSK and DPSK systems. Therefore, as long as the two systems are consistent there should be little impact on the comparison. To determine how much of the noise and interference passes through the matched filter requires a knowledge of its transfer characteristics.

For this system the basic signal shape has the form,

$$d(t) = \cos(\omega_c t) \text{Rect}\left(\frac{t-T/2}{T}\right) \quad (2.1.11)$$

where the  $\text{Rect}()$  function of equation 2.1.2 has been used. The received signal (no noise or interference) can be written in terms of  $d(t)$ ,

$$s(t) = \sum_{k=-\infty}^{\infty} a_k d(t-kT) \quad (2.1.12)$$

where  $a_k = +/- A$  depending on whether the transmitted signal phase is 0 or  $\pi$  radians. The matched filter is matched to this basic signal shape  $d(t)$ . Therefore, the matched filter impulse response is

$$m(t) = a d(T-t) \quad (2.1.13)$$

$$= a \cos(\omega_c t - \omega_c T) \text{Rect}\left(\frac{t-T/2}{T}\right)$$

where the even symmetry of the  $\cos()$  and  $\text{Rect}()$  functions has been used to write it in this form [6]. The transfer function of the matched filter is easily found from this equation using the Fourier transform and noting  $(\omega_c T)$  is merely a phase shift. So for the case where  $a=1$ ,

$$M(f) = \frac{T}{2} \left\{ \text{sinc}\left[(f-f_c)T\right] e^{-j\pi(f+f_c)T} + \text{sinc}\left[(f+f_c)T\right] e^{-j\pi(f-f_c)T} \right\} \quad (2.1.14)$$

using the well known "sinc" function. See figure 2.5 for the magnitude spectrum,  $|M(f)|$ . As suggested earlier it is possible to implement this by using a correlation processor which is reset or "dumped" at the end of each bit period so there will be little or no ISI [8].

Since the noise is zero mean gaussian it is only necessary to determine its average power (variance) in order to specify its first order probability density. Using the noise equivalent bandwidth of this filter or standard matched filter results one can easily find the amount of white noise power passed by the filter (see section 8.2.3). The amount of harmonic interference which is passed is not

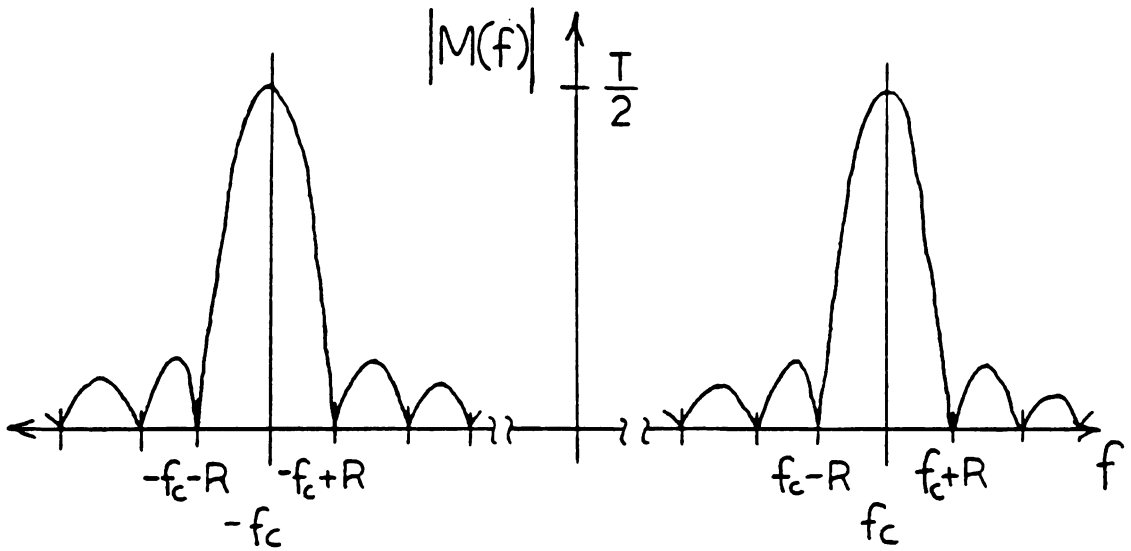


Figure 2.5 Magnitude Spectrum of Matched Filter Response

so obvious since the filter has an infinite absolute bandwidth. Due to the rapid decay of the side lobes of the matched filter's transfer function, however, a convenient choice for the filter bandwidth is the width of the main lobe. As shown in the figure this bandwidth is  $BW=(2/T)=2R$  where  $R$  is the system baud. Obviously if the bit rate  $R$  is known then it is possible to determine how many 60 Hz spectral lines lie within the main lobe. This means that  $U(t)$  can be written as

$$U(t) = s_M(t) + \sum_{i=1}^m h_{iM}(t) + n_M(t) \quad (2.1.15)$$

where the  $h_{iM}(t)$  are the filtered counterparts of the input signals  $h_i(t)$ , and only the  $m$  harmonics inside the main lobe are retained.

At this point  $U(t)$  passes into the detection portion of the receiver. The detection process relies on sampling  $U(t)$  at the appropriate times and comparing this to a threshold level. A sample/hold (S/H) device is used to sample the matched filter output at  $t=kT$  ( $k$  an integer) and saves it so it is available to the threshold detector during the next integration cycle. This results in the decision variable,

$$U = S + \sum_{i=1}^m H_i(\varphi_i) + N \quad (2.1.16)$$

where these variables represent the values of the variables in the previous equation at the sample times. Generally the level on the threshold detector can be set as desired and then any  $U$  value above this level is a "1" and below is a "0". It is possible to calculate

the average probability of a bit error given the input signal and system parameters. This probability is the bit error rate (BER).

## 2.2. Derivation of BER with Two Harmonic Interferers

It is possible to derive a closed form expression for the BER of this system. This is based on the decision variable  $U$  of equation 2.1.16. The actual derivation of the form of  $U$  is carried out in section 8.2 for the general case of  $m$  interferers and is found to be,

$$U = S + \sum_{i=1}^m H_i(\varphi_i) + N(0, \eta T/4)$$

with

$$S = +/- \frac{AT}{2}$$

$$H_i(\varphi_i) = P_i \cos(\Delta_i T(k - \frac{1}{2}) + \varphi_i)$$

and

$$P_i = \frac{B_i T}{2} \operatorname{sinc}\left(\frac{\Delta_i T}{2\pi}\right)$$

and  $N(0, \eta T/4)$  is a normally distributed random variable with a mean of zero and variance of  $\eta T/4$ . Even though this expression is for  $m$  interferers its use will be limited to the cases  $m=0,1,2$  for reasons explained in section 2.3. Of particular interest is the  $m=2$  case which applies to the very low bit rate system under consideration. This is explained in section 2.4.

$U$  can be more simply expressed as a normally distributed random variable which has mean  $[S + \sum_{i=1}^m H_i(\varphi_i)]$ . Defining  $\Delta \equiv \sum_{i=1}^m H_i(\varphi_i)$  this can be written as



$$U = N[(S + \Delta), \frac{\eta T}{4}]. \quad (2.2.1)$$

It is this random variable which determines the BER of the system. Note that the value of  $S$  depends on whether a "0" or a "1" is transmitted. If a "1" is sent then  $S = \frac{+AT}{2}$  and if a "0" then  $S = \frac{-AT}{2}$ . The probability density functions (pdf) for  $U$  given "0" is sent,  $p(U|0)$ , and  $U$  given "1" is sent,  $p(U|1)$ , are shown in figure 2.6.

The variable  $V_T$  shown in this figure is the threshold level. This is the decision level setting in the threshold detector. If  $U > V_T$  then a "1" is assumed to have been sent, and if  $U < V_T$  a "0" is assumed. To select a proper  $V_T$  value it is necessary to know how the random variable  $\Delta$  is distributed. Assuming  $H_i(\varphi_i)$  and  $H_j(\varphi_j)$  ( $i \neq j$ ) are independent random variables it can be shown that the probability density function of  $[H_i(\varphi_i) + H_j(\varphi_j)]$  will have a zero mean value since the pdfs of  $H_i(\varphi_i)$  and  $H_j(\varphi_j)$  are zero mean. Therefore,  $\Delta$  will have a zero mean. For that reason a good and convenient choice for  $V_T$  is  $V_T = 0$  since that lies halfway between the average mean values of  $p(U|0)$  and  $p(U|1)$ .

There are two possible errors in the decision process, i.e. detection of a "1" when a "0" was sent or vice versa. So the total probability of making an error given values of the random variables  $\varphi_i$  is

$$P_e(\varphi_1, \dots, \varphi_m) = P(U < V_T | 1)P(1) + P(U > V_T | 0)P(0). \quad (2.2.2)$$

Assuming that transmission of a "0" or "1" is equally likely and sub-

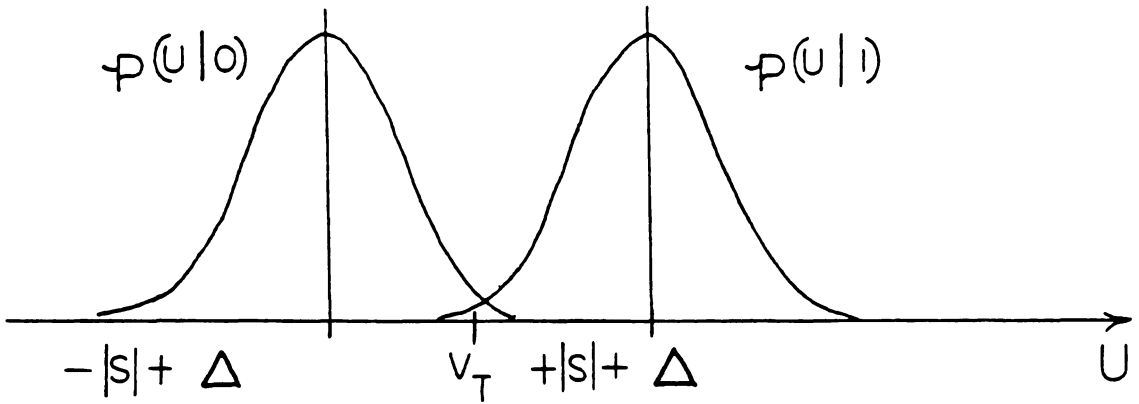


Figure 2.6 Conditional PDFs for Detection Variable U

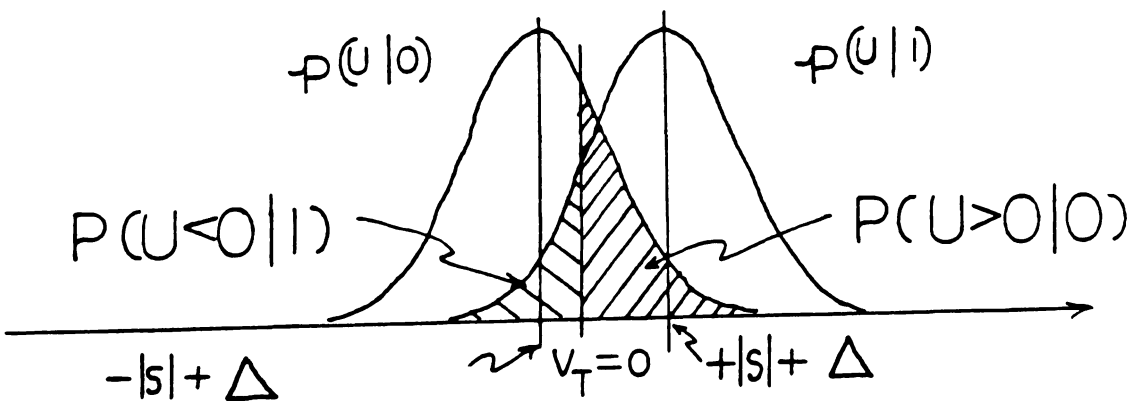


Figure 2.7 Probabilities for Calculation of  $P_e(\varphi_1, \dots, \varphi_m)$

stituting  $V_T = 0$  gives

$$P_e(\varphi_1, \dots, \varphi_m) = 0.5 \left\{ P(U < 0 | 1) + P(U > 0 | 0) \right\}. \quad (2.2.3)$$

Also,

$$P(U < 0 | 1) = \int_{-\infty}^0 N(|S| + \Delta, \frac{\eta T}{4}) dU \quad (2.2.4)$$

$$P(U > 0 | 0) = \int_0^{\infty} N(-|S| + \Delta, \frac{\eta T}{4}) dU. \quad (2.2.5)$$

These probabilities are indicated as cross-hatched regions in figure 2.7.

It is notationally convenient to use Q functions here [5].

Recalling  $|S| = \frac{AT}{2}$ , equation 2.2.3 can be written

$$P_e(\varphi_1, \dots, \varphi_m) = \frac{1}{2} \left\{ Q \left\{ \frac{AT/2 + \Delta}{\sqrt{\eta T/4}} \right\} + Q \left\{ \frac{AT/2 - \Delta}{\sqrt{\eta T/4}} \right\} \right\}. \quad (2.2.6)$$

The BER is defined as the average probability of error and this is found by averaging  $P_e(\varphi_1, \dots, \varphi_m)$  over the joint probability density function of  $\varphi_i$ , ie.  $p(\varphi_1, \dots, \varphi_m)$ .

$$BER = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} P_e(\varphi_1, \dots, \varphi_m) p(\varphi_1, \dots, \varphi_m) d\varphi_1 \dots d\varphi_m. \quad (2.2.7)$$

Recall that the  $\varphi_i$  are uniformly distributed over  $(-\pi, \pi)$  and it is assumed that they are independent so,

$$p(\varphi_1, \dots, \varphi_m) = p(\varphi_1) \dots p(\varphi_m) = \left[ \frac{1}{2\pi} \right]^m. \quad (2.2.8)$$

Using equations 2.2.6 - 2.2.8 gives the final form for the BER expression:

$$\text{BER} = \left[ \frac{1}{2\pi} \right]^m \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \left\{ Q \left\{ \frac{AT/2 + \Delta}{\sqrt{\eta T/4}} \right\} + Q \left\{ \frac{AT/2 - \Delta}{\sqrt{\eta T/4}} \right\} \right\} d\varphi_1 \dots d\varphi_m \quad (2.2.9)$$

This expression is in closed form. However, it is not very useful because Q functions cannot be evaluated directly, they must be approximated. So in order to use this equation for computer calculations an approximate form is necessary. This simplification is outlined in the next section.

### 2.3. Computer Calculation of BER

To make use of the BER expression in the previous section it is necessary to approximate the integration by a discrete summation. The Q functions can then be "looked up" or approximated by the computer at each discrete value of  $\varphi_i$ .

Let  $\varphi_i$  take on N discrete values (evenly spaced) from  $-\pi$  to  $\pi$ . For N even and  $n_i$  an integer define

$$\varphi_i \equiv 2\pi \frac{n_i}{N}, \quad -N/2 \leq n_i \leq (N/2 - 1). \quad (2.3.1)$$

Now the BER can be found in a manner analogous to the continuous case, ie.

$$P_e(\varphi_1', \dots, \varphi_m') = \frac{1}{2} \left\{ Q\left\{ \frac{AT/2 + \Delta'}{\sqrt{|\eta T/4|}} \right\} + Q\left\{ \frac{AT/2 - \Delta'}{\sqrt{|\eta T/4|}} \right\} \right\} \quad (2.3.2)$$

where  $\Delta' = \Delta$  evaluated at the discrete phase values  $\varphi_i'$ . The approximate BER is now found by averaging  $P_e(\varphi_1', \dots, \varphi_m')$  over the  $(N)^m$  combinations of  $\varphi_i'$  values.

$$\text{BER} \sim \frac{1}{(N)^m} \sum_{n_1=-N/2}^{N/2-1} \dots \sum_{n_m=-N/2}^{N/2-1} P_e(\varphi_1', \dots, \varphi_m'). \quad (2.3.3)$$

This yields

$$\text{BER} \approx \frac{1}{2(N)^m} \sum_{n_1=-N/2}^{N/2-1} \dots \sum_{n_m=-N/2}^{N/2-1} \left\{ Q\left\{ \frac{AT/2 + \Delta'}{\sqrt{|\eta T/4|}} \right\} + Q\left\{ \frac{AT/2 - \Delta'}{\sqrt{|\eta T/4|}} \right\} \right\}. \quad (2.3.4)$$

This can be implemented on a computer using an N value of about 100 in order to get a good approximation to the continuous case.

However, computational limitations restrict the practical use of this BER expression only to cases where  $m=0,1,2$ . One other complication is the presence of the Q functions. Fortunately there is an excellent approximation available (see section 8.5). Using this approximation it is a simple matter to calculate the BER given all of the necessary parameters. However, it is unlikely that the parameters in the equation will be known directly. It is much more likely that certain ratios of signal and noise powers or amplitudes will be available. For that reason a more useful form for the BER is derived.

A common parameter which is used is  $E_b/\eta$  where  $E_b$  is the average received energy per bit and  $\eta$  is the single-sided power spectral density of the white noise. For BPSK modulation with a received signal amplitude  $A$ ,  $E_b$  is found to be

$$E_b = \frac{A^2 T}{2}. \quad (2.3.5)$$

Also since  $E_b/\eta$  represents a type of signal to noise ratio (snr) it is useful to define

$$\text{snr} \equiv \frac{E_b}{\eta} = \frac{A^2 T}{2\eta}. \quad (2.3.6)$$

Assuming that the ratio of the signal amplitude to each of the harmonic interferers' amplitudes is known at the receiver, define for  $i=1,2,\dots,m$

$$\gamma_i \equiv \frac{A}{B_i}. \quad (2.3.7)$$

The next step is to modify the arguments of the Q functions so they will be functions of  $\gamma$  and snr. These arguments can be written as

$$\frac{AT/2 \pm \Delta'}{\sqrt{|\eta T/4}} = \frac{AT}{2} \frac{2}{\sqrt{|\eta T}} \pm \frac{2 \Delta'}{\sqrt{|\eta T}} \quad (2.3.8)$$

where

$$\Delta' = \sum_{i=1}^m \frac{B_i T}{2} \operatorname{sinc}\left(\frac{\Delta_i T}{2\pi}\right) \cos\left(\left(k-\frac{1}{2}\right)\Delta_i T + 2\pi \frac{n_i}{N}\right). \quad (2.3.9)$$

The first term is easily changed to

$$\frac{AT}{2} \frac{2}{\sqrt{|\eta T}} = \sqrt{\left| \frac{A^2 T^2}{2\eta} \right| \sqrt{2}} = \sqrt{2 \operatorname{snr}}. \quad (2.3.10)$$

The second term is a little more involved due to the  $\Delta'$  expression. This can be handled by defining for  $i=1,2,\dots,m$

$$D_i = \operatorname{sinc}\left(\frac{\Delta_i T}{2\pi}\right) \cos\left(\left(k-\frac{1}{2}\right)\Delta_i T + 2\pi \frac{n_i}{N}\right). \quad (2.3.11)$$

Then the second term can be rewritten as

$$\frac{2 \Delta'}{\sqrt{|\eta T}} = \frac{2}{\sqrt{|\eta T}} \frac{T}{2} \sum_{i=1}^m B_i D_i. \quad (2.3.12)$$

This can be modified as desired by noting

$$\frac{T}{\sqrt{|\eta T}} B_i D_i \frac{A}{A} = \frac{D_i}{\gamma_i} \sqrt{\left| \frac{A^2 T}{\eta} \right|} = \frac{D_i}{\gamma_i} \sqrt{2 \operatorname{snr}}. \quad (2.3.13)$$

This means that the final form for the BER approximation is found by combining these equations to get

$$\operatorname{BER} \approx \frac{1}{2} \frac{1}{(N)^m} \sum_{n_1=-N/2}^{N/2-1} \dots \sum_{n_m=-N/2}^{N/2-1} \left\{ Q\left( \sqrt{2 \operatorname{snr}} \left( 1 + \sum_{i=1}^m \frac{D_i}{\gamma_i} \right) \right) \right\}$$

$$+ Q\left\{\sqrt{2\text{snr}}\left(1 - \sum_{i=1}^m \frac{D_i}{\gamma_i}\right)\right\}. \quad (2.3.14)$$

From this development it is obvious that the BER is a function of

$$\text{snr}, \gamma_i, T, \omega_c, \omega_i, m.$$

However, some care must be used here since  $T$  actually determines how many harmonics (value of  $m$ ) must be included in the calculation. That means the variables are not independent as might be implied above. Note also that the BER is not a function of  $k$ . The  $(k\Delta_i T)$  terms represent constant phase shifts at the sample times and since the BER is averaged over all phase values  $(-\pi, \pi)$  they will have no effect.

