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Formulation of an Isotropic Plate with Temperature Dependent Material Elastic Properties and Its Design Application

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ABSTRACT

The material elastic properties, such as Young's modulus E and Poisson's ratio υ, vary as temperature changes. A classical plate theory has been modified to consider variable E and υ such that the resulting formulation is applicable to analyze the non-homogenous plate problem.

This paper recommends an essential approach in a process of plate formulation, which would decouple the existing available coupled plate formulation. As such, an analytical solution effort could be simplified and cost-effective. In addition, a realistic "margin of safety" could be established by a design evaluation procedure as presented in this paper.

1.0 INTRODUCTION

It is well known that the Young's modulus E and Poission's ratio υ vary as temperature changes. [1]. When there exists a temperature gradient across the plate thickness, its material elastic properties vary accordingly. A plate with such characteristics is classified as "non-homogenous plate." A design problem on variable material elastic properties could be found, for example, in reactor pressure vessel, and ASME Section III components installed in nuclear power plant facilities; and re-entry space vehicles. Because they are subjected to the effects of an elevated temperature field which is generated either by nuclear radiation or aerodynamic heating. Therefore, there is a need to modify the classical plate theory by considering the variable material properties instead of constant E and υ in its formulation such that a non-homogeneous plate problem could be appropriately analyzed by an applicable formulation.

Several different formulations have been developed for non-homogenous plate application [4, 5]. However, those existing available formulations are coupled for differential equations between lateral bending and in-plane stretching problems, as well as their corresponding boundary condition except where the entire edge is clamped. Such coupled conditions would cause analytical difficulty and time-consuming in obtaining a closed form solution.

The purpose of this paper is to present an essential approach in formulation process which would decouple the above-mentioned unfavorable coupled situation. Also, a design evaluation procedure based on a consideration of temperature-dependent allowable stress varying along the plate thickness is recommended.

An example is presented to illustrate the design application of the uncoupled formulation for a given actual distribution function of E(z) and υ(z). [7]. The "Margin of Safety" is obtained in accordance with the design evaluation procedure as recommended here. A comparison of the merits from various design evaluation approaches is also presented.

2.0 METHODOLOGY AND APPROACH

The formulation of a non-homogenous elastic thin plate, which is subjected to an arbitrary lateral load in a presence of thermal field, is based on a principle of stationary potential energy in conjunction with three-dimensional theory of elasticity. An application of calculus of variation procedure results in a governing differential equations for lateral bending and in-plane stretching problem, and their associated boundary conditions.

An essential approach in formulation process is to establish a condition which would decouple an otherwise coupled plate formulation.

3.0 FORMULATION OF A NON-HOMOGENOUS ELASTIC THIN PLATE

The principle of stationary potential energy is expressed mathematically as follows:

$$\delta \Pi = \delta (u - w) = o \qquad (3\text{-}1)$$

Eq. (3-1) states that an elastic system is in stable equilibrium condition when the potential energy (Π) of the system is a minimum; where

Π = Potential energy of the system;
U = The elastic strain energy, which is induced by mechanical and thermal load, stored in a deformed plate and
W = The work potential done by the external forces during deformation.

3.1 Establish The Expression Of Potential Energy Of The Elastic System.

3.1.1 The elastic strain energy (u) of the deformed plate is given as follows [1]

$$U = \iiint_V \tfrac{1}{2} (\sigma_{xx} \epsilon_{xx} + \sigma_{yy} \epsilon_{yy} + \sigma_{zz} \epsilon_{zz} + \sigma_{xy} \epsilon_{xy} + \sigma_{xz} \epsilon_{xz} + \sigma_{yz} \epsilon_{yz})\, dv + \iiint_V (\sigma_{xx}^T \epsilon_{xx} + \sigma_{yy}^T \epsilon_{yy} + \sigma_{zz}^T \epsilon_{zz})\, dv \qquad (3\text{-}2)$$

Where

σ_{ij} = the stress components induced by mechanical load;
ϵ_{ij} = strain components referred to the underformed state; and
σ_{ij}^T = the induced thermal stress components that occur when the elastic system is subjected to the thermal field T (x,y,z) and is restrained from thermal expansion on the external boundary.

a) Strain - displacement relationship for a classical elastic thin plate.

