Abstract

LING, XIAOLI. Pooled versus Reserved Inventory Policies in a Two-echelon Supply Chain.(Under the direction of Dr. Shu-Cheng Fang, Dr. Henry L.W. Nuttle and Dr. Xiuli Chao.)

We consider a two-echelon supply chain with two retailers and one supplier. The retailers order from the supplier who makes all the decisions and bears all the inventory risk. Throughout this thesis, we consider two different inventory systems: a reserved inventory system and a pooled inventory system. With the reserved inventory system, the supplier keeps separate inventories for each retailer. In contrast, the pooled inventory is shared by the two retailers, and the supplier makes the inventory decision based on the joint demand.

First, assuming a fixed wholesale price, we analyze the supplier's decisions in the reserved and the pooled inventory systems. We compare the profit of the supplier and retailers in the two systems under normally distributed demands. We also analyze the scenarios in which the retailers have service level requirements. We find that regardless of whether retailers have service level requirements, under the normally distributed demands, the profits of the supplier and retailers all increase after pooling inventory.

Then, we analyze the scenarios in which the wholesale prices are supplier's decision variables and demand is a function of the wholesale price. We analyze both the additive and multiplicative demands models. We again analyze the supplier's decision on the inventory levels and wholesale price with and without service level requirements. We also compare the profits of the retailers and supplier for the different policies under normally distributed demands. We find that the supplier's profit always increases after pooling the inventory while the retailers's profit may increase or decrease, depending on the system parameters.

This thesis first studies the single period model, and then extends the results to finite

and infinite horizon problem with either backlogging or lost sales assumptions. We give mathematical analysis of the pooled and the reserved inventory systems for the infinite horizon case and use simulation methods to compare these two different inventory systems for the finite horizon case.

POOLED VERSUS RESERVED INVENTORY POLICIES IN A TWO-ECHELON SUPPLY CHAIN

by
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Biography

Xiaoli, Ling was born in Yuhang, China on Jan 28, 1975. She entered the Department of Automation at Tsinghua University in 1993 and obtained her B.S. and M.S. in Automation in 1998 and 2001, respectively. In August 2001, she started her Ph.D. study in Industrial Engineering at North Carolina State University.

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Chapter 1

Introduction

1.1 Background

Supply chain management is about matching supply and demand, particularly with inventory management. Too much supply leads to inefficient investment and needless handling cost, while too little supply generates lost sales. The former is the inventory risk while the latter is the supply risk. In reality, most supply chains cannot match supply and demand perfectly. All of the firms in a supply chain will bear some supply risks, but firms can avoid or the decrease inventory risk.

Consider an electronics manufacturing service provider (EMS), who holds inventory of cpu chips for two or more original equipment manufacturers (OEM). The current inventory policy dictated by the OEM is to keep each company's inventory physically separate (reserved inventory.) Is this the most profitable inventory policy for the EMS? Is it the most profitable inventory policy for the OEMs?

In general, in this research, we are interested in knowing whether a supplier should pool inventory or reserve separate inventory for customers. If so, is this policy also beneficial for retailers? Additionally, how about when the retailers have service level requirements? We explore those questions for a two-echelon supply chain.

We consider a supply chain for a single product with a single supplier and two retailers as shown in Figure 1.1. In Chapters 1-5 we focus on one single period or selling season. In Chapters 6 and 7 we examine infinite and finite horizon scenarios. We associate customer region with each retailer and model customer demands as uncertain. During the selling season, retailers receive orders from the customers, place an order to the supplier and receive product for which they pay a unit wholesale price without delay. The supplier manufactures product and holds it in inventory at his own expense until an order comes from the retailers, i.e, the supplier bears all the inventory risks. The supplier has only one chance to produce before the season starts. When a stock-out occurs at the supplier, sales are lost. The objective of the supplier is to maximize his single-period profit. Profits of retailers are maximized when they receive their full order. However, they do not have control over inventory decisions.

In this thesis, we will look at two different inventory policies for the supplier:

1. Reserved inventory policy: the supplier holds, separately, reserved inventories for retailers 1 and 2;
2. Pooled inventory policy: the supplier holds one central inventory, which is shared by the two retailers.

For the reserved inventory system, at the beginning of the period, the supplier stocks x_1, x_2 respectively for retailers 1 and 2 at manufacturing cost c per unit. After retailer i $(i = 1, 2)$ observes local demand, she places an order with the supplier. Retailer i receives the inventory immediately and pays a wholesale price w_i $(w_i > c > 0)$ for each unit received. Let m_i be the markup on the wholesale price retailer i $(i = 1, 2)$ charge, i.e., the retail price is $p_i = w_i + m_i$ at retailer i. If the stock x_i of the supplier cannot satisfy the order from retailer i, the unsatisfied portion results in lost sales. The supplier takes on the task of doing inventory replenishment and bears the inventory risk. Units remaining at the end of the season are disposed at unit cost h $(|h| \leq c)$. Note that h may be negative, in which case

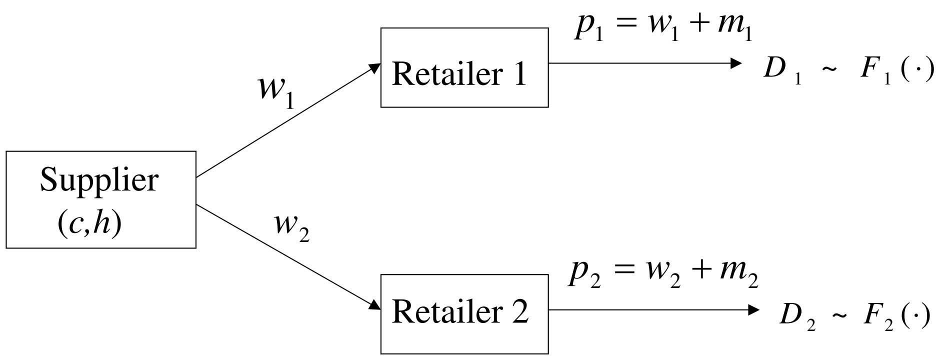

Figure 1.1: Two-echelon Supply Chain with One Supplier and Two Retailers

it represents a per-unit salvage value.

For the pooled inventory system, the supplier only has one central distribution center, and the two retailers share the stock at this central distribution center. At the beginning of the period, the supplier stocks x_p for retailers 1 and 2 with manufacturing cost c per unit. After the retailers observe their demands, they place orders with the supplier and pay a wholesale price w_p for each unit received. If the stock x_p of the supplier cannot satisfy the combined order, the unmet portion of the order is lost sales. We assume the markup is m_p for both of the retailers. In the pooled inventory case, when inventory cannot satisfy the total demand, the supplier needs to allocate the product to the retailers There are a number of papers discussing inventory allocation for difference scenarios (Cachon and Lariviere 1999). Our model focuses on the impact of the different policies on the profit of the supplier and the total profit of the retailers. Hence, we regard the two retailers as one joint retailer and thus need not consider the allocation policy in detail.

We denote retailer 1's and retailer 2's demands as D_1 and D_2, respectively. D_1 and D_2 are random variables with independent distributions. Let $F_i(\cdot)$ and $f_i(\cdot)$ denote the CDF and PDF of demand D_i $(i = 1, 2)$, respectively. Let D_p be the joint demand for the retailers with PDF $f_p(\cdot)$ and CDF $F_p(\cdot)$. Note that the joint demand $D_p = D_1 + D_2$, and $F_p(\cdot)$ is the convolution of $F_1(\cdot)$ and $F_2(\cdot)$.

In this thesis, we will analyze the scenarios in which the supplier charges the retailers a fixed wholesale price and scenarios in which the wholesale prices are the supplier's decision variables. With the fixed wholesale price, the parameters of the demands are exogenous to the system. However with the wholesale price as variables, we assume all the markup are the same, i.e., $m_1 = m_2 = m_p = m$ and the demands at each retailer are treated as functions of the wholesale prices charged the retailers by the supplier.

Penalty costs associated with shortages are often hard to estimate with accuracy. It is therefore common practice for the supplier to try to maximize his profit while satisfying a minimum service level requirement for each retailer. Thus the service level requirement represents implicit shortage costs, e.g., loss of goodwill. Throughout this thesis, the service level requirements are measured by the probability of no stock-out. We denote ρ_i as the minimum acceptable probability of no stock-out for retailer i in the reserved inventory case and ρ_p as the minimum acceptable probability of meeting the retailers' joint demand in the pooled inventory case.

1.2 Objective of the Thesis

As noted in the previous section, the reserved inventory scenario and the pooled inventory scenario are two important scenarios for supply chains in which the supplier risks the disposal cost. There is need for a more comprehensive study, particularly regarding the decision of the supplier under different scenarios. Therefore, the main objective of this thesis will be to model, analyze, and compare the impact of the reserved and pooled inventory policies.

More specifically, the aims of this thesis are:

1. To formulate problems involving the reserved inventory and pooled inventory system

including scenarios in which

- Wholesale prices are fixed,
    - Retailers have no service level requirements,
    - Retailers have service level requirements,
- Wholesale prices are decision variables,
    - Retailers have no service level requirements,
    - Retailers have service level requirements.
- Study the problem under the infinite horizon and the finite horizon.

2. To provide conditions that guarantee the existence and uniqueness of optimal solutions for these cases in which wholesale prices are decision variables.

3. To compare the results and impact of the reserved inventory and the pooled inventory policies under the different scenarios such as wholesale price, service level requirements.

1.3 Outline

This thesis is organized as follows

Chapter 2 gives a brief literature review of related work.

Chapter 3 focuses on the case in which the wholesale price is fixed. Several models are provided and the results under the pooled inventory and reserved inventory policies are compared.

In Chapter 4, we analyze the pooled inventory and reserved inventory models under the additive demand model. The optimality conditions and results are stated. Comparative results of the pooled inventory case and the reserved inventory case are stated.

In Chapter 5, we provide similar analysis of the systems under multiplicative demand model. The optimality conditions and results are stated. Comparative results of the pooled inventory case and the reserved inventory case are stated.

In Chapter 6, we extend the analysis to the infinite horizon problem and provide comparative results of the pooled inventory case and the reserved inventory case are stated.

In Chapter 7, we use simulation methods to analyze the systems over a finite horizon. Comparative results of the pooled inventory case and the reserved inventory case are stated.

Finally, Chapter 8 summarizes all of our results to date, and states our future research directions.

Chapter 2

Literature Review

This chapter provides a brief overview of literature on models involving inventory pooling and single product and single period models involving pricing decision.

2.1 Inventory Pooling

A key aspect of our research is the analysis of the effect of pooling inventory in the supply chain system. The literature on inventory pooling can be classified into three categories: component commonality; inventory transshipment in single echelon supply chains; inventory pooling in multi-echelons supply chain.

2.1.1 Component Commonality

If end products share common components, safety stock can be reduced and service levels maintained by pooling inventory of the common parts. The work-to-date on component commonality concentrates mainly on changes in safety inventory levels and does not consider the benefit of pooling to the suppliers and the retailers in the supply chain. Baker, Magazine, and Nuttle (1986) study a two product system with service level constraints where the objective is to minimize the total safety stock. They show that total safety stock drops after pooling while total stock of specialized parts increases. Gerchak, Magazine and Gamble

(1988) extend these results to a profit maximization setting. Finally, Gerchak and Henig (1986) analyze a model in a multi-period setting and resolve the optimal policy for the infinite horizon models.

2.1.2 Inventory Transshipment in Single Echelon Supply Chains

Inventory transshipment involves transferring inventory from one member to another of the same echelon of a supply chain in the event of a stock-out at the latter. This work is different from our work in that it concentrates on one echelon only.

The most relevant papers in this stream are those of Rudi, Kapur and Pyke (2001) and Dong and Rudi (2002) in which both the retailers' and supplier's profits are considered. Transshipment creates a virtual centralization of the inventory by utilizing the benefit of inventory pooling within the same inventory echelon. Seifert and Thonemann (1999) and Seifer et al. (2001) model single-directional transshipment from physical to internet retailers. Anupindi et al. (2001) consider a very general decentralized transshipment model where multiple retailers not only stock inventory internally but also jointly stock it at multiple, jointly owned warehouse locations, which is similar to Anupindi and Bassok (1999b).

2.1.3 Inventory Pooling in Multi-echelon Supply Chains

There are several papers that, like our work, investigate the benefits of pooling inventory in supply chain with more than one echelon. Anupindi and Bassok (1999a) consider a two-level supply chain with a single supplier and two retailers. Unlike our model, the inventory decision is made by the retailers without service level requirements, and the retailers bear all the inventory risk. They model a system in which a fraction of the customers are willing to wait for a delivery from another retailer (market search). They show that under this setting, the manufacturer may not always benefit from inventory pooling because total sales may drop. They also discuss the possibility of optimizing wholesale prices or introducing holding cost subsidies as methods for coordinating the supply chain. However, in their model, demand is exogenous and not sensitive to price.

As in our work, Netessine and Rudi (2003) consider two supply chain strategies, traditional operation and drop shipping. With traditional operation, the retailer holds the inventory purchased from the supplier, while with drop shipping, the supplier holds the inventory. Although they also consider a two-echelon system, the second echelon consists of a collection of identical retailers. The retailers are only intermediaries between the end customer and the supplier and function as a single joint retailer. Netessine and Rudi compare the traditional channel and drop shipping strategy under normally distributed demands and find that the supply chain's profit may be higher or lower with drop shipping relative to the traditional strategy.

Cachon (2004) considers the "push contract," in which the retailer bears all the inventory risk and the "pull contract," in which the supplier bears all the inventory risk because only the supplier holds inventory. The retailer replenishes as needed during the season. His study focuses on identifying Pareto-optimal price-only contracts and studies supply chain efficiency under such contracts. However, since there is only one retailer, the benefits of inventory pooling are not reflected and in addition he only considers the case of exogenous demands .

Of the existing literature, the work that is closest to ours is that of John Bartholdi, and Kemahlioglu (2003). They consider two retailers whose inventory is provided by a common supplier who bears all the inventory risk. They find that the total system profit will increase after pooling the inventory. In addition, using the Shapely value to allocate the additional profit, they analyze various schemes by which the supplier may pool inventory. By allocating the Shapely value, they could coordinate the whole supply chain. However, they only consider the scenarios in which the wholesale price is fixed and all the demands are not price-sensitive. We will show the optimal inventory and pricing policies when the wholesale prices are decision variables under the reserved and the pooled inventory systems. We also will analyze the comparative results for these scenarios.

2.2 Pricing Decision in Single-period Models

In our work to date, we have focused on single-product, single-period scenarios, first in which prices are fixed and later in which wholesale prices are decision variables. In this section, we discuss the literature on single production during single period models in which wholesale prices are decision variables. The single-product, single-period newsboy problem has a rich history which can be traced back to Edgeworth (1888). In its basic formulation, a decision maker facing random demand for a product for one period must decide how many units of the product to stock in order to maximize his expected profit. The optimal solution to this problem is to strike a balance between the expected shortage cost and leftover cost. Porteus (1990) provides an excellent review. Typically, the focus of the extensive literature is on operational efficiency to minimize expected cost. Parameters such as market price and wholesale price are usually exogenous and fixed.

Whitin (1955) was the first to formulate a newsboy model with price effects. In his model, selling price and stocking quantity are set simultaneously. Whitin assumes that the demand is a random variable depending on the unit-selling price. However, the price is a decision variable rather than an external parameter. Mills (1959) refines the formulation by explicitly specifying mean demand as a function of selling price. Subsequent works by Zabel (1970) and Young (1978) revisited the same single-period model under alternative specifications of the demand function. All of these works focus on single-echelon problem.

Another relevant stream of literature on pricing policy is in channel coordination. The most relevant paper in this stream is Wang et al. (2004). They consider a two-echelon supply chain with one supplier and one retailer where the supplier decides on the retail price and delivery quantity for his product, and for each unit sold, the retailer deducts a percentage from the selling price and remits the balance to the supplier. However, since there is only one retailer, there is no pooled inventory analysis. Bernstein and Federgruen (2001) investigate a two-echelon supply chain with a single supplier servicing a network of competing retailers. They assume that any retailers' random demands depend not only on its own retail price but on those of the other retailers as well. The retailers make all the decisions. Although there is more than one retailer in the second echelon, the authors focus on how to coordinate the whole supply chain system and do not consider the pooled inventory.

Chapter 3

Problem with Fixed Wholesale Price

In the next three chapters, we analyze and compare the impact of the inventory pooling for a two-echelon supply chain as is described in Chapter 1. Here, we assume that the demands at the retailers are exogenous. We denote retailer 1's and retailer 2's demands as D_1 and D_2, respectively. D_1 and D_2 are independent random variables. Let $F_i(\cdot)$ and $f_i(\cdot)$ denote the CDF and PDF of D_i ($i = 1, 2$), respectively. We assume that D_1 and D_2 are nonnegative continuous random variables. Continuity of demand is a common abstraction that is used to simplify the analysis since in practice demand is discrete. Let D_p be the joint demand for the retailers with PDF $f_p(\cdot)$ and CDF $F_p(\cdot)$. Note that the joint demand $D_p = D_1 + D_2$, and $F_p(\cdot)$ is the convolution of $F_1(\cdot)$ and $F_2(\cdot)$.

We present and analyze the decisions of the supplier for both the reserved inventory policy and the pooled inventory policy. We also compare the effect on the profit of the supplier and the total expected sales and profit of the retailers of those two different policies when demands are normally distributed. For each policy, we consider scenarios under the following conditions

- Retailers have no service level requirements,
- Retailers have service level requirements.

3.1 Retailers Have No Service Level Requirements

In this section, we will analyze the reserved inventory case and the pooled inventory case when the retailers do not impose service level requirements on the supplier.

3.1.1 Reserved Inventory Case

For the reserved inventory case, the supplier keeps two separate inventories, one for each retailer. The objective of the supplier is to maximize his expected profit, which is defined as the expected revenue less the expected disposal and manufacturing costs. Let $\Pi_r(x_1, x_2)$ denote the supplier's expected profit for any chosen inventory levels (x_1, x_2), which can be written as

$$\begin{aligned} \Pi_r(x_1, x_2) \quad &= \quad E[w_1 \min(x_1, D_1) - h(x_1 - D_1)^+ - cx_1 \\ &\quad\quad +w_2 \min(x_2, D_2) - h(x_2 - D_2)^+ - cx_2], \end{aligned}$$

and the optimization problem for the supplier is

$$\max_{x_1 \geq 0, x_2 \geq 0} \Pi_r(x_1, x_2).$$

Recall that D_1 and D_2 are independent random variables. Hence, we can rewrite $\Pi_r(x_1, x_2)$ as

$$\Pi_r(x_1, x_2) \quad = \quad \Pi_{r1}(x_1) + \Pi_{r1}(x_2),$$

where

$$\Pi_{r1}(x_1) = E[w_1 \min(x_1, D_1) - h(x_1 - D_1)^+ - cx_1],$$

and

$$\Pi_{r2}(x_2) = E[w_2 \min(x_2, D_2) - h(x_2 - D_2)^+ - cx_2].$$

Hence, the optimization problem for the supplier can be separated into two separate problems, one for each retailer. For retailer i, the supplier sets the inventory level to maximize his profit by solving the following problem

$$\max_{x_i \geq 0} \Pi_{ri}(x_i), \qquad i = 1, 2. \tag{3.1}$$

Let x_i^* be the optimal solution of problem (3.1). Since there is no service level constraint, the analysis of the inventory decision and the profit of the supplier are similar to that of a single location newsvendor problem(see Silver 2000). It is well known that the profit-maximizing inventory levels for the supplier are

$$x_i^* = F_i^{-1}(\frac{w_i - c}{w_i + h}).$$

Since $|h| \leq c$ and $w > c > 0$, hence $0 \leq \frac{w_i - c}{w_i + h} \leq 1$, which indicates that $F_i^{-1}(\frac{w_i-c}{w_i+h})$ is feasible. In addition, due to the nonnegativity of demand, $F_i^{-1}(\cdot)$ is nonnegative. Thus $x_i^* \geq 0$. Since the retailers do not hold inventory, their profits are proportional to the expected sales. Let $\pi_{ri}(x_i)$ be the profit of retailer i ($i = 1, 2$) when the supplier keeps inventory level x_i for her. We have

$$\pi_{ri}(x_i) = E[m_i \min(x_i, D_i)], \quad i = 1, 2.$$

We use π_{r1}^*, π_{r2}^* and Π_r^* to denote the optimal profit of retailer 1, retailer 2 and the supplier, respectively, when the supplier holds optimal inventory level x_i^* for retailer i. In addition, we define the retailers' total expected retail profit π_r^* as $\pi_{r1}^* + \pi_{r2}^*$.

3.1.2 Pooled Inventory Case

For the pooled inventory system, there is only one distribution center from which the two retailers share the stock. Recall that we let x_p be the total inventory level and D_p be the joint demand at the retailers. The supplier charges retailers w_p for each unit that retailers received. The markup price is m_p fir both of the retailers. Let $\Pi_p(x_p)$ be the supplier's expected profit when the inventory level is chosen as x_p in the pooled inventory case. We have

$$\Pi_p(x_p) \quad = \quad E[w_p \min(x_p, D_p) - h(x_p - D_p)^+ - cx_p].$$

The supplier sets the inventory level to x_p^* so as to maximize his profit by solving the following problem

$$\max_{x_p \geq 0} \Pi_p(x_p). \tag{3.2}$$

Since this problem is equivalent to problem (3.1), hence the optimal stocking policy is

$$x_p^* = F_p^{-1}(\frac{w_p - c}{w_p + h}).$$

Let $\pi_p(x_p)$ be the expected total retail profit given the supplier's inventory level x_p, which can be written as

$$\pi_p(x_p) = E[m_p \min(x_p, D_p)].$$

Similarly, when the supplier's inventory level is x_p^*, we use π_p^* to denote the total optimal profit of the retailers and Π_p^* to denote the optimal profit of the supplier.

3.1.3 Comparative Results

In general, we cannot say whether the retailer's profit will increase or decrease after pooling inventory. With the pooled inventory case, retailers' profits may decrease because the total inventory in the supply chain usually decreases. Gerchak and Mossamn (1992) derive conditions which include the demand distributions and cost parameters, under which pooling inventory may result in increased inventory. The phenomenon of retailers' profits decreasing was first observed by Anupindi and Bassok (1999a) in a different setting, where the retailers pay the holding cost, and it is their decision whether to pool inventory or not. We illustrate that the retailers' total expected profit can decrease after pooling the inventory with the following example.

Example 3.1.1 *Consider a system with two retailers and assume that demands D_1 and D_2 at the retailers are independent and uniformly distributed between [0,100]. The CDF of the demand at each retailer is*

$$F(u) = \begin{cases} \frac{u}{100} & 0 \leq u < 100 \\ 1 & u \geq 100. \end{cases}$$

Let $D_p = D_1 + D_2$, then the CDF of the random variable D_p is

$$F_p(u) = \begin{cases} (\frac{u^2}{10000}) - \frac{u^2}{20000} & 0 \leq u < 100 \\ -\frac{u^2}{20000} + \frac{2u}{100} - 1 & 100 \leq u < 200 \\ 1 & u \geq 200. \end{cases}$$

If the critical ratios $\frac{w_1 - c}{w_1 + h} = \frac{w_2 - c}{w_2 + h} = \frac{w_p - c}{w_p + h} = 0.8$, then in the reserved inventory case, the optimal inventory levels are

$$x_1^* = x_2^* = F^{-1}(0.8) = 80,$$

the expected sales at each retailer is 48, and the total expected sales is 96. In the pooled inventory case, $x_p^* = F_p^{-1}(0.8) = 137$, and the expected total sales is 95. Note that the expected profit of the retailer is proportional to the expected sales. Therefore, the total expected profit of the retailers drops after pooling the inventory.

Due to its mathematical tractability, the normal distribution appears to be the distribution of choice in modelling multi-location inventory problems. In addition, many random distributions can be approximated by the normal distribution. Although the range of the normal distributed variable is from $-\infty$ to $+\infty$, the mean value of the normal distribution is large enough relative to its variance. In this case, the relative demand values will almost surely be nonnegative. Alfaro and Corbett (2003) perform a simulation study of the pooling effect, comparing the impact of the normal distribution with several nonnormal distributions. They conclude that the effect of pooling does not vary much between the different distributions.

Suppose D_1 and D_2 are independently distributed normal random variables with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. Let $\Phi(\cdot)$ denote the CDF and $\phi(\cdot)$ the PDF of the standard normal distribution. In addition, we denote by $R(\cdot)$ the right-hand unit normal linear loss function, which is defined as follows (see Zipkin 2000)

$$R(x) = \int_x^\infty (u - x)\phi(u)du.$$

From Zipkin, we know that $R(x)$ is a nonnegative and nonincreasing function of x, and

$$R(x) + x \geq 0 \text{ for any } x. \tag{3.3}$$

The following theorems characterize the impact of the two policies on the profits of the supplier and retailers.

Theorem 3.1.1 *Assume that $w_1 = w_2 = w_p = w$. If D_1 and D_2 are independently and normally distributed random variables, the supplier's optimal profit is increased when the inventory is pooled, i.e., $\Pi_p^* \geq \Pi_r^*$.*

Proof. Under the reserved inventory scenario, the optimal inventory levels are given by

$$x_1^* = \mu_1 + \sigma_1 \Phi^{-1}(\frac{w-c}{w+h}),$$

and

$$x_2^* = \mu_2 + \sigma_2 \Phi^{-1}(\frac{w-c}{w+h}).$$

For the pooled inventory scenario, $D_p \sim N(\mu_1+\mu_2, \sqrt{\sigma_1^2+\sigma_2^2})$. Hence the optimal total inventory level for the pooled inventory scenario is

$$x_p^* = \mu_1 + \mu_2 + \sqrt{\sigma_1^2+\sigma_2^2}\Phi^{-1}(\frac{w-c}{w+h}).$$

However the excess stock is $(x_i^* - D_i)^+ = x_i^* - \min(x_i^*, D_i),\ i = 1, 2$. Hence the supplier's optimal profits in the reserved inventory case and the pooled inventory case are

$$\begin{aligned}
\Pi_r^* &= (w+h)E[(\min(x_1^*, D_1) + \min(x_2^*, D_2)] - (h+c)(x_1^* + x_2^*), \\
&= (w+h)\left(\mu_1 + \mu_2 - (\sigma_1+\sigma_2)R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right)\right) \\
&\quad -(h+c)\left(\mu_1+\mu_2+(\sigma_1+\sigma_2)\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
\Pi_p^* &= (w+h)E[\min(x_p^*, D_p)\} - (h+c)x_p^*] \\
&= (w+h)\left(\mu_1+\mu_2-\sqrt{\sigma_1^2+\sigma_2^2}R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right)\right) \\
&\quad -(h+c)\left(\mu_1+\mu_2+\sqrt{\sigma_1^2+\sigma_2^2}\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right).
\end{aligned}$$

Calculating the difference of Π_r^* and Π_p^*, we have

$$\begin{aligned}
\Pi_r^* - \Pi_p^* &= (w+h)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right) R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right) \\
&\quad +(h+c)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right)\Phi^{-1}\left(\frac{w-c}{w+h}\right) \\
&\leq (c+h)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right) R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right) \\
&\quad +(h+c)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right)\Phi^{-1}\left(\frac{w-c}{w+h}\right) \\
&= (c+h)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right)\left(R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right)+\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right) \\
&\leq 0.
\end{aligned}$$

The first inequality follows from the fact that $w > c$, $R(\cdot) > 0$ and the inequality $\sqrt{\sigma_1^2+\sigma_2^2} \leq \sigma_1+\sigma_2$ for any positive σ_1 and σ_2. The second inequality is seen by applying equation (3.3). Hence, the supplier's profit is increased after pooling the inventory, i.e., $\Pi_r^* \leq \Pi_p^*$. ■

Theorem 3.1.2 *Assume that $w_1 = w_2 = w_p = w$ and $m_1 = m_2 = m_p = m$. If D_1 and D_2 are independently and normally distributed random variables, the retailers' total expected profit is increased when the inventory is pooled, i.e., $\pi_p^* \geq \pi_r^*$.*

Proof. The expected profits of the retailers 1 and 2 in the reserved inventory case are given by

$$\pi_i^* = mE\min(x_i^*, D_i), \quad i = 1, 2,$$

and the total expected profit of the retailers in the pooled inventory case is given by

$$\pi_p^* = mE\min(x_p^*, D_p).$$

For the normally distributed demands, we have

$$\pi_{ri}^* = m\left(\mu_i - \sigma_i R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right)\right), \quad i = 1, 2,$$

and

$$\pi_p^* = m\left(\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2} R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right)\right).$$

Therefore,

$$\begin{aligned} \pi_p^* - \pi_r^* &= \pi_p^* - \pi_{r1}^* - \pi_{r2}^* \\ &= m\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right) R\left(\Phi^{-1}\left(\frac{w-c}{w+h}\right)\right). \end{aligned}$$

Since $R(\cdot)$ is nonnegative and the inequality $\sigma_1 + \sigma_2 \geq \sqrt{\sigma_1^2 + \sigma_2^2}$, we have

$$\pi_p^* - \pi_r^* \geq 0.$$

Hence the total expected profit is increased after pooling the inventory. ■

Theorem 3.1.3 *Assume that $w_1 = w_2 = w_p = w$, $m_1 = m_2 = m_p = m$. If D_1 and D_2 are independently normally distributed random variables, and the critical ratio $\frac{w-c}{w+h} \geq 0.5$, then the supplier's optimal inventory level in the pooled inventory case is less than the optimal total inventory levels in the reserved inventory case, i.e., $x_p^* \leq x_1^* + x_2^*$. Otherwise, $x_p^* > x_1^* + x_2^*$.*

Proof. We know that the optimal total inventory level in the reserved inventory case is given by

$$x_1^* + x_2^* = \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}(\frac{w-c}{w+h}).$$

However the optimal inventory level in the pooled inventory case is given by

$$x_p^* = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}(\frac{w-c}{w+h}).$$

The fact that $\Phi^{-1}(\cdot)$ is a monotone nondecreasing function and $\Phi^{-1}(0.5) = 0$ implies that $\Phi^{-1}(\frac{w-c}{w+h}) \geq 0$ provided the critical ratio $\frac{w-c}{w+h} \geq 0.5$. Hence $x_p^* \leq x_1^* + x_2^*$ follows the inequality $\sigma_1 + \sigma_2 \geq \sqrt{\sigma_1^2 + \sigma_2^2}$. Otherwise, when $\frac{w-c}{w+h} < 0.5$, we have $x_p^* > x_1^* + x_2^*$. ■

Under normally distributed demand, when the profit margin of the supplier $w - c$ is large or when the disposal cost h is small relative to the wholesale price w, the supplier's total optimal inventory levels in both the reserved inventory and the pooled inventory cases are greater than the mean value of the demands, and the total optimal inventory level drops after pooling the inventory. When the profit margin of the supplier $w - c$ is small or when the disposal cost h is large relative to the wholesale price w, the supplier's total optimal inventory levels in both the reserved inventory and the pooled inventory cases are less than the mean value of the demands, and the total optimal inventory level increases after pooling the inventory. However, the profit of the supplier is always increased after pooling the inventory. Although the supplier may keep less inventory in the pooled inventory case, the retailer's total expected sales will increase because the probability of stock-out drops due to sharing the inventory. Hence, pooling inventory is beneficial for both the supplier and the retailers.

3.2 Retailers Have Service Level Requirements

In this section, we consider scenarios in which the wholesale price is fixed but the retailers impose service level requirements on the supplier. Throughout this section, we use Type-1 service, i.e., we use the probability of no stock-out as the service level. In Section 3.2.1, we consider the reserved inventory case. We analyze the pooled inventory case in Section 3.2.2. Comparative results are presented in Section 3.2.3.