Computer software that calculates BER values using the above equation is written in the "C" language and has been run on a VAX 11/780 computer. Note that the Q functions are calculated using the approximation given in section 8.5. This software provides the results of the next section.

As mentioned earlier, the derivation presented here is of restricted computational usefulness since a maximum of two interferers can be included. If more interferers must be included (eg. larger  $R$  desired) then the corresponding computer computations will become prohibitively time consuming. For that reason an alternate derivation of a BER formula for the multiple interferer case is presented in section 4. This derivation uses greater approximations to arrive at a very simple BER expression that is easily implemented on the



computer. A PLC system simulation is also used to check the validity of this BER expression. All of this is detailed in section 4.

#### 2.4. Results for BPSK Modulation with Two Harmonic Interferers

Equation 2.3.14 of section 2.3 is used to calculate the BER for BPSK modulation. To make these results as realistic as possible the software is run using the parameters of an existing PLC system. The system is one introduced by the Westinghouse corporation and uses the following parameters:

$$T = \frac{1}{R} = \frac{1}{76.22} \text{ (sec)}$$

(R is the system baud)

$$\omega_c = (12.5 \text{ kHz}) 2\pi \text{ (rad/s)}$$

$$\omega_1 = (208) (60) 2\pi \text{ (rad/s)}$$

(ie. the 208<sup>th</sup> harmonic)

$$\omega_2 = (209) (60) 2\pi \text{ (rad/s)}$$

(ie. the 209<sup>th</sup> harmonic).

There are two harmonics included (m=2) since the main lobe width of the matched filter is  $2R=152.4$  Hz.  $\omega_1$  and  $\omega_2$  are the harmonics closest to the transmission frequency  $\omega_c$ . Using these parameters the BER is calculated as a function of snr with  $\gamma_1$  and  $\gamma_2$  as parameters.

To simplify the presentation of results, assume that the harmonics have about the same size of received amplitudes,  $B_1$  and  $B_2$ . This means that it is possible to use

$$\gamma = \gamma_1 = \gamma_2 \tag{2.4.1}$$

in place of both  $\gamma_1$  and  $\gamma_2$ . To be a useable system the signal ampli-

tude needs to be somewhat larger than the harmonic amplitudes. This means that the  $\gamma$  values of interest are roughly in the range

$$1 < \gamma < 5. \quad (2.4.2)$$

Using these assumptions the BER vs snr curves are shown in figure 2.8. Note that the curve for  $\gamma = 10^6$  is included for comparison to an essentially harmonic-free system. Comparing this to a standard BPSK BER vs  $(E_b/\eta)$  curve shows this result to be in good agreement [4]. That indicates the software is running properly. These curves show how the 60 Hz harmonics degrade system performance compared to a system with only gaussian noise.

From these curves it is obvious that an increase in the snr or  $\gamma$  will reduce the BER. However, these two quantities are both functions of the signal amplitude A. That means that the snr and  $\gamma$  are dependent quantities and an increase in one causes the other to also increase. An increase in the transmitted signal power will, therefore, result in a greater decrease in BER than expected if only the change in snr or  $\gamma$  is considered separately. To see how to take this into account look at the relation between snr and  $\gamma$  (see equations 2.3.6 and 2.3.7 .)

$$\gamma = \frac{A}{B} \text{ and } \text{snr} = \frac{A^2 T}{2\eta}.$$

So  $\gamma$  can be written as a function of snr,

$$\gamma = \frac{1}{B} \sqrt{\frac{2\eta \text{snr}}{T}} = c \sqrt{\text{snr}} \quad (2.4.3)$$

where

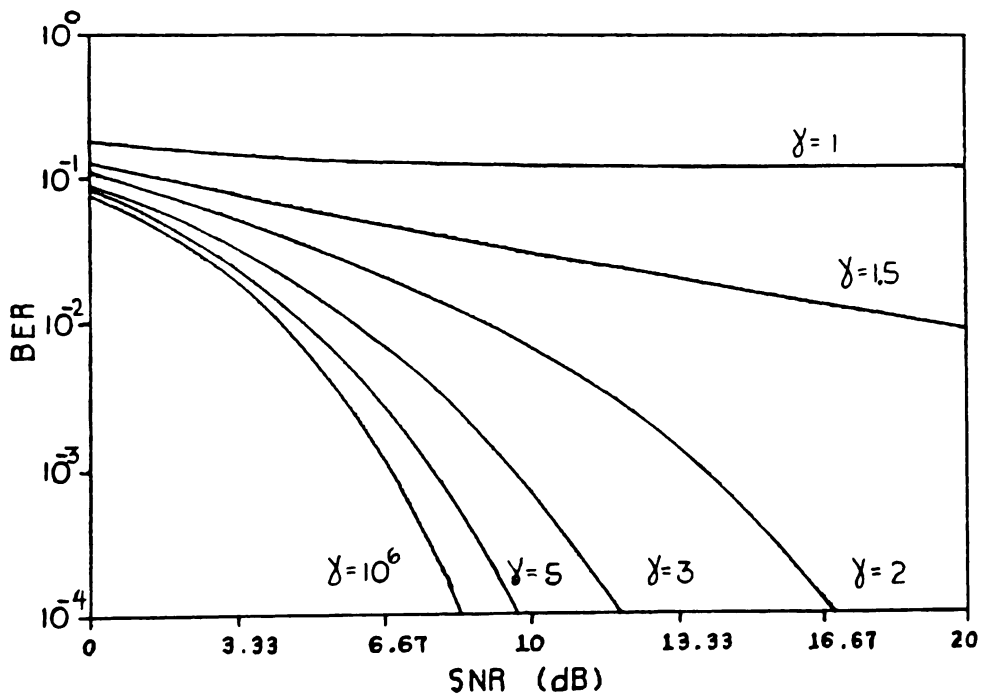


Figure 2.8 BER vs SNR (dB) for BPSK with  $R=76.22$  bps and  $f_c=12.5$  kHz

$$c = \sqrt{\frac{2\eta}{B^2 T}}. \quad (2.4.4)$$

This will be more useful if the snr is converted to dB as used in the BER vs  $\text{snr}_{\text{dB}}$  plot. Making this conversion and taking the square root yields

$$\gamma = c \cdot 10^{\left(\frac{\text{snr}_{\text{dB}}}{20}\right)}. \quad (2.4.5)$$

A plot of  $\gamma$  vs  $\text{snr}_{\text{dB}}$  is shown in figure 2.9.

Assuming that  $\eta$  and  $B$  stay constant, this curve can be used to see the true effect of an increase in signal amplitude by using the following procedure.

1) Assume the initial  $\gamma$  and  $\text{snr}_{\text{dB}}$  are known. Then the constant  $c$  can be calculated by using

$$c = \gamma / 10^{\left(\frac{\text{snr}_{\text{dB}}}{20}\right)}. \quad (2.4.6)$$

2) Use the BER vs  $\text{snr}_{\text{dB}}$  curves to find the initial BER.

3) To see the effect of an increase in snr on the BER use the  $\gamma$  vs  $\text{snr}_{\text{dB}}$  curve with the  $c$  value calculated in step 1. That is, select the new  $\text{snr}_{\text{dB}}$  value and determine the corresponding  $\gamma$  value. Then use the BER vs  $\text{snr}_{\text{dB}}$  plot and locate the BER corresponding to the new  $\gamma$  and  $\text{snr}_{\text{dB}}$  values. For example, using the BER vs  $\text{snr}_{\text{dB}}$  curves in figure 2.8 select the initial point

$$\text{snr}_{\text{dB}} = 2 \text{ and } \gamma = 1.5.$$

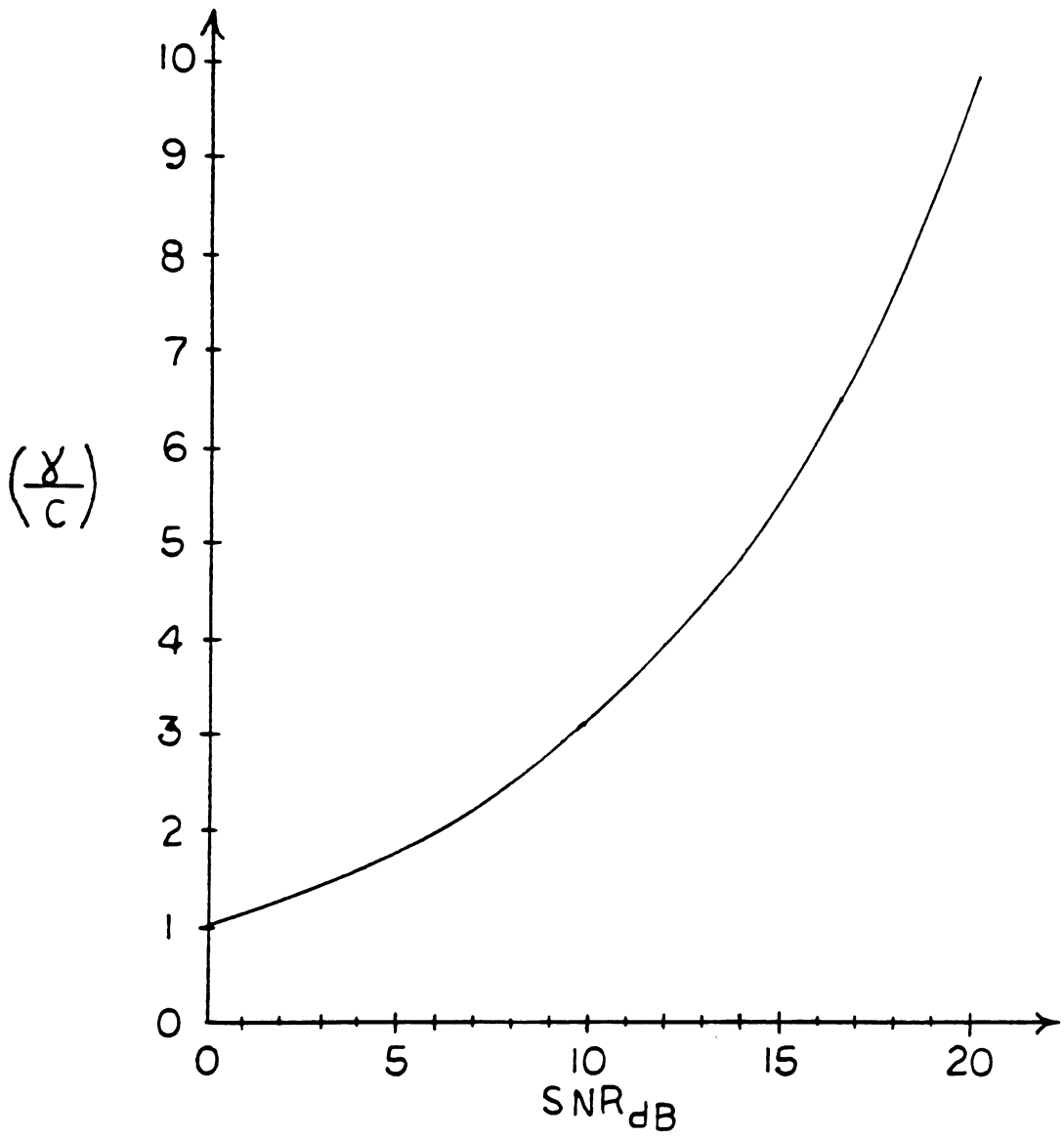


Figure 2.9 Curve for Calculation of Effect of Change in Transmitted Power on BER

The corresponding value of  $c$  is found from equation 2.4.6,

$$c = 1.5 / 10^{2/20} = 1.2.$$

The initial BER is found to be approximately  $10^{-1}$ . Increase the snr by 4 dB so

$$\text{snr}' = 6 \text{ dB}$$

Then the  $\gamma$  vs  $\text{snr}_{\text{dB}}$  curve gives

$$\gamma' \approx 2c = 2.4$$

This gives the new BER value of approximately  $17.8 \times 10^{-3}$ . Without making the adjustment from  $\gamma$  to  $\gamma'$  the new BER would have been thought to be approximately  $28.1 \times 10^{-3}$ . Therefore, it is obviously important to make this adjustment when using the BER vs  $\text{snr}_{\text{dB}}$  curves. Further conclusions will be deferred until section 6.

### 3. Differential Phase Shift Keying (DPSK) Transmission for PLC

#### 3.1. System Description

DPSK is a modulation technique which is closely related to the binary phase shift keying (BPSK) technique discussed in section 2.1. Both transmit the same type of signal, but they encode and decode the data differently. In BPSK the "0"s and "1"s are directly encoded (typically) into 0 or  $\pi$  radian phase shifts of the transmitted signal. On the other hand, DPSK uses a differential encoding scheme. That means that it is the difference between adjacent data bits that is encoded. For binary data this results in a transmitted phase of 0 radians when the adjacent bits are the same and  $\pi$  radians when they are different. See figure 3.1 for an example of a DPSK signal. A DPSK system is usually used because it requires a simpler receiver than a BPSK system. In particular, a BPSK system requires a demodulating signal at the receiver which has exactly the same frequency and phase as the modulating signal at the transmitter (ie. a coherent detector). This is not easy to achieve and in practice requires a lot of extra circuits just to accomplish this synchronization. At times it is not even possible to use BPSK because of a channel's time-varying phase distortions. Using DPSK solves these problems because the demodulating signal for the present bit period is the signal from the previous period (stored by a delay device). Since both signals are affected the same way by any channel phase distortions (provided the distortion varies slowly compared to the bit period) the DPSK system will still be operable with such a channel.



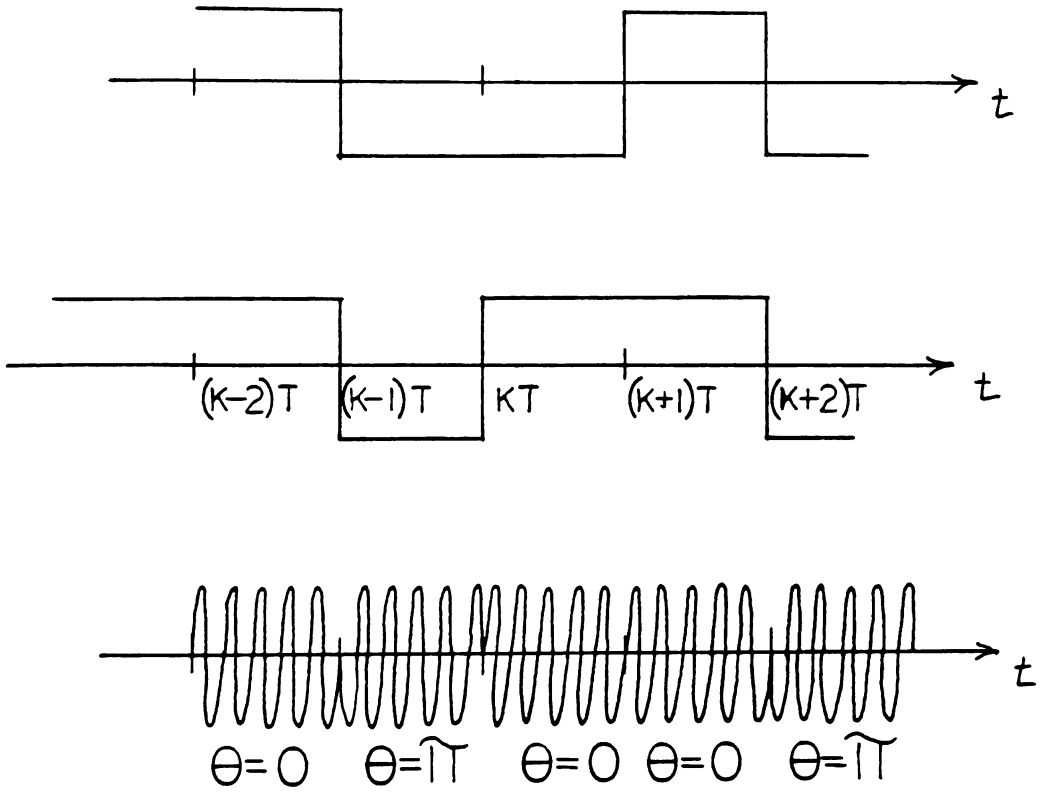


Figure 3.1 Binary Data, Differentially Encoded Form, Resulting DPSK Waveform

The block diagram for the proposed DPSK transmission system is shown together with the BPSK system in figure 2.2. Note that this is identical to the BPSK system except for:

- (1) type of encoder used in the transmitter
- (2) type of demodulation (includes everything after the matched filter).

The previous discussion in section 2.1 on general data transmission systems still applies to DPSK. So rectangular pulses and a "correlation processor" type matched receiver filter will again be used for reasons discussed in that section. This filter will again be reset or "dumped" at the end of each bit period so there will be no ISI.

The two differences mentioned above are a direct result of the differential encoding scheme used by the DPSK system. The encoder merely performs the desired encoding scheme. The resulting transmitted signal is identical to the BPSK case except the data must now be extracted differently. This data extraction occurs after the matched filter. There the present data signal,  $U(t)$ , is multiplied with the previous signal out of the delay device,  $U(t-T)$ , to yield  $W(t)$ . A low pass filter (LPF) then removes all but the baseband portion of this product to produce  $V(t)$ . Then a sampler and threshold detector are used as in the BPSK system. This description of the system operation is purposely vague because of the mathematical complexities introduced by this non-linear system. In particular, the random noise and harmonic interference is present in both the present and delayed signals and results in an output signal with products of ran-

dom variables. For that reason the pertinent details will be brought out as needed in sections 3.2 and 5 (multiple interferer DPSK case) and the references will be heavily cited.