Let u (x, y, z), v (x, y, z) and w (x, y, z) be the displacement components at any point (x, y, z) of the plate, where x and y are the coordinates in the reference plane, and z is the coordinate along the thickness of the plate (h) as shown in Fig. I. The strain components are expressed as follows [2]:

$$
\begin{aligned}
\epsilon_{xx} &= u_{,x} & \epsilon_{xy} &= \epsilon_{yx} = u_{,y} + v_{,x} \\
\epsilon_{yy} &= v_{,y} & \epsilon_{yz} &= \epsilon_{zy} = v_{,z} + w_{,y} \\
\epsilon_{zz} &= w_{,z} & \epsilon_{zx} &= \epsilon_{xz} = w_{,x} + u_{,z}
\end{aligned}
\tag{3-3}
$$

b) The assumptions for the classical elastic thin plate are as follows:

i) $\sigma_{zz} << \sigma_{xx}, \quad \sigma_{zz} << \sigma_{yy}$ (3-4)

ii) $\epsilon_{zz} \cong 0$ (3-5)

This implies that

$w\,(x,y,z) \cong \hat{w}\,(x, y)$ (3-6)

iii) The plane section hypothesis leads to the expression of

$u\,(x,y,z) = \hat{u}\,(x,y) - z\,\hat{w}_{,x}\,(x,y)$ (3-7)

$v\,(x,y,z) = \hat{v}\,(x,y) - z\,\hat{w}_{,y}\,(x,y)$ (3-8)

and also implies that

$\epsilon_{xz} = 0, \epsilon_{yz} = 0$ (3-9)

where $\hat{u}$ (x,y), $\hat{v}$ (x,y) and $\hat{w}$ (x,y) are the displacements of the reference plane in x, y and z directions.

iv) Material follows Hook's Law

In view of Eq. (3-7) and (3-8) Eq. (3-3) become

$$
\begin{aligned}
\epsilon_{xx} &= \hat{u}_{,x} - z\hat{w}_{,xx} = \hat{\epsilon}_{xx} - z\hat{w}_{,xx} \\
\epsilon_{yy} &= \hat{v}_{,y} - z\hat{w}_{,yy} = \hat{\epsilon}_{yy} - z\hat{w}_{,yy} \\
\epsilon_{xy} &= \hat{u}_{,y} + \hat{v}_{,x} - 2z\hat{w}_{,xy} = \hat{\epsilon}_{xy} - 2z\hat{w}_{,xy}
\end{aligned}
\tag{3-10}
$$

Noting assumption of Eq. (3-4), and (3-5), σ_{ij} and σ_{ij}^T can be expressed as follows (p.p 256 in [1])

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\upsilon^2}\,(\epsilon_{xx} + \upsilon\,\epsilon_{yy}) \\
\sigma_{yy} &= \frac{E}{1-\upsilon^2}\,(\epsilon_{yy} + \upsilon\,\epsilon_{yy}) \\
\sigma_{xy} &= \frac{E}{2\,(1+\upsilon)}\,\epsilon_{xy} \\
\sigma_{xx}^T &= \sigma_{yy}^T = -\frac{E}{1-\upsilon^2}\,(1+\upsilon)\propto T
\end{aligned}
\tag{3-11}
$$

Where $\propto$ is the linear thermal coefficient and a function of x, y and z.
In view of Eq. (3-5) and (3-9), and substituting Eq. (3-11) into Eq. (3-2), we obtain

$$U = \tfrac{1}{2} \iiint_V \frac{E}{1-\upsilon^2} \left[\epsilon_{xx}^2 + \epsilon_{yy}^2 + 2\upsilon\, \epsilon_{xx}\, \epsilon_{yy} + \tfrac{1}{2}(1-\upsilon)\, \epsilon_{xy}^2\right] dv - \iiint_V \frac{E\alpha T}{1-\upsilon} (\epsilon_{xx} + \epsilon_{yy})\, dv \qquad (3\text{-}12)$$

Introducing Eq. (3-10) into (3-12), U can be expressed in terms of $\hat{\epsilon}_{ij}$ and $\hat{w}_{,ij}$ as follows