3.2.1 Reserved Inventory System

In this scenario, the retailers are powerful enough to require the supplier to use a reserved-inventory policy, i.e., the supplier maintains separate inventory levels (x_1, x_2) for retailers 1 and 2, respectively. In addition each retailer has a minimum service level requirement. We assume that the retailers 1 and 2 have the minimum service level requirements ρ_1 and ρ_2, respectively. Let D_1 and D_2 be independent random variables with CDF $F_1(\cdot)$ and $F_2(\cdot)$ that represent the demands at retailers 1 and 2, respectively. Then given the inventory levels x_1 and x_2, the probability of no stock-out at retailer i is

$$P(D_i \leq x_i) = F_i(x_i), \qquad i = 1, 2.$$

Hence, the objective of the supplier is to maximize his profit while satisfying the service level requirements of the retailers. Keeping the same notation as in Section 3.1, the supplier's maximization problem is given by

$$\begin{aligned} & \max_{x_1 \geq 0, x_2 \geq 0} \Pi_r(x_1, x_2) \\ s.t. \quad & F_1(x_1) \geq \rho_1 \\ & F_2(x_2) \geq \rho_2 \end{aligned} \tag{3.4}$$

Similarly, we can separate problem (3.4) into the following two independent problems

$$\begin{aligned} & \max_{x_1 \geq 0} \Pi_{r1}(x_1) \\ s.t. \quad & F_1(x_1) \geq \rho_1, \end{aligned}$$

and

$$\max_{x_2 \geq 0} \Pi_{r2}(x_2)$$
$$s.t. \quad F_2(x_2) \geq \rho_2.$$

Without service level requirements, the supplier's problem is the same as the problem in Section 3.1. The optimal inventory level for retailer i corresponds to a service level of $\frac{w_i - c}{w_i + h}$, which we called the critical ratio. For the problem with service levels constraints, since $E[w_i \min(x_i, D_i) - h(x_i - D_i)^+ - cx_i]$ is a convex function of x_i and $F_i(x_i)$ is nondecreasing in x_i, the optimal inventory level x_i^* is $F_i^{-1}\left(\max(\rho_i, \frac{w_i - c}{w_i + h})\right)$.

Since the retailers do not hold any inventory, given the inventory level x_i, retailer i's expected profit $\pi_{ri}(x_i)$ can be written as

$$\pi_{ri}(x_i) = E[m_i \min(x_i, D_i)], \qquad i = 1, 2.$$

We use π_{ri}^* to denote the optimal profit of retailer i when the supplier holds x_i^* products for retailer i.

When the wholesale price is fixed, the higher service level requirements by retailers may mean that the supplier must hold more inventory. While higher inventory levels means higher expected sales, the supplier bears higher inventory holding risk when he maintains higher inventory levels for the retailers.

3.2.2 Pooled Inventory System

Now consider the supply chain when the supplier pools the inventory but must satisfy a joint service level requirement of the retailers. We still keep the same notation in Section 3.1. In this case, the objective of the supplier is to maximize his profit, subject to satisfying all demand with probability ρ_p, i.e., the supplier sets his inventory level by solving following problem

$$\max_{x_p \geq 0} \Pi_p(x_p)$$
$$s.t. \quad F_p(x_p) \geq \rho_p$$

where the supplier's expected profit is

$$\Pi_p(x_p) = E[w_p \min{(x_p, D_p)} - h(x_p - D_p)^+ - cx_p].$$

Recall that the optimal inventory level is $F_p^{-1}(\frac{w_p - c}{w_p + h})$ for the pooled inventory case in the absence of a service level constraint. The convexity of $E[w_p \min{(x_p, D_p)} - h(x_p - D_p)^+ - cx_p]$ and the fact that the CDF $F_p(\cdot)$ is nondecreasing implies that the optimal inventory level x_p^* is

$$F_p^{-1}(\max(\frac{w_p - c}{w_p + h}, \rho_p)).$$

Given the supplier's inventory level x_p, the total expected profit of the retailers is

$$\pi_p = E[m_p \min(x_p, D_p)].$$

We use π_p^* to denote the optimal total expected profit of the retailers when the supplier's inventory level is x_p^*.

3.2.3 Comparative Results

As was the case without service level requirements, inventory pooling by the supplier may or may not lead to increased expected retail sales. Example 3.2.1 illustrates an increase in total expected sales while Example 3.2.3 illustrates a decrease.

Example 3.2.1 *Consider a system with pooled inventory in which demands D_1 and D_2 at the retailers are independent and uniformly distributed between [0,100], and the CDF of the* *demand at each retailer is*

$$F(u) = \begin{cases} \frac{u}{100} & 0 \le u < 100 \\ 1 & u \ge 100. \end{cases}$$

Let $D_p = D_1 + D_2$*. The CDF of the random variable* D_p *is*

$$F_p(u) = \begin{cases} (\frac{u^2}{10000}) - \frac{u^2}{20000} & 0 \le u < 100 \\ -\frac{u^2}{20000} + \frac{2u}{100} - 1 & 100 \le u < 200 \\ 1 & u \ge 200. \end{cases}$$

If the critical ratios $\frac{w_1 - c}{w_1 + h} = \frac{w_2 - c}{w_2 + h} = \frac{w_p - c}{w_p + h} = 0.6$ *and all the service level requirements are 0.45, i.e.,* $\rho_1 = \rho_2 = \rho = 0.45$*, then in the reserved inventory case, the optimal inventory levels are*

$$x_1^* = x_2^* = F^{-1}(\max(0.45, 0.6)) = 60,$$

the expected sales of each retailer is 42*, and the expected total sales is* 84*. In the pooled inventory case,* $x_p^* = F_p^{-1}(0.6) = 111$*, and the total expected sales is* 88*. Therefore, the total expected profit of the retailers is increased after pooling the inventory.*

Example 3.2.2 *Continue to assume that the demands* D_1 *and* D_2 *at the retailers are independent and uniformly distributed between [0,100], that the critical ratios* $\frac{w_1 - c}{w_1 + h} = \frac{w_2 - c}{w_2 + h} = \frac{w_p - c}{w_p + h} = 0.8$ *and all the service level requirements are* 0.7*, i.e.,* $\rho_1 = \rho_2 = \rho = 0.7$*. Then in the reserved inventory case, the optimal inventory levels are*

$$x_1^* = x_2^* = F^{-1}(0.8) = 80,$$

the expected sales at each retailer is 48*, and the expected total sales is* 96*. In the pooled inventory case,* $x_p^* = F_p^{-1}(0.8) = 137$*, and the expected total sales is* 95*. Therefore, the total* *expected profit of the retailers drops after pooling the inventory.*

Under generally distributed demands, with higher inventory levels, the expected service level provided to the retailers and their expected sales also increase. If the required service level exceeds the critical ratio, the supplier loses money by providing a higher service level. We now examine the impact of the reserved inventory and the pooled inventory policies on the profit of the supplier and retailers. We will show the results both for the case when the retailers have the same service level requirements and for the case when they have different service level requirements.

Same level service requirements

For the case of normally distributed demands, we can provide a detailed comparison of the reserved and pooled inventory cases. We assume that D_i $(i = 1, 2)$ is independently and normally distributed random variables with mean μ_i, standard deviation σ_i, respectively. In addition, we suppose that all the service level requirements are the same, i.e., $\rho_1 = \rho_2 = \rho_p = \rho$. We have the following theorems.

Theorem 3.2.1 *Assume that $w_1 = w_2 = w_p = w$ and $\rho_1 = \rho_2 = \rho_p = \rho$. If D_1 and D_2 are independently and normally distributed random variables, the supplier's optimal profit is increased when the inventory is pooled, i.e., $\Pi_p^* \geq \Pi_r^*$.*

Proof. Under the reserved inventory scenario, the optimal inventory levels are given by

$$x_1^* = \mu_1 + \sigma_1 \Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right),$$

and

$$x_2^* = \mu_2 + \sigma_2 \Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

For the pooled inventory scenario, $D_p \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. Hence the optimal total inventory level for the pooled inventory scenario is

$$x_p^* = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

The supplier's optimal profits in the reserved inventory case and the pooled inventory case are

$$\begin{aligned}
\Pi_r^* &= (w+h)E[(\min(x_1^*, D_1) + \min(x_2^*, D_2)] - (h+c)(x_1^* + x_2^*), \\
&= (w+h)\left[\mu_1 + \mu_2 - (\sigma_1 + \sigma_2)R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right] \\
&\quad -(h+c)\left[\mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
\Pi_p^* &= (w+h)E[\min(x_p^*, D_p)] - (h+c)x_p^*\} \\
&= (w+h)\left[\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2}R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right] \\
&\quad -(h+c)\left[\mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right].
\end{aligned}$$

Calculating the difference of Π_r^* and Π_p^*, we have

$$\begin{aligned}
\Pi_r^* - \Pi_p^* &= (w+h)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right) R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h},\rho\right)\right)\right)\\
&\quad +(h+c)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h},\rho\right)\right)\\
&\leq (c+h)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right) R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h},\rho\right)\right)\right)\\
&\quad +(h+c)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h},\rho\right)\right)\\
&= (c+h)\left(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1+\sigma_2\right)\left[R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h},\rho\right)\right)\right)\right.\\
&\quad \left.+\Phi^{-1}\left(\max\left(\frac{w-c}{w+h},\rho\right)\right)\right]\\
&\leq 0.
\end{aligned}$$

The first inequality follows from the fact that $w > c$, $R(\cdot) > 0$, and $\sqrt{\sigma_1^2+\sigma_2^2} \leq \sigma_1+\sigma_2$ for any positive σ_1 and σ_2. The second inequality is seen by applying equation (3.3). Hence, the supplier's profit is increased after pooling the inventory, i.e., $\Pi_r^* \leq \Pi_p^*$. ■

Theorem 3.2.2 *Assume that $w_1 = w_2 = w_p = w$, $m_1 = m_2 = m_p = m$ and $\rho_1 = \rho_2 = \rho_p = \rho$. If D_1 and D_2 are independently and normally distributed random variables, the retailers' total expected profit is increased when the inventory is pooled, i.e., $\pi_p^* \geq \pi_r^*$.*

Proof. The expected profits of retailer i $(i = 1, 2)$ in the reserved inventory case are given by

$$\pi_{ri}^* = E[m \min(x_i^*, D_i)], \qquad i = 1, 2,$$

and the total expected profit of the retailers in the pooled inventory case is given by

$$\pi_p^* = E[m \min(x_p^*, D_p)].$$

For the normally distributed demands, we have

$$\pi_{ri}^* = m\left(\mu_i - \sigma_i R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right), \quad i = 1, 2,$$

and

$$\pi_p^* = m\left(\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2} R\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right).$$

Therefore,

$$\begin{aligned} \pi_p^* - \pi_r^* &= \pi_p^* - \pi_{r1}^* - \pi_{r2}^* \\ &= m\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right) R\left(\Phi^{-1}\left(\max(\frac{w-c}{w+h}, \rho\right)\right). \end{aligned}$$

Since $R(\cdot)$ is nonnegative and the inequality $\sigma_1 + \sigma_2 \geq \sqrt{\sigma_1^2 + \sigma_2^2}$, we have

$$\pi_p^* - \pi_r^* \geq 0.$$

Hence the total expected profit is increased after pooling the inventory. ■

Theorem 3.2.3 *Assume that $w_1 = w_2 = w_p = w$ and $\rho_1 = \rho_2 = \rho_p = \rho$. If D_1 and D_2 are independently normally distributed random variables, and the critical ratio $\max(\frac{w-c}{w+h}, \rho) \geq 0.5$, then the supplier's optimal inventory level in the pooled inventory case is less than the optimal total inventory in the reserved inventory case, i.e., $x_p^* \leq x_1^* + x_2^*$. Otherwise, $x_p^* > x_1^* + x_2^*$.*

Proof. We know that the optimal total inventory level in the reserved inventory case is given by

$$x_1^* + x_2^* = \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

However the optimal inventory level in the pooled inventory case is given by

$$x_p^* = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

The fact that $\Phi^{-1}(\cdot)$ is a monotone nondecreasing function and $\Phi^{-1}(0.5) = 0$ implies that $\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right) \geq 0$ provided $\max\left(\frac{w-c}{w+h}, \rho\right) \geq 0.5$. Hence $x_p^* \leq x_1^* + x_2^*$ follows from the inequality, $\sigma_1 + \sigma_2 \geq \sqrt{\sigma_1^2 + \sigma_2^2}$. Otherwise, when $\max\left(\frac{w-c}{w+h}, \rho\right) < 0.5$, we have $x_p^* > x_1^* + x_2^*$. ∎

Under identically and normally distributed demands, if the service level requirements in the reserved inventory case are the same as the joint service level requirement under the pooled inventory case, the supplier will get benefit from the pooled inventory policy. In addition, total expected sales are also increased by sharing the inventory between the two retailers. Because of the benefits of inventory pooling, both the retailers and the supplier will choose the pooled inventory policy. However, we find that the supplier's total inventory level may increase or decrease after pooling the inventory. Like the case without service level requirement, we find that the difference of the total optimal inventory level and the mean value of the demands is decreased after pooling the inventory.

Different Service Level Requirements

In the previous section, we have analyzed the supply chain in which the retailers have identical service level requirements in the pooled inventory system and supplier charges the retailers a common wholesale prices w_p. Here, for the pooled inventory system, we model the supply chain in which the retailers have different service level requirements and the wholesale price for retailer i is w_i $(i = 1, 2)$.

Let x_{pi} $(i = 1, 2)$ be the stock which the supplier keeps for retailer i before the selling season, but now assume that it is sharable, i.e., if the stock kept for retailer 1 runs out and stock is available in the inventory for retailer 2 after the demand of retailer 2 is satisfied, then this remaining stocking can be used to satisfy the unsatisfied demand at retailer 1. Let D_1, D_2 denote the demands at retailers 1 and 2, respectively. Under the pooled inventory policy, the probability of no stock-out at retailer 1 is

$$P(D_1 \leq x_{p1}) + P(D_1 > x_{p1}, D_2 < x_{p2}, D_1 + D_2 \leq x_{p1} + x_{p2}),$$

and at retailer 2 is

$$P(D_2 \leq x_{p2}) + P(D_2 > x_{p2}, D_1 < x_{p1}, D_1 + D_2 \leq x_{p1} + x_{p2}).$$

Let ρ_1 and ρ_2 be the service level requirements for retailers 1 and 2, respectively. The supplier can get the optimal inventory levels (x^*_{p1}, x^*_{p2}) by solving the following maximizing total profit problem,

$$\begin{aligned}
\max_{x_{p1}\geq 0, x_{p2}\geq 0} \Pi_p(x_{p1}, x_{p2}) &= \max_{x_{p1}\geq 0, x_{p2}\geq 0} E[w_1 \min(x_{p1}, D_1) - h(x_{p1} - D_1)^+ - cx_{p1} \\
&+ w_2 \min(x_{p2}, D_2) - h(x_{p2} - D_{p2})^+ - cx_{p2}] \\
&s.t.\ \ P(D_1 \leq x_{p1}) + P(D_1 > x_{p1}, D_2 < x_{p2}, D_1 + D_2 \leq x_{p1} + x_{p2}) \geq \rho_1 \\
&\quad\ \ P(D_2 \leq x_{p2}) + P(D_2 > x_{p2}, D_1 < x_{p1}, D_1 + D_2 \leq x_{p1} + x_{p2}) \geq \rho_2.
\end{aligned} \tag{3.5}$$

When ρ_1 and ρ_2 are the same in the pooled and reserved inventory scenarios, we have the following results.

Theorem 3.2.4 *The supplier's optimal inventory levels in the reserved inventory case is a feasible solution for supplier's maximization problem (3.5) in the pooled inventory case.*

Proof. Recall that (x^*_1, x^*_2) are the optimal inventory levels in the reserved inventory case. It's sufficient to show that (x^*_1, x^*_2) satisfy the constraints in problem (3.5). For the first constraint in equation (3.5)

$$\begin{aligned} & P(D_1 \leq x_1^*) + P(D_1 > x_1^*, D_2 < x_2^*, D_1 + D_2 \leq x_1^* + x_2^*) \\ \geq \; & P(D_1 \leq x_1^*) \\ = \; & \rho_1. \end{aligned}$$

Hence, (x_1^*, x_2^*) satisfies the first constraint. Similarly, we can prove that it also satisfies the second constraint. Hence (x_1^*, x_2^*) is a feasible solution for problem (3.5) ■

The objective function of the supplier's maximization problem in the reserved inventory case is the same as that in the pooled inventory case. However, Theorem 3.2.4 shows that the optimal solution of problem in the reserved inventory is a feasible solution for the problem in the pooled inventory case. Hence the optimal objective value, namely, the optimal profit of the supplier, in the pooled inventory case is at least as large as that in the reserved inventory case. We state this property in the following theorem.

Theorem 3.2.5 *The optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e.,* $\Pi_p^* \geq \Pi_r^*$.

Theorem 3.2.6 *The optimal inventory level for the supplier after pooling is smaller than before pooling, i.e.,* $x_{p1}^* + x_{p2}^* < x_1^* + x_2^*$.

Proof. We have

$$x_i^* = F_i^{-1}(\max(\frac{w_i - c}{w_i + h}, \rho_i)), \qquad i = 1, 2.$$

By Theorem 3.2.4, (x_1^*, x_2^*) is a feasible solution of the supplier's maximization problem in the pooled inventory case. $\Pi_p(x_{p1}, x_{p2})$ is a jointly concave function of x_{p1} and x_{p2}. Furthermore for any given x_{p2}, $\Pi_p(x_{p1}, x_{p2})$ is a decreasing function of x_{p1} when $x_{p1} \geq F_1^{-1}(\frac{w_1-c}{w_1+h})$. Similarly, for any given x_{p1}, $\Pi_p(x_{p1}, x_{p2})$ is a decreasing function of x_{p2} when

$x_{p2} \geq F_2^{-1}(\frac{w_2-c}{w_2+h})$. The expressions for x_1^* and x_2^* imply the following inequalities,

$$x_1^* \geq F_1^{-1}(\frac{w_1 - c}{w_1 + h}),$$

$$x_2^* \geq F_2^{-1}(\frac{w_2 - c}{w_2 + h}).$$

Since $\Pi_p(x_{p1}^*, x_{p2}^*) \geq \Pi_p(x_1^*, x_2^*)$, $x_{p1}^* \leq x_1^*$ and $x_{p2}^* \leq x_2^*$. ■

The results show that the supplier gets more profit in the pooled inventory case than in the reserved inventory case. In the pooled inventory case, the supplier produces less stock than in the reserved inventory case. However, we cannot say whether the expected profit of the retailer will increase or decrease after pooling inventory, which depends on the system parameters such as the demand parameters.

Chapter 4

Wholesale Prices are Decision Variables: Additive Demand Model

In the next two chapters, we consider the same problem as that in in Chapter 3, but the wholesale prices w_1 and w_2 are now decision variables. As before, we denote retailer 1's and retailer 2's random demands as D_1 and D_2, respectively. However, here we assume that all the markups are the same, i.e., $m_1 = m_2 = m_p = m$, and the demands are price-sensitive, and they are functions of the wholesale prices.

4.1 Problem Description

The way the price-sensitive random demand is modelled is very important. In this chapter, we consider an additive demand function of the following form (Mills 1959),

$$D_i(w_i) = y(w_i) + \epsilon_i, \qquad i = 1, 2.$$

where $y(w_i)$ is a deterministic and decreasing function of the product's wholesale price w_i, and ϵ_i is an independent random variable defined on the range [A,B] with CDF $G_i(\cdot)$, PDF

$g_i(\cdot)$ and mean value μ_i. In addition, we assume that

$$y(w_i) = a - bw_i \quad a > 0,\ b > 0.$$

For the pooled inventory case, we use $D_p(w_p)$ to be the total joint demand for the retailers when the wholesale price is w_p. Then

$$\begin{aligned} D_p(w_p) &= D_1(w_p) + D_2(w_p) \\ &= y(w_p) + y(w_p) + \epsilon_1 + \epsilon_2 \\ &= 2y(w_p) + \epsilon_p, \end{aligned}$$

where

$$\epsilon_p = \epsilon_1 + \epsilon_2.$$

The random variable ϵ_p is defined on the range [2A,2B] with mean value $\mu_p = \mu_1 + \mu_2$. We use $G_p(\cdot)$ and $g_p(\cdot)$ to denote the CDF and PDF of ϵ_p. Note that $G_p(\cdot)$ is the convolution of $G_1(\cdot)$ and $G_2(\cdot)$.

Specifying a feasible wholesale price range is common in the operations and economics literature (see Federguen and Heching 1999.) We assume that the set of feasible wholesale prices is confined to a finite interval $[c, w_{max}]$, where

- c: lowest possible unit wholesale price to be charged (which implies that the wholesale price should be greater than the manufacturing cost; otherwise, the supplier cannot get profit).
- w_{max}: highest possible unit wholesale price to be charged.

In order to assure that the feasible wholesale price guarantees the nonnegative demands, we require that $y(w_{max}) + A = a - bw_{max} + A \geq 0$, which in turn implies that $y(c) + A = a - bc + A \geq 0$.

In this chapter, we analyze the decisions of the supplier for both the reserved inventory case and the pooled inventory case. We also compare the results of the reserved inventory scenario and the pooled inventory scenario when ϵ_i $(i = 1, 2)$ are normally distributed. We do this for the following scenarios

- retailers have no service level requirements,
- retailers have service level requirements.

4.2 Retailers Have No Service Level Requirements

4.2.1 Reserved Inventory System

To maximize the expected supplier's profit, the supplier must choose the wholesale price and inventory level for each retailer. Let $\Pi_r(x_1, x_2, w_1, w_2)$ denote the supplier's expected profit when the supplier keeps inventory level x_i and charges w_i per unit for retailer i $(i = 1, 2)$. We have

$$\begin{aligned} \Pi_r(x_1, x_2, w_1, w_2) \quad = \quad & E[w_1 \min(x_1, D_1(w_1)) - h(x_1 - D_1(w_1))^+ - cx_1 \\ & + w_2 \min(x_2, D_2(w_2)) - h(x_2 - D_2(w_2))^+ - cx_2]. \end{aligned} \tag{4.1}$$

Recall that we have assumed that ϵ_1 and ϵ_2 are independent random variables. Thus D_1 and D_2 are also independent, and $\Pi_r(x_1, x_2, w_1, w_2)$ is separable, i.e.,

$$\Pi_r(x_1, x_2, w_1, w_2) = \Pi_{r1}(x_1, w_1) + \Pi_{r2}(x_2, w_2),$$

where

$$\begin{aligned} \Pi_{r1}(x_1, w_1) \quad &= \quad E[w_1 \min(x_1, D_1(w_1)) - h(x_1 - D_1(w_1))^+ - cx_1], \\ \Pi_{r2}(x_2, w_2) \quad &= \quad E[w_2 \min(x_2, D_2(w_2)) - h(x_2 - D_2(w_2))^+ - cx_2]. \end{aligned}$$

Hence, the supplier maximizes his profit by solving the following two problems

$$\max_{x_1 \geq 0, w_1 \in [c,\ w_{max}]} \Pi_{r1}(x_1, w_1), \tag{4.2}$$

$$\max_{x_2 \geq 0, w_2 \in [c,\ w_{max}]} \Pi_{r2}(x_2, w_2).$$

Due to the identical structures of Π_{r1} and Π_{r2}, in the rest of the section, we focus on problem (4.2).

Consider the following optimization problem

$$\max_{x_1, w_1 \in [c,\ w_{max}]} \Pi_{r1}(x_1, w_1). \tag{4.3}$$

The range of the wholesale price w_1 guarantees nonnegative demands, which implies that the optimal inventory level x_1^* is always nonnegative. Hence, problem (4.2) is equivalent to problem (4.3).

We define the expected excess stock, $\Lambda_1(x)$, and the expected shortage, $\Theta_1(x)$, when the inventory level is chosen as x and demand turns out to be ϵ_1 with PDF $g_1(\cdot)$. Specifically,

$$\Lambda_1(x) = \int_A^x (x-u) g_1(u) du,$$

and

$$\Theta_1(x) = \int_x^B (u-x) g_1(u) du.$$

From the definition of $\Theta_1(x)$, we know that it is a nonnegative function of x. Checking the first derivative of $\Theta_1(x)$ with respect to x, we have $\Theta_1'(x) = G_1(x) - 1 \leq 0$. Hence $\Theta_1(x)$ is decreasing in x. In addition, we find that $\Theta_1(x)$ and $\Lambda_1(x)$ satisfy the following equation

$$\Theta_1(x) = \Lambda_1(x) - x + \mu_1.$$

The supplier's profit from retailer 1, $\Pi_{r1}(x_1, w_1)$, can be written as

$$\begin{aligned}\Pi_{r1}(x_1, w_1) &= \int_A^{x_1-y(w_1)} \left[w_1 \left(y(w_1)+u\right) - h(x_1 - y(w_1) - u)\right] g_1(u)du \qquad (4.4)\\ &\quad + \int_{x_1-y(w_1)}^{B} w_1 x_1 g_1(u)du - cx_1 \\ &= I(w_1) - L(x_1, w_1)\end{aligned}$$

where

$$I(w_1) = (w_1 - c)(y(w_1) + \mu_1),$$

and

$$L(x_1, w_1) = (c+h)\Lambda_1(x_1 - y(w_1)) + (w_1 - c)\Theta_1(x_1 - y(w_1)).$$

$I(w_1)$ represents the supplier's riskless profit function, i.e., the profit of supplier for a given price w_1 when the demand variable ϵ_1 is replaced by its constant mean μ_1. Notice that without uncertainty on the demand side, the supplier can manufacture exactly the amount of inventory demanded. $L(x_1, w_1)$ is the loss function, which assesses an overage cost $c+h$ for each unit of the expected unused inventory $\Lambda_1(x_1 - y(w_1))$ and an underage cost $w_1 - c$ for each unit of $\Theta_1(x_1 - y(w_1))$ expected shortages. The following lemma gives some properties of $\Pi_{r1}(x_1, w_1)$ which will be used to solve the problem.

Lemma 4.2.1 *1. For a given w_1, $\Pi_{r1}(x_1, w_1)$ is concave in x_1.*

2. For a given w_1, the optimal inventory level is determined by

$$x_1^*(w_1) = y(w_1) + G_1^{-1}\left(\frac{w_1 - c}{w_1 + h}\right). \qquad (4.5)$$

Proof. Consider the first and second partial derivatives of $\Pi_{r1}(x_1, w_1)$ taken with respect to x_1

$$\begin{aligned}\frac{\partial \Pi_{r1}(x_1, w_1)}{\partial x_1} &= (-c-h) + (w_1+h)[1 - G_1(x_1 - y(w_1))], \\ \frac{\partial^2 \Pi_{r1}(x_1, w_1)}{\partial^2 x_1} &= -(w_1+h) g_1(x_1 - y(w_1)) < 0.\end{aligned}$$

Hence, for a given w_1, $\Pi_{r1}(x_1, w_1)$ is concave in x_1. The second part follows from $\frac{\partial \Pi_{r1}(x_1,w_1)}{\partial x_1} = 0$. ■

Lemma 4.2.1 shows that $\Pi_{r1}(x_1, w_1)$ is concave in x_1 for a given w_1. Thus, it is possible to reduce the original problem to an optimization problem over the single variable w_1 by first solving for the optimal value of x_1 as a function of w_1 and then substituting the result back into $\Pi_{r1}(x_1, w_1)$. Concavity of $\Pi_{r1}(x_1, w_1)$ in x_1 for a given w_1 allows us to use Zabel's method (1970) of first optimizing x_1 for a given w_1, and then searching over the resulting optimal trajectory to maximize $\Pi_{r1}(w_1, x_1^*(w_1))$.

Before we give the optimal solution of the optimization problem, we introduce the concept of the failure rate. For a random variable with CDF $F(\cdot)$ and PDF $f(\cdot)$, we use $h(u)$ to denote the generalized failure rate,

$$h(u) = \frac{uf(u)}{1 - F(u)},$$

and $r(u)$ to denote the classical failure rate,

$$r(u) = \frac{f(u)}{1 - F(u)}.$$

The classical failure rate gives roughly the percentage decrease in the probability of a stock-out from increasing the quantity stocked by one unit, while the generalized failure rate gives roughly the percentage decrease in the probability of a stock out from increasing the stocking quantity by 1%. An increasing failure rate (IFR) has appealing implications. As a supplier holds more stock for a retailer, the retailer's order quantity becomes less elastic, i.e., the probability of a stock-out becomes smaller. The IFR assumption is not restrictive because it applies for most common distributions. Distributions with IFR such as the normal or uniform distributions are clearly also IGFR (increasing general failure rate), but there are IGFR distributions that are not IFR.

Returning to our optimization problem, we can now give conditions and a procedure for calculating a unique optimal solution.

Theorem 4.2.1 *Let $r(\cdot)$ be the failure rate of the random variable ϵ_1. If $r'(z_1) > 0$ for each z_1, then (x_1^*, w_1^*) is uniquely determined by*

$$\begin{cases} 2b(w^0 - w_1) - \Theta_1(x_1 - y(w_1)) & = & 0, \\ G_1(x_1 - y(w_1)) & = & \frac{w_1 - c}{w_1 + h}. \end{cases}$$

where

$$w^0 = \frac{a + bc + \mu_1}{2b}.$$

Furthermore, the optimal wholesale price w_1^* satisfies

$$2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(w_1)) = 0,$$

and the optimal inventory level can be calculated by Lemma 4.2.1, i.e.,

$$x_1^* = y(w_1^*) + G_1^{-1}(\frac{w_1^* - c}{w_1^* + h}).$$

Proof. We know that the optimal solution (x_1^*, w_1^*) satisfies the following first-order conditions

$$
\begin{aligned}
\frac{\partial \Pi_{r1}(x_1, w_1)}{\partial x_1} &= (-c-h) + (w_1+h)[1 - G_1(x_1 - y(w_1))] \\
&= 0, \\
\frac{\partial \Pi_{r1}(x_1, w_1)}{\partial w_1} &= 2b(w^0 - w_1) - \Theta_1(x_1 - y(w_1)) \\
&\quad + b(-c-h) + b(w_1+h)[1 - G_1(x_1 - y(w_1))] \\
&= b(-c-h) + b(w_1+h)[1 - G_1(x_1 - y(w_1))] \\
&= 0.
\end{aligned}
$$

By Lemma 4.2.1, the optimal inventory level $x_1^*(w_1)$ for given w_1 is given by

$$x_1^*(w_1) = y(w_1) + G_1^{-1}(\frac{w_1 - c}{w_1 + h}).$$

Substituting $x_1^*(w_1)$ into equation (4.4), we get $\Pi_{r1}(x_1^*(w_1), w_1)$ as a function of w_1, namely,

$$
\begin{aligned}
\Pi_{r1}(x_1^*(w_1), w_1) &= (w_1 - c)(y(w_1) + \mu_1) \\
&\quad -(c+h)\Lambda_1(x_1^*(w_1) - y(w_1)) - (w_1 - c)\Theta_1(x_1^*(w_1) - y(w_1)).
\end{aligned}
$$

Taking the first derivative with respect to w_1, we have

$$
\begin{aligned}
\frac{d\Pi_{r1}(x_1^*(w_1), w_1)}{dw_1} &= 2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(w_1)) \\
&\quad + b(-c-h) + b(w_1+h)[1 - G_1(x_1^*(w_1) - y(w_1))] \\
&= 2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(w_1)).
\end{aligned}
$$

Defining $V(w_1) = 2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(w_1))$, and calculating the first derivative of V with respect to w_1, we get

$$\begin{aligned} V'(w_1) &= -2b + [1 - G_1(x_1^*(w_1) - y(w_1))](\frac{dx_1^*(w_1)}{dw_1} + b) \\ &= -2b + \frac{1 - G_1(x_1^*(w_1) - y(w_1))}{(w_1 + h)r(x_1^*(w_1) - y(w_1))}. \end{aligned}$$

The second derivative is

$$\begin{aligned} V''(w_1) &= -\frac{(1 - G_1(x_1^*(w_1) - y(w_1)))r'(x_1^*(w_1) - y(w_1))}{(w_1 + h)r^2(x_1^*(w_1) - y(w_1))}(\frac{dx_1^*(w_1)}{dw_1} + b) \\ & -\frac{1 - G_1(x_1^*(w_1) - y(w_1))}{(w_1 + h)^2 r(x_1^*(w_1) - y(w_1))} \\ & -\frac{g_1(x_1^*(w_1) - y(w_1))}{(w_1 + h)r(x_1^*(w_1) - y(w_1))}(\frac{dx_1^*(w_1)}{dw_1} + b), \end{aligned}$$

where

$$\frac{dx_1^*(w_1)}{dw_1} + b = \frac{1}{(w_1 + h)r(x_1^*(w_1) - y(w_1))} \geq 0.$$

Since $r'(\cdot) \geq 0$ and $r(\cdot) \geq 0$, we have

$$V''(w_1) \leq 0,$$

and thus $V(w_1)$ is unimodal. In addition, we assume that the wholesale price should be greater than c. When $w_1 = c$, by equation (4.5), the optimal inventory level is

$$x_1^*(c) = a - bc + A,$$

and

$$\begin{aligned} V(c) &= -2b(c - w^0) - \Theta_1(A) \\ &= a - bc + \mu_1 - \mu_1 + A \\ &= a - bc + A \geq 0. \end{aligned}$$

The inequality follows from the nonnegativity assumption. Furthermore

$$V(\infty) = -\infty.$$

Hence $V(w_1)$ has only one root. Therefore, given that ϵ_1 has an increasing failure rate, the problem has a unique solution given by its first order conditions. ■

The increasing failure rate of the random variable ϵ_1 guarantees the uniqueness of the optimal inventory and pricing policies. This failure rate condition is very common, and many distribution functions such as the normal and exponential distributions have increasing failure rates.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_i = w_i + m_i$ at retailer i $(i = 1, 2)$. Let $\pi_{ri}(x_i)$ be the profit of retailer i when the supplier keeps inventory level x_i for her. We have

$$\pi_{ri}(x_i) = E[m_i \min(x_i, D_i)], \qquad i = 1, 2.$$

4.2.2 Pooled Inventory System

For the pooled inventory case, the supplier sets up one common inventory x_p and charges each retailer a common unit wholesale price w_p. He sets the inventory level and wholesale price to maximize his expected profit.