### 3.2. Derivation of BER with Two Harmonic Interferers

This derivation is much more complicated than that for the BPSK system. This is due to the nonlinear nature of the DPSK receiver. It is possible, however, to use some results from other sections in order to somewhat simplify the work. Like the derivation of section 2.2, the BER depends on the value of a decision variable,  $V$ , in relation to some threshold level.  $V$  is labeled on the DPSK system diagram as are the intermediate signals of interest,  $U(t)$ ,  $W(t)$ , and  $V(t)$ . As in section 2.2, the actual derivation of the form of  $V$  is carried out in an appendix. This derivation is for the case where there are only two interferers ( $m=2$ ). This will allow comparison to the BPSK results of section 2.4 without the complexity of a more general derivation in terms of  $m$ . So from section 8.3,  $V$  is found to be,

$$\begin{aligned}
 V = N(0, \frac{A^2 T^3 \eta}{16}) + \frac{T^2}{8} \left\{ A_{k-1} A_{k-2} + B \sum_{i=1}^2 \operatorname{sinc} \left[ \frac{[\Delta_i T]}{2\pi} \right] \right. & \quad (3.2.1) \\
 \left. \left\{ A_{k-1} \cos[\epsilon_i - \Delta_i T] + A_{k-2} \cos[\epsilon_i] \right\} \right. & \\
 + B^2 \left\{ \sum_{i=1}^2 \operatorname{sinc}^2 \left[ \frac{[\Delta_i T]}{2\pi} \right] \cos(\Delta_i T) + 2 \operatorname{sinc} \left[ \frac{[\Delta_1 T]}{2\pi} \right] \operatorname{sinc} \left[ \frac{[\Delta_2 T]}{2\pi} \right] \right. & \\
 \left. \left. \cos \left[ \left( \epsilon_1 - \frac{\Delta_1 T}{2} \right) - \left( \epsilon_2 - \frac{\Delta_2 T}{2} \right) \right] \cos \left[ \frac{[(\Delta_1 + \Delta_2) T]}{2} \right] \right\} \right\} &
 \end{aligned}$$

Here for  $i=1,2$ ,  $\Delta_i = \omega_i - \omega_c$ ,  $\epsilon_i$  is a uniformly distributed random variable over  $(-\pi, \pi)$ , and  $A_{k-i} = +/- A$  depending on the transmitted bit in the  $(k-i)^{\text{th}}$  period. If there were no noise or harmonic

interference the value of  $V$  due to the signal only would be

$$V = \frac{T^2}{8} A_{k-1} A_{k-2}. \quad (3.2.2)$$

So the first term of the approximate expression for  $V$  is the noise contribution and all the other nonsignal terms represent the harmonic interference contribution. Using this information the BER can be derived in a manner analogous to the procedure of section 2.2 for BPSK.

For DPSK it is necessary to consider the signs of two successive bits to determine if an error is made in detection. Since each bit can take on +/- values there are four possible sign combinations for two bits. Label these as events A,B,C,D. The possibilities for the  $(k-2)^{\text{nd}}$  and  $(k-1)^{\text{st}}$  bit periods are

	$A_{k-1}$	$A_{k-2}$	$A_{k-1} A_{k-2}$
A:	A	A	$A^2$
B:	-A	-A	$A^2$
C:	A	-A	$-A^2$
D:	-A	A	$-A^2$

Events A and B imply transmission of a "1" (no sign difference) while events C and D imply transmission of a "0". Due to the symmetry of the possible values of  $A_{k-1} A_{k-2}$  about 0 the logical choice for a threshold value is  $V_T = 0$ . So errors occur if  $V < 0$  for events A and B, or  $V > 0$  for events C and D. So the conditional probability of error (assuming  $\epsilon_1$  and  $\epsilon_2$  are known for now) is given by

$$P_e(\epsilon_1, \epsilon_2) = P(V < 0 | A) P(A) + P(V < 0 | B) P(B) \quad (3.2.3)$$

$$+ P(V>0|C) P(C) + P(V>0|D) P(D).$$

Events A,B,C,D are assumed equiprobable so

$$P(A) = P(B) = P(C) = P(D) = \frac{1}{4}. \quad (3.2.4)$$

Now it is possible to determine the conditional probabilities in the previous equation if the expression for V is rewritten. It is possible to write V as one gaussian random variable with a fixed variance and a mean value that is a function of the event and the variables  $\epsilon_1$  and  $\epsilon_2$ . So for a representative event "E"

$$V = N(m_E, \sigma^2) \quad (3.2.5)$$

where

$$\begin{aligned} \sigma^2 &= \frac{A^2 T^3 \eta}{16} \\ m_E &= \frac{T^2}{8} \left\{ A_{k-1} A_{k-2} + B \sum_{i=1}^2 \operatorname{sinc} \left| \frac{\Delta_i T}{2\pi} \right| \right. \\ &\quad \left. \left\{ A_{k-1} \cos[\epsilon_i - \Delta_i T] + A_{k-2} \cos[\epsilon_i] \right\} \right. \\ &\quad + B^2 \left\{ \sum_{i=1}^2 \operatorname{sinc}^2 \left| \frac{\Delta_i T}{2\pi} \right| \cos(\Delta_i T) + 2 \operatorname{sinc} \left| \frac{\Delta_1 T}{2\pi} \right| \operatorname{sinc} \left| \frac{\Delta_2 T}{2\pi} \right| \right. \\ &\quad \left. \left. \cos \left[ \left( \epsilon_1 - \frac{\Delta_2 T}{2} \right) - \left( \epsilon_2 - \frac{\Delta_1 T}{2} \right) \right] \cos \left| \frac{(\Delta_1 + \Delta_2) T}{2} \right| \right\} \right\}. \end{aligned}$$

Given this information, for a representative event "E" and  $V_T = 0$ ,

$$P(V>0|E) = \int_0^{\infty} N(m_E, \sigma^2) dV = Q \left\{ \frac{-m_E}{\sigma} \right\} \quad (3.2.6)$$

$$P(V < 0 | E) = \int_{-\infty}^0 N(m_E, \sigma^2) dV = Q\left\{\frac{m_E}{\sigma}\right\}.$$

So now,

$$P_e(\epsilon_1, \epsilon_2) = \frac{1}{4} \left\{ Q\left\{\frac{m_A}{\sigma}\right\} + Q\left\{\frac{m_B}{\sigma}\right\} + Q\left\{\frac{-m_C}{\sigma}\right\} + Q\left\{\frac{-m_D}{\sigma}\right\} \right\}. \quad (3.2.7)$$

To simplify the equations to follow define

$$\Delta_x(\epsilon_1, \epsilon_2) = \sum_{i=1}^2 \text{sinc} \left[ \frac{\Delta_i T}{2\pi} \right] \cos(\Delta_i T) \quad (3.2.8)$$

$$+ 2 \text{sinc} \left[ \frac{\Delta_1 T}{2\pi} \right] \text{sinc} \left[ \frac{\Delta_2 T}{2\pi} \right] \cos \left[ \epsilon_1 - \epsilon_2 + \frac{(\Delta_2 - \Delta_1)T}{2} \right] \cos \left[ \frac{(\Delta_1 + \Delta_2)T}{2} \right]$$

$$\Delta_y(\epsilon_1, \epsilon_2) = \sum_{i=1}^2 \text{sinc} \left[ \frac{\Delta_i T}{2\pi} \right] \left[ \cos(\epsilon_i - \Delta_i T) + \cos(\epsilon_i) \right]$$

$$\Delta_z(\epsilon_1, \epsilon_2) = \sum_{i=1}^2 \text{sinc} \left[ \frac{\Delta_i T}{2\pi} \right] \left[ \cos(\epsilon_i - \Delta_i T) - \cos(\epsilon_i) \right].$$

Then using the proper signs for each event,

$$m_A, m_B = \frac{T^2}{8} \left\{ A^2 \pm A B \Delta_y(\epsilon_1, \epsilon_2) + B^2 \Delta_x(\epsilon_1, \epsilon_2) \right\} \quad (3.2.9)$$

"+" for  $m_A$  and "-" for  $m_B$

$$m_C, m_D = \frac{T^2}{8} \left\{ -A^2 \pm A B \Delta_z(\epsilon_1, \epsilon_2) + B^2 \Delta_x(\epsilon_1, \epsilon_2) \right\}$$

"+" for  $m_C$  and "-" for  $m_D$ .

With this information it is possible to compute the BER. Simply average the  $P_e(\epsilon_1, \epsilon_2)$  over the joint probability density function of  $\epsilon_1$  and  $\epsilon_2$ . This density is given in section 2.2 (using  $m=2$ ) as

$$p(\epsilon_1, \epsilon_2) = p(\epsilon_1) p(\epsilon_2) = \frac{1}{2\pi} \frac{1}{2\pi} = \frac{1}{4\pi}. \quad (3.2.10)$$

So

$$\text{BER} = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_e(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2. \quad (3.2.11)$$

Again this must be modified for computer computation as in section 2.2. This is carried out in the next section.



### 3.3. Computer Calculation of BER

In a manner closely analogous to section 2.2 the equation for the BER will be modified to use discrete summations in place of the integrals. So the computation will use  $N$  phase values evenly spaced from  $(-\pi, \pi)$ . To do this define for  $i=1, 2, \dots, N$  even, and  $n_1$  and  $n_2$  integer,

$$\epsilon'_i \equiv 2\pi \frac{n_i}{N}, \quad -N/2 \leq n_i \leq (N/2-1). \quad (3.3.1)$$

Then

$$\text{BER} \approx \frac{1}{N} \sum_{n_1=-N/2}^{N/2-1} \sum_{n_2=-N/2}^{N/2-1} P_e(\epsilon'_1, \epsilon'_2). \quad (3.3.2)$$

As in section 2.3 it is also necessary to modify  $P_e(\epsilon'_1, \epsilon'_2)$  so it uses  $\text{snr}$  and  $\gamma$  as parameters. Since  $P_e(\epsilon'_1, \epsilon'_2)$  is written in terms of  $Q$  functions with argument  $\frac{m}{\sigma}$  it is best to work on modifying that quantity. Using the expressions for  $\sigma$  and  $m_A, m_B, m_C, m_D$  from section 3.2 and using  $\epsilon'_i$  for  $\epsilon_i$  note,

$$\begin{aligned} \frac{m_A}{\sigma}, \frac{m_B}{\sigma} &= \frac{4}{AT \sqrt{|\eta T|}} \frac{T^2}{8} \left\{ A^2 \pm AB \Delta_y(\epsilon'_1, \epsilon'_2) + B^2 \Delta_x(\epsilon'_1, \epsilon'_2) \right\} \quad (3.3.3) \\ &= \frac{AT}{2 \sqrt{|\eta T|}} \pm \frac{BT}{2 \sqrt{|\eta T|}} \Delta_y(\epsilon'_1, \epsilon'_2) + \frac{B^2 T}{2A \sqrt{|\eta T|}} \Delta_x(\epsilon'_1, \epsilon'_2). \end{aligned}$$

Multiplying the second and third terms by  $(A/A)$  and rearranging,

$$\frac{m_A}{\sigma}, \frac{m_B}{\sigma} = \sqrt{\frac{|\text{snr}|}{2}} \left\{ 1 \pm \frac{\Delta_y(\epsilon'_1, \epsilon'_2)}{\gamma} + \frac{\Delta_x(\epsilon'_1, \epsilon'_2)}{\gamma^2} \right\}$$

"+" for  $\frac{m_A}{\sigma}$  and "-" for  $\frac{m_B}{\sigma}$ . Similarly,

$$\frac{-m_C}{\sigma}, \frac{-m_D}{\sigma} = \frac{4}{AT\sqrt{\eta T}} \frac{T^2}{8} \left\{ A^2 \pm AB \Delta_z(\varepsilon_1', \varepsilon_2') \right. \\ \left. - B^2 \Delta_x(\varepsilon_1', \varepsilon_2') \right\} \quad (3.3.4)$$

$$= \sqrt{\frac{|\text{snr}|}{2}} \left\{ 1 \pm \frac{\Delta_z(\varepsilon_1', \varepsilon_2')}{\gamma} - \frac{\Delta_x(\varepsilon_1', \varepsilon_2')}{\gamma^2} \right\}$$

"-" for  $\frac{-m_C}{\sigma}$  and "+" for  $\frac{-m_D}{\sigma}$ . So using

$$P_e(\varepsilon_1', \varepsilon_2') = \frac{1}{4} \left\{ Q\left\{\frac{m}{\sigma}\right\} + Q\left\{\frac{m}{\sigma}\right\} + Q\left\{\frac{-m}{\sigma}\right\} + Q\left\{\frac{-m}{\sigma}\right\} \right\} \quad (3.3.5)$$

$$\text{BER} \sim \frac{1}{4N^2} \sum_{n_1=-N/2}^{N/2-1} \sum_{n_2=-N/2}^{N/2-1} \quad (3.3.6)$$

$$\left\{ Q \left[ \sqrt{\frac{|\text{snr}|}{2}} \left( 1 + \frac{\Delta_y(\varepsilon_1', \varepsilon_2')}{\gamma} + \frac{\Delta_x(\varepsilon_1', \varepsilon_2')}{\gamma^2} \right) \right] \right. \\ + Q \left[ \sqrt{\frac{|\text{snr}|}{2}} \left( 1 - \frac{\Delta_y(\varepsilon_1', \varepsilon_2')}{\gamma} + \frac{\Delta_x(\varepsilon_1', \varepsilon_2')}{\gamma^2} \right) \right] \\ + Q \left[ \sqrt{\frac{|\text{snr}|}{2}} \left( 1 - \frac{\Delta_z(\varepsilon_1', \varepsilon_2')}{\gamma} - \frac{\Delta_x(\varepsilon_1', \varepsilon_2')}{\gamma^2} \right) \right] \\ \left. + Q \left[ \sqrt{\frac{|\text{snr}|}{2}} \left( 1 + \frac{\Delta_z(\varepsilon_1', \varepsilon_2')}{\gamma} - \frac{\Delta_x(\varepsilon_1', \varepsilon_2')}{\gamma^2} \right) \right] \right\}.$$

This expression has been coded onto the computer and this software produces the results of the next section. Also, as noted in

section 2.3 the derivation presented here is of restricted usefulness. Extending this derivation to include more than two harmonic interferers is possible but would prove to be prohibitively consuming of computer time. So a simpler alternative derivation for the multiple interferer case is presented in section 5.

### 3.4. Results for DPSK Modulation with Two Harmonic Interferers

Since the objective of this study is to compare BPSK and DPSK it is necessary to use the same system parameters here as were used in section 2.4. In fact, almost all of the comments in section 2.4 apply equally well to this section. These will simply be noted here. For further detail see section 2.4.

System parameters:

$$R = 1/T = 76.22 \text{ bits/sec.}$$

$$\omega_c = (12.5 \text{ kHz}) 2\pi \text{ rad/sec}$$

$$\omega_1 = (208)(60) 2\pi \text{ rad/sec}$$

$$\omega_2 = (209)(60) 2\pi \text{ rad/sec .}$$

Assumptions:

$$\gamma_1 = \gamma_2 = \gamma$$

range of interest is  $1 < \gamma < 5$  .

The discussion on the relation between  $\gamma$  and snr can also be used with these BER vs snr curves. Further assumptions were required to perform the DPSK BER derivation due to its complexity. These are mentioned in section 8.3 and the only special one to note is that on the basis of field data it is assumed [19]

gaussian noise power  $\ll$  harmonic interferer power .

This was used to discard a noise product term.

That assumption and some simplistic handling of other derivation difficulties have obviously impacted the quality of the resulting BER

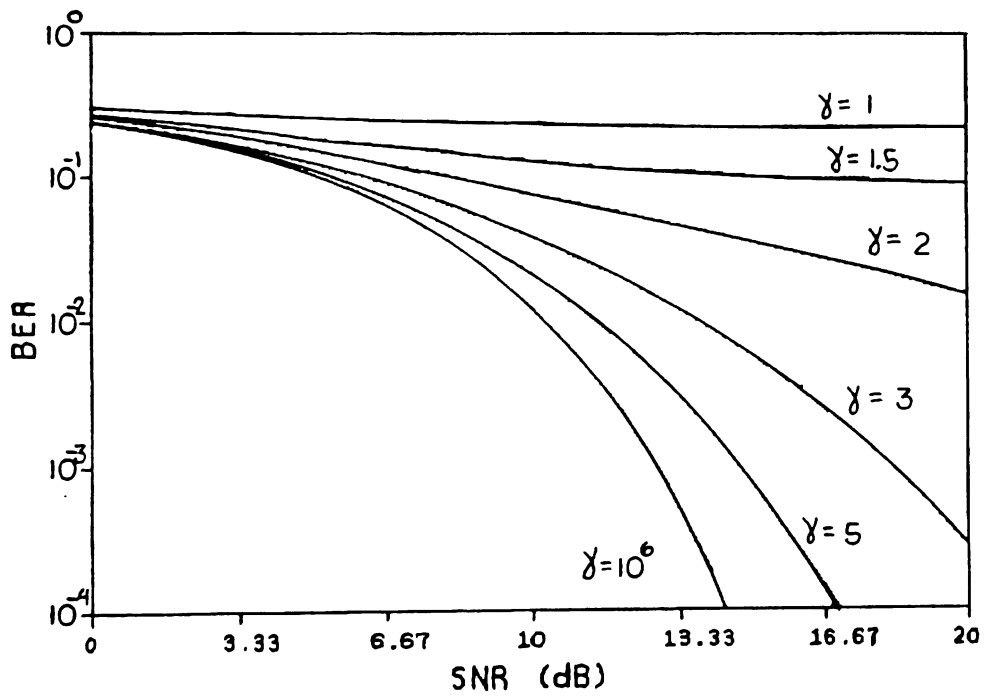


Figure 3.2 BER vs SNR (dB) for DPSK with  $R=76.22$  bps and  $f_c=12.5$  kHz

vs  $\text{snr}_{\text{dB}}$  curves shown in figure 3.2. Especially note that the  $\gamma = 10^6$  (no interference) curve will not match standard gaussian noise only results as the BPSK  $\gamma = 10^6$  curve did in section 2.4. A comparison to this standard will show that this curve is off by several dB. The argument for that is the suboptimum nature of this system and the assumptions made in the derivation. Those assumptions will only be valid as the harmonic noise begins to dominate the BER. Therefore, these curves should be adequate for the desired purpose of comparing BPSK and DPSK under the assumed power line communication conditions. Further conclusions will be deferred until section 6.

#### 4. BER Expression for BPSK with Many Harmonic Interferers

This section presents an alternate method for calculation of the BER for BPSK modulation when many harmonics are present in the signal's bandwidth. This alternative is prompted by the limitations inherent in the BER formula in sections 2.2 and 2.3. The BER derivation there is restricted to the case of two 60 Hz harmonic interferers. This is adequate for the example system which is used, however, computational limits prevent the extension of this derivation to include more than two interferers.

It is obvious that a need exists to be able to calculate BERs for the case of many interferers. Recall (see section 2.1) that the main lobe of the matched filter spectrum is taken as the filter's bandwidth. This lobe has a width of  $2R$  where  $R$  is the system baud. So as higher bit rate PLC systems are used more 60 Hz harmonics fall within the bandwidth of the signal. For example, a system with  $R=500$  bits/sec will have 16 or 17 harmonics within the main lobe of the filter. With this goal in mind an approximate and simple form for calculation of BER is derived.