$$\begin{aligned}
U = {} & \tfrac{1}{2} \iiint_V \frac{E}{1-\upsilon^2} \left[\hat{\epsilon}^2_{xx} + 2\upsilon\, \hat{\epsilon}_{xx}\, \hat{\epsilon}_{yy} + \hat{\epsilon}^2_{yy} + \tfrac{1}{2}(1-\upsilon)\, \hat{\epsilon}^2_{xy}\right. \\
& - \iiint_V \frac{Ez}{1-\upsilon^2} \left[\hat{\epsilon}^2_{xx}\, \hat{w}_{,xx} + \hat{\epsilon}_{yy}\, \hat{w}_{,yy} + \hat{\epsilon}_{xy}\, \hat{w}_{,xy}\right)\, dv \\
& - \iiint_V \frac{\upsilon E z}{1-\upsilon^2} \left[\hat{\epsilon}_{xx}\, \hat{w}_{,yy} + \hat{\epsilon}_{yy}\, \hat{w}_{,xx} - \hat{\epsilon}_{xy}\, \hat{w}_{,xy}\right)\, dv \qquad (3\text{-}13) \\
& + \tfrac{1}{2} \iiint_V \frac{Ez^2}{1-\upsilon^2} \left[\hat{w}_{,xx}^2 + 2\upsilon \hat{w}_{,xx}\, \hat{w}_{,yy} + \hat{w}^2_{,yy}\right) + 2(1-\upsilon)\, \hat{w}_{,xy}^2\right] dv \\
& - \iiint_V \frac{E\alpha T}{1-\upsilon} \left[\hat{\epsilon}_{xx} - z\hat{w}_{,xx} + \hat{\epsilon}_{yy} - z\hat{w}_{,yy}\right)\, dv
\end{aligned}$$

Separating the volume integral into two parts, one with respect to the undeformed thickness (h) and the other with respect to the undeformed domain in the x,y plane, Eq. (3-13) becomes

$$\begin{aligned}
U = {} & \tfrac{1}{2} k_1 \iint_R (\hat{\epsilon}_{xx}^2 + \hat{\epsilon}_{yy}^2 + \tfrac{1}{2} \hat{\epsilon}_{xy}^2)\, dA + k_2 \iint_R (\hat{\epsilon}_{xx}\, \hat{\epsilon}_{yy} - \tfrac{1}{4} \hat{\epsilon}_{xy}^2)\, dA \\
& - k_3 \iint_R (\hat{\epsilon}_{xx}\, \hat{w}_{,xx} + \hat{\epsilon}_{yy}\, \hat{w}_{,xy} + \hat{\epsilon}_{,xy}\, \hat{w}_{,xy})\, dA - k_4 \iint_R (\hat{\epsilon}_{xx}\, \hat{w}_{,yy} + \hat{\epsilon}_{yy}\, \hat{w}_{,xx} - \hat{\epsilon}_{,xy}\, \hat{w}_{,xy})\, dA + \\
& + \tfrac{1}{2} D_1 \iint_R (\hat{w}_{,xx} + \hat{w}_{,yy})^2\, dA + D_2 \iint_R (\hat{w}_{,xy}^2 - \hat{w}_{,xx}\, \hat{w}_{,yy})\, dA \qquad (3\text{-}14) \\
& - \iint_R N_T (\hat{\epsilon}_{xx} + \hat{\epsilon}_{yy})\, dA + \iint_R M_T (\hat{W}_{,xx} + \hat{W}_{,yy})\, dA
\end{aligned}$$

where

$$\begin{aligned}
& k_1 = \int_{-z_o}^{h-z_o} \frac{E}{1-\upsilon^2}\, dz, \quad k_2 = \int_{-z_o}^{h-z_o} \frac{\upsilon E}{1-\upsilon^2}\, dz \\
& k_3 = \int_{-z_o}^{h-z_o} \frac{Ez}{1-\upsilon^2}\, dz, \quad k_4 = \int_{-z_o}^{h-z_o} \frac{\upsilon E z}{1-\upsilon^2}\, dz \qquad (3\text{-}15) \\
& D_1 = \int_{-z_o}^{h-z_o} \frac{Ez^2}{1-\upsilon^2}\, dz, \quad D_2 = \int_{-z_o}^{h-z_o} \frac{\upsilon E z^2}{1-\upsilon^2}\, dz \\
& N_T(x,y) = \int_{-z_o}^{h-z_o} \frac{E\alpha T}{1-\upsilon}\, dz, \quad M_T(x,y) = \int_{-z_o}^{h-z_o} \frac{E\alpha T z}{1-\upsilon}\, dz
\end{aligned}$$

Where h is the thickness of the plate, z_o is unknown constant which locates the position of reference plan (its determination will be discussed later).