Let $\Pi_p(x_p, w_p)$ represent the supplier's expected profit when the wholesale price is w_p and the common inventory level is chosen as x_p. We have

$$\Pi_p(x_p, w_p) \quad = \quad E[w_p \min{(x_p, D_p(w_p))} - h(x_p - D_p(w_p))^+ - cx_p].$$

Recall that $D_p(w_p) = 2y(w_p) + \epsilon_p$ and ϵ_p is a random variable with mean μ_p and CDF $G_p(\cdot)$ and PDF $g_p(\cdot)$.

We define the expected excess stock $\Lambda_p(x)$ and the expected shortage $\Theta_p(x)$ when inventory level is chosen as x and random demand turns out to be ϵ_p as

$$\Lambda_p(x) = \int\limits_{2A}^{x} (x-u) g_p(u) du$$

and

$$\Theta_p(x) = \int\limits_{x}^{2B} (u-x) g_p(u) du.$$

We can rewrite the expected profit as

$$\begin{aligned} \Pi_p(x_p, w_p) \quad &= \quad \int_{2A}^{x_p - 2y(w_p)} \left[w_p \left(2y(w_p) + u\right) - h(x_p - 2y(w_p) - u)\right] g_p(u) du \\ &\quad + \int_{x_p - 2y(w_p)}^{2B} w_p x_p g_p(u) du - cx_p \\ &= \quad I_p(w_p) - L_p(x_p, w_p), \end{aligned}$$

where

$$I_p(w_p) = (w_p - c)(2y(w_p) + \mu_p)$$

and

$$L_p(x_p, w_p) = (c+h)\Lambda_p(x_p - 2y(w_p)) - (w_p - c)\Theta_p(x_p - 2y(w_p)).$$

Consequently, the expected profit again can be interpreted as the riskless profit, $I_p(w_p)$, less the expected loss due to the uncertainty, $L_p(x_p, w_p)$.

Due to the nonnegativity of demands, the supplier maximizes his profit by solving the following problem,

$$\max_{x_p, w_p \in [c, w_{max}]} \Pi_p(x_p, w_p)$$

Similar to Lemma 4.2.1, we have the following lemma concerning the properties of $\Pi_p(x_p, w_p)$.

Lemma 4.2.2 *1. For a given w_p, $\Pi_p(x_p, w_p)$ is concave in x_p.*

2. For a given w_p, the optimal inventory inventory level is determined by

$$x_p^*(w_p) = 2y(w_p) + G_p^{-1}(\frac{w_p - c}{w_p + h}).$$

The proof is similar to the proof of Lemma 4.2.1. So the details are omitted.

Similarly, we can get the following theorem on the uniqueness of the optimal solution of the problem.

Theorem 4.2.2 *Let $r(\cdot)$ be the failure rate of the random variable ϵ_p. If $r'(z_p) > 0$ for each z_p, then (x_p^*, w_p^*) is uniquely determined by*

$$\begin{cases} 4b(w_p - w^0) + \Theta_p(x_p - 2y(w_p)) = 0, \\ G_p(x_p - 2y(w_p)) = \frac{w_p - c}{w_p + h}, \end{cases}$$

where

$$w^0 = \frac{2a + 2bc + \mu_p}{4b}.$$

Furthermore, the optimal wholesale price w_p^* satisfies

$$4b(w^0 - w_p) - \Theta_p(x_p^*(w_p) - 2y(w_p)) = 0,$$

and the optimal solution of inventory level can be calculated by Lemma 4.2.2, i.e.,

$$x_p^* = 2y(w_p^*) + G_p^{-1}(\frac{w_p^* - c}{w_p^* + h}).$$

The proof follows that of Theorem 4.2.1. We thus omit the details.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_p = w_p + m_p$. Let $\pi_p(x_p)$ be the profit of the retailers when the supplier keeps inventory level x_p. We have

$$\pi_p(x_p) = E[m_p \min(x_p, D_p)].$$

We have derived procedures to calculate the optimal inventory and pricing policies for the reserved and the pooled inventory scenarios. In the following section, we show the results for the two scenarios when ϵ_1 and ϵ_2 are normally distributed.

4.2.3 Comparative Results

Recall that we defined the range of the random variable ϵ_i $(i = 1, 2)$ as $[A, B]$. A feasible condition for the wholesale price w is $y(w) + A = a - bw + A > 0$ in order to ensure nonnegative demands. As before, we use $\Phi(\cdot)$ to denote the cumulative distribution function, $\phi(\cdot)$ to denote the probability density function and $R(\cdot)$ to denote the right-hand unit normal linear loss function.

First we introduce a simple result that we will use in this and the following sections.

Lemma 4.2.3 *Assume that* $m_1 = m_2 = m_p = m$*. If* ϵ_1 *and* ϵ_2 *are independently, identically and normally distributed random variables, let* $\Theta_1(\cdot) = \Theta_2(\cdot) = \Theta(\cdot)$ *and* $G_1(\cdot) = G_2(\cdot) = G(\cdot)$*, then*

$$\Theta_p(2u) \leq 2\Theta(u) \text{ for any } u \in R.$$

Proof. Assume that ϵ_1 and ϵ_2 have the same mean μ and standard deviation σ and define

$$\Delta\Theta(u) = \Theta_p(2u) - 2\Theta(u).$$

A sufficient condition for $\Theta_p(2u) \leq 2\Theta(u)$ is $\Delta\Theta(u) \leq 0$. Calculating the first derivative of $\Delta\Theta(u)$ with respect to u, we get

$$\begin{aligned}\Delta\Theta'(u) &= -2(1 - G_p(2u)) + 2(1 - G(u)) \\ &= \Phi(\sqrt{2}(\frac{u-\mu}{\sigma})) - \Phi(\frac{u-\mu}{\sigma}).\end{aligned}$$

Since $\Phi(\cdot)$ is a nondecreasing function, we have

$$\Delta\Theta'(u) = \begin{cases} \geq 0 & \text{if } u \geq \mu \\ < 0 & \text{if } u < \mu. \end{cases}$$

We will show that $\Delta\Theta(u) \leq 0$. There are two cases.

Case 1 : $u \geq \mu$

$\Delta\Theta'(u) \geq 0$ indicates that $\Delta\Theta(u)$ is nondecreasing in u. Hence a sufficient condition of $\Delta\Theta(u) \leq 0$ is that $\Delta\Theta(+\infty) \leq 0$. However $\Theta(+\infty) = 0$ and $\Theta_p(+\infty) = 0$, hence $\Delta\Theta(u) \leq 0$ for $u \geq \mu$.

Case 2 : $u < \mu$

$\Delta\Theta'(u) < 0$ indicates that $\Delta\Theta(u)$ is decreasing in u. A sufficient condition for $\Delta\Theta(u) \leq 0$ is $\Delta\Theta(-\infty) = 0$. From Hadley and Whitin (1963), we have

$$\Theta(u) = \sigma\phi(\frac{u-\mu}{\sigma}) - (u-\mu)(1 - \Phi(\frac{u-\mu}{\sigma})).$$

Then

$$\begin{aligned}\Delta\Theta(u) &= \sqrt{2}\sigma\phi(\sqrt{2}(\frac{u-\mu}{\sigma})) - 2\sigma\phi(\frac{u-\mu}{\sigma}) \\ &\quad +2(u-\mu)(\Phi(\sqrt{2}(\frac{u-\mu}{\sigma})) - \Phi(\frac{u-\mu}{\sigma})).\end{aligned}$$

When u approaches $-\infty$, the first two terms go to 0 and $\Phi(\sqrt{2}(\frac{u-\mu}{\sigma})) - \Phi(\frac{u-\mu}{\sigma})$ converges to 0 with exponential speed. Hence the third term also converges to 0. Therefore $\Theta_p(2u) \leq 2\Theta(u)$ is obtained for $u < \mu$. ■

For the case in which ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, we can provide a detailed comparison of the reserved and pooled inventory cases. Corresponding to the set of theorems in Section 3.1.3, we have the following theorems.

Theorem 4.2.3 *Assume that $m_1 = m_2 = m_p = m$. If ϵ_1 and ϵ_2 are independently, identically and normally distributed variables, then the supplier will charge the retailers an identical wholesale price in the reserved inventory case, which is greater than or equal to the optimal wholesale price in the pooled inventory case, i.e., $w_1^* = w_2^* \geq w_p^*$.*

Proof. By Theorem 4.2.1 and Theorem 4.2.2, the optimal wholesale prices w_1^*, w_2^* and w_p^* satisfy following equations

$$\begin{aligned}V_i(w_i) &= 4b(w^0 - w_i) - 2\Theta_i(G_i^{-1}(\frac{w_i - c}{w_i + h})) = 0, \quad i = 1, 2, \\ V_p(w_p) &= 4b(w^0 - w_p) - \Theta_p(G_p^{-1}(\frac{w_p - c}{w_p + h})) = 0,\end{aligned}$$

and V_i, V_p are unimodal functions. Here $G_1(\cdot) = G_2(\cdot)$ guarantees $w_1^* = w_2^*$. Furthermore, the normal distribution has an increasing failure rate, which satisfies the conditions of Theorems 4.2.1 and 4.2.2. Hence V_i and V_p have unique solutions. Assume that ϵ_i $(i = 1, 2)$ has the mean value μ and standard deviation σ. A sufficient condition for $w_p^* \leq w_i^*$ is

$V_p(w) \leq V_i(w)$ for any w, which is equivalent to the following inequalities

$$\begin{aligned} V_p(w) - V_i(w) \leq 0, \quad i = 1, 2 \\ \Theta_p(G_p^{-1}(\frac{w-c}{w+h})) - 2\Theta_i(G_i^{-1}(\frac{w-c}{w+h})) \leq 0, \\ \Theta_p(2\mu + \sqrt{2}\sigma\Phi^{-1}(\frac{w-c}{w+h})) - 2\Theta_i(\mu + \sigma\Phi^{-1}(\frac{w-c}{w+h})) \leq 0. \end{aligned}$$

Define

$$\Delta V_i(Z_\alpha) = \Theta_p(2\mu + \sqrt{2}\sigma Z_\alpha) - 2\Theta_i(\mu + \sigma Z_\alpha)$$

where $Z_\alpha = \Phi^{-1}(\frac{w-c}{w+h})$.

Next we show that $\Delta V_i(Z_\alpha)$ is a monotone nondecreasing function of Z_α. Taking the first derivative of $\Delta V_i(Z_\alpha)$ with respect to Z_α, we obtain

$$\begin{aligned} \Delta V_i'(Z_\alpha) &= -\sqrt{2}\sigma(1 - G_p(2\mu + \sqrt{2}\sigma Z_\alpha)) + 2\sigma(1 - G_i(\mu + \sigma Z_\alpha)) \\ &= (2 - \sqrt{2})\sigma(1 - \Phi(Z_\alpha)) \geq 0. \end{aligned}$$

Hence, a sufficient condition for $\Delta V_i(Z_\alpha) \leq 0$ for any Z_α is that $\Delta V_i(+\infty) \leq 0$. Here

$$\Delta V_i(Z_\alpha)\,|_{Z_\alpha \to +\infty} = 0.$$

Hence $\Delta V_i(Z_\alpha) < 0$. Therefore, the wholesale price in the pooled inventory case is less than that in the reserved inventory case. ∎

Theorem 4.2.4 *Assume* $m_1 = m_2 = m_p = m$. *If* ϵ_1 *and* ϵ_2 *are independently, identically and normally distributed variables, then the optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e.,* $\Pi_p^* \geq \Pi_r^*$.

Proof. If ϵ_1 and ϵ_2 are independently, identically and normally distributed variables, Π_{r1}

is the same as Π_{r2} and hence they have same optimal solution and objective value. If we can show that $\Pi_p(2x_1, w_1) \geq 2\Pi_{r1}(x_1, w_1)$ for any (x_1, w_1), then $\Pi_p^* \geq \Pi_p(2x_1^*, w_1^*) \geq 2\Pi_{r1}(x_1^*, w_1^*) = \Pi_r^*$. Note that

$$\begin{aligned}
2\Pi_{r1}(x_1, w_1) - \Pi_p(2x_1, w_1) &= -2(c+h)\Lambda_1(x_1 - y(w_1)) - 2(w_1 - c)\Theta_1(x_1 - y(w_1)) \\
&\quad +(c+h)\Lambda_p(2x_1 - 2y(w_1)) + (w_1 - c)\Theta_p(2x_1 - 2y(w_1)) \\
&= (h + w_1)\left\{\Theta_p(2x_1 - 2y(w_1)) - 2\Theta_1(x_1 - y(w_1))\right\}.
\end{aligned}$$

Applying Lemma 4.2.3, we have

$$\Theta_p(2x_1 - 2y(w_1)) - 2\Theta_1(x_1 - y(w_1)) \leq 0.$$

Since $h + w_1 > 0$, this implies that $2\Pi_{r1}(x_1, w_1) - \Pi_p(2x_1, w_1) \leq 0$ for any (x_1, w_1). ■

Theorem 4.2.5 *Assume that $w_1 = w_2 = w_p = w$ and $m_1 = m_2 = m_p = m$. If ϵ_1 and ϵ_2 are independently, identically and normally distributed variables, then the retailers' total expected profit in the pooled inventory case is at least as large as that in the reserved inventory case, i.e., $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$.*

Proof. As before, given the inventory levels x_1, x_2 and x_p, we have

$$\begin{aligned}
\pi_{ri}(x_i) &= E[m \min(x_i, D_i)] \quad i = 1, 2, \\
\pi_p(x_p) &= E[m \min(x_p, D_p)].
\end{aligned}$$

Under the reserved inventory case, given the same wholesale price w, the optimal inventory levels are the same, i.e., $x_1 = x_2$. Furthermore,

$$\begin{aligned}
\pi_{r1}(x_1) + \pi_{r2}(x_2) &= m(2y(w) + 2\mu - 2\Theta_1(G_1^{-1}(\frac{w-c}{w+h}))), \\
\pi_p(x_p)) &= m(2y(w) + 2\mu - \Theta_p(G_p^{-1}(\frac{w-c}{w+h}))).
\end{aligned}$$

From the proof of Theorem 4.2.3, we know that

$$2\Theta_1(G_1^{-1}(\frac{w-c}{w+h})) \geq \Theta_p(G_p^{-1}(\frac{w-c}{w+h}));$$

therefore, $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$. ■

If the demands are normally distributed, these results indicate the supplier will always prefer to pool inventory. Under the pooled inventory policy, the supplier's inventory level will decrease because the variance of the total demands is decreased after pooling the inventory. If the wholesale prices are the same under these two policies, the retailers will also prefer the pooled inventory policy.

4.3 Retailers Have Service Level Requirements

Now we consider scenarios in which the retailers impose minimum service level requirements on the supplier. Because of the structure of the assumed demand function, the supplier may use the wholesale price to control the demands so as to meet service level requirements. We keep the same notation as in Section 4.2. Let ρ_1 and ρ_2 be the retailers' service level requirements in the reserved inventory system and ρ_p be the joint service level requirement in the pooled inventory system.

4.3.1 Reserved Inventory System

For the reserved inventory scenario, the supplier needs to solve the following problem,

$$\begin{aligned} &\max_{x_1,x_2\geq 0; w_1\in[c,w_{max}], w_2\in[c,w_{max}]} \Pi_r(x_1,x_2,w_1,w_2) \\ s.t. \quad & P(D_1(w_1)\leq x_1)\geq \rho_1 \\ & P(D_2(w_2)\leq x_2)\geq \rho_2 \end{aligned}$$

where

$$\Pi_r(x_1,x_2,w_1,w_2)=\Pi_{r1}(x_1,w_1)+\Pi_{r2}(x_2,w_2),$$

and

$$\begin{aligned} \Pi_{r1}(x_1,w_1) &= E[w_1\min(x_1,D_1(w_1))-h(x_1-D_1(w_1))^+-cx_1], \qquad (4.6) \\ \Pi_{r2}(x_2,w_2) &= E[w_2\min(x_2,D_2(w_2))-h(x_2-D_2(w_2))^+-cx_2]. \end{aligned}$$

Similar to Section 4.2.1, this problem is separable, so that the supplier needs to solve two problems with the same structure, one each for retailers 1 and 2. Again, we analyze the problem for retailer 1. For retailer 1, the objective of the supplier is to solve the following problem

$$\begin{aligned} &\max_{x_1\geq 0, w_1\in[c,w_{max}]} \Pi_{r1}(x_1,w_1) \qquad (4.7) \\ s.t. \quad & P(D_1(w_1)< x_1)\geq \rho_1. \end{aligned}$$

The method which is stated in Section 4.2.1 does not work well for the problem with service level requirements. We introduce another variable which we call the "stocking factor" defined as

$$z_1=x_1-y(w_1).$$

Substituting for x_1 in equation (4.7), the problem of choosing a price w_1 and a inventory level x_1 is equivalent to choosing a price w_1 and a stocking factor z_1. The expected profit becomes

$$\begin{aligned} \Pi_{r1}(z_1, w_1) &= \int_A^{z_1} \{w_1 (y(w_1) + u) - h(z_1 - u)\} g_1(u) du \\ &\quad + \int_{z_1}^B w_1 (y(w_1) + z_1) g_1(u) du - c (y(w_1) + z_1) \\ &= (w_1 - c)(y(w_1) + \mu_1) - (c + h)\Lambda_1(z_1) - (w_1 - c)\Theta_1(z_1), \end{aligned}$$

and the service level constraint becomes

$$G_1(z_1) \geq \rho_1.$$

Hence, the supplier's optimization problem is equivalent to the following problem

$$\begin{aligned} \max_{z_1, w_1 \in [c,\, w_{max}]} \quad & \Pi_{r1}(z_1, w_1) \\ s.t. \quad & z_1 = x_1 - y(w_1) \\ & G_1(z_1) \geq \rho_1. \end{aligned} \tag{4.8}$$

Considering the optimization problem without the first constraint, we have

$$\begin{aligned} \max_{z_1, w_1 \in [c, w_{max}]} \quad & \Pi_{r1}(z_1, w_1) \\ s.t. \quad & G_1(z_1) \geq \rho_1. \end{aligned} \tag{4.9}$$

Given the optimal solution (z_1^*, w_1^*) for problem (4.9), define the optimal inventory level as

$$x_1^* = z_1^* + y(w_1^*).$$

In addition, (z_1^*, w_1^*, x_1^*) is the optimal solution for problem (4.8). Hence these two problems are equivalent, i.e., the supplier only needs to solve problem (4.9).

We will analyze problem (4.9) without service level constraints first. Based on the optimal solution of the problem without constraints, then we derive the optimal solution with service level constraints.

The following lemma gives some properties of the problem without constraints.

Lemma 4.3.1 *1. For a given z_1, $\Pi_{r1}(z_1, w_1)$ is concave in w_1.*

2. For a given w_1, $\Pi_{r1}(z_1, w_1)$ is concave in z_1.

3. For a given z_1, the optimal price is determined by

$$w_1^*(z_1) = w^0 - \frac{\Theta_1(z_1)}{2b},$$

where $w^0 = \frac{a+bc+\mu_1}{2b}$.

4. For a given w_1, the optimal stocking factor is determined by

$$z_1^*(w_1) = G_1^{-1}(\frac{w_1 - c}{w_1 + h}).$$

Proof. Consider the first and second partial derivatives of $\Pi_{r1}(z_1, w_1)$ taken with respect to z_1 and w_1,

$$
\begin{aligned}
\frac{\partial \Pi_{r1}(z_1,w_1)}{\partial z_1} &= (-c-h)+(w_1+h)\int_{z_1}^{B} g_1(u)du,\\
\frac{\partial^2 \Pi_{r1}(z_1,w_1)}{\partial^2 z_1} &= -(w_1+h)g_1(z)<0,\\
\frac{\partial \Pi_{r1}(z_1,w_1)}{\partial w_1} &= y(w_1)+\mu_1+(w_1-c)\frac{dy(w_1)}{w_1}-\Theta_1(z_1)\\
&= 2b(w^0-w_1)-\Theta_1(z_1),\\
\frac{\partial^2 \Pi_{r1}(z_1,w_1)}{\partial^2 w_1} &= -2b<0,
\end{aligned}
$$

where $w^0 = \frac{a+bc+\mu_1}{2b}$. The first two statements follow from the negativity of the second derivatives. The last two statements follow from $\frac{\partial \Pi_{r1}(z_1,w_1)}{\partial w_1}=0$ and $\frac{\partial \Pi_{r1}(z_1,w_1)}{\partial z_1}=0$. ■

Substituting $w_1^* = w_1(z_1)$ into $\Pi_{r1}(z_1,w_1)$, we have

$$
\Pi_{r1}(z_1,w_1^*(z_1)) \quad = \quad (w_1^*-c)(y(w_1^*)+\mu_1)-(c+h)\Lambda_1(z_1)-(w_1^*(z_1)-c)\Theta_1(z_1),
$$

and we can translate the optimization problem to a maximization problem over a single variable z_1. The optimal inventory and pricing policies for the reserved inventory case are to hold $x_1^* = y(w_1^*(z_1^*))+z_1^*$ units to sell at the unit price w_1^*, where w_1^* is determined by Lemma 4.3.1. The following theorem addresses how to determine the optimal stocking factor z_1^*.

Theorem 4.3.1 *Let $r(\cdot)$ be the failure rate of the random variable ϵ_1. If $2r(z_1)^2+r'(z_1)>0$ for each z_1, then z_1^* is the unique optimal solution within the region [A,B] that satisfies*

$$
-(c+h)+(\frac{a+bc+\mu_1}{2b}+h-\frac{\Theta_1(z_1)}{2b})(1-G_1(z_1))=0
$$

Proof. Applying the chain rule

$$\begin{aligned}\frac{d\Pi_{r1}(z_1, w_1^*(z_1))}{dz_1} &= \frac{\partial\Pi_{r1}(z_1, w_1^*(z_1))}{\partial z_1} + \frac{\Pi_{r1}(z_1, w_1^*(z_1))}{\partial w_1}\frac{dw_1^*(z_1)}{dz_1} \\ &= -(c+h) + (\frac{a+bc+\mu_1}{2b} + h - \frac{\Theta_1(z_1)}{2b})(1 - G_1(z_1)).\end{aligned}$$

Let

$$V(z_1) = -(c+h) + (\frac{a+bc+\mu_1}{2b} + h - \frac{\Theta_1(z_1)}{2b})(1 - G_1(z_1)).$$

The optimal z_1^* should satisfy $V(z_1) = 0$. Now, consider the first and second derivative $V(z_1)$ with respect to z_1. We have

$$\begin{aligned}\frac{dV(z_1)}{dz_1} &= \frac{d}{dz_1}(\frac{d\Pi_{r1}(z_1, w_1^*(z_1))}{dz_1}) \\ &= -\frac{g_1(z_1)}{2b}\left(2b(\frac{a+bc+\mu_1}{2b} + h) - \Theta_1(z_1) - \frac{1-G_1(z_1)}{r(z_1)}\right),\end{aligned}$$

and

$$\frac{d^2V(z_1)}{dz_1^2} = \frac{dV(z_1)/dz_1}{g_1(z_1)}g_1'(z_1) - \frac{g_1(z_1)}{2b}\left\{2[1-G_1(z_1)] + \frac{[1-G_1(z_1)]r'(z_1)}{r(z_1)^2}\right\}.$$

If z_1 is local minimum or maximum, then it always satisfies $\frac{dV(z_1)}{dz_1} = 0$ and its second derivative is

$$\frac{d^2V(z_1)}{dz_1^2} = -\frac{g_1(z_1)[1-G_1(z_1)]}{2br(z_1)^2}\left[2r(z_1)^2 + r'(z_1)\right].$$

If the distribution function satisfies $2r(z_1)^2 + r'(z_1) > 0$, then $\frac{d^2V(z_1)}{dz_1^2} < 0$ at $V'(z_1) = 0$ which implies that $V(z_1)$ is a quasi-concave function. Hence, there are at most two roots for $V(z_1) = 0$. Furthermore, $V(B) = -(c+h) < 0$. $G(A) > 0$ plus $G(B) < 0$ guarantee the uniqueness of z_1^* and $A < z_1^* < B$. A sufficient condition for $G(A) > 0$ is

$$\begin{aligned} 2bV(A) &= -2b(c+h) + \left(\frac{a+bc+\mu_1}{2b} + 2bh - (\mu_1 - A) \right) \\ &= a - bc + A > 0. \end{aligned}$$

Hence, there is a unique optimal solution z_1^* . ■

The condition $2r(z_1)^2 + r'(z_1) > 0$ guarantees that $\Pi_{r1}(z_1, w_1^*(z_1))$ is a quasi-concave in z_1. A sufficient condition is that $r'(z_1) > 0$ which implies that the distribution has an increasing failure rate.

Now, we have an optimal solution z_1^* for the unconstrained problem. Returning to the original problem (4.9) with service level constraints, since $G_1(\cdot)$ is a nondecreasing function and $\Pi_{r1}(z_1, w_1^*(z_1))$ is a quasi-concave function on range $[c, w_{max}]$, the optimal stocking factor for the problem with service level constraints is $\max(z_1^*, G_1^{-1}(\rho_1))$.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the markup is m_i at retailer i ($i = 1, 2$). Let $\pi_{ri}(x_i)$ be the profit of retailer i when the supplier keeps inventory level x_i for her. We have

$$\pi_{ri}(x_i) = E[m_i \min(x_i, D_i)], \quad i = 1, 2.$$

4.3.2 Pooled Inventory System

For the pooled inventory case, the supplier will set up one common inventory x_p and charge each retailer a common unit wholesale price w_p. We face the problem

$$\begin{aligned} \max_{x_p \geq 0, w_p \in [c, w_{max}]} \quad & \Pi_p(x_p, w_p) \\ s.t. \quad & P(D_p \leq x_p) \geq \rho_p \end{aligned}$$

where

$$\begin{aligned}\Pi_p(x_p, w_p) &= \int_{2A}^{x_p-2y(w_p)} \left[w_p\left(2y(w_p)+u\right) - h(x_p - 2y(w_p) - u)\right] g_p(u)du \\ &\quad + \int_{x_p-2y(w_p)}^{2B} w_p x_p g_p(u)du - cx_p.\end{aligned}$$

For this case, define the stocking factor as $z_p = x_p - 2y(w_p)$ and let

$$\Lambda_p(z_p) = \int_{2A}^{z_p} (z_p - u)g_p(u)du, \ \Theta_p(z_p) = \int_{z_p}^{2B} (u - z_p)g_p(u)du.$$

The supplier's expected profit can then be written as

$$\begin{aligned}\Pi_p(z_p, w_p) &= \int_{2A}^{z_p} \left[w_p\left(2y(w_p)+u\right) - h(z_p - u)\right] g_p(u)du \\ &\quad + \int_{z_p}^{2B} \left[w_p\left(2y(w_p)+z_p\right)\right] g_p(u)du - c\left(2y(w_p)+z_p\right) \\ &= (w_p - c)(2y(w_p)+\mu_p) - (c+h)\Lambda_p(z_p) - (w_p - c)\Theta_p(z_p).\end{aligned}$$

The supplier's objective is to maximize his expected profit, i.e.,

$$\begin{aligned}&\max_{x_p \geq 0, w_p \in [c, w_{max}]} \Pi_p(x_p, w_p) \\ &\quad s.t. \ \ P(D_p \leq x_p) > \rho_p.\end{aligned}$$

Following arguments similar to those used in the previous section, it is straightforward to verify that $\Pi_p(z_p, w_p)$ is concave in z_p when w_p is given and is concave in w_p when x_p is given. As a consequence, the first-order conditions uniquely determine w_p^* at any x_p. Specifically, we have the following lemma.

Lemma 4.3.2 *1. For a given z_p, $\Pi_p(z_p, w_p)$ is concave in w_p.*

2. For a given w_p, $\Pi_p(z_p, w_p)$ is concave in z_p.

3. *For a given z_p, the optimal price is determined by*

$$w_p^*(z_p) = w^0 - \frac{\Theta_p(z_p)}{4b},$$

where $w^0 = \frac{2a+2bc+\mu_p}{4b}$.

4. *For a given w_p, the optimal stocking factor is determined by*

$$z_p^*(w_p) = G_p^{-1}(\frac{w_p - c}{w_p + h}).$$

We also have the following sufficient condition for uniqueness of the supplier's optimal solution z_p^*.

Theorem 4.3.2 *Let $r_p(\cdot)$ be the failure rate of the random variable ϵ_p. If $2r_p(z_p)^2 + r_p'(z_p) > 0$ for each z_p, then z_p^* is a unique optimal solution within the region [2A,2B] that satisfies*

$$-(c+h) + (\frac{2a+2bc+\mu_p}{4b} + h - \frac{\Theta_p(z_p)}{4b})(1 - G_p(z_p)) = 0.$$

Since the proof of the theorem is similar to the proof of Theorem 4.3.1, we omit the details here.

If the random variable ϵ_p has an increasing failure rate, we now have a procedure to get the optimal stocking factor z_p^* for the problem without service level requirements. Turning back to the problem with service level requirement, $G_p(\cdot)$ is a nondecreasing function and $\Pi_p(z_p, w_p^*(z_p))$ is a quasi-concave function on range $[c, w_{max}]$. Hence the optimal stocking factor for the pooled inventory problem with service level requirements is $\max(z_p^*, G_p^{-1}(\rho_p))$.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall the markup is m_p at the retails. Let $\pi_p(x_p)$ be the profit of the retailers when the supplier keeps inventory level x_p for them. We have

$$\pi_p(x_p) = E[m_p \min(x_p, D_p)].$$

4.3.3 Comparative Results

For normally distributed demands, we can provide a detailed comparison of the reserved and pooled inventory cases. We again use $\Phi(\cdot)$ to denote the cumulative distribution function and $\phi(\cdot)$ to denote the probability density function of the standard normal distribution.

In order to compare the results, we assume that the service level requirement of each retailer under the reserved inventory case and the joint service level requirement in the pooled inventory case are identical, i.e., $\rho_1 = \rho_2 = \rho_p = \rho$.

Lemma 4.3.3 *Assume that $\rho_1 = \rho_2 = \rho_p = \rho$ and $m_1 = m_2 = m_p = m$. For both the reserved $(i = 1, 2)$ and the pooled $(i = p)$ inventory systems, the optimal stocking factor $z_i^*(w_i)$ is nondecreasing in w_i.*

Proof. From the lemmas in the previous sections, we know that given the wholesale price w_i, the optimal stocking factor $z_i^*(w_i)$ can be written as

$$z_i^*(w_i) = G_i^{-1}(\max(\frac{w_i - c}{h + w_i}, \rho).$$

Since $G_i^{-1}(\cdot)$ is nondecreasing and it is easy to show that $\frac{w_i - c}{h + w_i}$ is nondecreasing in w_i, we know $G_i^{-1}(\max(\frac{w_i - c}{h + w_i}, \rho))$ is nondecreasing in w_i, i.e., z_i^* is nondecreasing in w_i. ■

Theorem 4.3.3 *Assume that $w_1 = w_2 = w_p = w$, $\rho_1 = \rho_2 = \rho_p = \rho$ and $m_1 = m_2 = m_p = m$ and that ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables. If $\max(\frac{w-c}{w+h}, \rho) \leq 0.5$, then the sum of the optimal stocking factors for retailers 1* *and 2 in the reserved inventory case are at least as large as the optimal stocking factor in the pooled inventory case, i.e.,* $z_p^*(w) \le z_1^*(w) + z_2^*(w)$. *Otherwise,* $z_p^*(w) > z_1^*(w) + z_2^*(w)$.