##### 4.1. Derivation of BER

See section 8.4 for the derivation of the decision variable  $U$  for this case. This derivation yields,

$$U = S + N(0, \sigma_v^2)$$

where

$$\sigma_v^2 = \frac{\eta T}{4} + \frac{1}{2} \left(\frac{T}{2}\right)^2 \sum_{j=1}^m B_j^2 \operatorname{sinc}^2\left(\frac{\Delta_j T}{2\pi}\right).$$

Following the procedure of section 2.2 in deriving the probability of a bit error yields

$$P_e = \frac{1}{2} \left\{ P(U > V_T | 0) + P(U < V_T | 1) \right\}. \quad (4.1.1)$$

Note that for this case there is even symmetry about  $U = 0$  for the conditional probabilities  $p(U|0)$  and  $p(U|1)$ . Therefore, select  $V_T = 0$  and use the symmetry to get

$$P_e = \frac{1}{2} \left\{ 2 P(U > 0 | 0) \right\} = \frac{1}{2} \left\{ 2 P(U < 0 | 1) \right\} \quad (4.1.2)$$

so

$$P_e = P(U > 0 | 0) = P(U < 0 | 1). \quad (4.1.3)$$

This probability of error is the same as the BER since there are no further random variables to average  $P_e$  over. Using Q functions  $P_e$  can be written

$$P_e = \text{BER} = Q \left\{ \frac{\sqrt{\frac{A^2 T^2}{2}}}{\sqrt{4\sigma_v^2}} \right\}. \quad (4.1.4)$$

It is useful to write this in terms of (see equations 2.3.7 and 2.3.8)

$$\text{snr} = \frac{A^2 T}{2\eta} \text{ and } \gamma = \frac{A}{B}$$

as in the BER derivation in section 2.3. To do this it is necessary to assume that all the  $B_j$  are equal, ie.  $B_j = B$ . It is also useful to define



$$a^2 \equiv \sum_{j=1}^m \text{sinc}^2\left(\frac{\Delta_j T}{2\pi}\right). \quad (4.1.5)$$

So now

$$\begin{aligned} \frac{A^2 T^2}{4\sigma_v^2} &= \frac{A^2 T^2}{4\left(\frac{\eta T}{4} + \frac{B^2 a^2 T^2}{8}\right)} \\ &= \frac{1}{\left\{\frac{\eta}{2} + \frac{B^2 a^2}{2}\right\} \left\{\frac{1}{A T} + \frac{1}{2A^2}\right\}} \\ &= \frac{1}{\left\{\frac{1}{2} \frac{1}{\text{snr}} + \frac{a^2}{2} \frac{1}{\gamma}\right\}} \end{aligned} \quad (4.1.6)$$

Therefore the final form for this BER approximation is found from the previous three equations as

$$\begin{aligned} \text{BER} &= Q \left\{ \sqrt{\frac{2}{\frac{1}{\text{snr}} + \frac{a^2}{\gamma^2}}} \right\} \\ &= f(\text{snr}, \gamma, a^2). \end{aligned} \quad (4.1.7)$$

#### 4.2. Computer Results for BPSK

Figure 4.1 shows the resulting curves for BER vs snr with  $\gamma$  as a parameter for a typical system using

$$f_c = 6000 \text{ Hz} \quad \text{and} \quad R = 500 \text{ bits/sec.}$$

Note that the discussion of the relation between  $\gamma$  and snr in section 2.4 applies here as well. Therefore, the  $\gamma$  vs  $\text{snr}_{\text{dB}}$  curve of figure 2.8 can be used as in section 2.4.

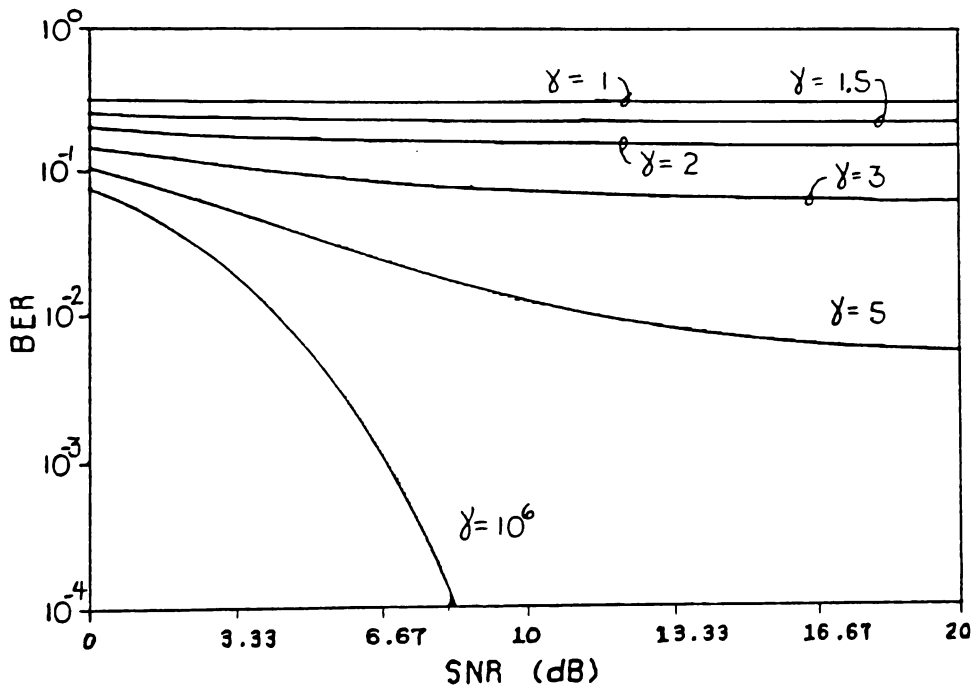


Figure 4.1 BER vs SNR (dB) for BPSK with  $R=500$  bps and  $f_c=6$  kHz

This PLC system receiver has been computer simulated and run using typical recorded power line noise [16]. BERs have been calculated using simulated data and these results are used for comparison with the BER calculation method presented here. In particular, four received signal levels are used in the test. Using information obtained about the average white noise and harmonic noise powers used in the simulation, the  $\gamma$  and snr values are calculated. The BER vs  $\text{snr}_{\text{dB}}$  curves are then used to determine the corresponding analytical BER values. The comparison between experimental and analytical results is shown in figure 4.2.

Unfortunately this figure shows that the simulation does not seem to completely support the analytical BER expression derived here. This could be for any number of reasons. A likely possibility is that the background noise in the simulation may not be gaussian-distributed. Since the BER depends on the area of the "tails" of the noise probability distribution any change in the distribution shape could have a large effect on the BER. Sensitivity to distribution shape will be more pronounced as the snr is increased since then less of the "tail" is included. A check of the results supports this possible explanation. Figure 4.2 shows how the two results diverge as the snr increases.

These limited results seem to indicate the BER approximation presented here may be adequate for low snr and  $\gamma$  values. It should also be somewhat accurate if the actual noise in the PLC system follows a gaussian distribution more closely than the example used here.

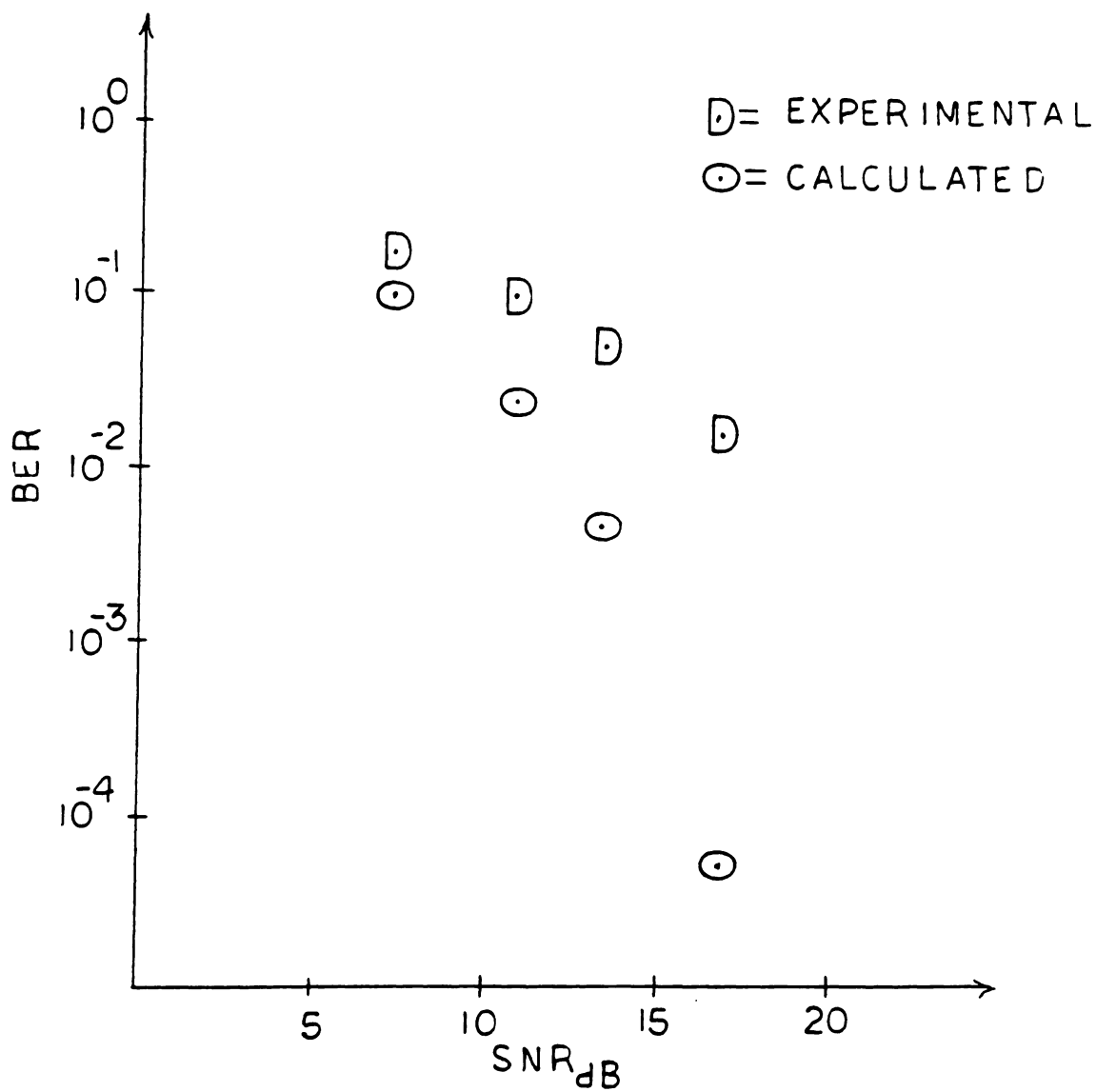


Figure 4.2 Comparison of Experimental and Analytical Results for BPSK with  $R=500$  bps and  $f_c=6$  kHz

Furthermore, field results indicate the low snr case is the most interesting one since for high snr values there seems to be no problem in successfully transmitting the required amount of data. So these results may indeed be useful.

## 5. BER Expression for DPSK for Many Harmonic Interferers

As was true for the BPSK analysis, there exists a need for a DPSK BER expression that encompasses the multiple interferer case. Fortunately, the results for this case require only a small modification of some standard results. This is, of course, dependent on rather broad assumptions. In particular, the validity of using the Central Limit Theorem to describe the probability distribution of the harmonic interference.

### 5.1. Derivation of BER

Using the results of section 2.1 one finds that the signal plus noise and interference at the matched filter output has the form

$$y(t) = s_M(t) + n_M(t) + i_M(t) \quad (5.1.1)$$

which are the filtered forms of the variables  $s(t)$ ,  $n(t)$ , and  $i(t)$ . As discussed in section 8.4, if the Central Limit Theorem is allowed (with respect to the interference) then the noise and interference can be combined into one gaussian process. Define this as

$$\psi(t) \equiv n_M(t) + i_M(t). \quad (5.1.2)$$

So at any time  $t = t_1$  the variable  $\psi(t_1)$  is gaussian distributed with a zero mean and variance (see equation 8.4.23)

$$\sigma_\psi^2 = \frac{\eta T}{4} + \frac{1}{2} \left(\frac{T}{2}\right)^2 \sum_{j=1}^m B_j^2 \operatorname{sinc}^2\left(\frac{\Delta_j T}{2\pi}\right).$$

So now,

$$y(t) = s_M(t) + \psi(t). \quad (5.1.3)$$

As shown in section 8.2.1  $s_M(t)$  has the approximate form,

$$s_M(t) \sim \frac{A}{2} (t - (k-1)T) \cos(\omega_c t + \theta) \quad (5.1.4)$$

valid for  $(k-1)T \leq t < kT$ . This is for an individual pulse received at  $t = (k-1)T$ . Of course, the correlation processor is sampled at  $t = kT$  and then dumped to start the processing cycle for the next data bit. At the sample times the filtered signal has the value, (using  $\omega_c T = 2\pi n$ ,  $n$  an integer, as assumed in section 8.2)

$$s_M(kT) = \pm A \left(\frac{T}{2}\right). \quad (5.1.5)$$

Now the expression for  $y(t)$  has the form of the signal used in the standard DPSK analysis of Lucky, Salz, and Weldon ie. a signal plus gaussian distributed noise which has passed through a front-end narrowband filter [10]. There is only one real difference, namely, the value of the data signal component at the sampling times is now  $\left(\frac{T}{2}\right) A$  instead of just  $A$ . Note that there are several different variations of this derivation [5],[8],[14],[15]. The most important difference is the type of front-end receiver filter which is used in each derivation. In particular, Lucky, Salz, and Weldon do not specify the type of filter and hence develop the most general result. The others present a result based on the assumption of a matched filter and white gaussian distributed noise.

The result of Lucky, Salz, and Weldon for the BER is

$$\text{BER} = \frac{1}{2} \exp\left[-\frac{A^2}{2\sigma^2}\right] \quad (5.1.6)$$

where  $A$  is the received signal amplitude at the sampling times and  $\sigma^2$

is the variance of the noise. To make use of this for this study simply substitute  $(\frac{AT}{2})$  for  $A$  and  $\sigma_{\psi}^2$  for  $\sigma^2$ . This gives

$$\text{BER} = \frac{1}{2} \exp \left\{ \frac{-A^2 T^2}{8 \sigma_{\psi}^2} \right\}. \quad (5.1.7)$$

It is useful to put this in terms of snr and  $\gamma$  as in section 4.1. As was done there assume  $B_j \sim B$  and define

$$a^2 = \sum_{j=1}^m \text{sinc}^2 \left( \frac{\Delta_j T}{2\pi} \right).$$

So

$$\begin{aligned} \frac{A^2 T^2}{8 \sigma_{\psi}^2} &= \frac{A^2 T^2}{8 \left\{ \frac{\eta T}{4} + \frac{T^2 B^2 a^2}{8} \right\}} \\ &= \frac{1}{\left\{ \frac{2\eta T}{A^2 T^2} + \frac{T^2 B^2 a^2}{A^2 T^2} \right\}} \\ &= \frac{1}{\left\{ \frac{1}{\text{snr}} + \frac{a^2}{\gamma^2} \right\}}. \end{aligned} \quad (5.1.8)$$

So the final form is

$$\text{BER} = \frac{1}{2} \exp \left\{ \frac{-1}{\left\{ \frac{1}{\text{snr}} + \frac{a^2}{\gamma^2} \right\}} \right\}. \quad (5.1.9)$$

## 5.2. Computer Results for DPSK

Figure 5.1 shows the resulting curves for BER vs snr with  $\gamma$  as a parameter for the system used in section 4.2 for BPSK, ie.



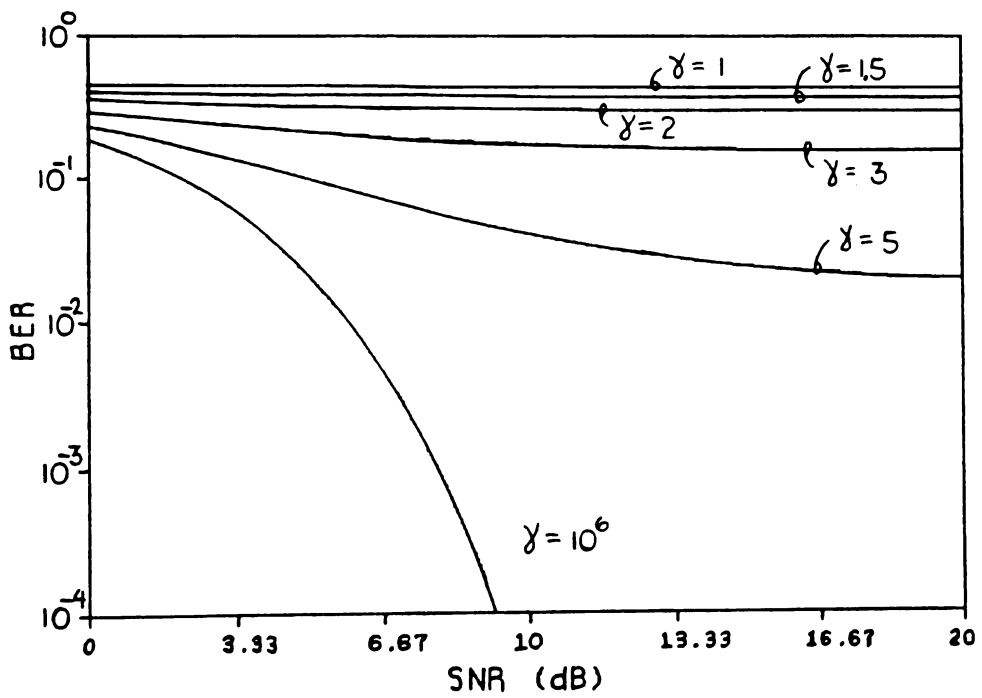


Figure 5.1 BER vs SNR (dB) for DPSK with  $R=500$  bps and  $f_c=6$  kHz

$$f_c = 6000 \text{ Hz and } R = 500 \text{ bits/sec .}$$

Note that the discussion of the relation between  $\gamma$  and snr in section 2.4 applies here as well. Therefore, the  $\gamma$  vs  $\text{snr}_{\text{dB}}$  curve of figure 2.8 can be used as discussed in that section.

## 6. Comparison of BPSK and DPSK Modulation for Two PLC Systems

### 6.1. Comparison for Two Interferer Case

Refer to sections 2.4 and 3.4 and figures 2.8, 2.9, and 3.2.

For convenience in comparison figure 6.1 shows both the BPSK and DPSK results for the representative  $\gamma$  values, 1,2,3,5. These results show that the harmonic interference has little effect on BER until its magnitude approaches the same order as the signal magnitude, ie.  $\gamma < 10$ . Then as  $\gamma$  increases the result is a devastating increase in the BER. (For the DPSK case this degree of degradation is qualitatively confirmed by Jones for just a single interferer which is at the signal frequency [9].) Note that an acceptable BER for this low bit rate, noncritical application could be about  $10^{-3}$ . This appears unachievable for  $\gamma < 2$  for BPSK and  $\gamma < 3$  for DPSK due to the apparent asymptotic behavior of the curves for low  $\gamma$  values. Thus it would appear that both systems have unacceptable operation in this environment especially since increasing the snr seems to have little effect when  $\gamma$  is very small. However, this conclusion is misleading due to the dependency of  $\gamma$  and snr mentioned in section 2.4. Both are functions of the signal amplitude, A, so the previous comments are based on the assumption that A remains constant while  $\eta$  and B vary. If that were not so then an increase in snr implies an increase in  $\gamma$  which will have a large effect on the BER.

