*Transactions of the 17th International Conference on Structural Mechanics in Reactor Technology (*SMiRT 17)
Prague, Czech Republic, August 17 –22, 2003
Paper # J01-1

Vibration of Cylindrical Shells with Oblique Ends

X. J. Hu and D. Redekop

Department of Mechanical Engineering, University of Ottawa, Canada

ABSTRACT

A theoretical solution is presented to the problem of natural vibrations of a circular cylindrical shell with oblique ends. Two geometric cases are considered. The first case involves a shell with a lower perpendicular end and an upper oblique end. Both ends have clamped supports. The second case involves a shell with both ends oblique and clamped. The shell of the latter case is assumed symmetric about a mid-length perpendicular plane. A solution is developed based on the Budiansky shell theory. In the solution the surface of the shell is first developed onto a plane, and the resulting irregular domain is then mapped, using blending functions, onto a square parent domain. The differential quadrature method is next applied to the parent domain to obtain numerical results. Validation and convergence studies are carried out. The results found using the current method are compared with results found using the finite element method. Finally results from a parametric study are presented, and conclusions are drawn.

KEY WORDS: Differential quadrature method, finite element method, shell theory, vibration analysis, cylindrical shell.

INTRODUCTION

Cylindrical shells with one or two oblique ends (Fig. 1) appear in pressure vessel and piping networks in the form of mitre-bends, 'hillside' nozzles, and trunnion pipe supports [1-6]. These shells are geometrically irregular as not all boundary lines coincide with coordinate lines. The formulation of analytical or semi-analytical solutions poses a formidable challenge. In the survey on shell vibrations by Leissa [7] it is reported that no solution had been presented for this geometry by 1973. The authors are unaware of any solution to the problem subsequent to this date.

In this paper the differential quadrature method (DQM) is applied to the title problem. The curved shell surface is developed onto a plane, and blending functions are used to map the geometrically irregular domain onto a square parent domain. Such blending functions have been used earlier by Malik and Bert [8] for the analysis of a plate vibration problem. Convergence, validation, and parametric studies are conducted. Results obtained using the DQM are compared with results found using the finite element method (FEM), and conclusions are drawn.

GEOMETRY AND MAPPING

In view of the symmetry conditions assumed for the second geometric case it is sufficient to consider a single geometric model. The shell in this model has a radius r, mean height L', and thickness h (Figs. 1-2). A typical point P on the shell mid-surface is described by the cylindrical coordinates y, θ. Displacement components u, v, w (respectively in the axial, circumferential and normal directions), and stress resultants (Fig. 3) are defined in these physical coordinates. The base of the shell in the model lies in a plane perpendicular to the shell axis, while the top lies in an oblique plane that makes an angle of α with the shell axis. At the base of the shell either clamped support conditions are assumed, or conditions of symmetry. The top of the shell is assumed to be clamped.

The shell geometry, which in the cylindrical coordinate system is irregular, and continuous in the circumferential direction, is first considered developed onto a plane (Fig. 4). In the development process two artificial boundary lines are created at the former $\theta = \pm 180^0$ line. Symmetry conditions must be enforced on these two lines, comparable to the continuity conditions existing over the $\theta = \pm 180^0$ line in the original cylindrical shell. Planar coordinates x,y which have dimensions of length are used to describe positions on the developed surface. Non-dimensional coordinates for this surface are defined as $\theta = x / r$, $\psi = y / r$. The governing domain equations for the shell are initially written in these non-dimensional coordinates.