3.1.2 Work potential (w) of the external force (pp 157 in [1])

$$W = \iint_R q\,(x,y)\,\hat{w}\,dA - \int_B M^{o}_{n}(s)\,\hat{w}_{,n}ds + \int_B V_n^{o}(s)\,\hat{w}\,ds + \int_B (N_{xn}\,\hat{u} + N_{yn}\,\hat{v})\,ds \tag{3-16}$$

Where $M_n^o(s)$ and $V_n^o(s)$ are the prescribed bending moment and vertical line force acting along the plate boundary; N^o_{xn} and N^o_{yn} are the prescribed horizontal in-plan forces in x and y directions and they can be expressed as

$$N^o{}_{xn} = N_{xx}\cos\alpha + N_{xy}\sin\alpha,\quad N^o{}_{yn} = N_{xy}\cos\alpha + N_{yy}\sin\alpha \tag{3-17}$$

The positive sense of above quantities is shown in Fig. 2 and Fig. 3.

3.2 Perform Calculus Variation to Obtain Plate Formulation

Substituting Eq. (3-14) and (3-16) into Eq. (3-1) we have the expression of the principle of the stationary potential energy, and performing the indicated variational procedure, the results are a set of differential equations and boundary conditions for different edge conditions, which are presented in the following. (For detailed variational procedure, see p.p 10-14 in [3].

3.2.1 Governing differential equations.

$$\left.\begin{aligned}&(k_1\,\hat{\epsilon}_{xx} + k_2\,\hat{\epsilon}_{yy} - k_3\,\nabla^2\hat{w} - N_T)_{,x} + \tfrac{1}{2}(k_1 - k_2)\,\hat{\epsilon}_{xy,y} = 0\\ &\tfrac{1}{2}(k_1 - k_2)\,\hat{\epsilon}_{xy,x} + (k_1\,\hat{\epsilon}_{yy} + k_2\,\hat{\epsilon}_{xx} - k_3\,\nabla^2\hat{w} - N_T)_{,y} = 0\end{aligned}\right\} \tag{3-18}$$

$$D_1\nabla^4\hat{w} + \nabla^2 M_T - k_3\,(\nabla^2\hat{u})_{,x} - k_3\,(\nabla^2\hat{v})_{,y} - q = 0 \tag{3-19}$$

Eqs. (3-18) and (3-19) are the differential equations for in-plane stretching and lateral bending problems. Note that they are coupled with each other, i.e, the in-plane stretching response affects that of the lateral bending and vice versa.

3.2.2 Boundary Conditions

The boundary conditions for clamped, simply supported and free edges can be found in pp. 13 and 14 of [3].

3.3 Decouple The Plate Formulation

The most essential approach of this paper is to select a reference plane location other than at the middle plane of the plate as used in classical plate theory and others [4,5], and as such it decouples the above plate formulation.

It is obvious that if we let $k_3 = 0$ then Eqs. (3-18) and (3-19) become uncoupled as follows:

$$\left.\begin{aligned}(k_1\,\hat{\epsilon}_{xx} + k_2\,\hat{\epsilon}_{yy} - N_T)_{,x} + \tfrac{1}{2}(k_1 - k_2)\,\hat{\epsilon}_{xy,y} = 0\\ \tfrac{1}{2}(k_1 - k_2)\,\hat{\epsilon}_{xy,x} + (k_1\,\hat{\epsilon}_{yy} + k_2\,\hat{\epsilon}_{xx} - N_T)_{,y} = 0\end{aligned}\right\} \quad (3\text{-}20)$$

$$D_1\nabla^4\,\hat{w} + \nabla^2 M_T - q = 0 \quad (3\text{-}21)$$

The location (z_o) of the reference plane is therefore determined from $k_3 = 0$ for a given $E(z)$ and $\upsilon(z)$. The boundary condition for different support cases will be simplified, because of $K_3 = 0$, which can be found in pp. 20, 21 and 23 of [3].

3.4 Simplify The Formulation For In Plane Stretching Problem

Follow the approach used in classical plate theory to introduce the Airy stress function $\phi(x,y)$ for in-plane forces and define

$$\tilde{N}_{xx} = \phi_{,yy},\ \tilde{N}_{yy} = \phi_{,xx},\ \tilde{N}_{xy} = \phi_{,xy} \quad (3\text{-}22)$$

Where
$$\begin{aligned}\tilde{N}_{xx} &= k_1\hat{\epsilon}_{xx} + k_2\,\hat{\epsilon}_{yy} - N_T\\ \tilde{N}_{yy} &= k_2\hat{\epsilon}_{xx} + k_1\hat{\epsilon}_{yy} - N_T\\ \tilde{N}_{xy} &= \tfrac{1}{2}(k_1 - k_2)\,\hat{\epsilon}_{xy}\end{aligned} \quad (3\text{-}23)$$