Obviously, it will be helpful to have an alternative way of presenting the results which will also include the interdependence of

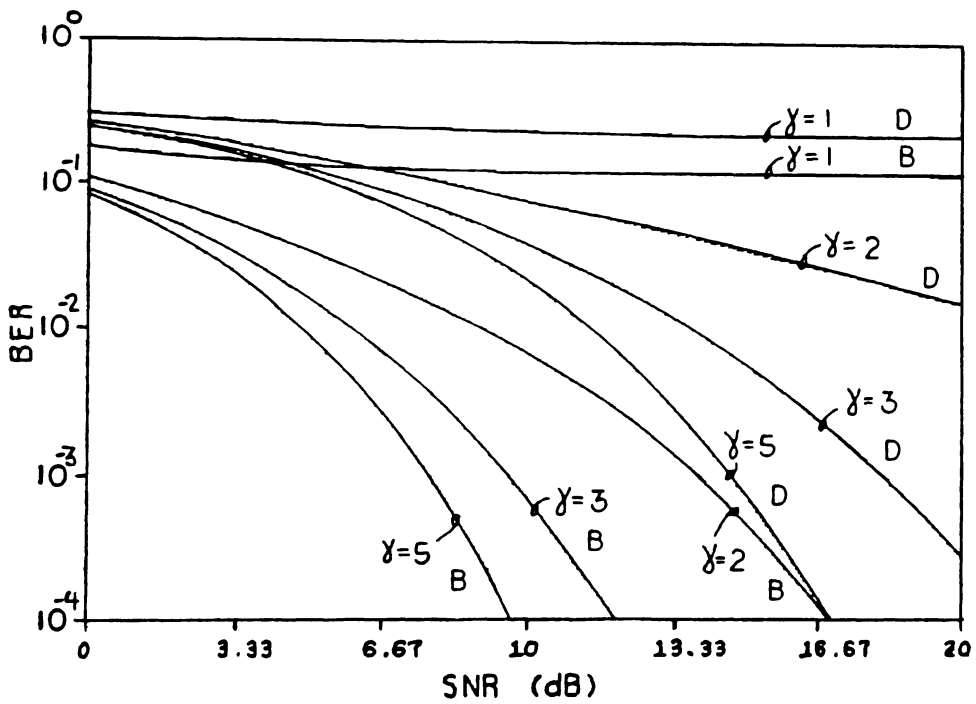


Figure 6.1 BER vs SNR (dB) Comparison of BPSK (B) and DPSK (D) with  $R=76.22$  bps and  $f_c=12.5$  kHz

$\gamma$  and snr. This can be accomplished by considering the parameter  $c$  of equation 2.4.4, ie.

$$c = \left[ \frac{2\eta}{B^2T} \right]^{1/2} .$$

For any given value of  $c$  there is a one-to-one correspondence between  $\gamma$  and snr (in dB) as given by equation 2.4.5, ie.

$$\gamma = c \cdot 10^{\left( \frac{\text{snr}}{20} \text{ dB} \right)} .$$

This relationship is illustrated in figure 2.9. It is possible to use this relationship and the data of figures 2.8 and 3.2 to create new plots of BER vs snr(dB) with  $c$  as a parameter. These are shown in figures 6.2 and 6.3. In this form the data is much easier to use because the parameter  $c$  is dependent only on the channel ( $\eta$  and  $B$  values) and the bit rate selected ( $T$  value), and not on the signal level. Furthermore, these curves are intuitively satisfying because an increase in snr is generally expected to improve the BER as these figures show.

Figures 6.2 and 6.3 provide two important points of comparison between BPSK and DPSK performance for this system. For a given channel (ie. value of  $c$ ) the BPSK system performance is 4-5 dB better than that of the DPSK system. That means to achieve the same BER the DPSK system must operate at a 4-5 dB larger snr. The other point to note is the slope of the curves. The steeper slope of the BPSK curves indicates that for a given increase in snr its BER will decrease more than that of the DPSK system.

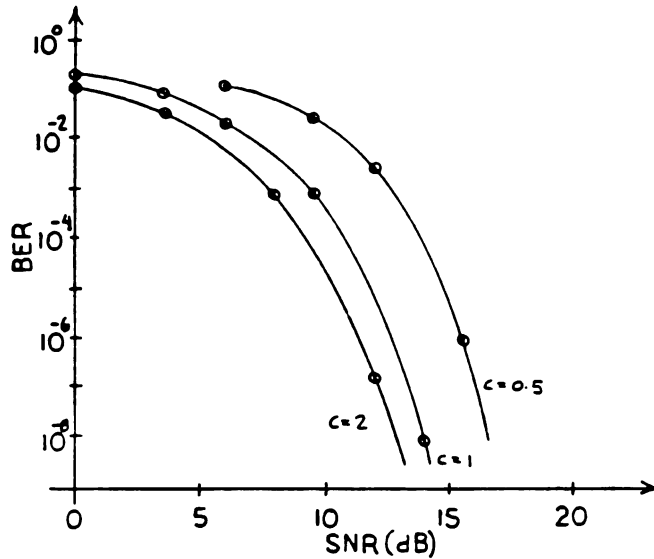


Figure 6.2 BER vs SNR (dB) for BPSK with parameter  $c=[2\eta/B^2T]^{1/2}$  and  $R=76.22$  bps and  $f_c=12.5$  kHz

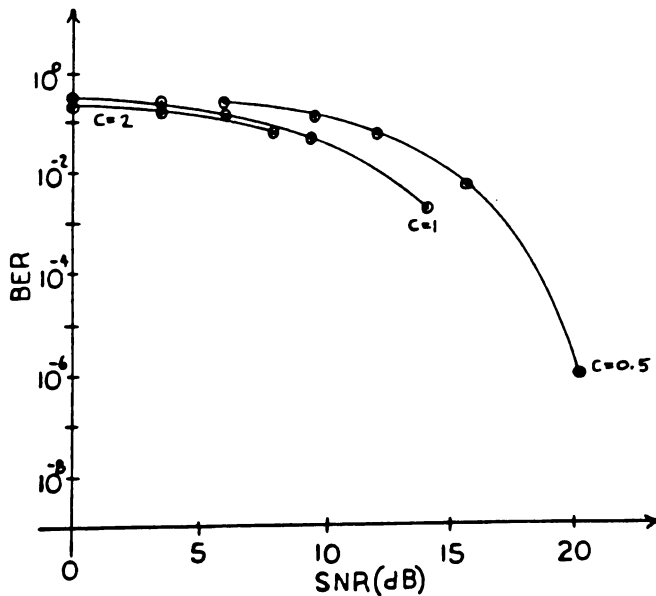


Figure 6.3 BER vs SNR (dB) for DPSK with parameter  $c=[2\eta/B^2T]^{1/2}$  and  $R=76.22$  bps and  $f_c=12.5$  kHz

There are two final notes. First, these curves obviously can be used to replace the procedure of section 2.4 for determining the effect of an increase in snr on the BER. Second, although the value of  $c$  varies with time as the channel changes, it may be possible to compute some average value for  $c$ . Then these curves can be used to determine the general level of snr necessary to achieve a desired BER.

## 6.2. Comparison for Nineteen Interferer Case

Refer to sections 4.2 and 5.2 and figures 2.9, 4.1 and 5.1.

For convenience in comparison figure 6.4 shows both the BPSK and DPSK results for the representative  $\gamma$  values, 1,2,5,8,10. The general comments made in section 6.1 are applicable here as well. Figures 6.5 and 6.6 are derived from figure 2.9 and figures 4.1 and 5.1 respectively. These correspond to figures 6.2 and 6.3 except they are for the 19 interferer case. As in section 6.1 a discussion of the significance of figure 6.4 as well as figures 6.5 and 6.6 is included.

Inspection of figure 6.4 shows the performance for the range of interest ( $1 \leq \gamma \leq 5$ ) to be considerably worse than for the case in section 6.1. This is expected since there are many more interferers in the signal's bandwidth and also the 100<sup>th</sup> harmonic lies right at the carrier frequency. However, it is not really proper to make a direct performance comparison since these sets of curves represent two different systems. From figure 6.4 it is apparent that neither the BPSK or DPSK system can achieve an acceptable BER ( $\sim 10^{-3}$ ) for the range,  $1 \leq \gamma \leq 5$ . So it will be necessary for greater signal power to be used to increase  $\gamma$  to an acceptable level. That is the reason for the inclusion of the  $\gamma=8,10$  curves. It is again helpful to display the data in a more useable form. This is shown in figures 6.5 and 6.6.



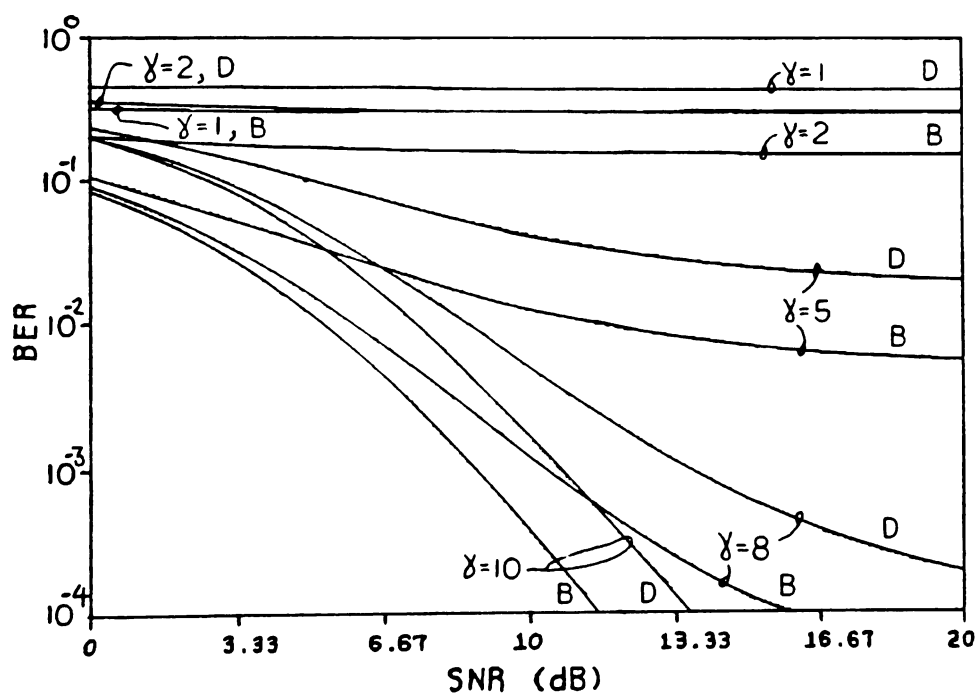


Figure 6.4 BER vs SNR (dB) Comparison of BPSK (B) and DPSK (D) with  $R=500$  bps and  $f_c=6$  kHz

Inspection of these two figures shows the performance of the BPSK and DPSK systems to be very similar. The BPSK system has a roughly 1-2.5 dB performance advantage for these three values of  $c$ . Furthermore, the two sets of curves have similar slopes for corresponding values of  $c$ . That means that neither system enjoys a real advantage as far as the rate of BER decrease with increasing snr is concerned. This is in contrast to the comparison made in section 6.1 where the BPSK system had a distinct performance advantage.

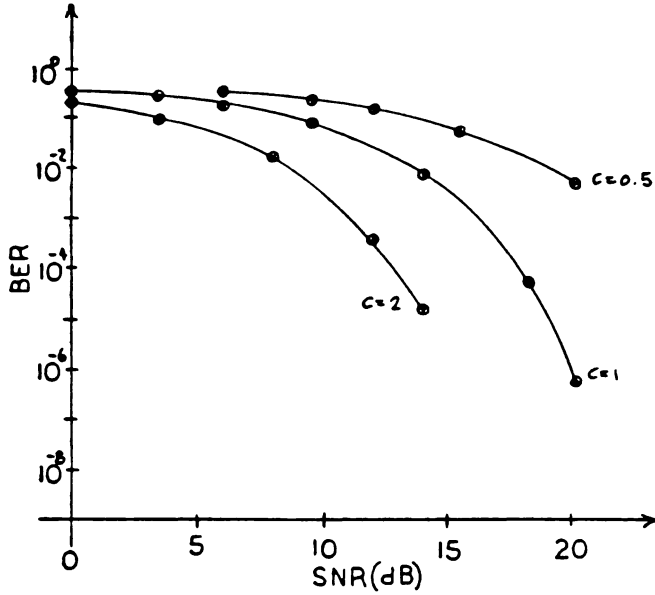


Figure 6.5 BER vs SNR (dB) for BPSK with parameter  $c=[2\eta/B^2T]^{1/2}$  and  $R=500$  bps and  $f_c=6$  kHz

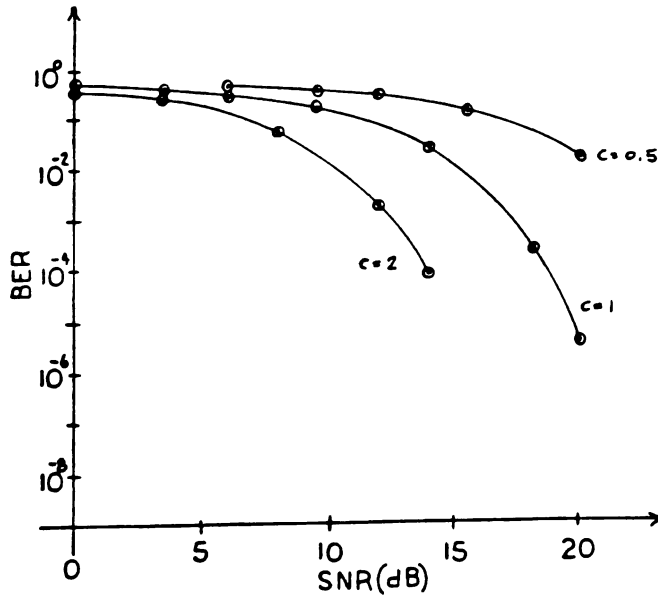


Figure 6.6 BER vs SNR (dB) for DPSK with parameter  $c=[2\eta/B^2T]^{1/2}$  and  $R=500$  bps and  $f_c=6$  kHz

### 6.3. Conclusions

The objective of this study was to compare BPSK and DPSK modulation performance in a power line communications medium. To that end two specific systems were selected for the comparison. The low bit rate (76.22 bps) system represents a typical system of today, while the higher baud (500 bps) system represents a system for the future. The bit error rate (BER) derivations for these systems proved to be quite different in approach and were heavily dependent on very specific assumptions about the harmonic interference present in the communication channel. Computer approximations were necessary for the computation of the BER for the 2 interferer case. Based on these computations, curves of BER vs snr (in dB) were presented with the parameter  $\gamma$  representing the effect of the harmonic interference. These curves were compared in sections 6.1 and 6.2. It was also necessary to present this data in another form to properly include the interdependence of  $\gamma$  and snr. This was accomplished by using the parameter  $c$ . This parameter effectively describes the channel by acting as a measure of the relative strengths of harmonic interference and white background noise. The resulting curves provided a basis for comparing the performance of the BPSK and DPSK systems.

As described in section 6.1, the curves of figures 6.2 and 6.3 indicate that the BPSK system is superior for the 76.22 baud system. Not only is there a 4-5 dB direct performance advantage for BPSK, but the slope of the curves indicates that for a given increase in signal power ( $\eta$  assumed constant), the BPSK BER will decrease more than the

DPSK BER. Based on this information the BPSK system's performance advantage appears to outweigh the greater equipment complexity required for coherent detection.

For the 500 baud system the curves of figures 6.5 and 6.6 are used to compare the two systems. These curves indicate that there is not much difference in performance between the BPSK and DPSK systems. The BPSK system has about a 1-2.5 dB advantage over DPSK and there is almost no difference in the slopes of the two sets of curves. It should be noted that the rather broad assumptions made about the harmonic interference could be the reason for the closer performances of BPSK and DPSK in this case than in the previous one. Nevertheless, based just on these results a good argument can be made for using the less sophisticated DPSK system and still achieving BER performance comparable to that of the BPSK system. This conclusion contrasts that of section 6.1, but simply shows the subjective nature of the decision as well as the possible effect of using two different BER derivation approaches.

There are many possibilities for further study in this problem. For instance, no consideration has been made here for phase jitter, frequency drift, channel fading [12], double errors for DPSK [18], or Nyquist pulse shaping [14]. It would also be helpful to see the results of a Monte Carlo simulation of these systems for comparison. This work has concentrated on the BER derivations which proved to be quite difficult due to a lack of reference material in this area. These equations could be used as a good starting point for evaluating

some of the system parameter shifts mentioned above.

There are also many other more sophisticated solution approaches that could be attempted, however, these rely on relatively sophisticated mathematics. In particular, the work on general quadratic detectors looks promising [3],[11] for an analytical study of this problem. There are also some statistical simulation studies that could be applied to this type of low bit rate system [2],[13]. These rely on the use of extreme-value theory and could be used in place of a Monte Carlo type simulation.

## 7. LIST OF REFERENCES

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## 8. Appendices

### 8.1. 60 Hz Harmonic Noise

This section presents some details and assumptions about the nature of the 60 Hz harmonic noise in a PLC system [20].

There are several common sources of 60 Hz harmonic noise in the PLC system. This study will assume the noise caused by power conversion and electronic control equipment (AC/DC converters, rectifiers, inverters, voltage controllers) to be the primary source of interference. This assumption is made because the main application of PLC will be to communicate with residential customers, and in residential areas the predominant noise sources are those listed above.

With this in mind, consider the type of noise generated by these devices. They all operate by turning on and off abruptly and are triggered by the 60 Hz power signal. For example, figure 8.1 shows a typical current waveform of a light dimmer (voltage controller). This operation results in a periodic train of positive and negative voltage spikes with a period equal to that of the 60 Hz power signal. The user adjusts the trigger level and hence the delay  $\tau$  shown in the figure. The other devices operate in a similar manner. They trigger at some particular level of the sine wave and produce a periodic impulse train. These waveforms have a period either equal to, or some integer multiple of, the period of the 60 Hz power signal.