Proof. Given the wholesale price w, the optimal stocking factor z_i^* $(i = 1, 2)$ in the reserved inventory system can be written as

$$z_i^* = G_i^{-1}(\max(\frac{w-c}{w+h}, \rho)), \quad i = 1, 2;$$

the optimal stocking factor z_p^* in the pooled inventory system can be written as

$$z_p^* = G_p^{-1}(\max(\frac{w-c}{w+h}, \rho)).$$

Assume ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables with mean μ and standard deviation σ, therefore,

$$\begin{aligned} z_i^*(w) &= \mu + \sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)), \;\; i = 1, 2 \\ z_p^*(w) &= 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)). \end{aligned}$$

The difference of $z_1^*(w) + z_2^*(w)$ and $z_p^*(w)$ is

$$z_1^*(w) + z_2^*(w) - z_p^*(w) = (2 - \sqrt{2})\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)).$$

When $\max(\frac{w-c}{w+h}, \rho) \le 0.5$, we have $\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)) \le 0$. Hence, $z_1^*(w) + z_2^*(w) \ge z_p^*(w)$. When $\max(\frac{w-c}{w+h}, \rho) > 0.5$, we have $\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)) > 0$, and $z_1^*(w) + z_2^*(w) < z_p^*(w)$. ■

Theorem 4.3.4 *Assume that* $\rho_1 = \rho_2 = \rho_p = \rho$ *and* $m_1 = m_2 = m_p = m$.*If* ϵ_1 *and* ϵ_2 *are independently, identically and normally distributed random variables, then the optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved* *inventory case, i.e.,* $\Pi_p^* \geq \Pi_r^*$.

Proof. Assume ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables with mean value μ and standard deviation σ. Then Π_{r1} is the same as Π_{r2} and hence they have the same optimal solution and objective value. If we can show that $\Pi_p(z_p^*(w_1), w_1) \geq 2\Pi_{r1}(z_1^*(w_1), w_1)$ for any w_1, then $\Pi_p^* \geq \Pi_p(z_p^*(w_1^*), w_1^*) \geq 2\Pi_{r1}(z_1^*(w_1^*), w_1^*) = \Pi_r^*$.
Note that

$$
\begin{aligned}
z_i^*(w_1) &= \mu + \sigma\Phi^{-1}(\max(\frac{w_1 - c}{w_1 + h}, \rho)), \quad i = 1, 2, \\
z_p^*(w_1) &= 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w_1 - c}{w_1 + h}, \rho)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
2\Pi_{r1}(z_1^*(w_1), w_1) - \Pi_p(z_p^*(w_1), w_1) &= -2(c+h)\Lambda_1(z_1^*(w_1)) - 2(w_1 - c)\Theta_1(z_1^*(w_1)) \\
&\quad + (c+h)\Lambda_p(z_p^*(w_1)) + (w_1 - c)\Theta_p(z_p^*(w_1)) \\
&= (h + w_1)\left\{\Theta_p(z_p^*(w_1)) - 2\Theta_1(z_1^*(w_1))\right\}.
\end{aligned}
$$

From the proof of Theorem 4.2.3, we know that

$$
2\Theta_1(\mu + \sigma\Phi^{-1}(\max(\frac{w_1 - c}{w_1 + h}, \rho))) \geq \Theta_p(2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w_1 - c}{w_1 + h}, \rho))).
$$

In addition, $(w_1 + h) > 0$. Thus $2\Pi_{r1}(z_1^*(w_1), w_1) - \Pi_p(z_p^*(w_1), w_1) \leq 0$ for any w_1. ■

Theorem 4.3.5 *Assume that* $w_1 = w_2 = w_p = w$, $\rho_1 = \rho_2 = \rho_p = \rho$ *and* $m_1 = m_2 = m_p = m$. *If* ϵ_1 *and* ϵ_2 *are independently, identically and normally distributed variables, then the retailers' total expected profit in the pooled inventory case is at least as large as that in the* *reserved inventory case, i.e.,* $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$.

Proof. Given the inventory levels x_1, x_2 and x_p in the reserved and pooled inventory systems, the expected profits of the retailers are

$$\pi_{ri}(x_i) = E[m \min(x_i, D_i)] \quad i = 1, 2,$$

and

$$\pi_p(x_p) = E[m \min(x_p, D_p)].$$

Given the identical wholesale price $w_1 = w_2 = w_p = w$, the optimal stocking factors are

$$z_1^*(w_1) = z_2^*(w_2) = \mu + \sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)),$$

and

$$z_p^*(w) = 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)).$$

Furthermore, the optimal inventory levels are $x_i = z_i^*(w) + y(w)$ $(i = 1, 2)$ and $x_p = z_p^* + 2y(w)$. Hence

$$\begin{aligned} \pi_{r1}(x_1) + \pi_{r2}(x_2) &= my(w)[2\mu - 2\Theta_1(z_1^*(w))], \\ \pi_p(x_p) &= my(w)([2\mu - \Theta_p(z_p^*(w))]. \end{aligned}$$

From the proof of Theorem 4.2.3, we know that

$$2\Theta_1(\mu + \sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho))) \geq \Theta_p(2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho))).$$

Therefore, $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$. ∎

If the demands are normally distributed, the supplier will always prefer to the pooled inventory policy. Under the pooled inventory policy, the supplier's profit will increase. If the wholesale prices are the same under those two policies, the retailers will also prefer to the pooled inventory policy.

In closing this chapter, we note that the results for one supplier and two retailers extend in a straight forward to the case of one supplier and N retailers with join demand $D_p = \sum_{i=1}^{N} D_i$. All the details are omitted.

Chapter 5

Wholesale Prices are Decision Variables: Multiplicative Demand Model

5.1 Problem Description

In Chapter 4, we characterized the decisions and performance of the reserved and the pooled inventory systems. Specifically, we derived the optimal inventory and pricing policy, which allowed us to characterize analytical properties of, and gain insights into the performance of the reserved and the pooled inventory system under normally distributed demands. However, these properties and insights were based on a specific demand function, namely, an additive demand function. It is both interesting and important to know if those properties and insights hold for a different demand model.

In this chapter, we consider the following multiplicative model,

$$D_i(w_i) = y(w_i)\epsilon_i, \quad i = 1, 2,$$

where $y(w_i)$ is a deterministic and decreasing function of the product's wholesale price w_i, and ϵ_i is a random variable defined on the range [A,B] with CDF $G_i(\cdot)$, PDF $g_i(\cdot)$, mean value μ_i and standard deviation σ_i. In addition, let $y(w_i)$ take the form of

$$y(w_i) = aw_i^{-b} \qquad \text{where} \quad a > 0, b > 1.$$

In order to guarantee the nonnegative demands, we require that $B > A \geq 0$.

In the pooled inventory case, we use $D_p(w_p)$ to denote the total joint demand for the retailers with wholesale price w_p. Hence we have

$$\begin{aligned} D_p(w_p) &= D_1(w_p) + D_2(w_p) \\ &= y(w_p)(\epsilon_1 + \epsilon_2) \\ &= y(w_p)\epsilon_p, \end{aligned}$$

where

$$\epsilon_p = \epsilon_1 + \epsilon_2.$$

The random variable ϵ_p is defined on the range [2A,2B] with mean value $\mu_p = \mu_1 + \mu_2$. We use $G_p(\cdot)$ and $g_p(\cdot)$ to denote the CDF and PDF of ϵ_p. $G_p(\cdot)$ is the convolution of $G_1(\cdot)$ and $G_2(\cdot)$.

In this chapter, as before, we present and analyze the decisions of the supplier for both the reserved inventory case and the pooled inventory case. We also compare the results for the reserved and the pooled inventory scenarios when ϵ_1 and ϵ_2 are normally distributed. Within the analysis, we consider the following scenarios:

- retailers have no service level requirements,
- retailers have service level requirements.

5.2 Retailers Have No Service Requirements

5.2.1 Reserved Inventory Case

To maximize his expected profit, for each retailer, the supplier needs to choose both the wholesale price and inventory level, and these decisions must be taken before the demand has been observed. Note that the wholesale price has a lower bound, the wholesale price must be greater than manufacturing cost c; otherwise, the supplier cannot earn profit.

Following Petruzzi and Data (1999), we define a stocking factor for retailer i ($i = 1, 2$) as $z_i = x_i/y(w_i)$. As before, we define $\Pi_{ri}(z_i, w_i)$ as the expected profit associated with retailer i when the supplier uses z_i stocking factor for retailer i and sells each unit at wholesale price w_i. As the description in Chapter 4, the supplier's problem can be separated into two problems

$$\max_{z_1, w_1 \geq c} \Pi_{r1}(z_1, w_1),$$

$$\max_{z_2, w_2 \geq c} \Pi_{r2}(z_2, w_2),$$

where

$$\begin{aligned} \Pi_{ri}(z_i, w_i) &= \int_A^{z_i} \{w_i\,(y(w_i)u) - hy(w_i)(z_i - u)\}\, g_i(u)du && (5.1) \\ &\quad + \int_{z_i}^{B} w_i y(w_i) z_i g_i(u)du + cy(w_i)z_i \\ &= (w_i - c)(y(w_i)\mu_i) - y(w_i)(c+h)\Lambda_i(z_i) - y(w_i)(w_i - c)\Theta_i(z_i), && i = 1, 2. \end{aligned}$$

Due to the identical structure of Π_{r1} and Π_{r2}, we focus on the following problem in the remainder of the section.

$$\max_{z_1, w_1 \geq c} \Pi_{r1}(z_1, w_1).$$

Lemma 5.2.1 introduces some properties of $\Pi_{r1}(z_1, w_1)$, which is useful to solve the problem.

Lemma 5.2.1 *1. For a given w_1, $\Pi_{r1}(z_1, w_1)$ is concave in z_1.*

2. For a given z_1, $\Pi_{r1}(z_1, w_1)$ is a quasi-concave function in w_1.

3. For a given z_1, the optimal price is determined by

$$w_1^*(z_1) = w^0 + \frac{b}{b-1}\frac{(c+h)\Lambda_1(z_1)}{\mu_1 - \Theta_1(z_1)},$$

where $w^0 = \frac{bc}{b-1}$.

4. For a given w_1, the optimal stocking factor is determined by

$$z_1^*(w_1) = G_1^{-1}(\frac{w_1 - c}{w_1 + h}).$$

Proof. Consider the first derivatives of $\Pi_{r1}(z_1, w_1)$ taken with respect to z_1 and w_1, we have

$$\begin{aligned}
\frac{\partial \Pi_{r1}(z_1, w_1)}{\partial z_1} &= y(w_1)(-c-h) + y(w_1)(w_1+h)(1 - G_1(z_1)), \\
\frac{\partial \Pi_{r1}(z_1, w_1)}{\partial w_1} &= a w_1^{-(b+1)}(b-1)(\mu_1 - \Theta_1(z_1))\left(-w_1 + \frac{bc}{b-1} + \frac{b}{b-1}\frac{(c+h)\Lambda_1(z_1)}{\mu_1 - \Theta(z_1)}\right).
\end{aligned}$$

Calculate the second partial derivatives of $\Pi_{r1}(z_1, w_1)$ with respect to z_1,

$$\frac{\partial^2 \Pi_{r1}(z_1, w_1)}{\partial^2 z_1} = -y(w_1)(w_1+h)g_1(z_1) \le 0.$$

$\frac{\partial^2 \Pi_{r1}(z_1, w_1)}{\partial^2 z_1} \le 0$ implies the first statement. Since $\mu_1 - \Theta_1(z_1) \ge A > 0$, $\frac{\partial \Pi_{r1}(z_1, w_1)}{\partial w_1} = 0$ if and only if $-w_1 + \frac{bc}{b-1} + \frac{b}{b-1}\frac{(c+h)\Lambda_1(z_1)}{\mu_1 - \Theta_1(z_1)} = 0$, which implies that there is only one local minimum or maximum,

$$w_1^*(z_1) = w^0 + \frac{b}{b-1}\frac{(c+h)\Lambda_1(z_1)}{\mu_1 - \Theta_1(z_1)},$$

where $w^0 = \frac{bc}{b-1}$. Checking the second derivative of $\Pi_{r1}(z_1, w_1)$ with respect to z_1 at this local optimal point, we have

$$\frac{\partial^2 \Pi_{r1}(z_1, w_1)}{\partial^2 w_1}\Big|_{w_1 = w_1^*(z_1)} = -a w_1^{-(b+1)}(b-1) < 0.$$

Hence, given a fixed z_1, $\Pi_{r1}(z_1, w_1)$ is a quasi-concave function in w_1 . The third part follows from equation $\frac{\partial \Pi_{r1}(z_1,w_1)}{\partial w_1} = 0$. The fourth part follows from $\frac{\partial \Pi_{r1}(z_1,w_1)}{\partial z_1} = 0$. ■

Substituting $w_1 = w_1^*(z_1)$ into equation (5.1) for $i = 1$, the optimization problem becomes a maximization problem over a single variable z_1, i.e.,

$$\max_{z_1} \Pi_{r1}(z_1, w_1^*(z_1)), \tag{5.2}$$

where

$$\Pi_{r1}(z_1, w_1^*(z_1)) \quad = \quad y(w_1^*(z_1))\left[\mu_1(w_1^*(z_1) - c) - (c+h)\Lambda_1(z_1) - (w_1^*(z_1) - c)\Theta_1(z_1)\right].$$

Assume z_1^* is the optimal solution of problem (5.2). Then the optimal inventory and pricing policy for the reserved inventory case is to stock $x_1^* = y(w_1^*)z_1^*$ units for retailer 1 to sell to her at the unit price w_1^*, where w_1^* is determined by Lemma 5.2.1. The following theorem addresses the determination of z_1^*.

Theorem 5.2.1 *Let $r(\cdot)$ be the failure rate of random variable ϵ_1. If $r(z_1)$ satisfies the following inequality*

$$2r(z_1)^2 + r'(z_1) > 0,$$

there is a unique optimal solution z_1^* within the region [A,B], which satisfies

$$(-c-h)/(1-G_1(z_1)) + w_1^*(z_1) + h = 0.$$

Proof. By the chain rule

$$\begin{aligned}
\frac{d\Pi_{r1}(z_1, w_1^*(z_1))}{dz_1} &= \frac{\partial \Pi_{r1}(z_1, w_1^*(z_1))}{\partial z_1} + \frac{\partial \Pi_{r1}(z_1, w_1^*(z_1))}{\partial w_1^*(z_1)} \frac{dw_1^*(z_1)}{dz_1} \\
&= -y(w_1^*(z_1))(1-G_1(z_1))((-c-h)/(1-G_1(z_1)) + w_1^*(z_1) + h).
\end{aligned}$$

Define

$$V(z_1) = (-c-h)/(1-G_1(z_1)) + w_1^*(z_1) + h, .$$

The optimal solution z_1^* satisfies $V(z_1^*) = 0$. Now, considering the first derivative and second derivative of $V(z_1)$ with respect to z_1, we have

$$\begin{aligned}
\frac{dV(z_1)}{dz_1} &= \frac{b}{b-1}\left(\frac{(c+h)(G_1(z_1)z_1 - \Lambda_1(z_1))}{(\mu_1 - \Theta_1(z_1))^2}\right) - \frac{(c+h)r(z_1)}{1-G_1(z_1)}, \\
\frac{d^2V(z_1)}{dz_1^2} &= \frac{b}{b-1}(-2)(1-G_1(z_1))\left(\frac{(c+h)(G_1(z_1)z_1 - \Lambda_1(z_1))}{(\mu_1 - \Theta_1(z_1))^3}\right) \\
&\quad + \frac{b}{b-1}g_1(z_1)\left(\frac{(c+h)z_1}{(\mu_1 - \Theta_1(z_1))^2}\right) - (c+h)(\frac{r'(z_1)}{1-G_1(z_1)} + \frac{r^2(z_1)}{1-G_1(z_1)} \\
&= -(c+h)(\frac{b-2}{b-1}\frac{r(z_1)}{\mu_1 - \Theta_1(z_1)} + \frac{2r(z_1)^2 + r(z_1)'}{1-G_1(z_1)}) \\
&\quad -(\frac{2(1-G_1(z_1)}{\mu_1 - \Theta_1(z_1)} + r(z_1))\frac{dV}{dz_1}.
\end{aligned}$$

The points whose values are local minimum or maximum always satisfy

$$\frac{dV(z_1)}{dz_1} = 0,$$

and at those points, we have

$$\frac{d^2V(z_1)}{dz_1^2} = -(c+h)(\frac{b-2}{b-1}\frac{r(z_1)}{\mu_1 - \Theta_1(z_1)} + \frac{2r(z_1)^2 + r(z_1)'}{1-G_1(z_1)}).$$

If the distribution function satisfies $2r(z_1)^2 + r'(z_1) > 0$, then $\frac{d^2V(z_1)}{dz_1^2} < 0$ at those points which satisfy $V'(z_1) = 0$. Hence $V(z_1)$ is a quasi-concave function, and there are at most two roots for $V(z_1) = 0$. Furthermore, $V(B) = -\infty < 0$. $V(A) = w(z_1) + c > 0$ guarantees the uniqueness of z_1^*. ■

Since the retailers do not hold any inventory, the expected profit of each retailer is proportional to her expected sales. As before, let $\pi_{ri}(x_i)$ be the expected total retail profit given the supplier's inventory level x_i for retailer i $(i = 1, 2)$, which can be written as

$$\pi_{ri}(x_i) = E[m_i \min(x_i, D_i)] \text{ for } i = 1, 2.$$

5.2.2 Pooled Inventory Case

Now, consider the pooled inventory scenario in which the supplier will set up one common inventory x_p and charge each retailer a common wholesale price w_p to maximize his expected profit.

For this case, we define the inventory stocking factor as $z_p = x_p/y(w)$. Then the expected profit of the supplier can be written as

$$\begin{aligned} \Pi_p(z_p, w_p) &= \int_A^{z_p} \{w_p\,(y(w_p)u) - hy(w_p)(z_p - u)\}\, g_p(u)du \\ &\quad + \int_{z_p}^{B} w_p y(w_p) z_p g_p(u)du + cy(w_p)z_p \\ &= (w_p - c)(y(w_p)\mu_p) - y(w_p)(c+h)\Lambda_p(z_p) - y(w_p)(w_p - c)\Theta_p(z_p). \end{aligned} \tag{5.3}$$

The supplier's objective is to maximize his expected profit by solving the following problem

$$\max_{z_p, w_p \geq c} \Pi_p(z_p, w_p).$$

Lemma 5.2.2 *1. For a given w_p, $\Pi_p(z_p, w_p)$ is concave in z_p.*

2. For a given z_p, $\Pi_p(z_p, w_p)$ is a quasi-concave function in w_p.

3. For a given z_p, the optimal price is determined by

$$w_p^*(z_p) = w^0 + \frac{b}{b-1} \frac{(c+h)\Lambda_p(z_p)}{\mu_p - \Theta_p(z_p)},$$

where $w^0 = \frac{bc}{b-1}$.

4. For a given w_p, the optimal stocking factor is determined by

$$z_p^*(w_p) = G_p^{-1}(\frac{w_p - c}{w_p + h}).$$

The proof of the above theorem can be carried out by following a procedure similar to that for Theorem 5.2.1. Thus we omit the details.

Substituting $w_p = w_p^*(z_p)$ into equation (5.3), the optimization problem becomes a maximization problem over a single variable z_p

$$\max_{z_p} \Pi_p(z_p, w_p^*(z_p)), \tag{5.4}$$

where

$$\Pi_p(z_p, w_p^*(z_p)) = y(w_p^*(z_p)) \left[\mu_p(w_p^*(z_p) - c) - (c+h)\Lambda_p(z_p) - (w_p^*(z_p) - c)\Theta_p(z_p)\right].$$

Assume z_p^* is the optimal solution of problem (5.4). Then the optimal stocking and pricing policy for the reserved inventory case is to stock $x_p^* = y(w_p^*)z_p^*$ units and to sell to retailers at the unit wholesale price w_p^*, where w_p^* is determined by Lemma 5.2.2 and the following theorem gives the conditions for a unique optimal solution z_p^*.

Theorem 5.2.2 *Let $r(z_p)$ be the failure rate of random variable ϵ_p. If $r(z_p)$ satisfies the following inequality*

$$2r(z_p)^2 + r'(z_p) > 0,$$

there is a unique optimal solution z_p^* within the region [2A,2B] which satisfies

$$(-c-h)/(1-G_p(z_p)) + w_p^*(z_p) + h = 0.$$

The proof of the theorem is similar to the proof of Theorem 5.2.1, so the details are omitted.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_p = w_p + m_p$. Let $\pi_p(x_p)$ be the profit of the retailers when the supplier keeps inventory level x_p. We have

$$\pi_p(x_p) = E[m_p \min(x_p, D_p)].$$

Section 5.2.1 and this section show how to solve the problem under the reserved and the pooled inventory cases. In the next section, we present some comparative results.

5.2.3 Comparative Results

We defined the range of the random component of the demand ϵ_i as $[A, B]$ with $A \geq 0$, and $y(w_i) \geq 0$ $(i = 1, 2)$, which implies nonnegative the demand demand D_i. Here, we again use normal distribution to approximate the other demand distribution. As before, we use Φ to denote the cumulative distribution function and ϕ to denote the probability density function of the standard normal distribution.

Lemma 5.2.3 *Assume that $m_1 = m_2 = m_p = m$. For both the reserved inventory case* *(i=1,2) and the pooled inventory case (i=p), the optimal stocking factor $z_i^*(w_i)$ is nondecreasing in w_i.*

Proof. By the lemmas in the previous sections, we know that $z_i^*(w_i)$ satisfies the following equations

$$1 - G_i(z_i^*) = \frac{c+h}{h+w_i}.$$

Calculating the first derivative of z_i^* with respect to w_i, we have

$$\frac{dz_i^*}{dw_i} = \frac{c+h}{(h+w_i)^2 g_i(z_i)}.$$

The nonnegativity of $\frac{c+h}{(h+w_i)^2}$ and $g_i(z_i)$ imply that $\frac{dz_i^*}{dw_i} \geq 0$. Hence $z_i^*(w_i)$ is nondecreasing in w_i. ■

Theorem 5.2.3 *Assume that $w_1 = w_2 = w_p = w$ and $m_1 = m_2 = m_p = m$. if ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables. If $\frac{w-c}{w+h} \geq 0.5$, then the sum of the optimal stocking factors for the retailers 1 and 2 in the reserved inventory case is at least as large as the optimal stocking factor in the pooled inventory case, i.e., $z_p^*(w) \leq z_1^*(w) + z_2^*(w)$. Otherwise, $z_p^*(w) > z_1^*(w) + z_2^*(w)$.*

Proof. We know that

$$G_1(z_1^*) = G_2(z_2^*) = G_p(z_p^*) = 1 - \frac{c+h}{h+w}.$$

Therefore,

$$\begin{aligned} z_i^*(w) &= \mu + \sigma\Phi^{-1}(\frac{w-c}{w+h}) \text{ for } i = 1, 2, \\ z_p^*(w) &= 2\mu + \sqrt{2}\sigma\Phi^{-1}(\frac{w-c}{w+h}), \\ z_1^*(w) + z_2^*(w) - z_p^*(w) &= (2-\sqrt{2})\sigma\Phi^{-1}(\frac{w-c}{w+h}). \end{aligned}$$

When $\frac{w-c}{w+h} \geq 0.5$, then $\Phi^{-1}(\frac{w-c}{w+h}) \geq 0$, and hence $z_1^*(w) + z_2^*(w) \geq z_p^*(w)$. When $\frac{w-c}{w+h} < 0.5$, then $\Phi^{-1}(\frac{w-c}{w+h}) < 0$, and hence $z_1^*(w) + z_2^*(w) \geq z_p^*(w)$. ■

Theorem 5.2.4 *Assume that $m_1 = m_2 = m_p = m$. If ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, the optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e., $\Pi_p^* \geq \Pi_r^*$.*

Proof. when ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, Π_{r1} is the same as Π_{r2} and hence they have the same optimal value. If we can show that $\Pi_p(2z_1, w_1) \geq 2\Pi_{r1}(z_1, w_1)$ for any (z_1, w_1), then $\Pi_p^* \geq \Pi_p(2z_1^*, w_1^*) \geq 2\Pi_{r1}(z_1^*, w_1^*) = \Pi_r^*$.

Note that

$$\begin{aligned} 2\Pi_{r1}(z_1, w_1) - \Pi_p(2z_1, w_1) &= y(w_1)(-2(c+h)\Lambda_1(z_1) - 2(w_1 - c)\Theta_1(z_1) \\ &\quad +(c+h)\Lambda_p(2z_1) + (w_1 - c)\Theta_p(2z_1)) \\ &= y(w_1)(h + w_1)\left[\Theta_p(2z_1) - 2\Theta_1(z_1)\right]. \end{aligned}$$

From Lemma 4.2.3, we have $\Theta_p(2z_1) - 2\Theta_1(z_1)) \leq 0$. In addition $y(w_1)(h + w_1) > 0$, which implies that $2\Pi_{r1}(z_1, w_1) - \Pi_p(2z_1, w_1) \leq 0$ for any (z_1, w_1). ■

Theorem 5.2.5 *Assume that $w_1 = w_2 = w_p = w$ and $m_1 = m_2 = m_p = m$. If ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, then the retailers'* *total expected profit in the pooled inventory case is at least as large as that in the reserved inventory case, i.e.,* $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$.

Proof. As before, given the inventory level x_1, x_2 and x_p, we have the expected profit of retailer as

$$\begin{aligned}\pi_{ri}(x_i) &= E[m\min(x_i, D_i)] \text{ for } i = 1, 2,\\ \pi_p(x_p) &= E[m\min(x_p, D_p)].\end{aligned}$$

The optimal stocking factors given by the wholesale price $w_1 = w_2 = w_p = w$ are

$$z_1^*(w) = z_2^*(w) = \mu + \sigma\Phi^{-1}(\frac{w-c}{w+h}),$$

and

$$z_p^*(w) = 2\mu + \sqrt{2}\sigma\Phi^{-1}(\frac{w-c}{w+h}).$$

Furthermore, given wholesale price w, the optimal inventory levels are $x_i = y(w)z_i^*(w)$ ($i = 1, 2$) and $x_p = y(w)z_p^*(w)$. Hence, we have

$$\begin{aligned}\pi_{r1}(x_1) + \pi_{r2}(x_2) &= my(w)[2\mu - 2\Theta_1(z_1^*(w))],\\ \pi_p(x_p) &= my(w)[2\mu - \Theta_p(z_p^*(w))].\end{aligned}$$

From the proof of Theorem 4.2.3, we know that

$$2\Theta_1(\mu + \sigma\Phi^{-1}(\frac{w-c}{w+h})) \geq \Theta_p(2\mu + \sqrt{2}\sigma\Phi^{-1}(\frac{w-c}{w+h})).$$

Therefore, $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$. ■

In the absence of service level requirements, when the demands are normally distributed, the comparative results indicate that the supplier will always prefer to pool inventory. If the wholesale prices are the same under the two policies, the retailers will also prefer the pooled inventory policy.

5.3 Retailers Have Service Level Requirements

5.3.1 Reserved Inventory Case

In this scenario, the retailers are powerful enough to require the supplier to use a reserved-inventory policy, i.e., the supplier maintains a separate inventory level for each retailer. In addition the retailers have minimum service level requirements. We assume that retailers 1 and 2 have service level requirements ρ_1 and ρ_2, respectively. We maintain the same notations as in Section 5.2.1. Hence, for retailer i, the supplier maximizes his profit by solving the following problem

$$\begin{aligned} &\max_{x_i, w_i \geq c} \Pi_{ri}(x_i, w_i) \\ s.t.\quad & P(D_i(w_i) < x_i) \geq \rho_i \quad \text{for} \;\; i = 1, 2. \end{aligned}$$

Introducing the stocking factor $z_i = x_i / y(w_i)$, the problem becomes

$$\begin{aligned} &\max_{z_i, w_i \geq c} \Pi_{ri}(z_i, w_i) \\ s.t.\quad & G_i(z_i) \geq \rho_i \quad \text{for} \;\; i = 1, 2. \end{aligned}$$

Recall that without service level requirements, we can get the optimal stocking factor z_i^* $(i = 1, 2)$ by the procedure provided in Section 5.2.1. For the problem with service levels constraints, since $G_i(z_i)$ is nondecreasing in z_i and $\Pi_{ri}(z_i, w_i^*(z_i))$ is a quasi-concave function over range $[c, +\infty]$. Hence the optimal stocking factor for the supplier with service level requirement is $\max(z_i^*, G_i^{-1}(\rho_i))$.

Since the retailers do not hold any inventory, the expected profit of each retailer is proportional to her expected sales . As before, let $\pi_{ri}(x_i)$ be the expected profit of retailer i given the supplier's inventory level x_i for retailer i ($i = 1, 2$), which can be written as

$$\pi_{ri}(x_i) = E[m_i \min(x_i, D_i)] \quad \text{for} \quad i = 1, 2.$$

5.3.2 Pooled Inventory Case

For the pooled inventory case, the supplier will set up one common inventory x_p and charge each retailer a common unit wholesale price w_p. Furthermore, the retailers have a joint service level requirement ρ_p. We maintain the same notation as in Section 5.2.2. The supplier maximizes his profit by solving the following problem

$$\begin{aligned} & \max_{x_p, w_p \geq c} \Pi_p(x_p, w_p) \\ s.t. \quad & P(D_p(w_p) < x_p) \geq \rho_p \end{aligned}$$

Introducing the stocking factor $z_p = x_p / y(w_p)$, the problem becomes

$$\begin{aligned} & \max_{x_p, w_p \geq c} \Pi_p(z_p, w_p) \\ s.t. \quad & P(D_p(w_p) < x_p) \geq \rho_p. \end{aligned}$$

Recall that without service level requirements, we can get the optimal stocking factor z_p^* by the procedure provided in Section 5.2.1. For the problem with service levels constraints, since $G_p(z_p)$ is nondecreasing in z_p and $\Pi_p(z_p, w_p^*(z_p))$ is a quasi-concave function on range $[c, +\infty]$, the optimal stocking factor for the problem with service level requirement is $\max(z_p^*, G_p^{-1}(\rho_p))$.

Since the retailers do not hold any inventory, the expected profits of the retailers are proportional to their expected sales. Let $\pi_p(x_p)$ be the expected total retail profit given the supplier's inventory level x_p, which can be written as

$$\pi_p(x_p) = E[m_p \min(x_p, D_p)].$$

5.3.3 Comparative Results

In this section, we analyze the impact of pooling inventory under normally distributed demands when the supplier needs to satisfy the retailer's service level requirements. Assume that ϵ_1 and ϵ_2 are independently, identically and normally distributed with the mean value μ and standard deviation σ. In addition, assume that the service level requirement of each retailer under the reserved inventory case and the joint service level in the pooled inventory case are the same, i.e., $\rho_1 = \rho_2 = \rho_p = \rho$.

Lemma 5.3.1 *Assume that* $w_1 = w_2 = w_p = w$, $\rho_1 = \rho_2 = \rho_p = \rho$ *and* $m_1 = m_2 = m_p = m$. *For both the reserved inventory case (i=1,2) and the pooled inventory case (i=p), the optimal stocking factor* $z_i^*(w_i)$ *is nondecreasing in* w_i.

Proof. We know that $z_i^*(w_i)$ satisfies the following equations

$$G_i(z_i^*) = \max(\frac{w_i - c}{w_i + h}, \rho),$$

where $\frac{w_i - c}{w_i + h}$ is increasing in w_i and G_i is a nondecreasing function. Hence $z_i^*(w_i)$ is nondecreasing with the increasing of w_i. ■

Theorem 5.3.1 *Assume thatw*$_1 = w_2 = w_p = w$, $\rho_1 = \rho_2 = \rho_p = \rho$ *and* $m_1 = m_2 = m_p = m$. *If* ϵ_1 *and* ϵ_2 *are independently, identically and normally distributed random variables and* $\max(\frac{w-c}{w+h}, \rho) \geq 0.5$, *then the sum of the optimal total stocking factors for the retailers 1 and 2 in the reserved inventory case is at least as large as the optimal stocking factor in the pooled inventory case, i.e.,* $z_p^*(w) \leq z_1^*(w) + z_2^*(w)$. *Otherwise,* $z_p^*(w) > z_1^*(w) + z_2^*(w)$.