$\phi(x,y)$ satisfies Eq. (3-20), and the application of compatibility equation leads to

$$\nabla^4\phi + \frac{k_1 - k_2}{k_1}\nabla^2 N_T = 0 \quad (3\text{-}24)$$

Note that if the temperature T(z) varies along the plate thickness only, the Eq. (3-24) and (3-21) reduce to

$$\nabla^4\phi = 0 \quad (3\text{-}25)$$

$$D_1\nabla^4\hat{w} = q \quad (3\text{-}26)$$

For detailed derivation, refer to pp.23 ~ 29, in [3].

The boundary conditions can be expressed in term of $\phi(x,y)$ accordingly. See pp. 28 & 29 in [3].

4.0 DESIGN EVALUATION PROCEDURE

When the material elastic properties vary as temperature changes, so as their allowable stresses as documented in [7]. Such special characteristics of the actual variation of allowable stress in terms of temperature distribution along the plate thickness is recommended to be the basis for the present design evaluation procedure.

5.0 EXAMPLE

Problem: To determine the Max. Stress and Margin of Safety (MS) for an infinite strip of plate subjected to a uniform lateral load q under isothermal condition and simply supported at x - 0, a (Fig. 5). The material elastic properties are listed in Table 1.

Solution: The distribution function of E(z) and υ(z) for ASTM (240) can be expressed as

$$E(z) = E_o \left[1-0.25 \left(\frac{z}{h} + \frac{z_o}{h}\right)^2\right]$$

$$\upsilon(z) = \upsilon_o \left(1 + 0.49 \frac{z}{h}\right)$$

Where $E_o = 28{,}3$ ksi, $\upsilon_0 = 0.32$, $z_o = 0.4$, $h = 1$

The summary of results are tabulated in Table 2

$$M.S = \frac{\text{Allowable}}{\text{Max. Stress}} = \frac{3\, s_m}{\sigma_1 - \sigma_2} = \frac{3 \times 13.5}{15} = 2.7 \text{ at } T = 700°F.$$

6.0 CONCLUSION

The formulation as presented in this paper is attractive and distinguishable from other available formulations because:

1) Uncoupled formulation simplifies analytical solution efforts.

2) The differential equation for the in-plane stretching problem is homogenous even if there is lateral loading versus a non-homogenous differential equation from other available formulations. [4,5] and

3) The design evaluation procedure recommended here would provide a realistic "Margin of Safety"

Table 1. E(z) and υ(z) v.s. Temperature

T°F	75°	300°	500°	700°	Ref
E(z),ksi	28.3	27.4	26.6	25.4	[7] p.43
υ(z)	0.241	0.292	0.304	0.323	[7] p.43
Sm(ksi)	16.7	16.7	14.8	13.5	[6] p.19

Table 2. Summary of Results

in-plane stretching problem	D.E	B.C at x = o,a	Stress at x = a/2, z = 0.6
	$\nabla^4\phi = 0$	$\phi_{,xx} = \phi_{,xy} = 0$	$N_{xx} = N_{yy} = N_{xy} = 0$
lateral bending	$D_1\nabla^4\hat{w} = 0$	$w = 0$ $D_1\hat{w}_{,xx} = 0$	$\sigma_{xx}{}' = 21.19$ ksi $\sigma_{yy}{}' = 6.77$ ksi For $a = 8' - 0$, $q = 1^{\#}/_{in}{}^2$

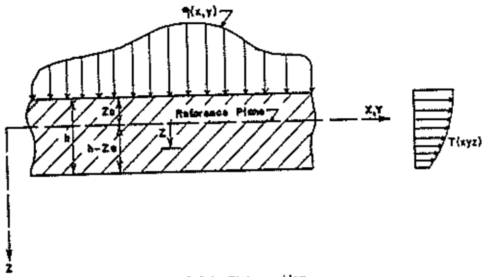

FIG. 1 Plate problem

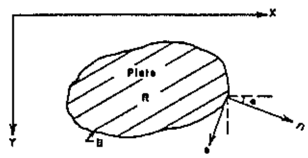

FIG. 2 Plate domain

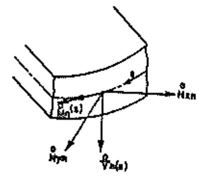

FIG. 3 Boundary forces

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