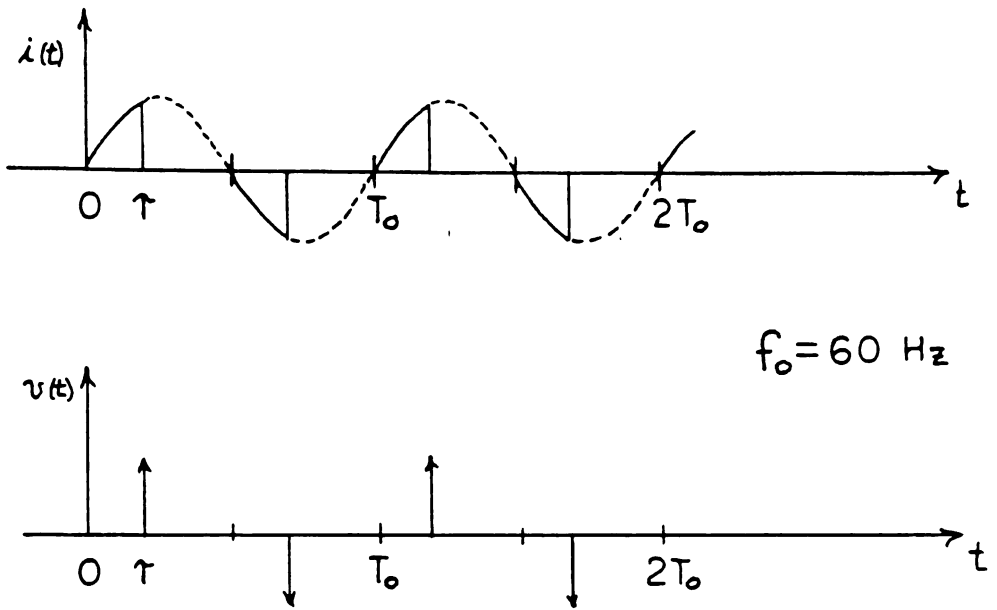


Figure 8.1 Typical Light Dimmer Current and Voltage Waveforms

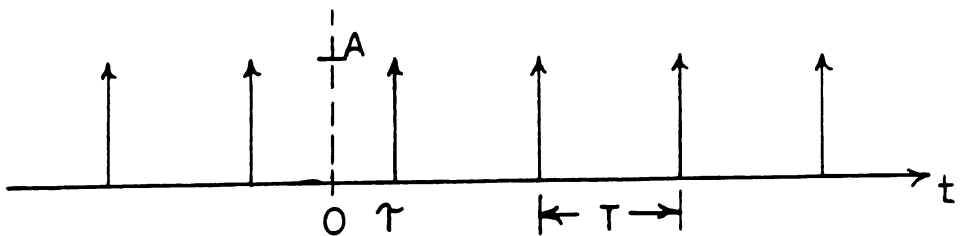


Figure 8.2 Impulse Train Model for Source of 60 Hz Harmonics

These impulses create the 60 Hz harmonics. To see this one need only look at the Fourier series of an impulse train [7]. To simplify the analysis, consider the impulse train generated by each individual device in the system to be of the form,

$$g(t) = A \sum_{i=-\infty}^{+\infty} \delta(t-iT-\tau) \quad (8.1.1)$$

where  $|\tau| < (T/2)$  ie. the periodic impulse train is infinite in extent and has a time shift of  $\tau$  with respect to the zero time reference. This is shown in figure 8.2. Note that the presence of both positive and negative spikes can be accounted for by considering them to be due to separate devices. This is not a bad model since in reality the devices trigger at slightly different positive and negative values of the 60 Hz power signal. That means there will be a slightly different delay and amplitude for the positive and negative impulse trains. It can be shown that the Fourier series of this function can be written

$$g(t) = \frac{A}{T} + \frac{2A}{T} \sum_{n=1}^{\infty} \cos(n\omega_0 t + \varphi_n) \quad (8.1.2)$$

where

$$\varphi_n \equiv -n\omega_0 \tau. \quad (8.1.3)$$

( $\omega_0 = \frac{2\pi}{T} = 2\pi \cdot 60$  and  $n$  is an integer.) Now consider  $m$  independent sources of impulsive noise of the form of  $g(t)$  corresponding to contributions from the individual residential sites. Call the resultant sum the signal of interest,  $i(t)$ .

$$i(t) = \sum_{j=1}^m g_j(t) = \sum_{j=1}^m \left( \frac{A_j}{T} + \frac{2A_j}{T} \sum_{n=1}^{\infty} \cos(n\omega_0 t + \varphi_{nj}) \right). \quad (8.1.4)$$

Each source has a different trigger delay  $\tau_j$  and amplitude  $A_j$ . Each delay causes a corresponding phase shift  $\varphi_{nj}$ . There is no way to predict this phase shift or amplitude and so they can be considered random variables. The actual distribution of these random variables are not known. However, it seems reasonable to assume a uniform distribution (equally likely) for the phase shift over the range  $-\pi \leq \varphi_{nj} < \pi$ . This corresponds to delays,  $\tau$ , from  $-T/2$  to  $T/2$ . It is not possible to make any assumption at this time about the amplitude distribution but fortunately it is not absolutely necessary for this research.

As mentioned in section 2.2 (refer to equation 2.2.8) it is useful to know how the phases of the different harmonics are related; in particular, the phases of adjacent harmonics.

Consider the  $n^{\text{th}}$  and  $(n+1)^{\text{st}}$  ( $n \neq 0$ ) harmonics of  $i(t)$ :

$$h_1(t) = \frac{2}{T} \sum_{j=1}^m A_j \cos(n\omega_0 t + \varphi_{nj}) = B_1 \cos(n\omega_0 t + \varphi_1) \quad (8.1.5)$$

$$\begin{aligned} h_2(t) &= \frac{2}{T} \sum_{j=1}^m A_j \cos((n+1)\omega_0 t + \varphi_{(n+1)j}) & (8.1.6) \\ &= B_2 \cos((n+1)\omega_0 t + \varphi_2). \end{aligned}$$

Both the phase ( $\varphi_1, \varphi_2$ ) and the amplitude ( $B_1, B_2$ ) of the harmonics are random variables. That is because they are the result of summing variables with random phase and amplitude. Since the  $\varphi_{nj}$  and  $\varphi_{(n+1)j}$  are uniformly distributed over  $(-\pi, \pi)$  it is reasonable to assume  $\varphi_1$

and  $\varphi_2$  are also distributed the same way. As for  $B_1$  and  $B_2$  no assumption can be made at this time about their probability distributions. For that reason they will be treated as given parameters when they are used in the derivations of section 2.2.

Note also that for each value of  $j$  (each source) the difference between  $\varphi_{nj}$  and  $\varphi_{(n+1)j}$  is

$$\varphi_{nj} - \varphi_{(n+1)j} = -n\omega_0 T_j - (-(n+1)\omega_0 T_j) = \omega_0 T_j. \quad (8.1.7)$$

This means that as the contributions of all of the independent sources are summed to get  $h_1(t)$  and  $h_2(t)$ , there is a random phase difference between corresponding terms in the summations. This suggests that the resultant phases,  $\varphi_1$  and  $\varphi_2$ , are independent random variables. This observation is useful in deriving the BER for BPSK modulation in section 2.2.

## 8.2. Derivation of Decision Variable U for BPSK System with Two Interferers

Reference the BPSK system block diagram in figure 2.2. It is possible to carry out this derivation in general terms by considering  $m$  interferers. Then the special case of  $m=2$  can be used in section 2.4.

The decision signal of interest is given by equation 2.1.15,

$$U(t) = s_M(t) + \sum_{i=1}^m h_{iM}(t) + n_M(t).$$

These are the matched filter outputs due to the input signal  $s(t) + \sum_{i=1}^m h_i(t) + n(t)$ .  $U(t)$  is sampled at  $t=kT$  to yield the decision variable,  $U$ . The value of  $U$  is used to decide if a "0" or "1" was transmitted for that bit period.

Since the signal and harmonics are at discrete frequencies it will be helpful to first derive the matched filter output for this case. Note that since the matched filter is dumped after each sample the signals from previous periods will not affect the present matched filter output (no ISI). So it is sufficient to consider the matched filter's response to just the present signal during each bit period.

To begin consider an input signal of the form,

$$x(t) = C \cos(\omega_1 t + \beta) \quad (k-1)T \leq t < kT. \quad (8.2.1)$$

The corresponding output of the matched filter will be given by

$$y(t) = x(t) * m(t) \quad (8.2.2)$$

(where "\*" indicates convolution). From equation 2.1.13 with  $a=1$ ,

$$m(t) = \cos\left[\omega_c (t-T)\right] \text{Rect}\left[\frac{t-T/2}{T}\right].$$

So  $y(t)$  for  $(k-1)T \leq t < kT$  can be written

$$y(t) = C \int_{-\infty}^{\infty} \cos(\omega_i \tau + \beta) \text{Rect}\left[\frac{\tau - (k-\frac{1}{2})T}{T}\right] \quad (8.2.3)$$

$$\cos[\omega_c (t-\tau-T)] \text{Rect}\left[\frac{t-\tau-T/2}{T}\right] d\tau.$$

Note that the first  $\text{Rect}()$  function is nonzero only for  $(k-1)T \leq \tau < kT$  and the second  $\text{Rect}()$  function is nonzero only for  $(t-T) \leq \tau < t$ . The product of these two functions is nonzero only for  $(k-1)T \leq t < (k+1)T$ . However, recall that the filter will be dumped at  $t=kT$  to reset it for the next incoming signal. That means the filter's response to the present signal that would correspond to  $t > kT$  can be thrown away. That leaves only the filter response for  $(k-1)T \leq t < kT$  which can be written,

$$y(t) = C \int_{(k-1)T}^t \cos(\omega_i \tau + \beta) \cos[\omega_c (t-\tau-T)] d\tau. \quad (8.2.4)$$

Apply the identity

$$\cos A \cos B = \frac{1}{2} \left[ \cos(A+B) + \cos(A-B) \right] \quad (8.2.5)$$

to this equation. Note that the first term will have a frequency of  $\omega = (\omega_i - \omega_c)$  and the second a frequency of  $\omega = (\omega_i + \omega_c)$ . The integration acts as a low pass filter to essentially remove the second term.

So define



$$\Delta_i \equiv \omega_i - \omega_c$$

and then

$$y(t) \approx \frac{C}{2} \int_{(k-1)T}^t \cos[\Delta_i \tau + \beta + \omega_c (t-T)] d\tau \quad (8.2.6)$$

$$= \frac{C}{2\Delta_i} \left\{ \sin[\Delta_i t + \beta + \omega_c (t-T)] - \sin[\Delta_i (k-1)T + \beta + \omega_c (t-T)] \right\}.$$

Use the identity

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

to rewrite this equation as

$$y(t) \approx \frac{C(t-(k-1)T)}{2} \operatorname{sinc} \left| \frac{[\Delta_i (t-(k-1)T)]}{2\pi} \right| \quad (8.2.7)$$

$$\cos \left| \frac{[\Delta_i (t+(k-1)T)]}{2} + \beta + \omega_c (t-T) \right|$$

for  $(k-1)T \leq t < kT$ . This result can be used to determine the contribution of  $s(t)$  and  $h_i(t)$  to  $U(t)$ . It will be helpful here to make the assumption that there are an integral number of carrier cycles in each bit period. This is a typical text book assumption for this type of system and will be used in the work to follow. The result is that  $\omega_c T = 2\pi n$  where  $n$  is an integer.

### 8.2.1. Input $s(t)$

$s(t) = A \cos(\omega_c t + \vartheta)$  and the corresponding output of the sample/hold is  $s_M(t)$ . Recall that  $A$  is the signal amplitude at the receiver and  $\vartheta = 0$  or  $\pi$  radians. For this case equation 8.2.7 can be used with  $\omega_i = \omega_c$ ,  $C = A$ ,  $\beta = \vartheta$ . So  $\Delta_i = 0$  and

$$s_M(t) = y(t) = \frac{A}{2} (t - (k-1)T) \cos(\omega_c t + \vartheta) \quad (8.2.8)$$

for  $(k-1)T \leq t < kT$ .

### 8.2.2. Input $h_i(t)$

$h_i(t) = B_i \cos(\omega_i t + \varphi_i)$  and the corresponding output of the sample/hold is  $h_{iM}(t)$ . Recall that  $B_i$  is the harmonic amplitude at the receiver and  $\varphi_i$  is a uniformly distributed random variable from  $(-\pi, \pi)$ . For this case equation 8.2.7 can be used with  $C=B_i$ ,  $\beta=\varphi_i$ , and  $\omega_i=\omega_i$ . So  $\Delta_i=\omega_i-\omega_c$  and

$$h_{iM}(t) = y(t) = P_i(t) \cos \left[ \frac{\Delta_i (t - (k-1)T)}{2} + \varphi_i + \omega_c t \right] \quad (8.2.9)$$

for  $(k-1)T \leq t < kT$  and

$$P_i(t) = \frac{B_i}{2} (t - (k-1)T) \operatorname{sinc} \left[ \frac{\Delta_i (t - (k-1)T)}{2\pi} \right].$$

### 8.2.3. Input $n(t)$

The matched filter is a linear device and so it is known that a gaussian distributed input will produce a gaussian distributed output. This filter does not supply any energy to the signal so the mean value of the distribution will remain equal to zero. It is, therefore, only necessary to determine the variance of the gaussian distribution in order for its first order probability density to be completely specified.

The correlation processor used here is functionally identical to a matched filter [8]. As mentioned in section 2.1 this means that its impulse response has the general form

$$m(t) = a d(T-t) \quad (8.2.10)$$

where  $T$  is the symbol period,  $d(t)$  is the signal to be detected, and  $a$  is an arbitrary constant.

Using the assumption of white input noise it can be shown that the variance (mean square power) of the noise at the filter output at the sampling times is

$$\sigma^2 = \frac{\eta}{2} a^2 E_d \quad (8.2.11)$$

where  $E_d$  is the energy in the finite-duration signal  $d(t)$  [6]. For this system  $a=1$  and  $E_d$  is found to be

$$E_d = \int_{kT}^{(k+1)T} d^2(t) dt = \int_{kT}^{(k+1)T} \cos^2(\omega_c t) dt. \quad (8.2.12)$$

Since  $\omega_c \gg 1$  this reduces to

$$E_d \approx \frac{T}{2}. \quad (8.2.13)$$

This means that the variance can be expressed approximately as

$$\sigma^2 = \frac{\eta T}{4}. \quad (8.2.14)$$

(Note that this could also be arrived at using the concept of noise equivalent bandwidth.) The output signal can now be expressed as a normally distributed random process with mean of zero and variance of  $\frac{\eta T}{4}$ . ("Normal distribution" is another name for gaussian distribution.) The notation for this is

$$N(0, \sigma^2) = N(0, \frac{\eta T}{4}). \quad (8.2.15)$$

Putting all of these terms together yields the decision signal

$$U(t) = \frac{A}{2} (t - (k-1)T) \cos(\omega_c t + \theta) + N(0, \frac{\eta T}{4}) \quad (8.2.16)$$

$$+ \sum_{i=1}^m P_i(t) \cos \left[ \frac{\Delta_i(t+(k-1)T)}{2} + \varphi_i + \omega_c t \right]$$

for  $(k-1)T \leq t < kT$ . This signal is sampled at  $t=kT$  to get the decision variable  $U$ . This can be written as

$$U = S + \sum_{i=1}^m H_i(\varphi_i) + N(0, \frac{\eta T}{4}) \quad (8.2.17)$$

where

$$S = +/- \frac{AT}{2}$$

(+/- for  $\vartheta = 0/\pi$  radians) and

$$H_i(\varphi_i) = P_i \cos \left[ \Delta_i T \left( k - \frac{1}{2} \right) + \varphi_i \right]$$

with

$$P_i = \frac{B_i T}{2} \operatorname{sinc} \left[ \frac{\Delta_i T}{2\pi} \right].$$

### 8.3. Derivation of Decision Variable V for DPSK System with Two Harmonic Interferers

Reference the DPSK system block diagram in figure 2.2.

The DPSK system is complicated by the presence of components not found in the BPSK system. In particular, the delay device, multiplier, and low pass filter (LPF). Therefore, it is necessary to consider the specific case of two interferers rather than the general case of  $m$  interferers because the derivation is quite involved. It is also very helpful to approach this problem methodically and to make all reasonable simplifying assumptions as the derivation progresses. For that reason the derivation has been broken down into several steps.

#### 8.3.1. Derivation of Multiplier Output, $W(t)$

As indicated in figure 2.2,  $W(t)$  is given by the product of two signals. Both are matched filter outputs and the first is due to the present input signal while the second is due to the previous bit period signal which has been delayed by one period. This can be expressed as

$$W(t) = U_{k-1}(t) U_{k-2}(t-T). \quad (8.3.1.1)$$

Here the subscripts on the  $U(t)$  indicate the period for which the signal applies. For example,  $U_{k-1}(t)$  is the output due to the input signal for  $(k-1)T \leq t < kT$ . The expression for  $U_{k-1}(t)$  is derived in section 8.2, see equation 8.2.16 with  $m=2$ . This can be written as

$$U_{k-1}(t) = \left[ u_1(t) + u_2(t) + u_3(t) + \right. \quad (8.3.1.2) \\ \left. + u_4(t) \right] \text{Rect} \left[ \frac{t - (k-1)T}{T} \right]$$

(the Rect() function emphasizes the finite time duration of the signal) . Here

$$u_1(t) = \frac{A_{k-1}}{2} (t - (k-1)T) \cos(\omega_c t)$$

( $A_{k-1} = +/- A$  depending on whether  $\vartheta = 0$  or  $\pi$  radians for the  $(k-1)^{\text{st}}$  period),

and for  $i=1,2$  (respectively),

$$u_2(t), u_3(t) = \frac{B_i}{2} (t - (k-1)T) \text{sinc} \left[ \frac{\Delta_i (t - (k-1)T)}{2\pi} \right] \\ \cos \left[ \frac{\Delta_i (t + (k-1)T)}{2} + \varphi_{i,k-1} + \omega_c t \right]$$

and  $u_4(t) =$  gaussian distributed random process with zero mean and variance of  $\frac{\eta T}{4}$ . This can be expressed as

$$u_4(t) = N_{k-1} \left( 0, \frac{\eta T}{4} \right).$$

Now  $U_{k-2}(t-T)$  can be found by simply substituting  $(k-2)$  everywhere for  $(k-1)$  in the  $U_{k-1}(t)$  expression and then substituting  $(t-T)$  for  $t$ . This gives

$$U_{k-2}(t-T) = \left[ u_5(t) + u_6(t) + u_7(t) \right. \quad (8.3.1.3)$$

$$+ u_8(t) \Big] \text{Rect} \left[ \frac{t - (k - \frac{1}{2})T}{T} \right]$$

where

$$u_5(t) = \frac{A_{k-2}}{2} (t - (k-1)T) \cos(\omega_c t)$$

and for  $i=1,2$  (respectively),

$$u_6(t), u_7(t) = \frac{B_i}{2} (t - (k-1)T) \text{sinc} \left[ \frac{\Delta_i (t - (k-1)T)}{2\pi} \right] \\ \cos \left[ \frac{\Delta_i (t + (k-3)T)}{2} + \varphi_{i,k-2} + \omega_c t \right]$$

(using  $\omega_c T = 2\pi n$ ,  $n$  an integer, as needed)

and  $u_8(t)$  = gaussian distributed random process with mean of zero and variance of  $\frac{\eta T}{4}$ . Again this can be expressed as

$$u_8(t) = N_{k-2} \left( 0, \frac{\eta T}{4} \right).$$

Note here that  $u_4(t)$  and  $u_8(t)$  are assumed to be uncorrelated noise since they come from different bit periods. Also since  $\varphi_1$  and  $\varphi_2$  cannot shift abruptly it is assumed that they will stay approximately constant over two successive bit periods. That means for  $i=1,2$

$$\varphi_{i,k-1} \approx \varphi_{i,k-2} = \varphi_i \quad (8.3.1.4)$$

where the subscripts indicate which bit period the phase is associated with.