Proof. We know that

$$G_1(z_1^*) = G_2(z_2^*) = G_p(z_p^*) = \max(\frac{w-c}{w+h}, \rho).$$

Therefore,

$$\begin{aligned}
z_i^*(w) &= \mu + \sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)), \quad i = 1, 2, \\
z_p^*(w) &= 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)), \\
z_1^*(w) + z_2^*(w) - z_p^*(w) &= (2-\sqrt{2})\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)),
\end{aligned}$$

When $\max(\frac{w-c}{w+h}, \rho) \geq 0.5$, we have $\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)) \geq 0$ and $z_1^*(w)+z_2^*(w) \geq z_p^*(w)$. Otherwise, when $\max(\frac{w-c}{w+h}, \rho) < 0.5$, we have $\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)) < 0$ and $z_1^*(w)+z_2^*(w) \geq z_p^*(w)$. ■

Theorem 5.3.2 *Assume that $w_1 = w_2 = w_p = w$, $\rho_1 = \rho_2 = \rho_p = \rho$ and $m_1 = m_2 = m_p = m$. If ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, then the optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e., $\Pi_p^* \geq \Pi_r^*$.*

Proof. when ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, Π_{r1} is the same as Π_{r2} and hence they have the same optimal solution and objective value. If we can show that $\Pi_p(z_p^*(w_1), w_1) \geq 2\Pi_{r1}(z_1^*(w_1), w_1)$ for any w_1, then $\Pi_p^* \geq \Pi_p(z_p^*(w_1), w_1^*) \geq 2\Pi_{r1}(z_1^*(w_1^*), w_1^*) = \Pi_r^*$.
Note that

$$z_i^*(w_1) = \mu + \sigma\Phi^{-1}(\max(\frac{w_1-c}{w_1+h}, \rho)), \quad i = 1, 2,$$

and

$$z_p^*(w_1) \quad = \quad 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w_1 - c}{w_1 + h}, \rho)).$$

Hence,

$$\begin{aligned} 2\Pi_{r1}(z_1^*(w_1), w_1) - \Pi_p(z_p^*(w_1), w_1) &= y(w_1)\left(-2(c+h)\Lambda_1(z_1^*(w_1)) - 2(w_1 - c)\Theta_1(z_1^*(w_1))\right. \\ &\quad \left.(c+h)\Lambda_p(z_p^*(w_1)) + (w_1 - c)\Theta_p(z_p^*(w_1))\right) \\ &= y(w_1)(h + w_1)\left\{\Theta_p(z_p^*(w_1)) - 2\Theta_1(z_1^*(w_1))\right\}. \end{aligned}$$

From the proof of Theorem 4.2.3, we have $\Theta_p(z_p^*(w_1)) - 2\Theta_1(z_1^*(w_1))) \leq 0$. In addition $y(w_1)(h + w_1) > 0$, which implies that $2\Pi_{r1}(z_1^*(w_1), w_1) - \Pi_p(z_p^*(w_1), w_1) \leq 0$ for any w_1. ■

Theorem 5.3.3 *Assume that $w_1 = w_2 = w_p = w$, $\rho_1 = \rho_2 = \rho_p = \rho$ and $m_1 = m_2 = m_p = m$. If ϵ_1 and ϵ_2 are independently, identically and normally distributed random variables, then the retailers' total expected profit in the pooled inventory case is at least as large as that in the reserved inventory case, i.e., $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$.*

Proof. As before, we have

$$\begin{aligned} \pi_{ri}(x_i) &= mE[\min(x_i, D_i)], \quad i = 1, 2 \\ \pi_p(x_p) &= mE[\min(x_p, D_p)]. \end{aligned}$$

The optimal inventory stock given by the wholesale price $w_1 = w_2 = w_p = w$ is

$$z_1^*(w) = z_2^*(w) = \mu + \sigma\Phi^{-1}(\max(\frac{w - c}{w + h}, \rho)),$$

and

$$z_p^*(w) = 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)).$$

Furthermore, given the wholesale price w, the optimal inventory levels are $x_i = y(w)z_i^*(w)(i = 1, 2)$ and $x_p = y(w)z_p^*(w)$. Hence

$$\begin{aligned}\pi_{r1}(x_1) + \pi_{r2}(x_2) &= my(w)(2\mu - 2\Theta_1(z)),\\ \pi_p(x_p) &= my(w)(2\mu - \Theta_p(z_p)).\end{aligned}$$

From the proof of theorem 4.2.3, we know that

$$2\Theta_1(\mu + \sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho))) \geq \Theta_p(2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho))).$$

Therefore, $\pi_{r1}(x_1) + \pi_{r2}(x_2) \leq \pi_p(x_p)$. ■

When the retailers have the same service level requirements under the reserved and pooled inventory cases, under normally distributed demands, the comparative results indicate that the supplier prefers the pooled inventory policy. If the wholesale prices are the same under the two policies, the retailers will also prefer the pooled inventory policy. We thus find that the comparative results which hold for the scenarios without service level requirements also hold for the scenarios with service level requirements. In addition, we also find that all the comparative results of the two policies under the additive demand model hold under the multiplicative model.

Chapter 6

Infinite Horizon Problem

In this chapter we extend the two-echelon model to allow procurement decisions to be made in each period of an infinite planning horizon. There are one supplier and two retailers; however, there is no inventory at the retailers. At the beginning of the period, the supplier needs to decide how many items to produce. After the retailers observe the demand, they will place an order with the supplier. We assume that there is zero lead time. That part of retailers' order that cannot be met with the supplier's on-hand stock could be lost or backlogged. We consider both cases in this chapter. The supplier bears all the inventory holding and penalty costs, which are charged at the end of each period. We assume the demands in each period are independently and identically distributed random variables. The objective of the supplier is to maximize e his total discounted profit over the infinite horizon.

When the lead time is zero, whether the excess demand is completely backlogged or lost, the optimal inventory policy is an order-up-to policy (see Clark 1960). We are interested in the profits of the retailers and supplier under the reserved and pooled inventory policy. To our knowledge, this problem has not been analyzed mathematically in the inventory theory literature so far. In this chapter, we formulate the dynamic program and provide the optimal policy under the reserved and pooled inventory systems. In addition, we compare these two inventory systems.

This chapter is organized as follows. In Section 6.1, we analyze the backlogging case, provide the supplier's optimal decisions under the reserved and pooled inventory systems, and show the comparative results under these two inventory systems. In Section 6.2, we analyze the lost sales case, provide the supplier's optimal decisions under the reserved, and pooled inventory systems and show the comparative results under these two inventory systems.

6.1 Infinite Horizon with Backlogging

First, we consider the reserved inventory system where the supplier keeps two inventory for each retailer. The sequence of events in a period is as follows.

1. At the beginning of period t, the supplier increases his inventory levels to $y_{i,t}$ $(i = 1, 2)$ from the current inventory levels $x_{i,t}$ $(i = 1, 2)$. Here we use $y_{i,t}, x_{i,t}$ $(i = 1, 2)$ to denote the parameters of the supplier's inventory corresponding to retailer i. Each increased unit of inventory incurs a manufacturing cost of c.

2. In period t, demand $D_{i,t}$ $(i = 1, 2)$ at retailer i is an independently and identically distributed random variable, with CDF $F_i(\cdot)$ and PDF $f_i(\cdot)$, with finite mean μ_i and variance σ_i. After observing the amount of demand, the retailer places an order $D_{i,t}$ to the supplier. Retailer i makes m_i dollars profit for each unit of demands. If the inventory at the supplier cannot satisfy the order, the unmet demands are fully backlogged. The supplier charges w_i for each unit which retailer i orders and his inventory level drops to $y_{i,t} - D_{i,t}$. $(y_{i,t} - D_{i,t})^-$ denotes the level of unmet demands at the end of the period, here, as usual, $x^- := max(-x, 0)$.

3. The supplier bears all risk cost. At the end of the period, an inventory holding cost of h is incurred by the supplier per unit of remaining inventories, or a penalty cost b is incurred per unit of unmet demands.

The pooled inventory system operates in a similar manner except that the supplier only maintains one pooled inventory. We let $y_{p,t}$ and $x_{p,t}$ to denote the parameters of the supplier's pooled inventory. $D_{p,t}$ is used to denote the total demand occurring at retailers. It can be expressed as $D_{p,t} = D_{1,t} + D_{2,t}$. Both of the retailers make m_p dollars profit for each unit of demands. The supplier charges w_p for each unit which retailers order.

For a thorough review of infinite horizon, discounted total profit maximization problems, we refer to Bertsekas(1995). In our problem, we assume that the discounted factor α is between 0 and 1, i.e., $0 \leq \alpha < 1$.

For a given policy π, the supplier will increase the inventory level from $x_{i,t}$ to $y^{\pi}_{i,t}$ for retailer i ($i = 1, 2$) in period t. Our objective is to find the optimal policy π, i.e., optimal order-up-to inventory level in each period . Let $g_{i,t}()$ represent the supplier's profit from retailer i in period t. Here $g_{i,t}()$ is a function of the initial inventory level $x_{i,t}$, the replenished inventory level $y^{\pi}_{i,t}$ and demand $D_{i,t}$ at retailer i. We denote the total expected, discounted profit that the supplier earns from retailer i ($i = 1, 2$) over the infinite horizon as $V^{\pi}_i(x_{i,1})$. Mathematically,

$$V^{\pi}_i(x_{i,1}) = \lim_{N\to\infty} E\left[\sum_{t=1}^{N} \alpha^{t-1} g_{i,t}(y^{\pi}_{i,t}, x_{i,t}, D_{i,t})\right].$$

If the profit function $g_{i,t}()$ is independent of time, i.e.,

$$g_{i,1}() = g_{i,2}() = \ldots = g_{i,N}() = \ldots,$$

we can drop the subscript of t, and,

$$V^{\pi}_i(x_{i,1}) = \lim_{N\to\infty} E\left[\sum_{t=1}^{N} \alpha^{t-1} g_{i}(y^{\pi}_{i,t}, x_{i,t}, D_{i,t})\right].$$

The supplier is interested in finding the optimal policy π from the set of all feasible policies Π and hence the optimal total expected discounted profit over the infinite horizon,

$$V_i^*(x_{i,1}) = \max_{\pi \in \Pi} V_i^\pi(x_{i,1}, D_{i,t}).$$

Dropping the time subscript for notational convenience, consider the reserved inventory system in which the supplier is in charge of inventory i for retailer i ($i = 1, 2$), making all ordering decisions to optimize the expected present value of profit incurred over an infinite horizon here. Each unit of leftover inventory at the end of a period incurs a holding cost. $(y_i - D_i)^+$ denotes the level of leftover inventory at the end of the period, as usual $x^+ := max(x, 0)$. Thus $hE(y_i - D_i)^+$ gives the expected holding cost for the supplier in each period. Similarly, $b(y_i - D_i)^-$ gives the backlog penalty for the supplier in each period. The expected holding and penalty costs incurred in a period can be expressed as follows

$$L(y_i) = hE(y_i - D_i)^+ + bE(y_i - D_i)^-.$$

Therefore, for each period, we use $U_i(\cdot)$ to denote the supplier's expected net profit from retailer i, which is given by

$$\begin{aligned} U_i(y_i, x_i) &= Eg_i(y_i, x_i, D_{i,t}) \\ &= E[w_i D_i - L(y_i) - c(y_i - x_i)] \qquad \text{for i=1,2.} \end{aligned}$$

The first term is the supplier's revenue, the second term is the holding and penalty costs, and the last term is the manufacturing cost. Moving cx_i outside of function $U_i(y_i, x_i)$, $U_i(y_i, x_i)$ can be expressed by the summation of a function of y_i and term cx_i, i.e., we have

$$U_i(y_i, x_i) = cx_i + G_i(y_i),$$

where

$$G_i(y_i) = E[w_i D_i - L(y_i) - cy_i].$$

Notice that $G_i(y_i)$ is a concave function of y_i.

Before we discuss the optimal policy of the infinite horizon problem, we state several assumptions. First we assume that all the cost and price parameters are nonnegative, i.e.,

$$w_i, m_i, c, h, b \geq 0 \quad \text{for i=1,2.}$$

In addition, we assume that the shortage cost is always greater than the manufacturing cost, i.e.,

$$b \geq c.$$

From Clark and Scarf (1960), we know that the optimal policy for this model is an order-up-to policy with a fixed policy parameter R_i (a nonnegative integer) for the inventories corresponding to retailer i ($i = 1, 2$). The policy operates as follows: assume we start at period $t = 1$ with inventory x_i ; at the beginning of each period, if retailers placed an order with the supplier in the previous period, the supplier replenishes his inventory level to R_i. The order-up-to policy implies that when the initial inventory level is less than R_i, the supplier will increase the inventory level to R_i. If the initial inventory level x_i is greater than R_i, the supplier will not make an order until the inventory level drops to or below R_i.

Without loss of generality, we assume that the initial inventory level for retailer i ($i = 1, 2$) at the beginning of the infinite horizon is less than R_i, i.e., $x_i < R_i$. Then R_i may be determined by solving the following equation,

$$\begin{aligned} V_i(x_i) &= \max_{y_i \geq x_i} \{cx_i + G_i(y_i) + \alpha E V_i(y_i - D_i)\} \quad i = 1, 2 \qquad (6.1) \\ &= cx_i + G_i(R_i) + \alpha E V_i(R_i - D_i). \end{aligned}$$

Over an infinite horizon, assume that the optimal order-up-to level R_i policy is applied in periods from the second period. Given this, in order to find the optimal inventory level y_i in the first period, we can solve the following optimization problem,

$$\begin{aligned} V_i(x_i) &= \max_{y_i \geq x_i}\{cx_i + G_i(y_i) + E\left[\lim_{N\to\infty}\sum_{t=2}^{N}\alpha^{t-1}g(y_{i,t}, x_{i,t})\right]\}. \\ &= \max_{y_i \geq x_i} E[cx_i + G_i(y_i) + \alpha c(y_i - D_i) + \alpha G_i(R_i) + \alpha^2 V_i(R_i - D_i)]. \end{aligned}$$

$E[cx_i + G_i(y_i) + \alpha c(y_i - D_i) + \alpha G_i(R_i) + \alpha^2 V_i(R_i - D_i)]$ is a concave function of y_i because $G(y_i)$ is a concave function and $\alpha c(y_i - D_i)$ is a linear function of y_i. Our problem is to maximize a concave function.

Letting $Z_i(y_i) = E\{cx_i + G_i(y_i) + \alpha c(y_i - D_i) + \alpha G_i(R_i) + \alpha^2 V_i(R_i - D_i)\}$, and taking the first derivative of $Z_i(y_i)$ with respect to y_i, we have

$$\begin{aligned} \frac{dZ_i(y_i)}{y_i} &= \frac{dG_i(y_i)}{y_i} + \alpha c \\ &= -(h+b)F_i(y_i) - c + b + \alpha c. \end{aligned}$$

Since R_i is also the optimal order-up-to level for the first period, which implies the following equation,

$$\frac{dZ_i(y_i)}{y_i}|_{y_i=R_i} = 0,$$

and we have the optimal order-up-to level

$$R_i = F_i^{-1}(\frac{b-(1-\alpha)c}{h+b}).$$

Assumptions $b, h, c \geq 0$ and $b > (1-\alpha)c > 0$ imply that $0 \leq \frac{b-(1-\alpha)c}{h+b} < 1$, which guarantees the feasibility of the solution of R_i. In addition, it is easy to show that $\frac{b-(1-\alpha)c}{h+b}$ is increasing in b and decreasing in h. Thus the optimal order-up-to inventory level is increasing in b and decreasing in h. This can be understood intuitively: when the penalty is higher, the supplier will maintain a higher inventory level in order to reduce the quantity of lost sales; when the holding cost is higher, the supplier will maintain a lower inventory level in order to reduce the holding cost.

Given the initial inventory levels x_1 and x_2, the supplier's optimal profit $\Pi_r(x_1, x_2)$ under the reserved inventory system can be expressed as

$$\Pi_r(x_1, x_2) = V_1(x_1) + V_2(x_2).$$

Similarly, for the pooled inventory system, the supplier needs to solve the following optimization problem to maximize his profit,

$$V_p(x_p) = \max_{y_p \geq x_p} \{cx_p + G_p(y_p) + \alpha E V_p(y_p - D_p)\}.,$$

where

$$G_p(y_i) = E[w_p D_p - L(y_p) - cy_p].$$

The optimal order-up-to level for the pooled inventory system is

$$R_p = F_p^{-1}(\frac{b - (1-\alpha)c}{h + b}).$$

The supplier's optimal profit under the pooled inventory system can be expressed as

$$\Pi_p(x_p) = V_p(x_p).$$

Moving cx_i back to the previous (Veinott 1965), we have $cx_i = c\alpha(y_i - D_i)$. For the infinite horizon, given the optimal order-up-to level R_i, we can rewrite the total expected discount profit as follows

$$\begin{aligned} V_i(x_i) &= E[w_i D_i + \alpha c(R_i - D_i) - cR_i - L(R_i) + \alpha V_i(R_i - D_i)] \qquad (6.2) \\ &= E[(w_i - \alpha c)D_i - (1-\alpha)cR_i - L(R_i) + \alpha V_i(R_i - D_i)]. \end{aligned}$$

From equation (6.2), we see that the expected discount total profit does not depend on the initial inventory level. Hence, we can simplify the notation $\Pi_r(x_1, x_2), V_1(x_1), V_2(x_2)$ as Π_r^*, V_1^*, V_2^*. From problem (6.2), the supplier's profit V_i^* from retailer i can be rewritten as

$$V_i^* = \frac{1}{1-\alpha} E[(w_i - \alpha c)D_i - (1-\alpha)cR_i - L(R_i)] \quad \text{for} \quad i = 1, 2.$$

Hence, the supplier's total expected profit Π_r^* can be written as

$$\Pi_r^* = \frac{1}{1-\alpha} E[(w_1 - \alpha c)D_1 + (w_2 - \alpha c)D_2 - (1-\alpha)c(R_1 + R_2) - L(R_1) - L(R_2)].$$

Similarly for the pooled inventory system, we have

$$\Pi_p^* = \frac{1}{1-\alpha} E[(w_p - \alpha c)D_p - (1-\alpha)cR_p - L(R_p)].$$

Since the retailers do not make any decisions, their profits depend on the decisions of the supplier. Their total profit π_r^* and π_p^* under the two different inventory systems are the same, namely

$$\pi_r^* = \pi_p^* = \sum_{i=0}^{\infty} \alpha^i m_i E[D_1 + D_2];$$

Now we will focus on the profit of the supplier under the reserved and pooled inventory systems. For the case of normally distributed demands, we can provide a detailed comparison of the reserved and pooled inventory cases. We assume that D_1 and D_2 are independently and normally distributed random variables with mean value μ_1 and μ_2 and standard deviations σ_1 and σ_2. Let $\Phi(\cdot)$ be the CDF of the standard normal distribution.

Theorem 6.1.1 *If D_1 and D_2 are independently and normally distributed random variables, and all the wholesale prices under the reserved and pooled inventory systems are the same, i.e., $w_1 = w_2 = w_p = w$, the supplier's optimal profit is increased when the inventory is pooled, i.e., $\Pi_p^* \geq \Pi_r^*$.*

Proof. Under the reserved inventory scenario, the optimal inventory levels are given by

$$R_1 = \mu_1 + \sigma_1 \Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right),$$

and

$$R_2 = \mu_2 + \sigma_2 \Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right).$$

Under the pooled inventory scenario, $D_p \sim N(\mu_1+\mu_2, \sqrt{\sigma_1^2+\sigma_2^2})$. Hence the optimal total inventory level for the pooled inventory scenario is

$$R_p = \mu_1 + \mu_2 + \sqrt{\sigma_1^2+\sigma_2^2}\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right).$$

The supplier's optimal profits in the reserved inventory case and the pooled inventory case are

$$\begin{aligned}
\Pi_r^* &= \frac{1}{1-\alpha}E[(w-\alpha c)(D_1+D_2)-(1-\alpha)c(R_1+R_2)-L(R_1)-L(R_2)], \\
&= \frac{1}{1-\alpha}E[(w-\alpha c-b)(D_1+D_2)-((1-\alpha)c+h)(R_1+R_2) \\
&\quad +(h+b)(\min(R_1,D_1)+\min(R_2,D_2))]
\end{aligned}$$

Recalling that

$$\min(R_i, D_i) = \mu_i - \sigma_i \Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right),$$

we have

$$\begin{aligned}\Pi_r^* &= \frac{1}{1-\alpha}\left((w-\alpha c-b)(\mu_1+\mu_2)-((1-\alpha)c+h)\left(\mu_1+\mu_2+(\sigma_1+\sigma_2)\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right.\\ &\left.+(h+b)(\mu_1+\mu_2-(\sigma_1+\sigma_2)\ R\left(\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right),\end{aligned}$$

where

$$R(x)=\int_x^\infty (z-x)d\Phi(z).$$

$$\begin{aligned}\Pi_p^* &= \frac{1}{1-\alpha}E[(w-\alpha c)D_p-(1-\alpha)cR_p-L(R_p)],\\ &= \frac{1}{1-\alpha}\left((w-\alpha c-b)(\mu_1+\mu_2)-((1-\alpha)c+h)\left(\mu_1+\mu_2+\sqrt{\sigma_1^2+\sigma_2^2}\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right.\\ &\left.+\ (h+b)(\mu_1+\mu_2-\sqrt{\sigma_1^2+\sigma_2^2}R\left(\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right).\end{aligned}$$

Calculating the difference $\Pi_r-\Pi_p$, we have

$$\begin{aligned}\Pi_r-\Pi_p &= \frac{1}{1-\alpha}\left(((1-\alpha)c+h)(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1-\sigma_2)\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right.\\ &\left.+\ (h+b)(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1-\sigma_2)R\left(\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right)\\ &< \frac{1}{1-\alpha}\left(((1-\alpha)c+h)(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1-\sigma_2)\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right.\\ &\left.((1-\alpha)c+h)\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_1-\sigma_2)R\left(\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right)\\ &= \frac{1}{1-\alpha}\left(((1-\alpha)c+h)(\sqrt{\sigma_1^2+\sigma_2^2}-(\sigma_1+\sigma_2))\left(\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right.\right.\\ &\left.\left.+R\left(\Phi^{-1}\left(\frac{b-(1-\alpha)c}{h+b}\right)\right)\right)\right)\\ &\leq 0.\end{aligned}$$

The first inequality follows from the assumption that $b>c$ which implies that $b>(1-\alpha)c$ for $0\leq\alpha<1$, and $R(\cdot)>0$ and $\sqrt{\sigma_1^2+\sigma_2^2}\leq(\sigma_1+\sigma_2)$ for any positive σ_1 and σ_2. The second inequality is shown by applying equation $R(x) + x \geq 0$ for any x (see Chapter 3). Hence, the supplier's profit is increased after pooling the inventory, i.e., $\Pi_r^* \leq \Pi_p^*$. ■

Theorem 6.1.2 *If D_1 and D_2 are independently and normally distributed random variables, and the critical ratio $\frac{b-(1-\alpha)c}{h+b} \geq 0.5$, then the supplier's optimal inventory level in the pooled inventory case is less than the optimal total inventory in the reserved inventory case, i.e., $R_p \leq R_1 + R_2$. Otherwise, $R_p > R_1 + R_2$.*

Proof. We know that the optimal total inventory level in the reserved inventory case is given by,

$$R_1 + R_2 = \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}(\frac{b-(1-\alpha)c}{h+b}).$$

However the optimal inventory level in the pooled inventory case is given by

$$R_p = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}((\frac{b-(1-\alpha)c}{h+b}).$$

The facts that $\Phi^{-1}(\cdot)$ is a monotone nondecreasing function and $\Phi^{-1}(0.5) = 0$ imply that $\Phi^{-1}((\frac{b-(1-\alpha)c}{h+b}) \geq 0$ provided $(\frac{b-(1-\alpha)c}{h+b}) \geq 0.5$. Hence $R_p \leq R_1 + R_2$ follows from the inequality, $\sigma_1 + \sigma_2 \geq \sqrt{\sigma_1^2 + \sigma_2^2}$. Otherwise, when $(\frac{b-(1-\alpha)c}{h+b}) < 0.5$, we have $R_p > R_1 + R_2$. ■

Under independently and normally distributed demands, the supplier will benefit from the pooled inventory policy, and the total profit of the retailer cannot increase from the pooled inventory policy. However, the supplier's optimal total inventory level may increase or decrease after pooling the inventory. When the penalty cost of lost sales is high enough or the holding inventory cost is low enough, the supplier will reduce the inventory level to a lower level at the beginning of the period after pooling the inventory. Otherwise, the supplier will raise inventory to a higher level after pooling the inventory.

6.2 Infinite Horizon with Lost-sales

We have discussed the optimal policy of the backlogging case and comparative results under the pooled inventory and the reserved inventory systems in the previous section. Now, we are interested in whether the same results hold when unsatisfied demand at each period is lost. In this section, we will to answer this question. We will keep the same notation as that in the previous section.

For the reserved inventory system, given any inventory level x_i of the inventory i ($i = 1, 2$) at the beginning of period, we define $V_i(x_i)$ as the maximum total discount expected profit. Thus we have the following dynamic equation

$$V_i(x_i) = \max_{y_i \geq x_i} E[cx_i + w_i \min(y_i, D_i) - cy_i - h(y_i - D_i)^+ - b(y_i - D_i)^- + \alpha V_i(y_i - D_i)^+].$$

It is well known that the order-up-to policy is still the optimal policy for this case (Clark 1960). As usual, we assume that the initial inventory level $x_i < R_i$. Assuming that the optimal order-up-to level is R_i, we have the following equation,

$$\begin{aligned} V_i(x_i) &= \max_{y_i \geq x_i} E[cx_i + w_i \min(y_i, D_i) - cy_i - h(y_i - D_i)^+ - b(y_i - D_i)^- + \alpha V_i(y_i - D_i)^+] \\ &= E[cx_i + w \min(R_i, D_i) - cR_i - h(R_i - D_i)^+ - b(R_i - D_i)^- + \alpha V_i(R_i - D_i)^+]. \end{aligned}$$

Applying the same approach as in the previous section, we consider the problem of deciding the optimal the order-up-to level y_i during the first period, while order-up-to level R_i is applied in the periods following the first period. The initial inventory level at the beginning of second period is $(y_i - D_i)^+$. We can write the problem as

$$\begin{aligned} V_i(x_i) &= \max_{y_i \geq x_i} E[cx_i + w_i \min(y_i, D_i) - cy_i - h(y_i - D_i)^+ - b(y_i - D_i)^- \\ &\quad +\alpha(c(y_i - D_i)^+ + w_i \min(R_i, D_i) - cR_i - h(R_i - D_i)^+ - b(R_i - D_i)^-) + \alpha V_i(R - D_i)^+)]. \end{aligned}$$

Let $U_i(y_i) = E[w_i \min(y_i, D_i) - cy_i + (\alpha c - h)(y_i - D_i)^+ - b(y_i - D_i)^-]$, which is a concave function. Taking the first derivative of $U_i(y_i)$ with respect to y_i, we have

$$\frac{dU_i(y_i)}{y_i} = w_i - c + b + (\alpha c - h - w_i - b)F_i(y_i).$$

Since R_i is also the optimal order-up-to level in the first period, it also satisfies the following equation

$$\frac{dU_i(y_i)}{y_i}|_{y_i=R_i} = 0,$$

which implies that

$$R_i = F_i^{-1}(\frac{w_i + b - c}{h + b + w_i - \alpha c}).$$

Under the optimal policy, the supplier's profit under the reserved inventory system can be expressed as

$$\Pi_r(x_1, x_2) = V_1(x_1) + V_2(x_2).$$

Applying the same approach to the pooled inventory system, the supplier needs to solve the following optimization problem to maximize his profit

$$V_p(x_p) = E[cx_p + w_p \min(R_p, D_p) - cR_p - h(R_p - D_p)^+ - b(R_p - D_p)^- + \alpha V_p(R_p - D_p)^+].$$

The optimal order-up-to level for the pooled inventory system is

$$R_p = F_p^{-1}(\frac{w_p + b - c}{h + b + w_p - \alpha c}).$$

The supplier's optimal profit under the pooled inventory system can be expressed as

$$\Pi_p(x_p) = V_p(x_p).$$

In the reserved inventory policy, given the optimal order-up-to level R_i ($i = 1, 2$) and initial inventory level x_i, we also can rewrite the supplier's expected discount profit from retailer i as follows

$$\begin{aligned} V_i(x_i) &= E[(\alpha c - h)(R_i - D_i)^+ + w_i \min(R_i, D_i) \\ &\quad -cR_i - b(R_i - D_i)^- + \alpha V_i(R_i - D_i)^+]. \end{aligned} \tag{6.3}$$

In the pooled inventory policy, given the optimal order-up-to level R_p and initial inventory level x_p, we also can rewrite the supplier's expected discount profit from retailer i as follows

$$\begin{aligned} V_p(x_p) &= E[(\alpha c - h)(R_p - D_p)^+ + w_p \min(R_p, D_p) \\ &\quad -cR_p - b(R_p - D_p)^- + \alpha V_p(R_p - D_p)^+]. \end{aligned} \tag{6.4}$$

From the supplier's profit expression, we see that the discounted total profit does not depend on the initial inventory level. Hence, we can simplify the notation $\Pi_r(x_1, x_2), V_1(x_1), V_2(x_2)$ as Π_r^*, V_1^*, V_2^*. From equation (6.3), under the reserved inventory system, the supplier's profit can be rewritten as

$$\Pi_r^* = V_1^* + V_2^*$$

where

$$V_i^* = \frac{1}{1-\alpha} E[(\alpha c - h)(R_i - D_i)^+ + w_i \min(R_i, D_i) - cR_i - b(R_i - D_i)^-], \quad \text{for } i = 1, 2.$$

In addition,under the pooled inventory system, the supplier's profit can be written as

$$\Pi_p^* = \frac{1}{1-\alpha} E[(\alpha c - h)(R_p - D_p)^+ + w_p \min(R_p, D_p) - cR_p - b(R_p - D_p)^-].$$

Since the retailers, they do not make any decisions and their profits depend on the decisions of the supplier. Let π_r^* and π_p^* be the retailers' total profit under the reserved and pooled inventory cases, respectively, which can be expressed as,

$$\begin{aligned}\pi_r^* &= \sum_{i=0}^{\infty} \alpha^i E[m_1 \min(R_1, D_1) + m_2 \min(R_2, D_2)];\\ \pi_p^* &= \sum_{i=0}^{\infty} \alpha^i E[m_p \min(R_p, D_p)].\end{aligned}$$

The total profit of the retailers under the two different cases is different under this scenario while they are same under the backlogging scenario.

When demands are normally distributed, we find that the results(that supplier's profit is increased after pooling inventory) still holds.