Now it is apparent that  $W(t)$  will contain sixteen separate terms. A double subscript will be used to identify these terms. For example,  $u_{15}(t) = u_1(t) u_5(t)$ . This yields

$$W(t) = ss(t) + sh(t) + hh(t) + hn(t) + sn(t) + nn(t) \quad (8.3.1.5)$$

where

$$ss(t) = \text{signal X signal} = u_{15}(t) \text{ Rect} \left[ \frac{\left| t - (k - \frac{1}{2})T \right|}{T} \right]$$

sh(t) = signal X harmonic

$$= \left[ u_{16}(t) + u_{17}(t) + u_{25}(t) + u_{35}(t) \right] \text{ Rect} \left[ \frac{\left| t - (k - \frac{1}{2})T \right|}{T} \right]$$

hh(t) = harmonic X harmonic

$$= \left[ u_{26}(t) + u_{27}(t) + u_{36}(t) + u_{37}(t) \right] \text{ Rect} \left[ \frac{\left| t - (k - \frac{1}{2})T \right|}{T} \right]$$

hn(t) = harmonic X noise

$$= \left[ u_{46}(t) + u_{47}(t) + u_{28}(t) + u_{38}(t) \right] \text{ Rect} \left[ \frac{\left| t - (k - \frac{1}{2})T \right|}{T} \right]$$

sn(t) = signal X noise

$$= \left[ u_{45}(t) + u_{18}(t) \right] \text{ Rect} \left[ \frac{\left| t - (k - \frac{1}{2})T \right|}{T} \right]$$

and

$$nn(t) = \text{noise X noise} = u_{48}(t) \text{ Rect} \left[ \frac{\left| t - (k - \frac{1}{2})T \right|}{T} \right].$$

It obviously will be advantageous to simplify this expression before



expanding all the components. The first step can be taken here. The term  $nn(t)$  is particularly troublesome in analytical work because it represents the product of random variables. Fortunately field tests indicate that the white background noise power in real PLC systems is typically 10 dB below the average harmonic noise levels [19]. Since all the other terms of  $W(t)$  include the much higher power harmonic and signal components the  $nn(t)$  term will be considered negligible and will be discarded from the analysis at this point. Further simplifications will take place in the next section.

### 8.3.2. Derivation of Low Pass Filter (LPF) Output, $V(t)$

In a typical DPSK system with only gaussian noise the LPF is simply used to remove double frequency terms from the multiplier output. The addition of the harmonic interference to the received signal makes its exact use less obvious. So it is necessary to look closely at how the LPF will affect  $W(t)$ , especially those product terms that include the gaussian noise ( $h_n(t)$  and  $s_n(t)$ ).

Note here that it will be necessary to work with power spectral densities (PSD) due to the random nature of these signals. In particular, it is necessary to discover how the multiplication affects the PSD of the original noise terms. Working in the frequency domain like this is necessary in order to easily see the effect of the LPF on the signal,  $W(t)$ .

It will be helpful to first establish some useful results that can be used later in the derivation. To begin note that the assumption of a white input noise means that at the matched filter input the noise PSD is,

$$G_n(f) = \frac{\eta}{2} \quad -\omega < f < \omega \quad . \quad (8.3.2.1)$$

Passing this through the matched filter with transfer function  $M(f)$  produces the output noise with

$$G_{n_M}(f) = G_n(f) |M(f)|^2 \quad (8.3.2.2)$$

by a well known theorem [8].  $|M(f)|^2$  is derived in section 8.4 and so

$$G_{n_M}(f) \approx \frac{\eta}{2} (T/2)^2 \left\{ \text{sinc}^2 [T(f-f_c)] + \text{sinc}^2 [T(f+f_c)] \right\}. \quad (8.3.2.3)$$

Therefore,  $G_{n_M}(f)$  consists of  $\text{sinc}^2(\cdot)$  functions centered at  $\pm f_c$ .

The next useful result is found in most communications texts [5]. Given a random process  $x(t)$  and a random variable  $\beta$  (independent of  $x(t)$ ) which is uniformly distributed over  $(-\pi, \pi)$  such that

$$z(t) = x(t) \cos(\omega_0 t + \beta) \quad (8.3.2.4)$$

it can be shown that the autocorrelation of  $z(t)$  is

$$R_z(\tau) = \frac{1}{2} R_x(\tau) \cos(\omega_0 \tau). \quad (8.3.2.5)$$

This represents modulation in the  $\tau$  domain. So in the frequency domain (by the Wiener-Kinchine theorem),

$$G_z(f) = \frac{1}{4} \left[ G_x(f-f_0) + G_x(f+f_0) \right]. \quad (8.3.2.6)$$

The final point for now is to find the PSD of a particular term that shows up later in the derivation,

$$x(t) = C n_M(t) \text{Rect} \left[ \frac{t - (k - \frac{1}{2})T}{T} \right]. \quad (8.3.2.7)$$

To begin first find the autocorrelation of  $x(t)$ ,

$$R_x(\tau) = E \left\{ x(t)x(t-\tau) \right\} \quad (8.3.2.8)$$

( $E$  = expected value)

$$= C^2 E \left\{ n_M(t)n_M(t-\tau) \right\} \text{Rect} \left[ \frac{t - (k - \frac{1}{2})T}{T} \right] \text{Rect} \left[ \frac{t - \tau - (k - \frac{1}{2})T}{T} \right]$$

$$= C^2 R_{n_M}(\tau) \text{Rect}\left[\frac{\tau}{2T}\right]$$

(ie.  $R_x(\tau) = 0$  for  $|\tau| > T$ ). Then transforming to the frequency domain,

$$G_x(f) = 2C^2 T G_{n_M}(f) * \text{sinc}(2fT) \quad (8.3.2.9)$$

where "\*" indicates convolution. Now it is time to proceed with the derivation of interest.

The place to begin is with the terms of  $W(t)$  labeled  $h_n(t)$ . It is possible to write out one function to represent any of these four terms.

$$u_a(t) = n_M(t) \frac{B_i}{2} (t-(k-1)T) \text{sinc}\left[\frac{\Delta_i(t-(k-1)T)}{2\pi}\right] \quad (8.3.2.10)$$

$$\cos\left[\frac{\Delta_i(t+(k-1)T-\alpha)}{2} + \varphi_i + \omega_c t\right]$$

where  $i=1,2$  and  $\alpha = 0, 2T$  as appropriate. This can be rewritten as

$$u_a(t) = n_M(t) \frac{B_i}{\Delta_i} \sin\left[\frac{\Delta_i(t-(k-1)T)}{2}\right] \quad (8.3.2.11)$$

$$\cos\left[\frac{\Delta_i(t+(k-1)T-\alpha)}{2} + \varphi_i + \omega_c t\right].$$

Since  $\varphi_i$  is random and uniformly distributed over  $(-\pi, \pi)$ , and the sine functions have a period of  $2\pi$  it is possible to subtract  $\pi/2$  from the argument of the cosine term without really affecting the final results. This allows the cosine term to be written as a sine term and then a trig identity is used to write  $u_a(t)$  in the more useable form

$$u_a(t) = \frac{n_M(t)B_i}{2\Delta_i} \left\{ \cos\left[\Delta_i\left((k-1)T-\frac{\alpha}{2}\right) + \varphi_i + \omega_c t\right] - \cos\left[\Delta_i\left(t-\frac{\alpha}{2}\right) + \varphi_i + \omega_c t\right] \right\} \quad (8.3.2.12)$$

So now  $u_a(t)$  can be written as a sum of terms of the form

$$z(t) = C n_M(t) \cos(\omega_0 t + \beta). \quad (8.3.2.13)$$

Here  $C$  is a constant, and

$$\omega_0 = \omega_c \quad \text{or} \quad \omega_0 = \Delta_i + \omega_c = \omega_i \quad (8.3.2.14)$$

and

$$\beta = \varphi_i + \Delta_i\left((k-1)T-\frac{\alpha}{2}\right) \quad \text{or} \quad \beta = \varphi_i - \Delta_i\frac{\alpha}{2}$$

depending on which term is involved. Since  $\omega_i = \omega_1$  or  $\omega_2$  and these are assumed to be the harmonics closest to  $\omega_c$ ,

$$\omega_0 \sim \omega_c$$

for all cases. Also note that the random variable  $\beta$  will be uniformly distributed over  $(-\pi, \pi)$  for reasons discussed above.

Now it is possible to determine the effect of the LPF on the  $hn(t)$  terms. Using the form derived for  $u_a(t)$ ,  $hn(t)$  is seen to be made up of a sum of terms of the form,

$$q(t) = x(t) \cos(\omega_0 t + \beta) \quad (8.3.2.15)$$

where  $\omega_0$  and  $\beta$  can take on the values described earlier and

$$x(t) = C n_M(t) \text{Rect}\left|\frac{t - (k-\frac{1}{2})T}{T}\right|. \quad (8.3.2.16)$$

So relying on the initial results presented in this section,

$$G_q(f) = \frac{1}{4} \left\{ G_x(f-f_0) + G_x(f+f_0) \right\} \quad (8.3.2.17)$$

and

$$G_x(f) = 2C^2T G_{n_M}(f) * \text{sinc}(2fT) \quad (8.3.2.18)$$

and

$$G_{n_M}(f) \approx D \left\{ \text{sinc}^2[T(f-f_c)] + \text{sinc}^2[T(f+f_c)] \right\} \quad (8.3.2.19)$$

(D is a constant). Now

$$\begin{aligned} G_x(f-f_0) &= 2C^2T G_{n_M}(f-f_0) * \text{sinc}[2T(f-f_0)] \quad (8.3.2.20) \\ &= 2C^2TD \left\{ \text{sinc}^2[T(f-f_0-f_c)] + \text{sinc}^2[T(f-f_0+f_c)] \right\} \\ &\quad * \text{sinc}[2T(f-f_0)]. \end{aligned}$$

Use  $f_0 \approx f_c$  as mentioned earlier to get a general idea about where  $G_x(f-f_0)$  is located in frequency.

$$\begin{aligned} G_x(f-f_0) &\approx 2C^2TD \left\{ \text{sinc}^2[T(f-2f_c)] + \text{sinc}^2[Tf] \right\} \quad (8.3.2.21) \\ &\quad * \text{sinc}[2T(f-f_c)] \end{aligned}$$

The convolution will result in some distorted waveform centered at frequencies of about  $f \approx f_c$  and  $f \approx 3f_c$ . The exact waveform is not important here since  $f_c \gg 1$  and so a LPF will remove both of these waveforms except for possibly some very small signal due to the side lobes which will be considered negligible. One can go through a similar development for the  $G_x(f+f_0)$  term to get

$$G_x(f+f_0) \approx 2C^2TD \left\{ \text{sinc}^2[T(f+2f_c)] + \text{sinc}^2[Tf] \right\} \quad (8.3.2.22)$$

$$* \text{sinc}[2T(f+f_c)]$$

This results in a distorted waveform centered at frequencies of about  $f \approx -f_c$  and  $f \approx -3f_c$ . So once again the LPF will remove these terms.

The result of all of this is that the LPF will remove all contributions of  $h_n(t)$  to  $V(t)$ . Therefore, discard the four terms associated with  $h_n(t)$  from further consideration in determining  $V(t)$ .

The next terms to consider are those labeled  $s_n(t)$ . These are the most complicated ones to deal with because they represent nonstationary random processes. However, since only the sampling times,  $t=kT$ , are of interest it should be possible to approximately solve this problem. In particular, it is necessary to determine the mean and variance of the random processes at  $t=kT$ . A very large assumption will also be made, namely, that at any given point in time the random processes above will be approximately gaussian distributed. This is necessary in order to attempt a solution.

To begin note that  $u_{45}(t)$  and  $u_{18}(t)$  are identical in form and can be represented by

$$u_b(t) = C n_M(t) (t-(k-1)T) \cos(\omega_c t) \quad (8.3.2.23)$$

where  $C = +/- (A/2)$ . The only difference is that the  $n_M(t)$  terms in  $u_{45}(t)$  and  $u_{18}(t)$  will be assumed uncorrelated. So consider  $s_n(t)$  composed of two terms of the form

$$z(t) = C n_M(t) (t-(k-1)T) \cos(\omega_c t) \text{Rect} \left\{ \frac{t-(k-\frac{1}{2})T}{T} \right\}. \quad (8.3.2.24)$$

Then assuming  $t$  is a known parameter (and referring to the earlier derivation of  $R_x(\tau)$ ),

$$R_z(t, t-\tau) = E \left\{ z(t) z(t-\tau) \right\} \quad (8.3.2.25)$$

$$= C^2 (t-(k-1)T) (t-\tau-(k-1)T) R_{n_M}(\tau) \text{Rect} \left\{ \frac{\tau}{2T} \right\}$$

$$\left[ \cos^2(\omega_c t) \cos(\omega_c \tau) + \cos(\omega_c t) \sin(\omega_c t) \sin(\omega_c \tau) \right].$$

Taking the Fourier transform with respect to  $\tau$  and with  $t$  as a parameter,

$$G_z(t, f) = C^2 (t-(k-1)T) G_{n_M}(f) * F \left\{ t-\tau-(k-1)T \right\} \quad (8.3.2.26)$$

$$* 2T \text{sinc}[2fT] * \left\{ \frac{1}{2} \cos^2(\omega_c t) [\delta(f-f_c) + \delta(f+f_c)] \right.$$

$$\left. + \frac{1}{2j} \cos(\omega_c t) \sin(\omega_c t) [\delta(f-f_c) - \delta(f+f_c)] \right\}.$$

The item of interest here is where the power spectrum of  $z(t)$  will be centered so we can determine the effect of the LPF. Assume that convolution with the  $F \left\{ t-\tau-(k-1)T \right\}$  term will distort the spectra but will not translate it in frequency. Obviously, this is also true of convolution with the "2T sinc(2fT)" term. First consider just

$$G(t, f) = G_{n_M}(f) * \frac{1}{2} \left\{ \delta(f-f_c) [\cos^2(\omega_c t) \right. \quad (8.3.2.27)$$



$$- j \cos(\omega_c t) \sin(\omega_c t)] + \delta(f+f_c) [\cos^2(\omega_c t) + j \cos(\omega_c t) \sin(\omega_c t)] \Big\}.$$

Due to the  $\delta$  functions this can be evaluated by inspection. Recalling the form of  $G_{n_M}(f)$  given earlier, the first term in  $G(t,f)$  would be

$$D_1 \left\{ \text{sinc}^2[T(f-2f_c)] + \text{sinc}^2[fT] \right\} \quad (8.3.2.28)$$

where

$$D_1 = \frac{\eta}{4} (T/2)^2 [\cos^2(\omega_c t) - j \cos(\omega_c t) \sin(\omega_c t)].$$

Obviously, the LPF would only pass the  $D_1 \text{sinc}^2(fT)$  portion. Similarly, the second term of  $G(t,f)$  is

$$D_2 \left\{ \text{sinc}^2[T(f+2f_c)] + \text{sinc}^2[fT] \right\} \quad (8.3.2.29)$$

where

$$D_2 = \frac{\eta}{4} (T/2)^2 [\cos^2(\omega_c t) + j \cos(\omega_c t) \sin(\omega_c t)].$$

Again only the  $D_2 \text{sinc}^2(fT)$  term would pass through the LPF. So at the output of the LPF the approximate result is

$$G_{\text{out}}(t) \approx (D_1 + D_2) \text{sinc}^2(fT) \quad (8.3.2.30)$$

(note  $D_2 = D_1^*$ )

$$= \frac{\eta}{2} (T/2)^2 \cos^2(\omega_c t) \text{sinc}^2(fT).$$

So consider at the LPF output,

$$G_{z_{\text{out}}}(t,f) \approx C^2 (t-(k-1)T) * F \left\{ t-\tau-(k-1)T \right\} \quad (8.3.2.31)$$

$$* 2T \operatorname{sinc}(2fT) * G_{\text{out}}(t, f).$$

So

$$R_{z_{\text{out}}}(t, \tau) \sim C^2 (t - (k-1)T) (t - \tau - (k-1)T) \operatorname{Rect}\left[\frac{\tau}{2T}\right] \quad (8.3.2.32)$$

$$\frac{\eta}{2} (T/2)^2 \cos^2(\omega_c t) \frac{1}{T} \operatorname{Tri}\left[\frac{\tau}{T}\right]$$

where

$$\operatorname{Tri}\left[\frac{\tau}{T}\right] \equiv 1 - \frac{|\tau|}{T}, \quad |\tau| < T \quad (8.3.2.33)$$

$$0, \quad |\tau| > T$$

is the standard triangular function. Now since  $z(t)$  will be zero mean the variance is given by (substitute back in for  $C$  too)

$$\sigma_z^2(t) = R_{z_{\text{out}}}(t, 0) \sim \frac{\eta T}{4} \frac{A^2}{8} (t - (k-1)T)^2 \cos^2(\omega_c t) \quad (8.3.2.34)$$

for a given time  $t$ . Now it is possible to complete the derivation. Consider the LPF output due to  $sn(t)$  once again but now in the form

$$sn_{\text{out}}(t) = [z_{45}(t) + z_{18}(t)] \quad (8.3.2.35)$$

where  $z_{45}(t)$  and  $z_{18}(t)$  correspond to  $z(t)$  with  $n_M(t)$  equal to  $u_4(t)$  and  $u_8(t)$  respectively. If at any instant of time  $z_{45}$  and  $z_{18}$  are gaussian distributed random variables which are uncorrelated then  $sn_{\text{out}}(t)$  will also be gaussian distributed at that time. Furthermore, it is well known that the mean and variance of  $sn_{\text{out}}(t)$  will equal the sum of the means and variances, respectively, of  $z_{45}(t)$  and  $z_{18}(t)$ . So  $sn_{\text{out}}(t)$  at any given time  $t_0$  will have a zero mean and a variance of  $2 \sigma_z^2(t_0)$ . Writing this in a standard notation,

$$s_{n_{out}}(t=t_0) = N[0, 2\sigma_z^2(t=t_0)]. \quad (8.3.2.36)$$

Now the rest of the terms of  $W(t)$  are all products of signal-signal, signal-harmonic, or harmonic-harmonic. The harmonic terms all have the uniformly distributed random phase,  $\varphi_i$ , mentioned earlier. For this work the  $\varphi_i$  will be assumed known at this stage. Later the final results for the BER will be obtained by averaging over the joint probability density of the  $\varphi_i$ . Therefore, this stage of the work is straightforward. Simply form the products and decide which of the resulting frequencies the LPF will pass and discard the rest.

First consider the signal-signal product terms,  $ss(t)$ . This would be the only output if no noise or harmonics were present.

$$ss(t) = \frac{A_{k-1}A_{k-2}}{4} (t-(k-1)T)^2 \cos^2(\omega_c t) \text{ Rect} \left[ \frac{t-(k-\frac{1}{2})T}{T} \right] \quad (8.3.2.37)$$

and

$$\cos^2(\omega_c t) = \frac{1 + \cos(2\omega_c t)}{2}.$$

The double frequency term is rejected by the LPF so at the LPF output

$$s_{s_{out}}(t) = \frac{A_{k-1}A_{k-2}}{8} (t-(k-1)T)^2 \text{ Rect} \left[ \frac{t-(k-\frac{1}{2})T}{T} \right]. \quad (8.3.2.38)$$

This output takes on a +/- sign depending on the four possible combinations of the signs of two successive bits.