Theorem 6.2.1 *If D_1 and D_2 are independently and normally distributed random variables, and all the wholesale prices under the reserved and pooled inventory systems are the same, i.e., $w_1 = w_2 = w_p = w$, the supplier's optimal profit is increased when the inventory is pooled, i.e., $\Pi_p^* \geq \Pi_r^*$.*

Proof. Under the reserved inventory scenario, the optimal inventory levels are given by

$$R_1 = \mu_1 + \sigma_1 \Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}),$$

and

$$R_2 = \mu_2 + \sigma_2 \Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}).$$

For the pooled inventory scenario, $D_p \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. Hence the optimal total inventory level for the pooled inventory scenario is

$$R_p = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}).$$

The supplier's optimal profits in the reserved and pooled inventory cases are

$$
\begin{aligned}
\Pi_r^* &= \sum_{i=1}^{2} \frac{1}{1-\alpha} E[(\alpha c - h)(R_i - D_i)^+ + w \min(R_i, D_i) - cR_i - b(R_i - D_i)^-] \\
&= \sum_{i=1}^{2} \frac{1}{1-\alpha} E[(\alpha c - h - c)R_i - (\alpha c - h - w - b)\min(R_i, D_i) - bD_i] \\
&= \sum_{i=1}^{2} \frac{1}{1-\alpha} [(\alpha c - h - c)(\mu_i + \sigma_i \Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c})) \\
&\quad -(\alpha c - h - w - b)(\mu_i - \sigma_i R(\Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}))) - b\mu_i]
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_p^* &= \frac{1}{1-\alpha} E\left[(\alpha c - h)(R_p - D_p)^+ + w \min(R_p, D_p) - cR_p - b(R_p - D_p)^-\right] \\
&= \frac{1}{1-\alpha} \left[(\alpha c - h - c)(\mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}))\right. \\
&\quad \left. -(\alpha c - h - w - b)(\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2} R(\Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}))) - b(\mu_1 + \mu_2)\right].
\end{aligned}
$$

Calculating the difference of Π_r and Π_p, we have

$$
\begin{aligned}
\Pi_r^* - \Pi_p^* &= \frac{1}{1-\alpha} \left[(\alpha c - h - c)\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right) \Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right. \\
&\quad \left. +(\alpha c - h - w - b)\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right) R\left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right)\right] \\
&< \frac{1}{1-\alpha} \left[(\alpha c - h - c)\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right) \Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right. \\
&\quad \left. +(\alpha c - h - c)\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right) R\left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right)\right] \\
&= \frac{1}{1-\alpha} \left[(\alpha c - h - c)\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right)\right. \\
&\quad \left. \left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right) + R\left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right)\right)\right] \\
&\leq 0.
\end{aligned}
$$

The assumption that $b > c, w \geq 0$ implies that $b+w > \alpha c$ for $0 \leq \alpha < 1$. The first inequality follows from $R(\cdot) > 0$ and the inequality $\sqrt{\sigma_1^2 + \sigma_2^2} \leq \sigma_1 + \sigma_2$ for any positive σ_1 and σ_2. The second inequality is shown by applying equation $R(x) + x \geq 0$ for any x (see Chapter 3). Hence, the supplier's profit is increased after pooling the inventory, i.e., $\Pi_r \leq \Pi_p$. ■

Theorem 6.2.2 *If D_1 and D_2 are independently and normally distributed random variables, and all the markup prices are the same, i.e., $m_1 = m_2 = m_p = m$, the total profit of the retailers is increased when the inventory is pooled, i.e., $\pi_p^* \geq \pi_r^*$.*

Proof. Assume R_i $(i = 1, 2)$ is the optimal order-up-to level for retailer i in the reserved inventory system and R_p is the optimal order-up-to level in the pooled inventory system, we first show that total expected sales in each period in the pooled inventory is higher than in the reserved inventory system.

$$
\begin{aligned}
& E[\min(R_1, D_1) + \min(R_2, D_2) - \min(R_p, D_p)] \\
= \; & \mu_1 + \mu_2 - (\sigma_1 + \sigma_2) R\left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right) \\
& - \left(mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2} R\left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right)\right) \\
= \; & (\sqrt{\sigma_1^2 + \sigma_2^2} - \sigma_1 + \sigma_2) R\left(\Phi^{-1}\left(\frac{w+b-c}{h+b+w-\alpha c}\right)\right) \\
\leq \; & 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
\pi_r^* &= \sum_{i=0}^{\infty} \alpha^i m E\{\min(R_1, D_1) + \min(R_2, D_2)\}; \\
&\leq \sum_{i=0}^{\infty} \alpha^i m E\{\min(R_p, D_p)\} \\
&= \pi_p^*,
\end{aligned}
$$

which completes the proof. ■

Theorem 6.2.3 *If D_1 and D_2 are independently and normally distributed random variables, and the critical ratio $\frac{w+b-c}{h+b+w-\alpha c} \geq 0.5$, then the supplier's optimal inventory level in the pooled inventory case is less than the optimal total inventory in the reserved inventory case, i.e., $R_p \leq R_1 + R_2$. Otherwise, $R_p > R_1 + R_2$.*

Proof. We know that the optimal total inventory level in the reserved inventory case is given by,

$$R_1 + R_2 = \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}).$$

However the optimal inventory level in the pooled inventory case is given by

$$R_p = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}((\frac{w+b-c}{h+b+w-\alpha c}).$$

The fact that $\Phi^{-1}(\cdot)$ is a monotone, nondecreasing function and $\Phi^{-1}(0.5) = 0$ implies that $\Phi^{-1}(\frac{w+b-c}{h+b+w-\alpha c}) \geq 0$ provided $\frac{w+b-c}{h+b+w-\alpha c} \geq 0.5$. Hence $R_p \leq R_1 + R_2$ follows from the inequality, $\sigma_1 + \sigma_2 \geq \sqrt{\sigma_1^2 + \sigma_2^2}$. Otherwise, when $\frac{w+b-c}{h+b+w-\alpha c} < 0.5$, we have $R_p > R_1 + R_2$. ■

Under identically and normally distributed demands, the supplier will benefit from the pooled inventory policy while the retailer's total profit will decrease after pooling the inventory. However, the supplier's total inventory level may increase or decrease after pooling the inventory. When the penalty cost of lost sales is high enough or the holding inventory cost is low enough, then the supplier will increase a lower inventory level at the beginning of the period after pooling the inventory. Otherwise, the supplier will raise inventory to a higher level after pooling the inventory.

Chapter 7

Finite Horizon Problem

Having analyzed the pooled and reserved inventory systems for an infinite horizon, we are interested in whether all the results hold for a finite horizon. In this chapter we try to answer this question.

This chapter is organized as follows. In Section 7.1, for the backlogging case, we analyze the supplier and retailers under the pooled and reserved inventory system. In Section 7.2, we give the corresponding results for the lost-sales case.

7.1 Finite Horizon with Backlogging

In our model in this section, there is a finite selling season for the retailers which is divided into T periods. When the supplier's inventory is not large enough to cover all of the retailers' orders, the shortage is backlogged. We will focus on the reserved inventory system first. As in the infinite case, we use index i ($i = 1, 2$) to distinguish retailers under the reserved inventory system.

Next we formulate the multi-period profit function. Note that in this section $x_{i,t}$ represents the initial inventory level reserved for retailer i at the beginning of period t and $y_{i,t}$ ($y_{i,t} \geq x_{i,t}$) represents the inventory level after ordering. The demands that occur at retailer i in each period are identically distributed and denoted simply D_i. We define $V_{i,t}(x_{i,t})$

as the supplier's maximum total discounted expected profit from retailer i from period t through the end of planning horizon (period T), given $x_{i,t}$ at the beginning of period t.

We have the following recursion equation

$$V_{i,t}(x_{i,t}) = \max_{y_{i,t} \geq x_{i,t}} \{cx_{i,t} + G_i(y_{i,t}) + \alpha EV_{i,t+1}(y_{i,t} - D_i)\}$$

for $t = T, \ldots, 1$, where

$$G_i(y_{i,t}) = E[w_i D_i - hE(y_{i,t} - D_i)^+ - bE(y_{i,t} - D_i)^- - cy_{i,t}]$$

and $V_{i,T+1}(x_{i,T+1}) = 0$ for all $x_{i,T+1}$.

For the pooled inventory system, $x_{p,t}$ represents the initial total inventory level at the beginning of period t, and $y_{p,t} \geq x_{p,t}$ is total inventory level after ordering. We define $V_{p,t}(x_{p,t})$ as the supplier's maximum total discounted expected profit from period t through the end of planning horizon, given $x_{p,t}$ and set $V_{p,T+1}(x_{p,T+1}) = 0$ for all $x_{p,T+1}$. As before, the total demand $D_p = D_1 + D_2$.

For the pooled inventory case, we have the following recursion equation,

$$V_{p,t}(x_{p,t}) = \max_{y_{p,t} \geq x_{p,t}} \{cx_{p,t} + G(y_{p,t}) + \alpha EV_{p,t+1}(y_{p,t} - D_p)\}$$

for $t = T, \ldots, 1$, where

$$G_p(y_{p,t}) = E[w_p D_p - hE(y_{p,t} - D_p)^+ - bE(y_{p,t} - D_p)^- - cy_{p,t}].$$

In this chapter, we will compare supplier's profit, and supplier's inventory levels after ordering under the reserved and the pooled inventory systems. Because the unsatisfied part of any order is fully backlogged, the total profit of retailers is the same under both of the systems.

Due to the difficulty in obtaining closed form expressions for the expected T-period profit and inventory level, we can only present numerical evidence to support our claims. We assume that the demands are normally distributed, and all the wholesale prices and markup price are the same, i.e., $w_1 = w_2 = w_p = w$, $m_1 = m_2 = m_p = m$. We use the parameter set listed in Table 7.1 as a "base case".

holding cost	h	5
purchasing cost	c	3
penalty cost	b	10
wholesale price	w	10
discount factor	α	0.95
mean	μ	50
standard deviation	σ	12
horizon	T	10
markup price	m	1

Table 7.1: Parameter Table

The values of the parameters satisfy the following assumptions of the problem

- Purchasing cost is less than penalty cost, i.e., $c < b$.
- Purchasing cost and penalty cost are less than the wholesale price, i.e., $c,\ b \leq w$; otherwise, the supplier cannot make profit.
- The demand should be nonnegative. For the normal distribution, we assume that the probability of demand in the range $[\mu - 4\sigma, \mu + 4\sigma]$ is nonzero; for values outside this range, the probability is 0.

Claim 7.1.1 *For the backlogging case, the profit of the supplier under the pooled inventory system is greater than under the reserved inventory system.*

In order to validate this claim, we compare the profits of the supplier with and without pooling under a wide range of parameter values. Specifically, we vary, one at a time, the discount factor α from 0 to 1, the penalty cost b from 3 to 10, the wholesale price from

10 to 20, the purchasing cost c from 1 to 10, the holding cost h from 0 to 20, the mean of demand μ from 40 to 60, and the standard deviation of demand σ from 0 to 15.(see Figures 7.1-7.7.)

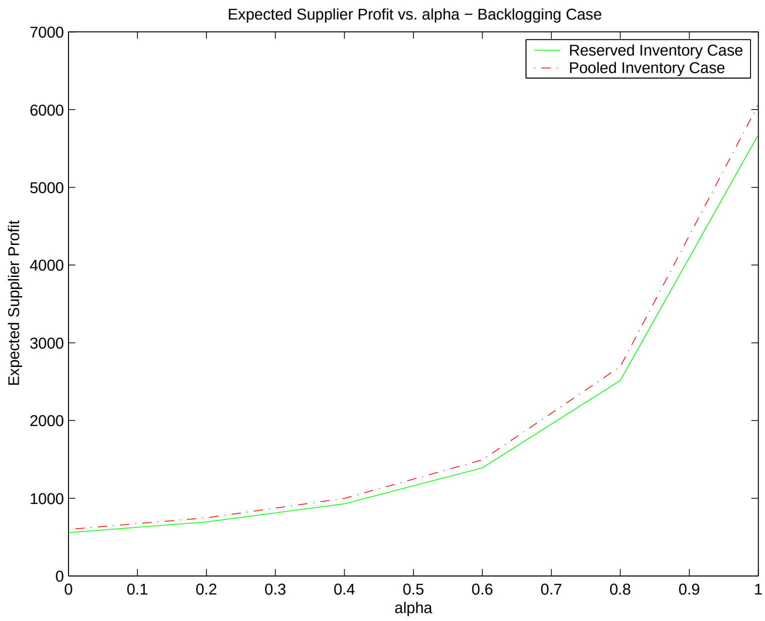

Figure 7.1: Supplier's Profit vs. Discount Factor α

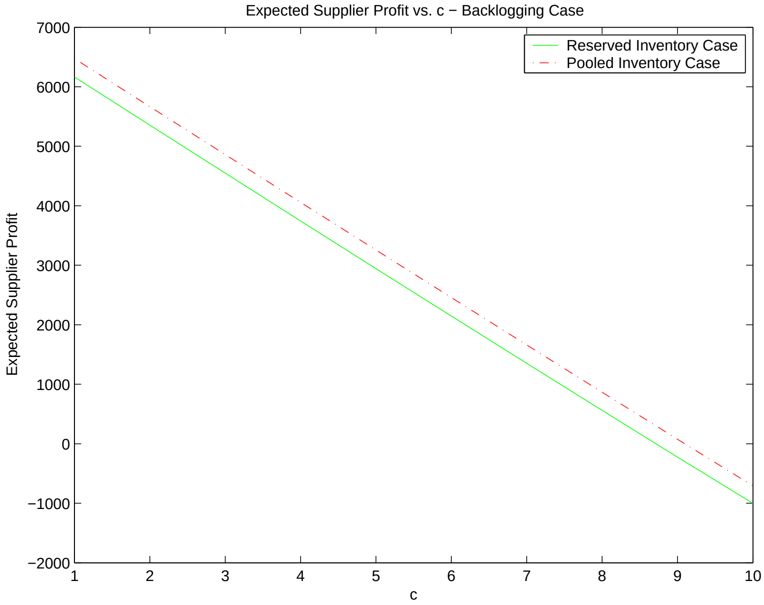

Figure 7.2: Supplier's Profit vs. Manufacturing Cost c

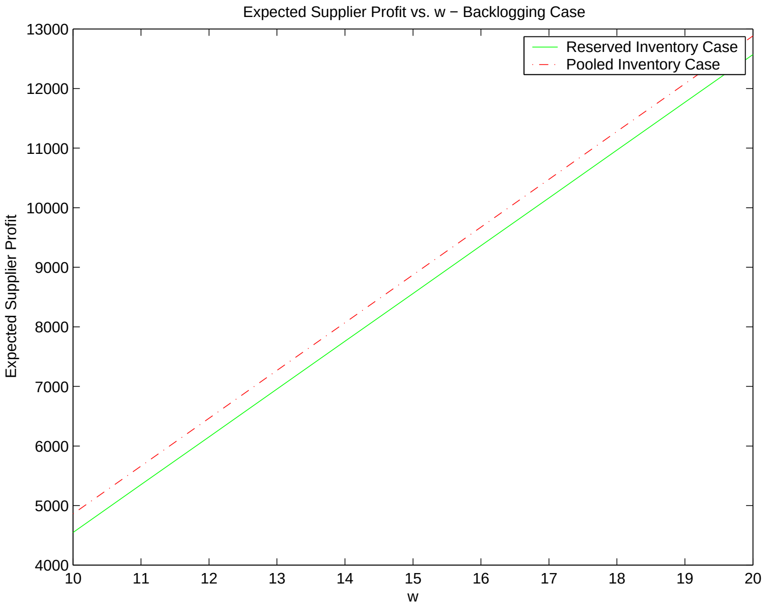

Figure 7.3: Supplier's Profit vs. Wholesale Price w

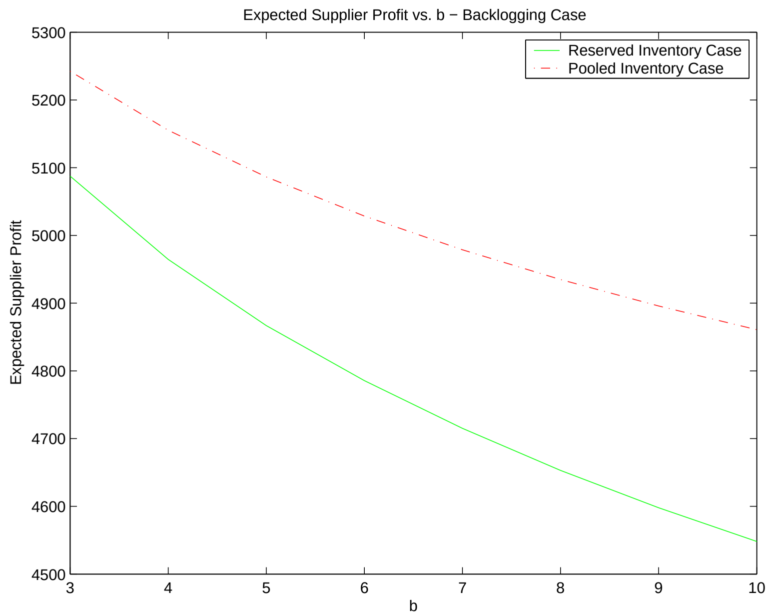

Figure 7.4: Supplier's Profit vs. Penalty Cost b

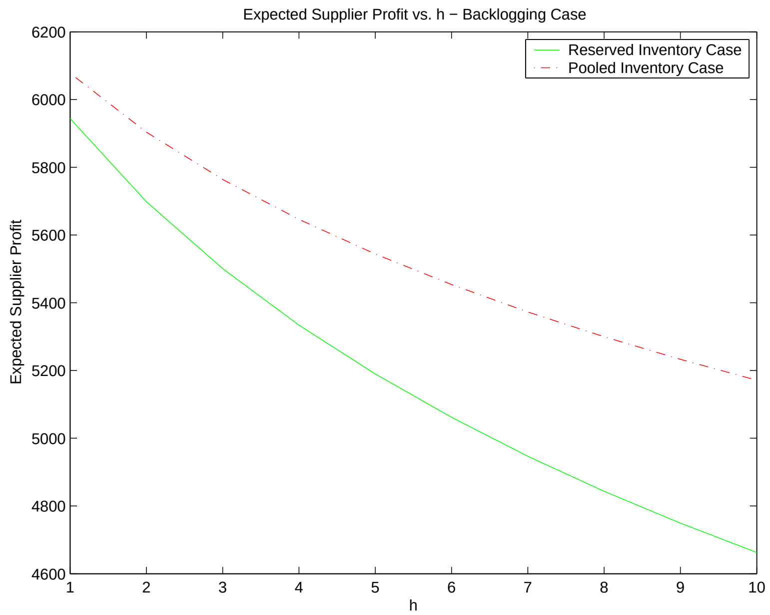

Figure 7.5: Supplier's Profit vs. Holding Cost h

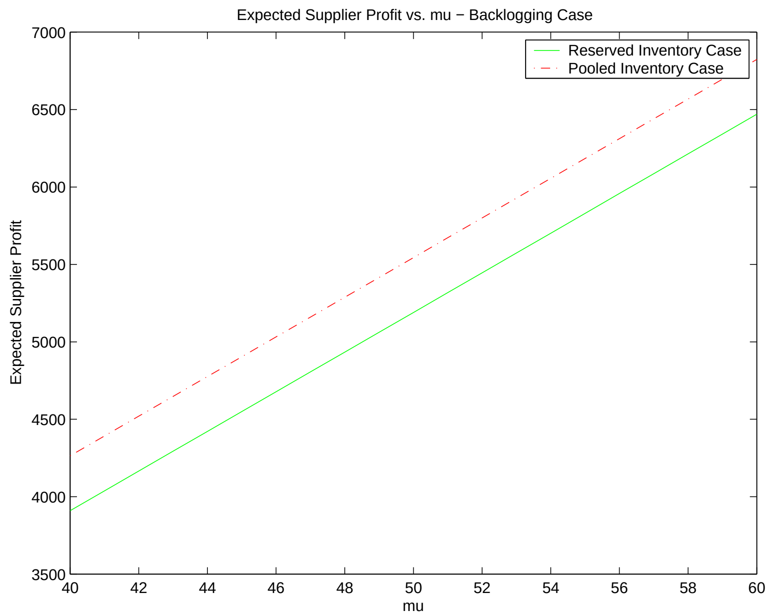

Figure 7.6: Supplier's Profit vs. Mean μ

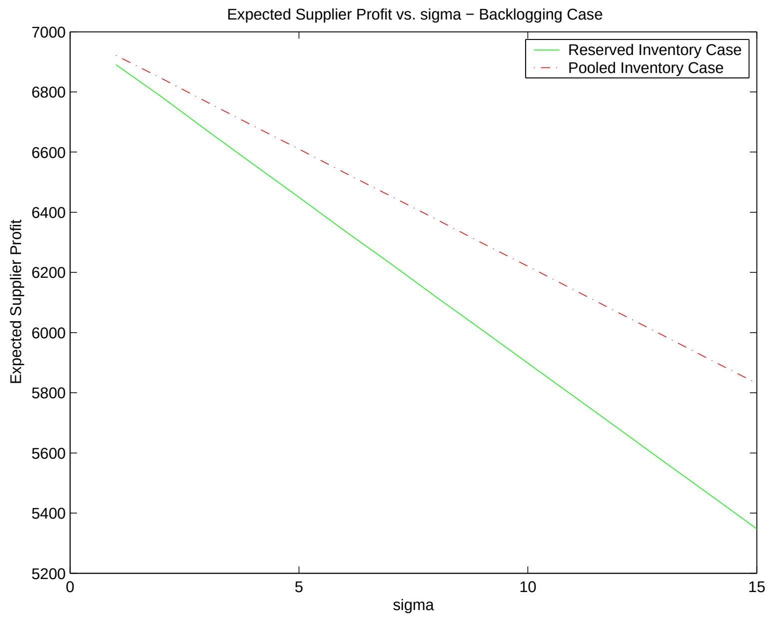

Figure 7.7: Supplier's Profit vs. Standard Deviation σ

The results in the figures all support Claim 7.1. However, we cannot say the total inventory level after ordering in the reserved inventory system will be greater or smaller than that in the pooled inventory system. Thus we have the following claim:

Claim 7.1.2 *The inventory level after ordering in the pooled inventory case may be greater or less than the total inventory in the reserved inventory case.*

Figure 7.8 shows that the total inventory level after ordering in period 1 for 10-period horizon in the reserved inventory system is less than that in the pooled inventory system when the discount factor α is small while it becomes larger than that in the pooled inventory system when α is large. A similar switch occurs as the penalty cost increases (see Figure 7.12). Figure 7.14 shows that the total inventory level after ordering in the reserved inventory system is less than that in the pooled inventory system when standard deviation σ is less than 1. For the other parameters, the total inventory level after ordering in period 1 in the reserved inventory system is greater than that in the pooled inventory system independent of parameter values of holding cost h, mean value μ, manufacturing cost c and wholesale price w(see Figures 7.9, 7.10, 7.11 and 7.13).

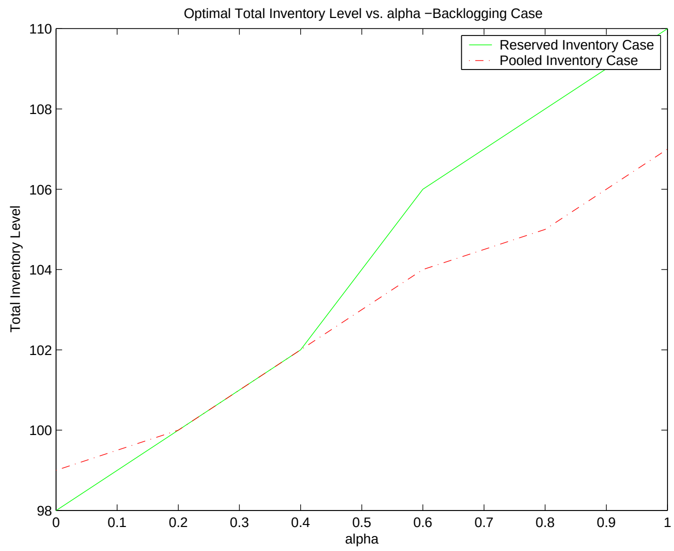

Figure 7.8: Supplier's Total Inventory Level vs. Discount Factor α

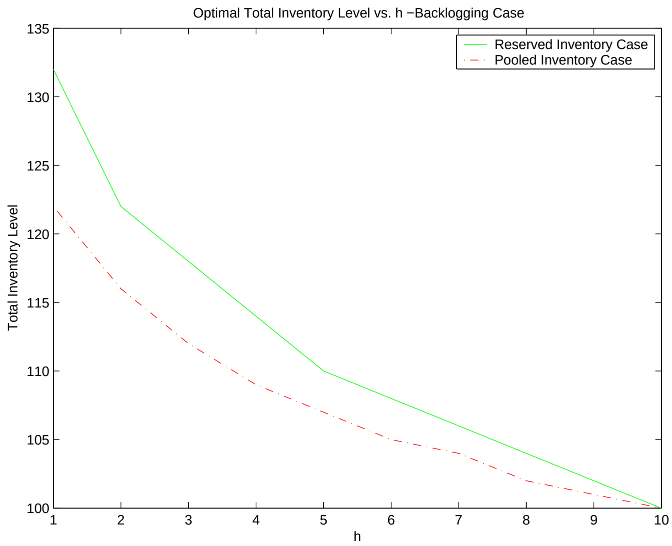

Figure 7.9: Supplier's Total Inventory Level vs. Holding Cost h

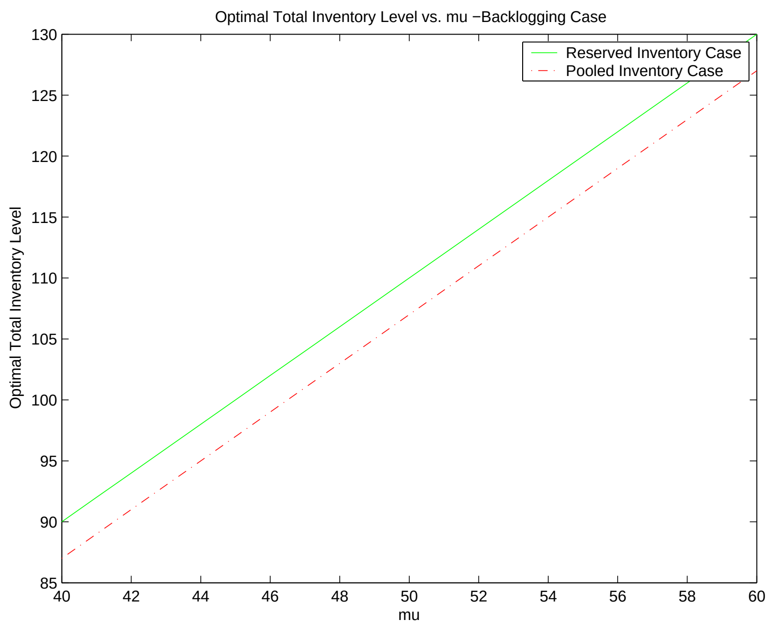

Figure 7.10: Supplier's Total Inventory Level vs. Mean μ

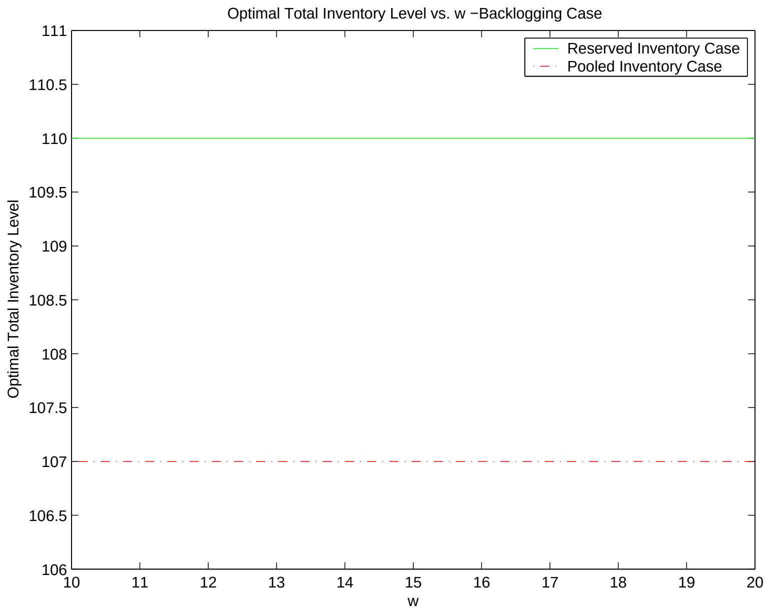

Figure 7.11: Supplier's Total Inventory Level vs. Wholesale Price w

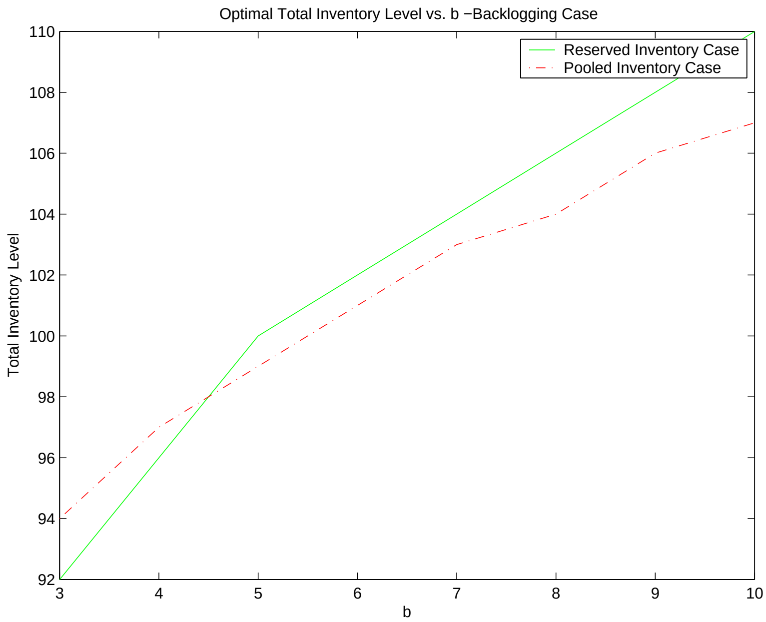

Figure 7.12: Supplier's Total Inventory Level vs. Penalty Cost b

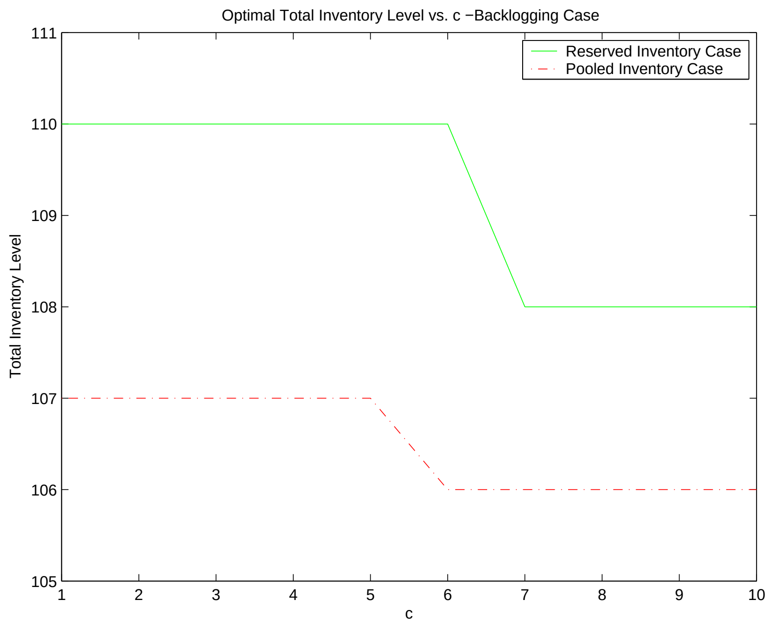

Figure 7.13: Supplier's Total Inventory Level vs. Manufacturing Cost c

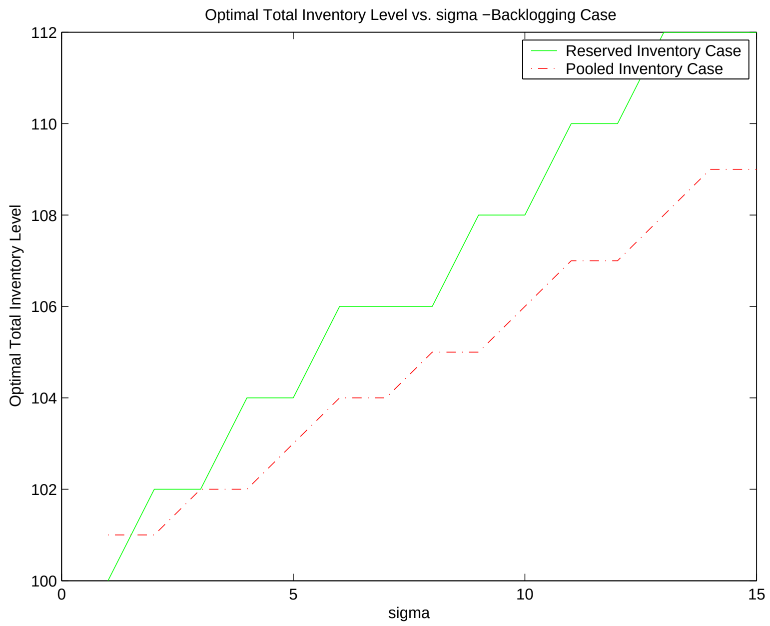

Figure 7.14: Supplier's Total Inventory Level vs. Standard Deviation σ

Dynamic inventory theory indicates that optimal inventory level $y_{p,t}$ in period t after ordering in a finite horizon is always nonincreasing as t approaches the end of horizon(period T) and not larger than the optimal inventory level y_p in the infinite horizon case, i.e., $y_{p,T} \leq y_{p,T-1} \ldots \leq y_1 \leq y_p$; equivalent results applies for y_{it} $i = 1, 2; t = 1, 2, \ldots, T$. Figures 7.15-7.21 verify this for a wide range of data. Specifically, Figures 7.15-7.21 show that the optimal pooled inventory level in period 10 in 10-period horizon is always consistently less than that in the infinite horizon for a wide range of data. Whereas Figure 7.22 shows that the optimal pooled inventory level in period 9 in 10-period horizon is consistently the same as that in the infinite horizon, which matches the mathematical results well. The results in Figure 7.22 indicate that convergence of $y_{p,t}$ to y_p occurs very quick. Similar results hold for the reserved inventory case.

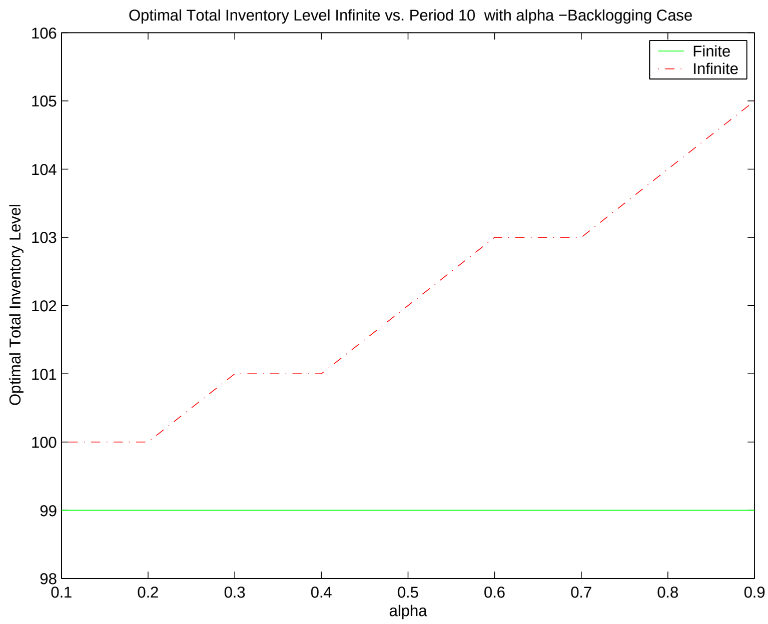

Figure 7.15: Supplier's Total Inventory Level for Period 10 vs. Discount Factor α

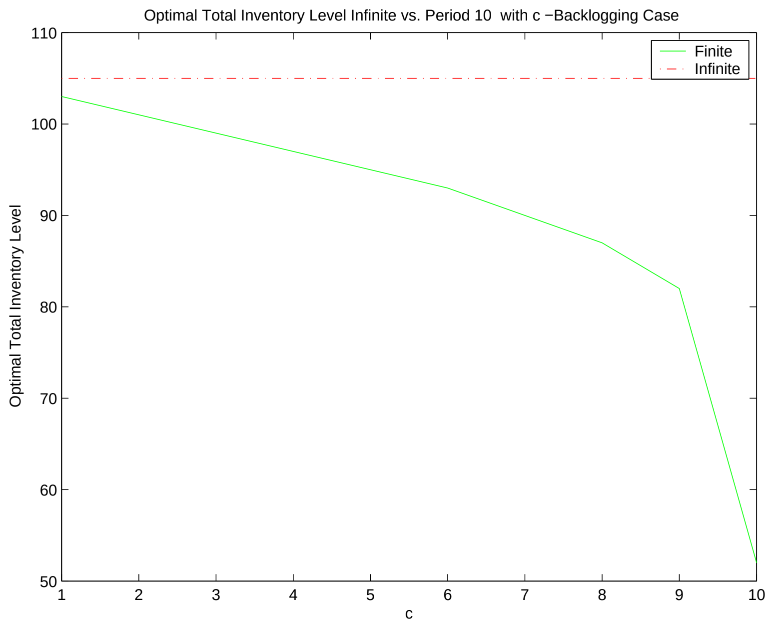

Figure 7.16: Supplier's Total Inventory Level for Period 10 vs. Manufacturing Cost c

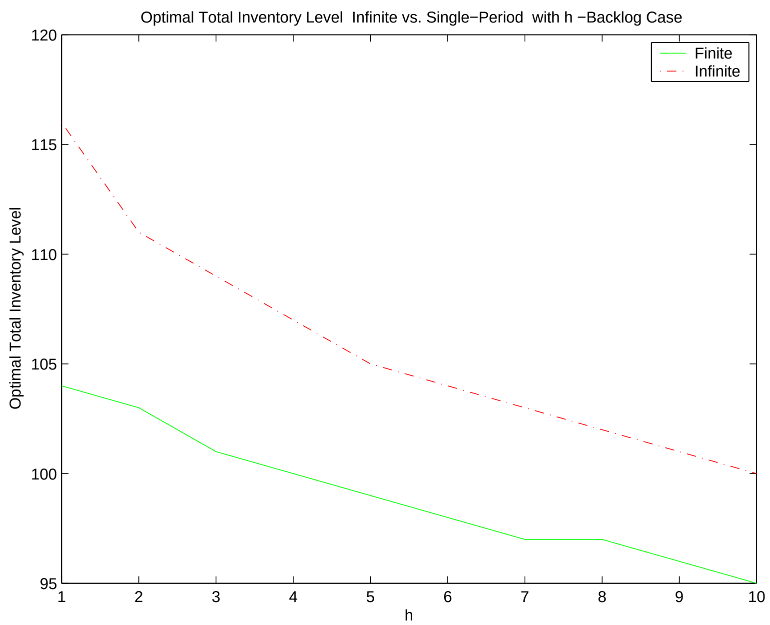

Figure 7.17: Supplier's Total Inventory Level for Period 10 vs. Holding Cost h

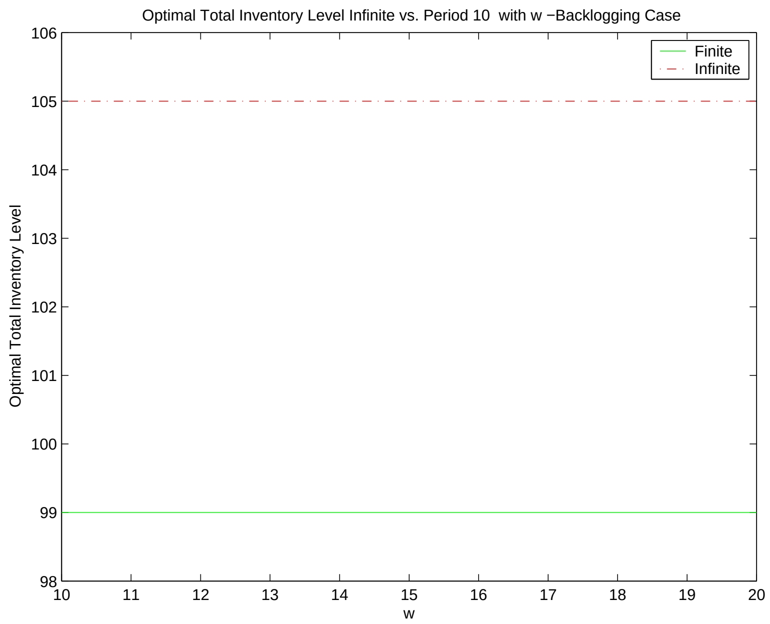

Figure 7.18: Supplier's Total Inventory Level for Period 10 vs. Wholesale Price w

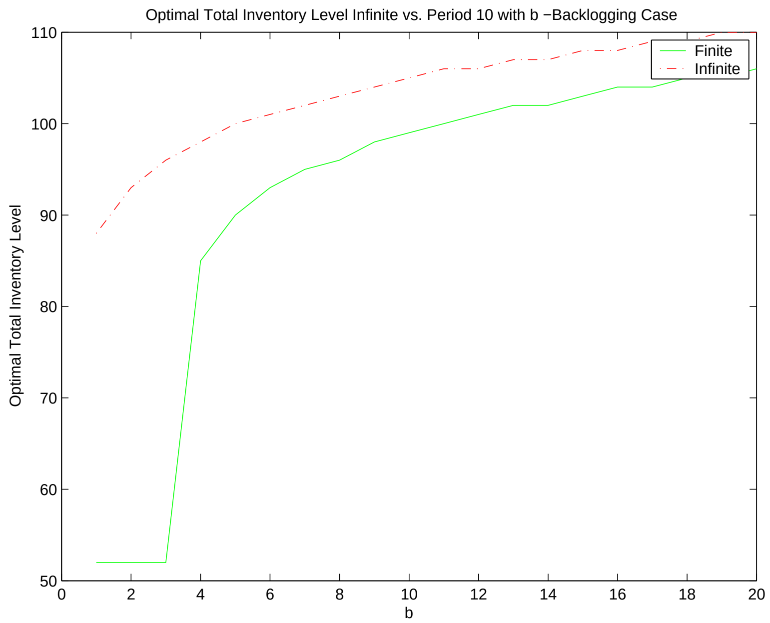

Figure 7.19: Supplier's Total Inventory Level for Period 10 vs. Penalty Cost b

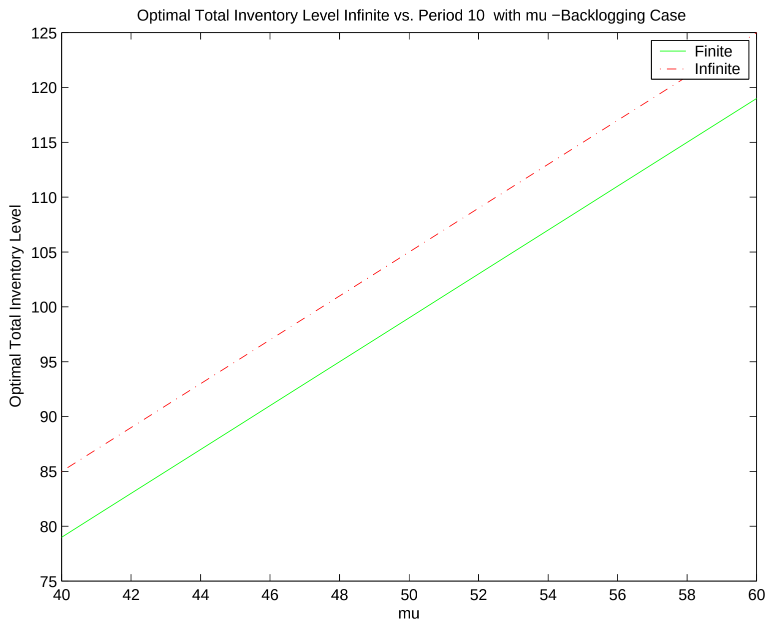

Figure 7.20: Supplier's Total Inventory Level for Period 10 vs. Mean μ

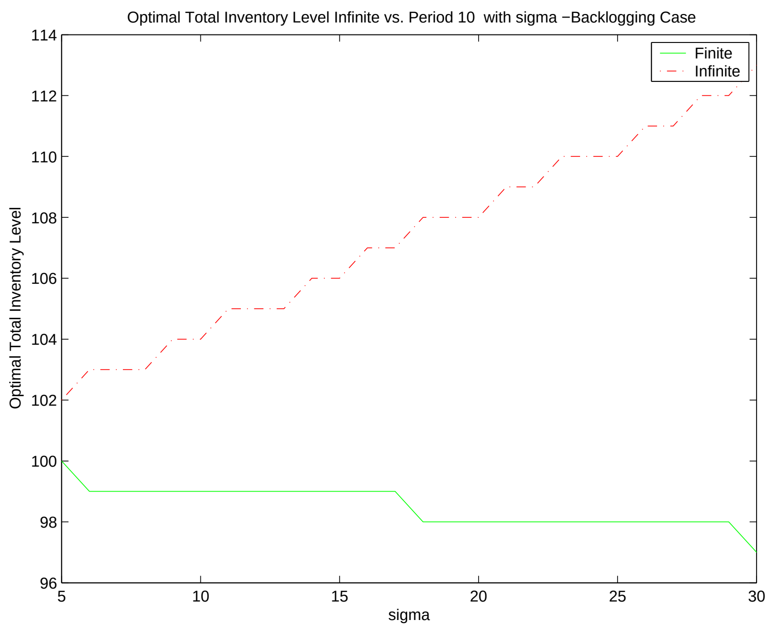

Figure 7.21: Supplier's Total Inventory Level for Period 10 vs. Standard Deviation σ

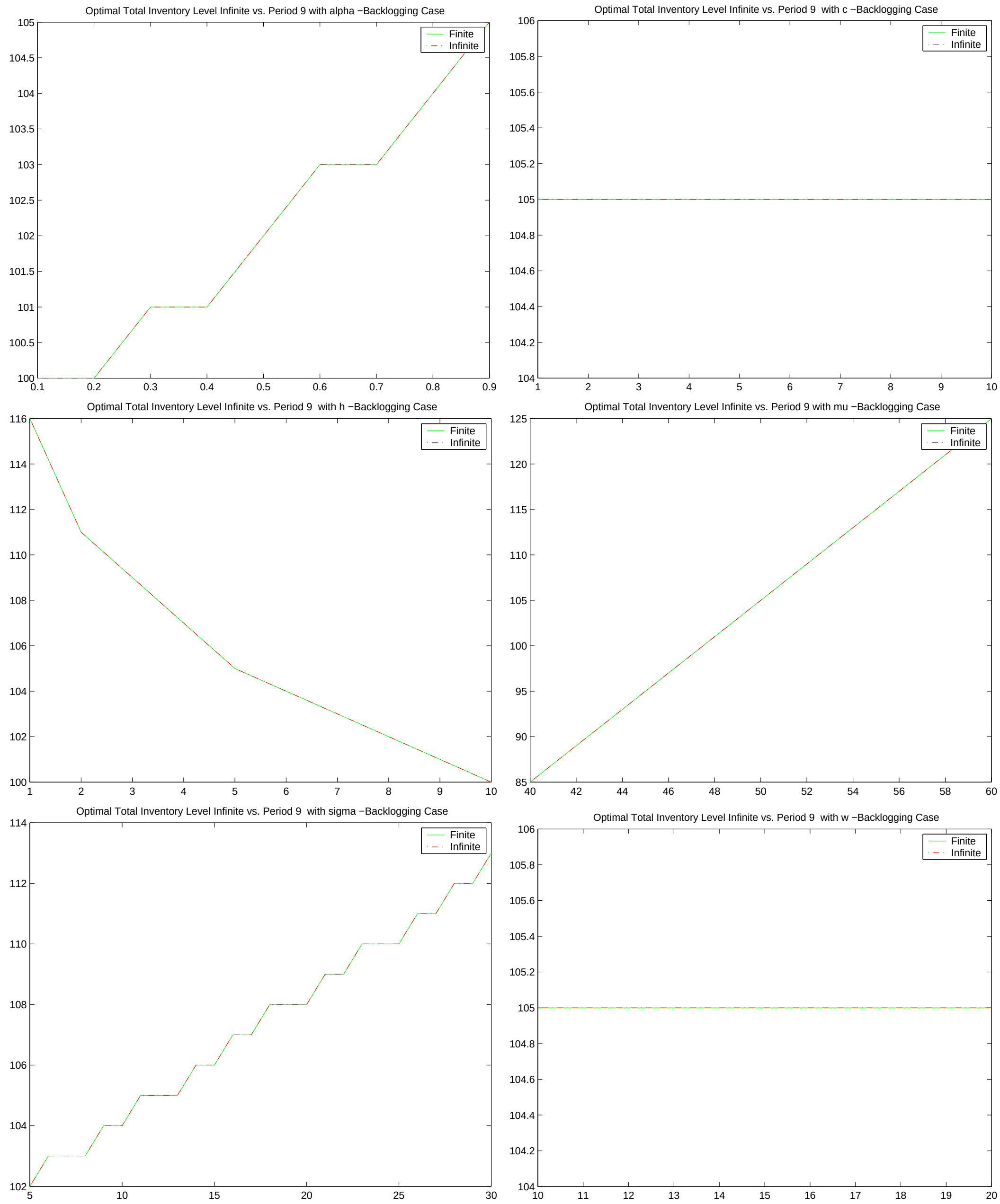

Figure 7.22: Supplier's Total Inventory Level for Period 9 vs. Infinite Horizon

7.2 Finite Horizon with Lost-sales

In the lost sales case, the retailer's unsatisfied demands are lost while the supplier bears the lost-sales cost of b per unit. We keep all the notation the same as in the previous section. For the reserved inventory system, we have the following recursion equation

$$V_{i,t}(x_{i,t}) \quad = \quad \max_{y_{i,t} \geq x_{i,t}} \{cx_{i,t} + G_i(y_{i,t}) + \alpha EV_{i,t+1}(y_{i,t} - D_i)^+\}$$

for $t = T, \ldots, 1$, where in this case,

$$G_i(y_{i,t}) = E[w_i \min(D_i, y_{i,t}) - hE(y_{i,t} - D_i)^+ - bE(y_{i,t} - D_i)^- - cy_{i,t}]$$

and $V_{i,T+1}(x_{i,T+1}) = 0$ for all $x_{i,T+1}$.

For the pooled inventory system, we have the recursion equation

$$V_{p,t}(x_{p,t}) \quad = \quad \max_{y_{p,t} \geq x_{p,t}} \{cx_{p,t} + G(y_{p,t}) + \alpha EV_{p,t+1}(y_{p,t} - D_p)^+\}$$

for $t = T, \ldots, 1$, where

$$G_p(y_{p,t}) = E[w_p \min(D_p, y_{p,t}) - hE(y_{p,t} - D_p)^+ - bE(y_{p,t} - D_p)^- - cy_{p,t}].$$

and $V_{p,T+1}(x_{p,T+1}) = 0$ for all $x_{p,T+1}$.

As before, we assume that the demands are normally distributed, and all the wholesale prices and markup price are the same, i.e., $w_1 = w_2 = w_p = w$, $m_1 = m_2 = m_p = m$. We examine the profits of the supplier and retailers and the total inventory level after ordering under the reserved and pooled inventory systems.

Claim 7.2.1 *The profit of the supplier under the reserved inventory system is less than that under the pooled inventory system.*

Using the same parameter values as in the previous section, Figures 7.23 to 7.29 verify the claim for a wide range of data.

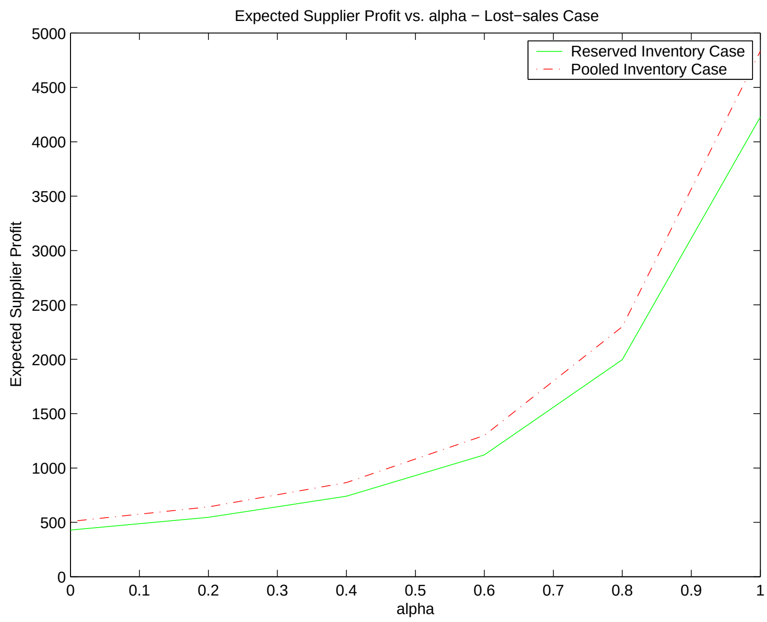

Figure 7.23: Supplier's Profit vs. Discount Factor α

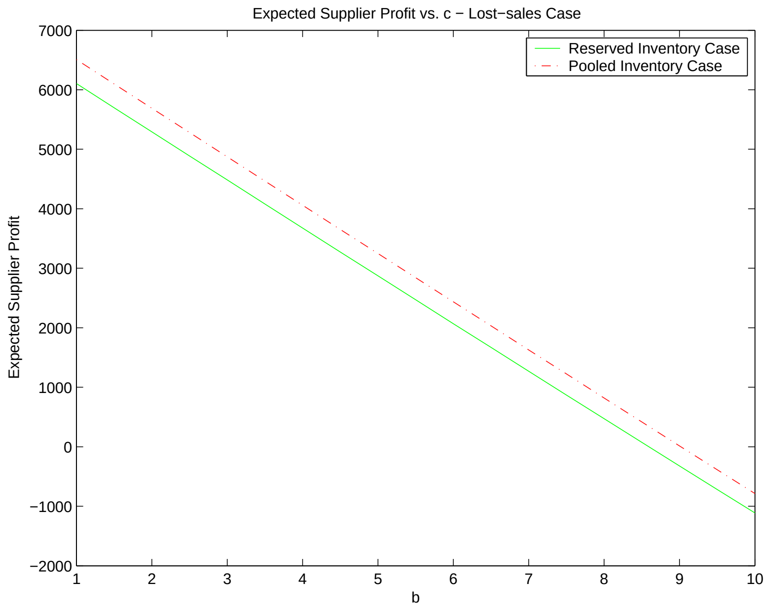

Figure 7.24: Supplier's Profit vs. Manufacturing Cost c

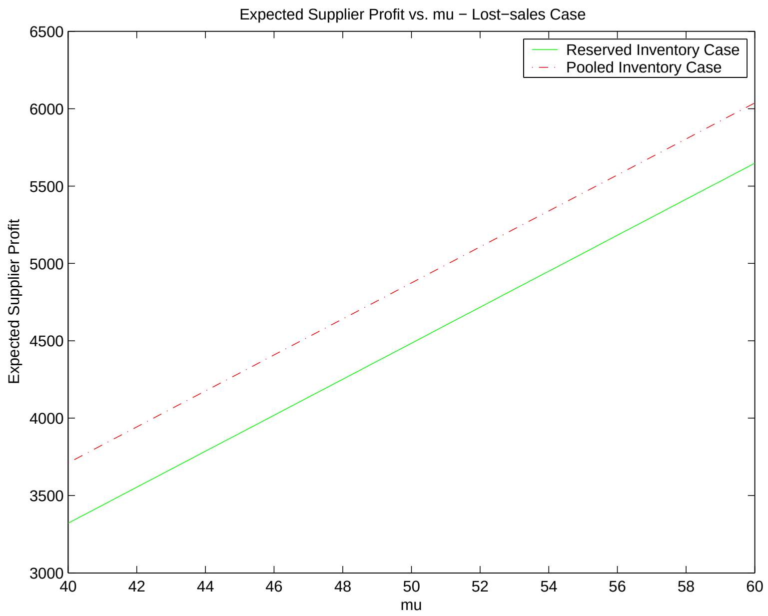

Figure 7.25: Supplier's Profit vs. Mean μ

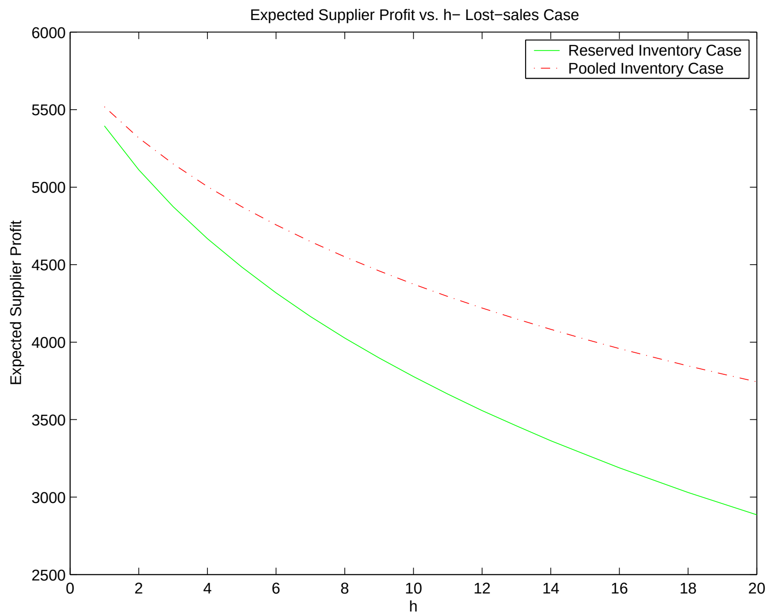

Figure 7.26: Supplier's Profit vs. Holding Cost h

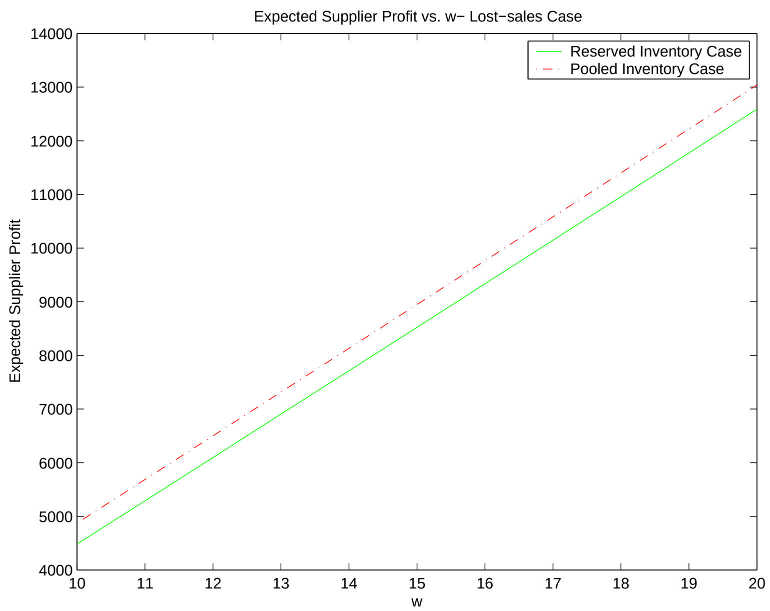

Figure 7.27: Supplier's Profit vs. Wholesale Price w

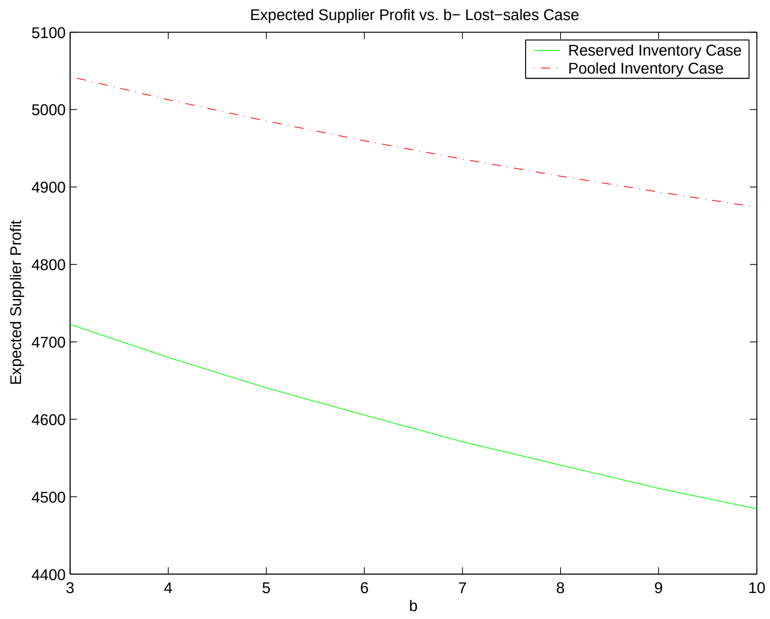

Figure 7.28: Supplier's Profit vs. Penalty Cost b

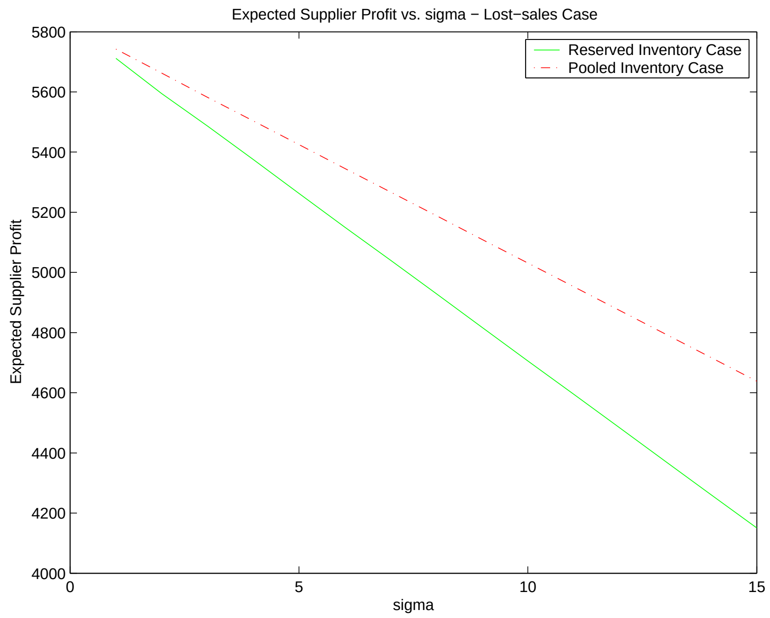

Figure 7.29: Supplier's Profit vs. Standard Deviation σ

While the total profit of the retailers is the same under the two different inventory systems when excess demands are backlogged, in the lost sales case, the amount of unsatisfied demands has an impact on the retailers' profit. Hence, the profits of the retailers are different under the two policies. We have the following claim

Claim 7.2.2 *The expected total profit of the retailers under the reserved inventory is less than that in the pooled inventory system.*

Figures 7.30-7.36 show this to be the case for the wide range of data (the same data as in the backlogging case).