Next consider the signal-harmonic product terms,  $sh(t)$ . Here the LPF will not really be affected by the  $\text{Rect}()$  function so leave it out for simplicity. Now each of the four terms of  $sh(t)$  can be written in the general form,

$$u_a(t) = C_a (t-(k-1)T)^2 \cos(\omega_c t) \text{sinc} \left| \frac{[\Delta_i(t-(k-1)T)]}{2\pi} \right| \quad (8.3.2.39)$$

$$\cos \left| \frac{[\Delta_i(t+(k-1)T-\alpha)]}{2} + \varphi_i + \omega_c t \right|.$$

$C_a = \frac{+/- AB_i}{4}$ ,  $i=1,2$  and  $\alpha = 0,2T$  as appropriate. Note that since  $\Delta_i = \omega_i - \omega_c$  is a low frequency the  $\text{sinc}()$  term can cause little frequency translation. So consider just the translation effect of  $\cos(\omega_c t)$  on the other cosine term. This other cosine term is at a frequency of

$$\omega = \frac{\Delta_i}{2} + \omega_c = \frac{\omega_i + \omega_c}{2}.$$

Multiplication by the  $\cos(\omega_c t)$  term will translate this to frequencies

$$\omega_L = \omega - \omega_c = \frac{\omega_i - \omega_c}{2} \approx 0 \quad (8.3.2.40)$$

and

$$\omega_H = \omega + \omega_c = \frac{\omega_i + 3\omega_c}{2} \gg 1.$$

and will cut the magnitudes in half. The  $\omega_L$  term is passed by the LPF and the  $\omega_H$  is rejected. So at the LPF output,

$$u_{a_{out}}(t) \sim \frac{C}{a} \frac{(t-(k-1)T)^2}{2} \operatorname{sinc} \left[ \frac{\Delta_i(t-(k-1)T)}{2\pi} \right] \cos \left[ \frac{\Delta_i(t+(k-1)T-a)}{2} + \varphi_i \right] \quad (8.3.2.41)$$

Use this result to write  $sh_{out}(t)$  as

$$sh_{out}(t) \sim \frac{(t-(k-1)T)^2}{8} \sum_{i=1}^2 B_i \operatorname{sinc} \left[ \frac{\Delta_i(t-(k-1)T)}{2\pi} \right] \left\{ A_{k-1} \cos \left[ \frac{\Delta_i(t+(k-3)T)}{2} + \varphi_i + \omega_c t \right] + A_{k-2} \cos \left[ \frac{\Delta_i(t+(k-1)T)}{2} + \varphi_i + \omega_c t \right] \right\} \quad (8.3.2.42)$$

The very last terms to consider are the harmonic-harmonic products,  $hh(t)$ . Again ignore the  $\operatorname{Rect}()$  function for now and look at the general form of the terms in  $hh(t)$ .

$$u_{ab}(t) = \frac{B_i B_j}{4} (t-(k-1)T)^2 \operatorname{sinc} \left[ \frac{\Delta_i(t-(k-1)T)}{2\pi} \right] \operatorname{sinc} \left[ \frac{\Delta_j(t-(k-1)T)}{2\pi} \right] \cos \left[ \frac{\Delta_i(t+(k-1)T)}{2} + \varphi_i + \omega_c t \right] \cos \left[ \frac{\Delta_j(t+(k-3)T)}{2} + \varphi_j + \omega_c t \right] \quad (8.3.2.43)$$

where  $i$  and  $j = 1$  or  $2$  as appropriate.

As in the  $sh_{out}(t)$  derivation the  $\operatorname{sinc}()$  terms will cause little frequency translation of the remaining terms. So only consider the frequency translation caused by the product of the two cosine terms. This product will result in two more cosine terms with frequencies of

$$\omega_L = \frac{\Delta_i}{2} + \omega_c - \frac{\Delta_j}{2} + \omega_c \quad (8.3.2.44)$$

$$= \frac{\omega_i - \omega_j}{2} = 0 \text{ if } i=j \text{ or } \approx 0 \text{ if } i \neq j$$

and

$$\begin{aligned} \omega_H &= \frac{\Delta_i}{2} + \omega_c + \frac{\Delta_j}{2} + \omega_c \\ &= \frac{\omega_i + \omega_j}{2} + \omega_c \approx 2\omega_c. \end{aligned}$$

Therefore, the LPF rejects the  $\omega_H$  cosine term and passes the  $\omega_L$  cosine term. So (including factor of 1/2 from the cosine product),

$$u_{ab\text{out}}(t) \approx \frac{B_i B_j}{8} (t - (k-1)T)^2 \operatorname{sinc} \left| \frac{\Delta_i (t - (k-1)T)}{2\pi} \right| \quad (8.3.2.45)$$

$$\operatorname{sinc} \left| \frac{\Delta_j (t - (k-1)T)}{2\pi} \right| \cos \left| \frac{(\Delta_i - \Delta_j)(t + (k-1)T)}{2} + \Delta_j T + (\varphi_i + \varphi_j) \right|.$$

Using the appropriate substitution of  $i, j$  values for each  $u_{ab}(t)$  term yields,

$$hh_{\text{out}}(t) \approx \frac{(t - (k-1)T)^2}{8} \left\{ \sum_{i=1}^2 B_i^2 \operatorname{sinc}^2 \left| \frac{\Delta_i (t - (k-1)T)}{2\pi} \right| \right\} \quad (8.3.2.46)$$

$$\begin{aligned} &\cos(\Delta_i T) + B_1 B_2 \operatorname{sinc} \left| \frac{\Delta_1 (t - (k-1)T)}{2\pi} \right| \operatorname{sinc} \left| \frac{\Delta_2 (t - (k-1)T)}{2\pi} \right| \\ &\left\{ \cos \left| \frac{(\Delta_1 - \Delta_2)(t + (k-1)T)}{2} + \Delta_2 T + \varphi_1 - \varphi_2 \right| \right\} \\ &+ \cos \left| \frac{(\Delta_1 - \Delta_2)(t + (k-1)T)}{2} - \Delta_1 T + \varphi_1 - \varphi_2 \right| \}. \end{aligned}$$

These results can all be combined to give

$$V(t) \approx ss_{\text{out}}(t) + sh_{\text{out}}(t) + hh_{\text{out}}(t) + hn_{\text{out}}(t) \quad (8.3.2.47)$$

$$+ sn_{out}(t) + nn_{out}(t)$$

where  $hn_{out}(t)$  and  $nn_{out}(t)$  have been determined to give a negligible contribution to the sum.

### 8.3.3. Derivation of Sample/Hold Output, $V$

$V(t)$  is sampled at  $t=kT$  in order to get  $V$ . The components of  $V(t)$  have been derived in the previous section so simply refer to them there. It is obvious that the expression for  $V(t)$  is complicated. For that reason it will be helpful to make two more observations before writing out the form of  $V$ . First, as used in the other BER derivations in this paper it is assumed

$$B_1 \approx B_2 = B. \quad (8.3.3.1)$$

Second, recall that the random variables  $\varphi_1$  and  $\varphi_2$  are uniformly distributed over  $(-\pi, \pi)$  and only the periodic sine and cosine functions are involved. Therefore, addition of a constant to  $\varphi_1$  or  $\varphi_2$  will result in a new random variable which is also uniformly distributed over  $(-\pi, \pi)$ . So for later use define for  $i=1,2$  and a given  $k$  value

$$\epsilon_i = \varphi_i + \Delta_i T(k - \frac{1}{2}). \quad (8.3.3.2)$$

Note that since the  $\varphi$  s are assumed independent the  $\epsilon$  s will also be independent. To continue the derivation simply substitute  $t=kT$  into the  $V(t)$  expression. The result can be written,

$$V = N(0, \frac{A^2 T^3}{16} \eta) + \frac{T^2}{8} \left\{ A_{k-1} A_{k-2} + B \sum_{i=1}^2 \operatorname{sinc} \left[ \frac{[\Delta_i T]}{2\pi} \right] \right. \quad (8.3.3.3)$$

$$\left. \left\{ A_{k-1} \cos \left[ \frac{[\Delta_i (2k-3)T]}{2} + \varphi_i \right] + A_{k-2} \cos \left[ \frac{[\Delta_i (2k-1)T]}{2} + \varphi_i \right] \right\} \right.$$

$$\left. + B^2 \left[ \sum_{i=1}^2 \operatorname{sinc}^2 \left[ \frac{[\Delta_i T]}{2\pi} \right] \cos(\Delta_i T) + 2 \operatorname{sinc} \left[ \frac{[\Delta_1 T]}{2\pi} \right] \operatorname{sinc} \left[ \frac{[\Delta_2 T]}{2\pi} \right] \right] \right.$$



$$\cos \left[ \left( \Delta_1 - \Delta_2 \right) \left( k - \frac{1}{2} \right) T + \frac{(\Delta_2 - \Delta_1)T}{2} + \varphi_1 - \varphi_2 \right] \cos \left[ \frac{(\Delta_1 + \Delta_2)T}{2} \right].$$

Using the new definition this can be written in a more useful form as,

$$V = N(0, \frac{A^2 T^3 \eta}{16}) + \frac{T^2}{8} \left\{ A_{k-1} A_{k-2} + B \sum_{i=1}^2 \operatorname{sinc} \left[ \frac{[\Delta_i T]}{2\pi} \right] \right. \quad (8.3.3.4)$$

$$\left. \left\{ A_{k-1} \cos[\varepsilon_i - \Delta_i T] + A_{k-2} \cos[\varepsilon_i] \right\} \right.$$

$$+ B \sum_{i=1}^2 \left[ \operatorname{sinc}^2 \left[ \frac{[\Delta_i T]}{2\pi} \right] \cos(\Delta_i T) + 2 \operatorname{sinc} \left[ \frac{[\Delta_1 T]}{2\pi} \right] \operatorname{sinc} \left[ \frac{[\Delta_2 T]}{2\pi} \right] \right.$$

$$\left. \left. \cos \left[ \left( \varepsilon_1 - \frac{\Delta_2 T}{2} \right) - \left( \varepsilon_2 - \frac{\Delta_2 T}{2} \right) \right] \cos \left[ \frac{(\Delta_1 + \Delta_2)T}{2} \right] \right\}.$$

This is the desired expression for V.

#### 8.4. Derivation of Decision Variable U Using Central Limit Theorem

To begin this derivation it is necessary to consider the initial signal and system assumptions from section 2.1. First consider the total harmonic interference present at the input to the receiver (see equations 2.1.7 and 2.1.8)

$$i(t) = h_1(t) + h_2(t) + \dots + h_m(t) + \dots + h_N(t)$$

where  $N$  is large and

$$h_j(t) = B_j \cos(\omega_j t + \varphi_j).$$

(For the  $n_j$ th harmonic.) The  $h_j(t)$ s are actually random processes due to the random nature of  $\varphi_j$ . So at any given time  $t_1$ ,  $h_j(t_1)$  is a random variable and, therefore,  $i(t_1)$  is a random variable. Recall in section 8.1 that it is argued that the  $\varphi_j$  of adjacent harmonics are independent random variables. It seems reasonable to extend this argument so that all the  $\varphi_j$  are assumed independent. In that case the  $h_j(t_1)$ s will also be independent random variables. This suggests that it may be possible to apply the Central Limit Theorem to determine the probability distribution for  $i(t_1)$  [15]. The Central Limit Theorem is stated below.

Central Limit Theorem:

Let  $X_1, X_2, \dots, X_N$  be independent random variables with means  $m_1, m_2, \dots, m_N$  and variances  $\sigma_1^2, \sigma_2^2, \dots, \sigma_N^2$  respectively. Then the distribution of the random variable

$$Z \equiv \sum_{j=1}^N X_j \quad (8.4.1)$$

approaches a Gaussian distribution as  $N$  becomes large with

$$\text{mean } m = \sum_{j=1}^N m_j \quad (8.4.2)$$

and

$$\text{variance } \sigma^2 = \sum_{j=1}^N \sigma_j^2 \quad (8.4.3)$$

provided

$$\lim_{N \rightarrow \infty} \frac{\sigma_j}{\sigma} = 0 \quad \text{for all } j. \quad (8.4.4)$$

The condition on  $\frac{\sigma_j}{\sigma}$  is used to insure that no one  $X_j$  will dominate the sum. Based on the assumption of independence of the  $h_j(t_1)$  it is obvious that the first part of the hypothesis is met. As for the other condition (equation 8.4.4) it is necessary to look at the  $\sigma_j$ .

For the case being considered let

$$h_j \equiv h_j(t_1) \quad (8.4.5)$$

and so

$$m_j = \overline{h_j} \text{ and } \sigma_j^2 = \overline{(h_j - m_j)^2}. \quad (8.4.6)$$

(The bar indicates an ensemble average.) It can be shown that the power spectral density of  $h_j(t)$  is

$$G_{h_j}(f) = \left(\frac{B_j}{2}\right)^2 \left\{ \delta(f - f_j) + \delta(f + f_j) \right\} \quad (8.4.7)$$

where  $f_j = n_j (60)$  ( $n_j$  specifies the harmonic of the 60 Hz power sig-

nal) [5]. By inspection it is obvious that  $m_j = 0$  since there is no DC term in  $G_{h_j}(f)$ . It is also easy to see that if  $m_j = 0$  then

$$\sigma_j^2 = \overline{h_j^2} = \int_{-\infty}^{\infty} G_{h_j}(f) df. \quad (8.4.8)$$

Due to the properties of dirac delta functions this can be solved by inspection.

$$\sigma_j^2 = \frac{B_j^2}{2}. \quad (8.4.9)$$

If the  $B_j$  are assumed to be approximately equal,  $B_j \approx B$ , (experimental measurements of typical PLC noise indicate that this is a reasonable assumption [16],[19]) then all the  $\sigma_j$  contribute almost equally to the total variance. So under this assumption the condition of equation 8.4.4 is met and

$$i(t_1) = \sum_{j=1}^N h_j$$

may be considered a gaussian distributed random variable with mean of 0 and variance

$$\sigma_{i(t_1)}^2 = \sum_{j=1}^N \sigma_j^2.$$

Based on this result it is possible to determine the distribution of the matched filter output random variable at the sample times, ie.

$$\Delta = H_1(\varphi_1) + H_2(\varphi_2) + \dots + H_m(\varphi_m) + \dots + H_N(\varphi_N). \quad (8.4.10)$$

Here each  $H_j(\varphi_j)$  corresponds to an input  $h_j$ .

Since the matched filter is a passive linear device a gaussian distributed input should produce a gaussian distributed output and

the mean will not change.  $i(t_1)$  has a zero mean and so that implies that  $\Delta$  will have a zero mean. The variance of  $\Delta$  ( $\sigma_\Delta^2$ ) can be found by noting

$$\sigma_\Delta^2 = \int_{-\infty}^{\infty} G_i(f) |M(f)|^2 df \quad (8.4.11)$$

where  $G_i(f)$  is the power spectral density of  $i(t)$  and  $M(f)$  is the matched filter transfer function. For this system (see equation 2.1.14)

$$M(f) = \frac{T}{2} \left\{ \text{sinc}[(f-f_c)T] e^{-j\pi(f+f_c)T} + \text{sinc}[(f+f_c)T] e^{-j\pi(f-f_c)T} \right\}$$

and

$$|M(f)|^2 = M(f) M(f)^* \quad (8.4.12)$$

$$\approx \left(\frac{T}{2}\right)^2 \left\{ \text{sinc}^2[T(f-f_c)] + \text{sinc}^2[T(f+f_c)] \right\}$$

if the cross product terms are neglected since the side lobes decay rapidly. Due to the orthogonality of the  $h_j(t)$  it is possible to get the PSD of  $i(t)$  by adding up the PSD of the  $h_j(t)$  [4]. So

$$G_i(f) = \sum_{j=1}^N G_{h_j}(f) \quad (8.4.13)$$

$$= \sum_{j=1}^N \left(\frac{B_j}{2}\right)^2 \left\{ \delta(f-f_j) + \delta(f+f_j) \right\}$$

Since  $G_i(f)$  and  $|M(f)|^2$  are even functions (by inspection),

$$\sigma_\Delta^2 = 2 \int_0^{\infty} G_i(f) |M(f)|^2 df \quad (8.4.14)$$

$$\approx \frac{T^2}{2} \int_0^{\infty} G_i(f) \text{sinc}^2[T(f-f_c)] df.$$

Using the properties of the dirac delta function gives

$$\sigma_{\Delta}^2 \approx \frac{T^2}{8} \sum_{j=1}^N B_j^2 \operatorname{sinc}^2 \left[ \frac{T(f_j - f_c)}{2} \right]. \quad (8.4.15)$$

To be practical one only includes those  $f_j$  falling in the main lobe of the  $\operatorname{sinc}^2(x)$  function. Assuming there are  $m$  such frequencies and using the variable  $\Delta_j = (\omega_j - \omega_c)$ ,

$$\sigma_{\Delta}^2 \approx \frac{1}{2} \left( \frac{T}{2} \right)^2 \sum_{j=1}^m B_j^2 \operatorname{sinc}^2 \left( \frac{\Delta_j T}{2\pi} \right). \quad (8.4.16)$$

Now using this approximation the decision variable  $U$  can be written

$$U = S + N(0, \sigma_{\Delta}^2) + N(0, \eta T/4). \quad (8.4.17)$$

It can be shown that the sum of two normally distributed random variables is also normally distributed [15]. ie. if

$$Z = X_1 + X_2 \quad (8.4.18)$$

where  $X_i$  is normally distributed with

$$\text{mean} = m_i \text{ and variance} = \sigma_i^2 \quad (8.4.19)$$

then  $Z$  is normally distributed with

$$\text{mean} = m = m_1 + m_2 \quad (8.4.20)$$

and

$$\text{variance} = \sigma^2 = \sigma_1^2 + \sigma_2^2. \quad (8.4.21)$$

So  $U$  can be written in simplified form as

$$U = S + N(0, \sigma_{\psi}^2) \quad (8.4.22)$$

where

$$\sigma_{\psi}^2 = \frac{\eta T}{4} + \frac{1}{2} \left(\frac{T}{2}\right)^2 \sum_{j=1}^m B_j^2 \operatorname{sinc}^2\left(\frac{\Delta_j T}{2\pi}\right). \quad (8.4.23)$$

### 8.5. Q Function Approximation

There is a very useful approximation for the Q function which can be derived from the following approximation for the "error function", erf(x) [1].

$$\text{erf}(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) \exp(-x^2) + \epsilon(x) \quad (8.5.1)$$

where

$$t = \frac{1}{1+px} \quad \text{and} \quad \epsilon(x) = \text{error term} \quad (8.5.2)$$

with

$$|\epsilon(x)| \leq 1.5 \times 10^{-7}$$

and with constants  $p = 0.3275911$ ,  $a_1 = 0.254829592$ ,  $a_2 = -0.284496736$ ,  $a_3 = 1.421413741$ ,  $a_4 = -1.453152027$ , and  $a_5 = 1.061405429$ . Note that by definition,

$$Q(z) \equiv \frac{1}{2} (1 - \text{erf}(z/\sqrt{2})). \quad (8.5.3)$$

So

$$Q(z) = \frac{1}{2} \left\{ (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) \exp(-z^2/2) + \epsilon(z/\sqrt{2}) \right\} \quad (8.5.4)$$

where now

$$t = \frac{1}{1+p(z/\sqrt{2})} \quad (8.5.5)$$

and the constants are the same as for the error function.