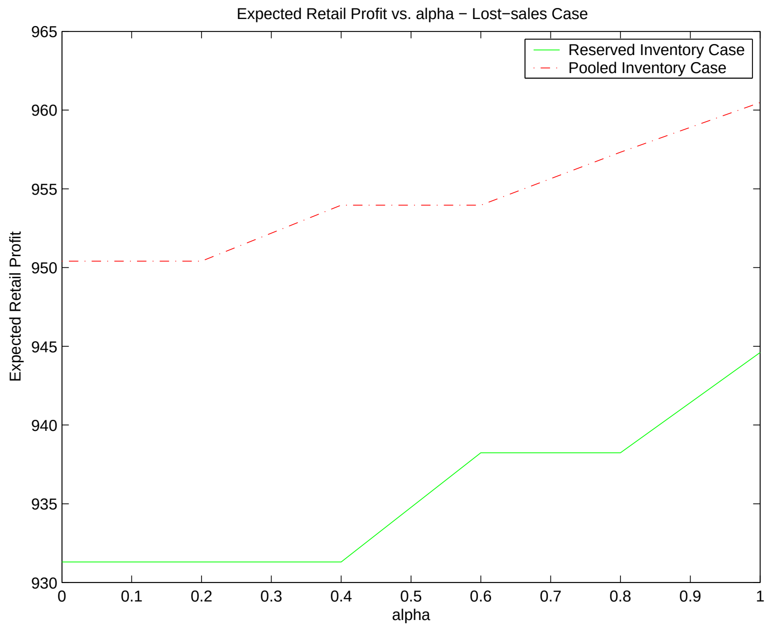

Figure 7.30: Retailer's Profit vs. Discount Factor α

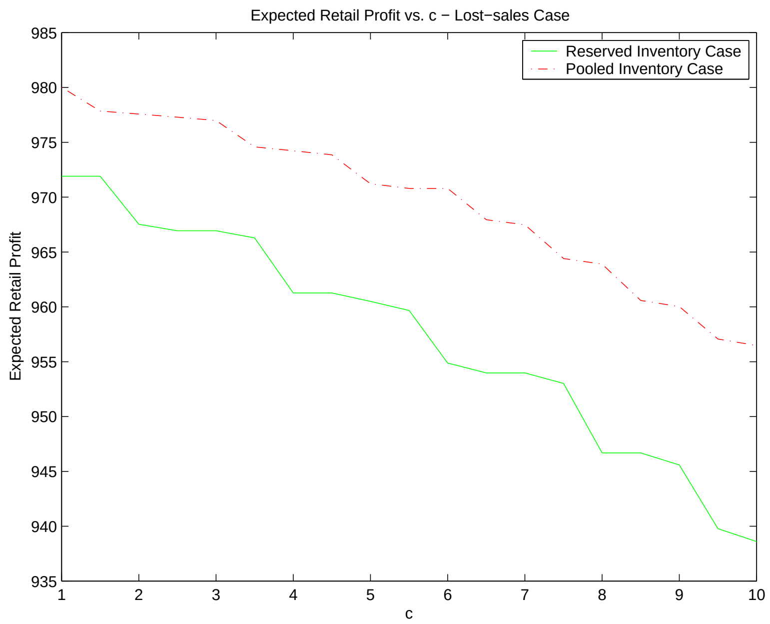

Figure 7.31: Retail Profit vs. Manufacturing Cost c

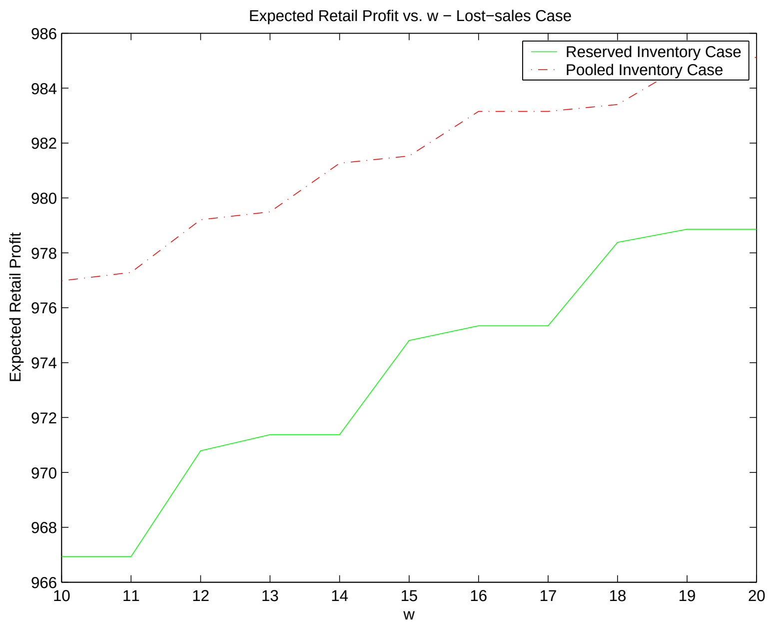

Figure 7.32: Retail Profit vs. Wholesale Price w

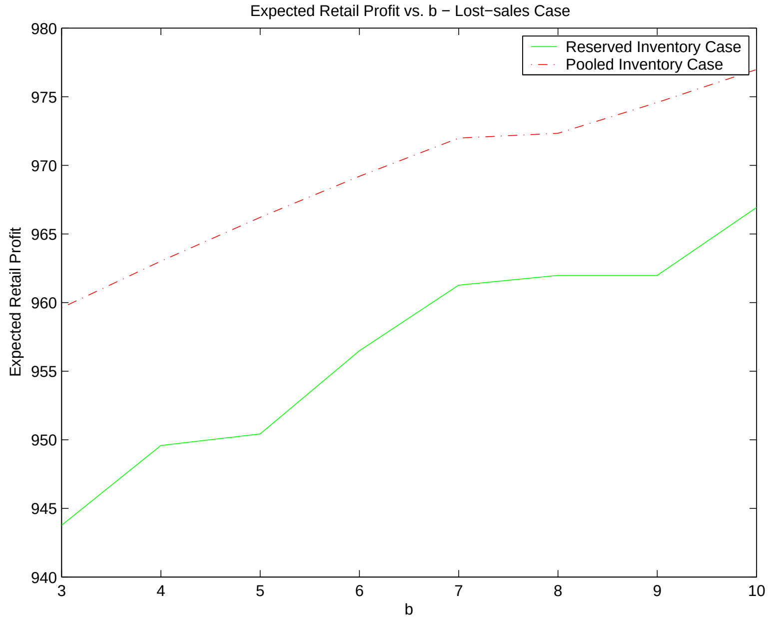

Figure 7.33: Retail Profit vs. Penalty Cost b

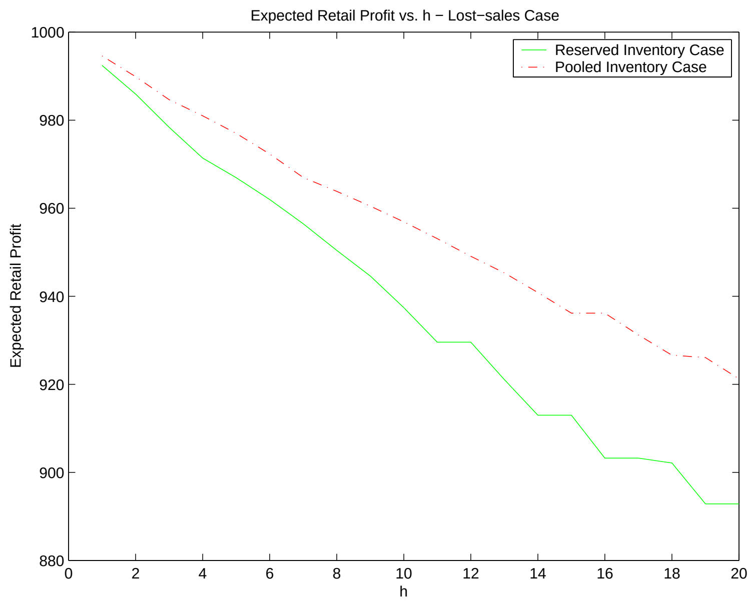

Figure 7.34: Retail Profit vs. Holding Cost h

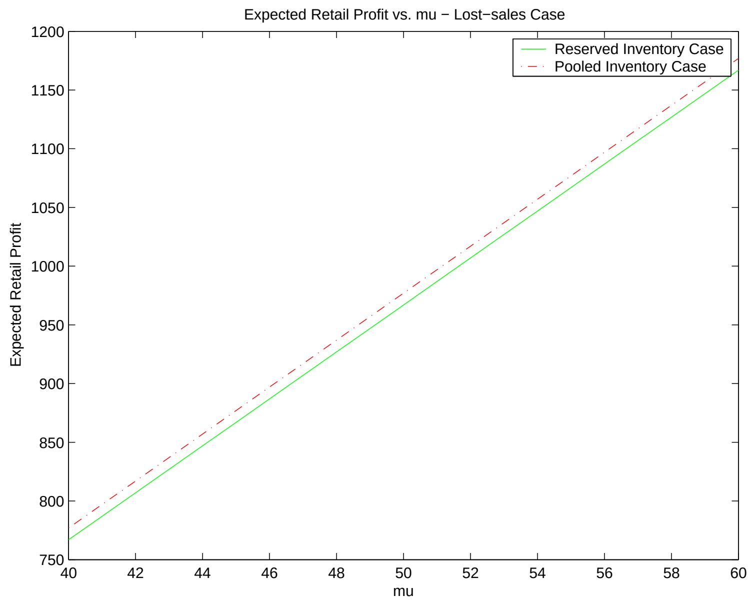

Figure 7.35: Retail Profit vs. Mean μ

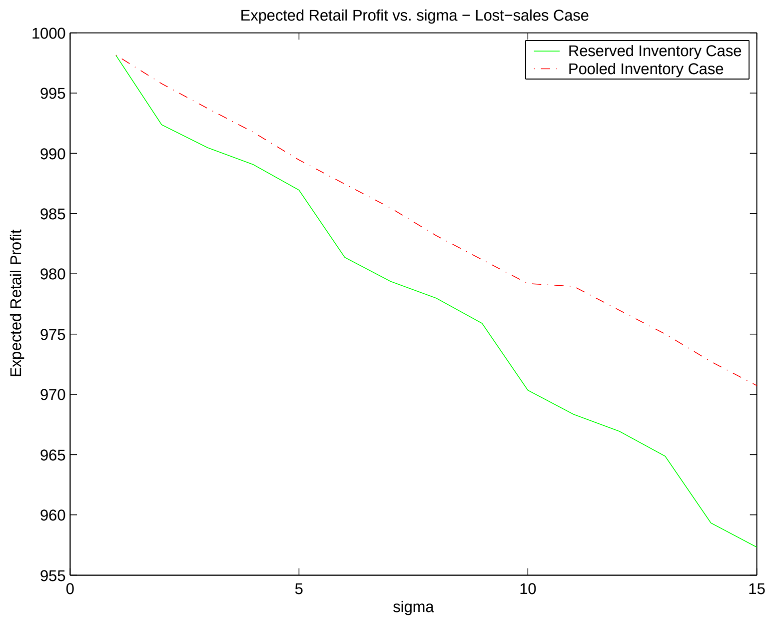

Figure 7.36: Retail Profit vs. Standard Deviation σ

For the backlogging case, we saw that the total inventory level in the pooled inventory system may or may not be greater than that in the reserved inventory system. In the lost-sales case, this is also true. Figure 7.39 shows that difference in total inventory level between the pooled inventory system and the reserved inventory system may be negative or positive depending on the value of holding cost h. Note that this differs from the backlogging case where the relation depending on discount factor α, penalty cost b and standard deviation σ.

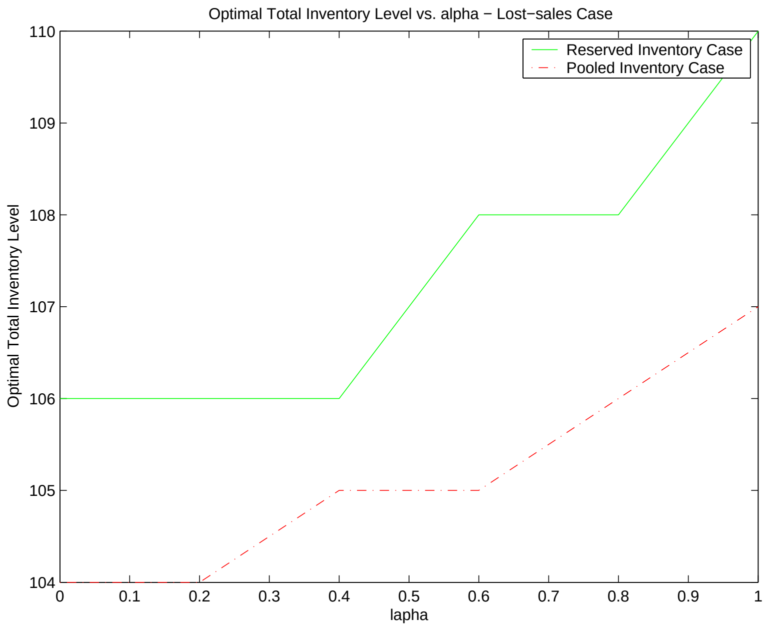

Figure 7.37: Supplier's Total Inventory Level vs. Discount Factor α

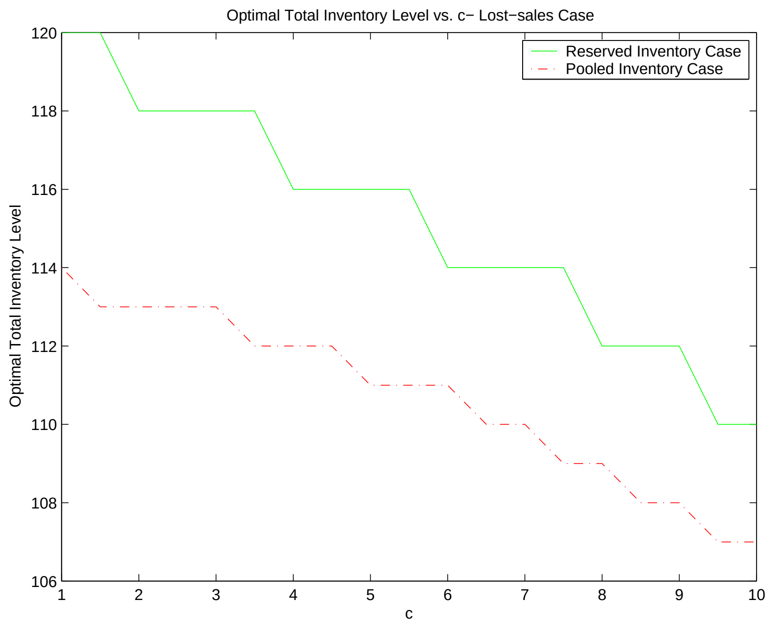

Figure 7.38: Supplier's Total Inventory Level vs. Manufacturing Cost c

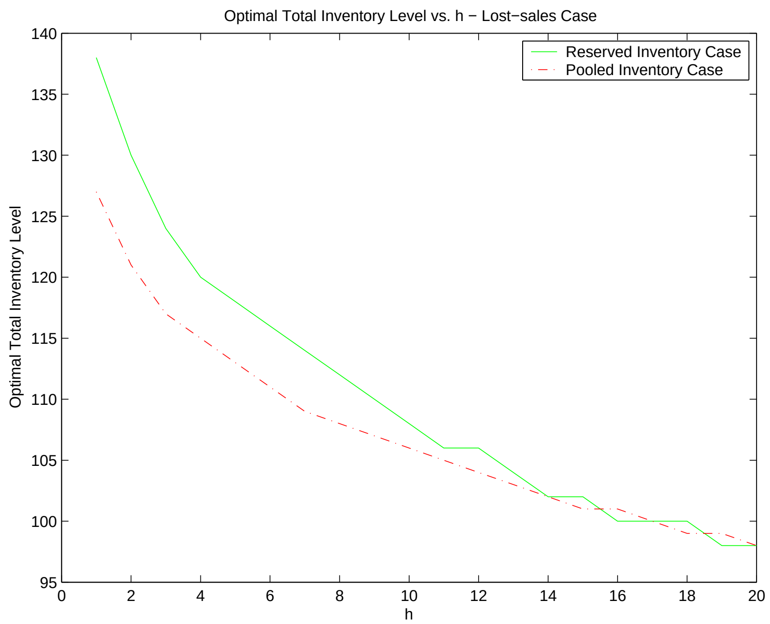

Figure 7.39: Supplier's Total Inventory Level vs. Holding Cost h

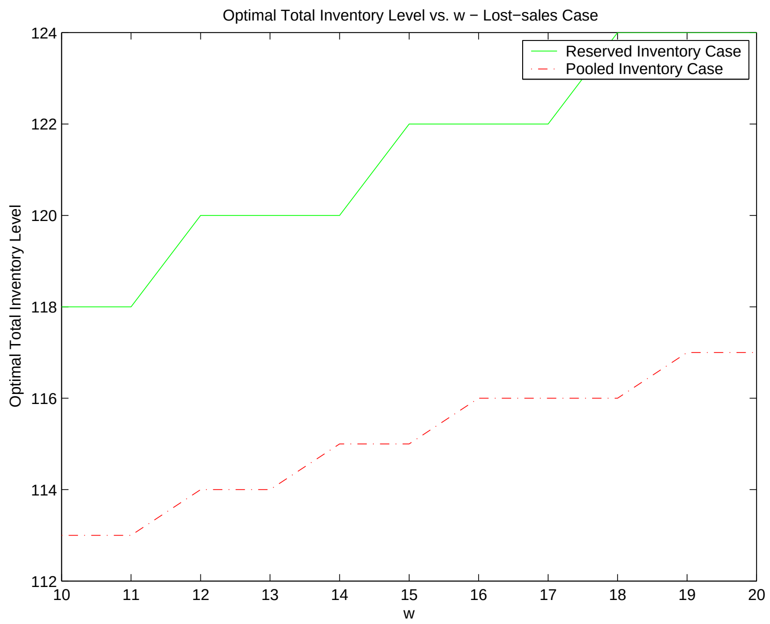

Figure 7.40: Supplier's Total Inventory Level vs. Wholesale Price w

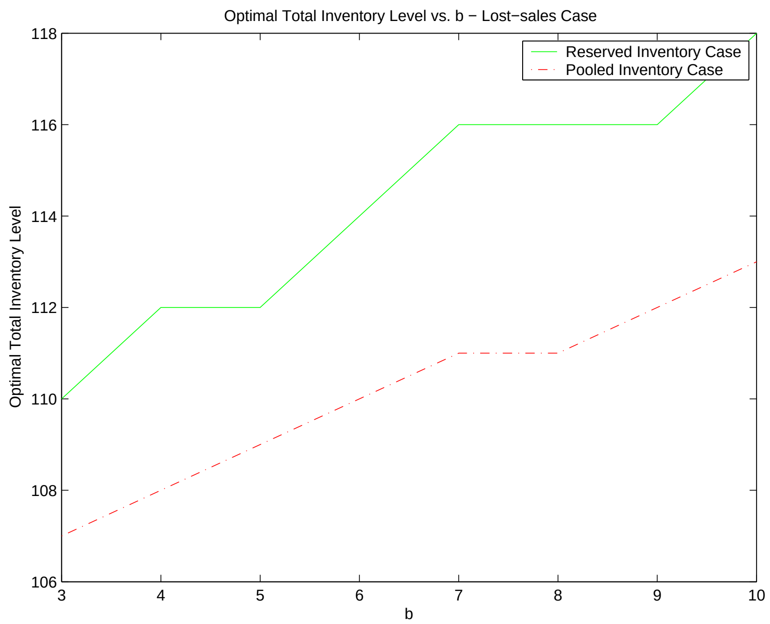

Figure 7.41: Supplier's Total Inventory Level vs. Penalty Cost b

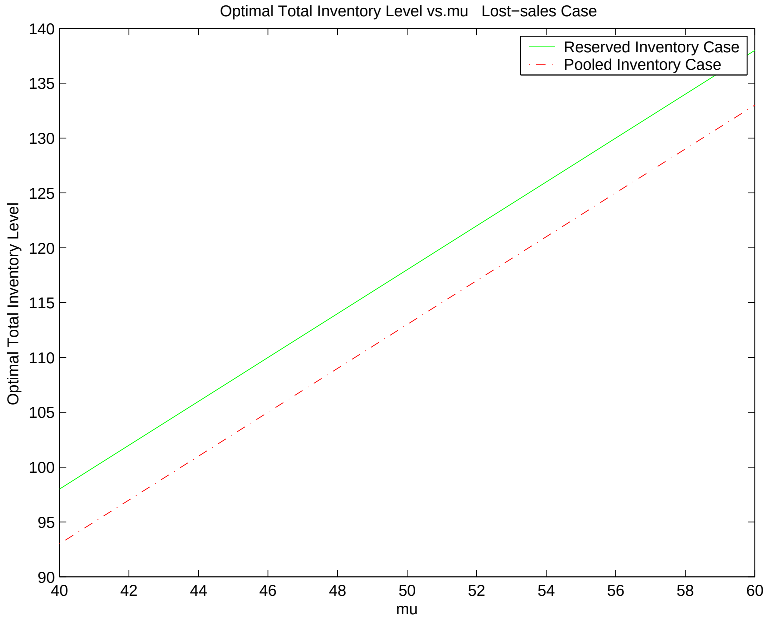

Figure 7.42: Supplier's Total Inventory Level vs. Mean μ

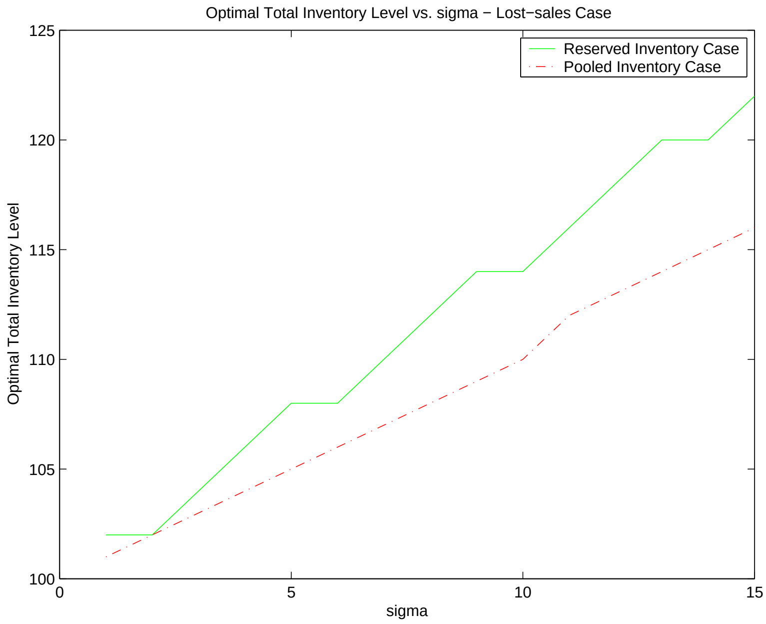

Figure 7.43: Supplier's Total Inventory Level vs. Standard Deviation σ

As in the backlogging case, the optimal inventory level $y_{p,t}$ in period t after ordering in a finite horizon is always nonincreasing as t approaches the end of horizon(period T) and not larger than the optimal inventory level y_p in the infinite horizon case, i.e., $y_{p,T} \leq y_{p,T-1} \ldots \leq y_1 \leq y_p$; equivalent results apply for y_{it} $i = 1, 2; t = 1, 2, \ldots, T$. Figure 7.44 shows that the optimal pooled total inventory level in period 10 of the 10-period horizon is less than that in the infinite horizon, which matches the mathematical results well. Figure 7.45 shows that the optimal pooled total inventory level in period 8 of the 10-period horizon is the same as that in the infinite horizon. Again similar results hold for the reserved inventory case.

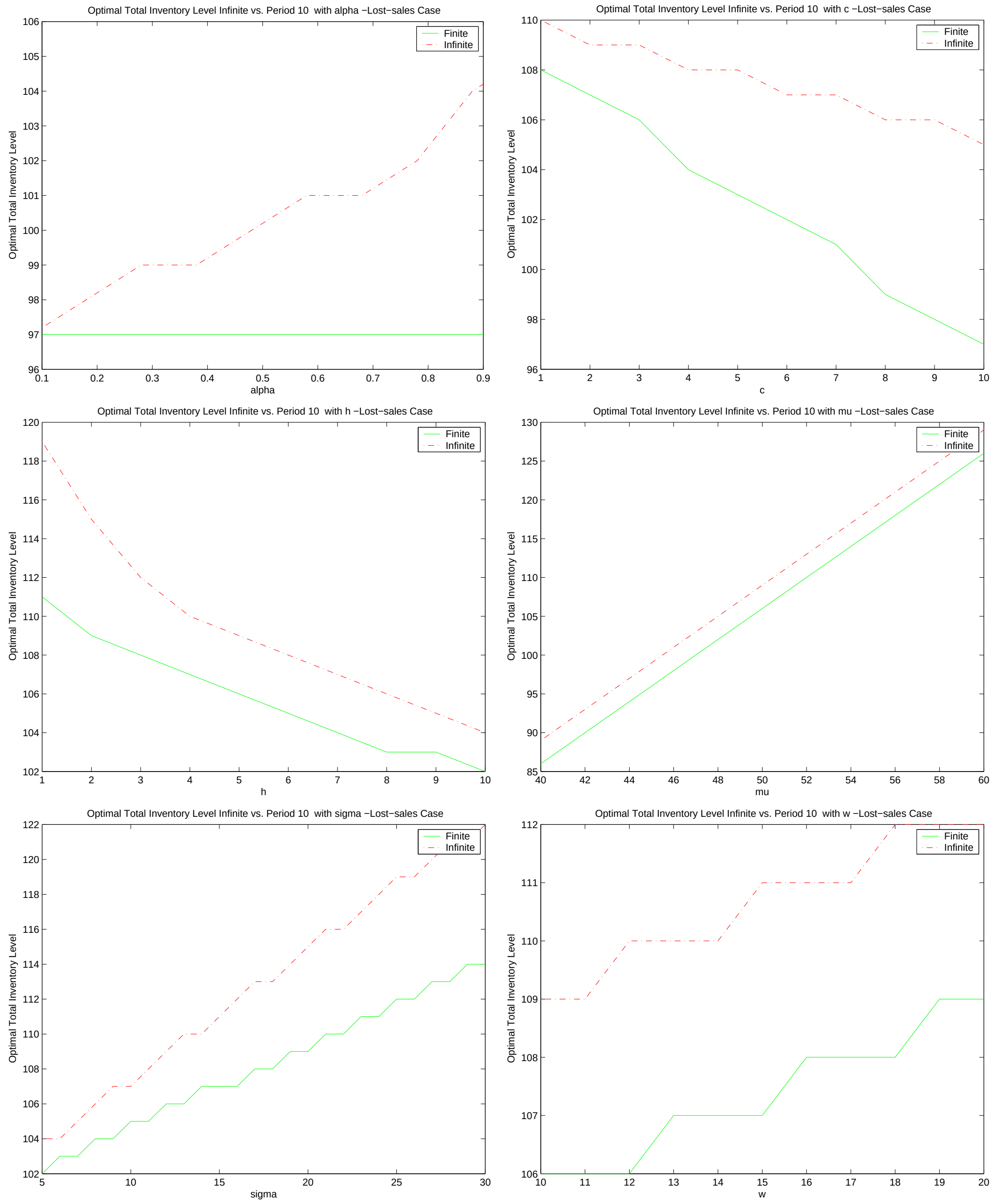

Figure 7.44: Supplier's Total Inventory Level for Period 10

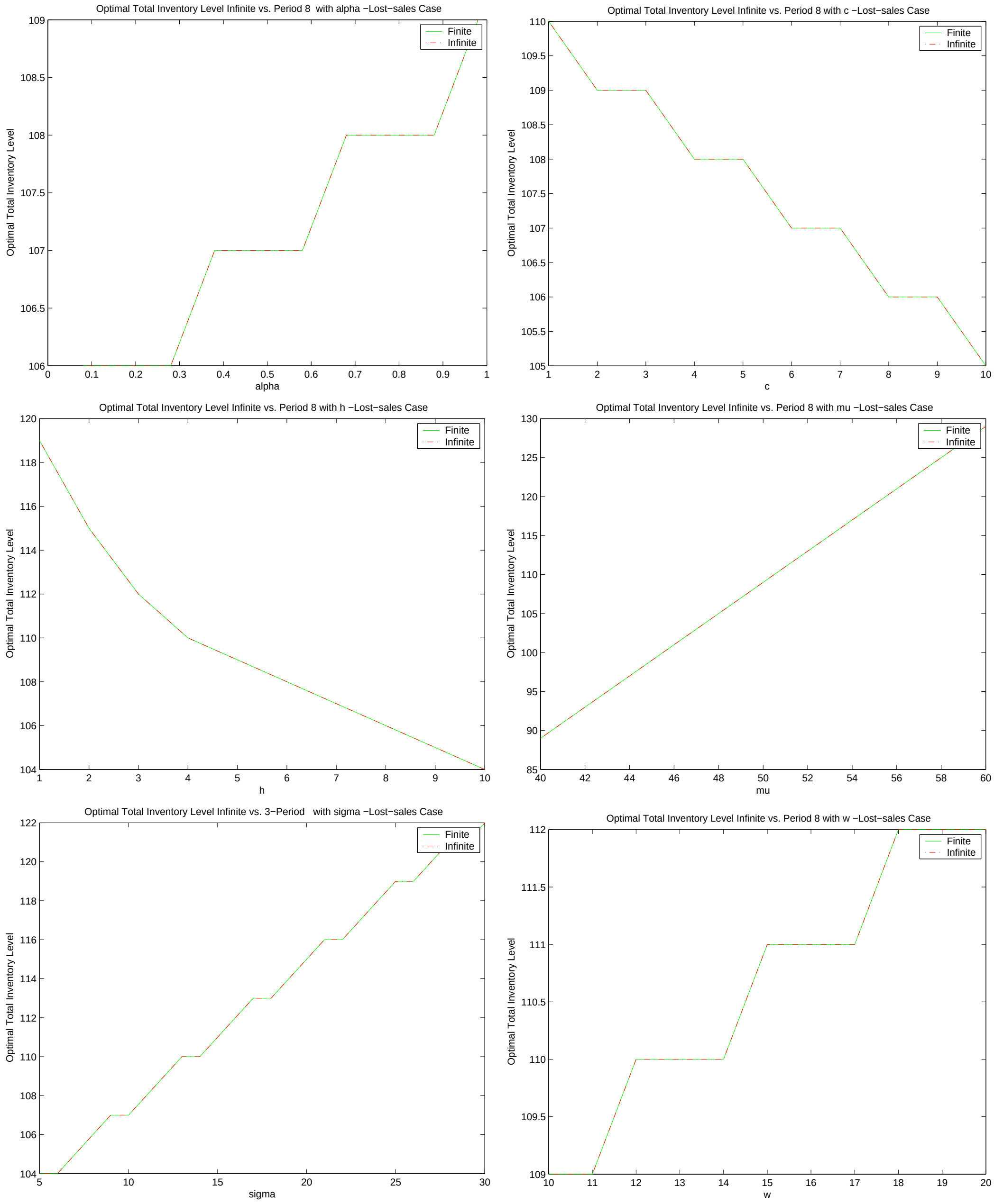

Figure 7.45: Supplier's Total Inventory Level for Period 8

Chapter 8

Conclusions and Future Research

8.1 Conclusions

In this thesis, we have studied a two-echelon supply chain with one supplier and two retailers. While most of our work focus on single period(or selling season) scenarios, in the last two chapters, we have considered multi-period and infinite horizon scenarios. The supplier bears the supply chain's inventory risk because only the supplier holds inventory while the retailer replenishes as needed during the season.

For the single period case, we considered scenarios in which the wholesale price is fixed, and we studied and compared the supplier's inventory decision under two policies: reserved inventory and pooled inventory. Under the first policy, the inventory is stocked in separate locations for each retailer while under the latter policy, inventory is centrally stocked by the supplier, and hence the supply chain may benefit from risk pooling. However, in general, whether the profit of the retailers and supplier increases or decreases upon inventory pooling depends on the problem parameters such as the demand parameters.

In order to obtain insight into the impact of the reserved inventory and the pooled inventory policies, we compared the profits of the supplier and retailers under those two policies assuming normal demand distributions. First we showed that the supplier's profit in the pooled inventory case is always greater than in the reserved inventory case. In addition, the total expected sales are also increased after pooling the inventory. In addition to the basic model, we also studied the case when the retailers have service level requirements, and we obtained similar conclusions.

Besides the model with fixed wholesale prices, we also developed inventory and pricing models for the supplier when the wholesale price is a decision and demand is price-sensitive. Note that these scenarios are much more complex. We analyzed two different demand models: an additive demand model and a multiplicative demand model. For normally distributed demands, we compared the results for the reserved inventory and the pooled inventory policies with and without service level requirements. The results shows that the supplier always prefers to pool inventory, while in general, this need not be so for the retailers. We also analyzed the retailers' profits under some specific conditions.

For the infinite horizon with fixed wholesale price and no service level requirements, we were able to show the comparative results for the single period case hold when the excess demand is backlogged and when it is lost. For the multi-period finite horizon case, we were unable to show the results mathematically. We were able to show numerically that the results also hold over a wide range of data.

Although a number of results are developed in this research, there is a long list of possible directions for future research.

8.2 Future Research

Some future research directions are:

1. Coordination between the supplier and retailers.

   In this thesis, we have demonstrated the benefit of the pooled inventory policy. This conclusion leads to the question of how to allocate the benefits between the retailers and supplier. In general a contract coordinates the retailer's and supplier's actions. Cachon (2002) gives an excellent review on contract coordination, and he analyzes buy back contracts, revenue sharing contracts, quantity flexibility contracts, sale rebate contracts and quantity discount contracts, etc. to coordinate the system. Wang (2004) also analyzes the consignment contract with revenue sharing in a supply chain with one supplier and one retailer. The supplier chooses the retail price and delivery quantity, and the supplier and retailer share the revenue. There is also another trend in literature on supply chains, namely, cooperative games. One approach is a snapshot allocation game (SAG), which is used by Anupindi et al. (2001). In SAG, the value of the game is calculated based on each realization of random demand. Another approach is the allocation game in expectation (AGE) by Anupindi et al. (1999b). Granot and Sosic (2002) mention the Shapley value as a profit-allocation mechanism that may induce optimal supply chain inventory decisions. Then Bartholdi et al. (2003) use the Shapely value as a value-sharing mechanism to affect the operational decisions of supply chain partners. It would be interesting to try to use these methods to allocate the benefits; however, the most difficult part will be deciding how to allocate the inventory under the pooled inventory policy.

2. Competition between the retailers.

   In our thesis, we have assumed that there is no competition between the retailers. What if the retailers are in competition for a common customer base? Chen et al. (2001) and Bernstein et al. (2001) consider a supply chain model with one supplier and multiple competing retailers in which each retailer makes decisions such as inventory level and retail price. Bernstein et al. (2002) consider a supply chain system in which the supplier decides how the retailers are replenished and bears all the production cost incurred in the supply chain. However each retailer chooses her own retail price, and demand at each retailer is determined by a general demand function of the prices charged by all retailers. It is possible to extend our model to a supply chain with one supplier and two competing retailers. Applying the demand model, we can analyze the scenarios under the reserved and the pooled inventory policies. However, the problem is quite complex.

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