Following the procedure used by Malik and Bert [8] for plates of irregular shape, a square parent domain (Fig. 5) is defined in the natural coordinates ξ, η, with $-1 \le \xi \le 1$, $-1 \le \eta \le 1$. Blending functions are then employed to develop mapping relations between the natural coordinates ξ, η and the developed coordinates, θ, ψ. The blending functions [8] are given by

$$s = \frac{1}{2}\{(1-\eta)\bar{s}_1(\xi) + (1+\xi)\bar{s}_2(\eta) + (1+\eta)\bar{s}_3(\xi) + (1-\xi)\bar{s}_4(\eta)\}$$
$$-\frac{1}{4}\{(1-\xi)(1-\eta)s_1 + (1+\xi)(1-\eta)s_2 + (1+\xi)(1+\eta)s_3 + (1-\xi)(1+\eta)s_4\}$$

where $s = \theta, \psi$. The $\bar{\theta}_i(\xi), \bar{\theta}_i(\eta), \bar{\psi}_i(\xi), \bar{\psi}_i(\eta)$, expressions are the parametric expressions for the edges of the developed surface, and the θ_i, ψ_i are the non-dimensional Cartesian coordinates of the corner points of the developed surface. The relations between the two sets of coordinates for the present case

$$\theta = \pi\xi; \quad \psi = \varsigma(1+\eta) \tag{1}$$

where $\varsigma = a + b\cos\phi$, $a = L'/2r$, $b = 0.5\tan\alpha$, and L' is the mean height of the shell. The product $\pi\xi$ is represented for clarity by ϕ, and the variables ϕ, η are used subsequently to describe the parent domain.

Using the chain rule of calculus, the transformation of derivatives from the θ, ψ system to the ϕ, η system can be determined as

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} + \frac{b(1+\eta)\sin\phi}{\varsigma}\frac{\partial}{\partial\eta}; \quad \frac{\partial}{\partial\psi} = \frac{1}{\varsigma}\frac{\partial}{\partial\eta} \tag{2}$$

These relations correspond to those used by Gill [5] for the static analysis of mitred bends. Relations for higher order and mixed derivatives are readily developed from these basic relations.

BOUNDARY CONDITIONS

For the condition of clamped supports at the base ($y = 0$) the boundary equations are given by

$$u = 0; \quad v = 0; \quad w = 0; \quad \frac{1}{r}\frac{\partial w}{\partial\psi} = 0 \tag{3}$$

where *u, v,* and *w* again represent the physical displacement components. The sign convention for these displacements and the accompanying stress resultants is given in Fig. 3. For the conditions of symmetry at the base the relations are

$$u = 0; \quad \frac{1}{r}\frac{\partial w}{\partial\psi} = 0; \quad N_{y\theta} + \frac{3}{2r}M_{y\theta} = 0; \quad \frac{1}{r}\left(\frac{\partial M_y}{\partial\psi} + 2\frac{\partial M_{y\theta}}{\partial\theta}\right) = 0 \tag{4}$$

For the clamped support conditions at the top of the shell the boundary relations are given [9] as

$$u = 0; \quad v = 0; \quad w = 0; \quad \frac{\partial w}{\partial\theta}\cos\beta + \frac{\partial w}{\partial\psi}\sin\beta = 0 \tag{5}$$

where β is the angle between the normal **n** to the shell boundary and the y-axis (Fig. 4). The conversion of the boundary conditions into a form suitable for the DQM is discussed in the following.

Displacement and force conditions must be satisfied on the artificial boundaries ($\theta = \pm 180^0$) created by developing the cylindrical shell onto a plane. These conditions in this study are satisfied by selecting trigonometric functions as the trial functions in the circumferential direction. The conditions are then satisfied automatically owing to the periodic nature of these functions. Using this approach there is no need for explicit equations in the DQM code corresponding to these conditions.

BUDIANSKY SHELL THEORY

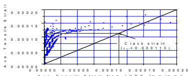
To determine the vibration characteristics of the shell the first-order linear version of the Budiansky shell theory [10] is used, in combination with the D'Alembert principle. The governing domain equations given in the non-dimensional cylindrical coordinates are

$$[L]\ \{U\} = \lambda\ \{U\} \tag{6}$$

where the operator L is defined by

$$[L]=\begin{bmatrix} \frac{\partial^2}{\partial\psi^2}+k_1\frac{\partial^2}{\partial\theta^2} & k_2\frac{\partial^2}{\partial\psi\partial\theta} & \nu\frac{\partial}{\partial\psi}+k_3\frac{\partial^3}{\partial\psi\partial\theta^2} \\ k_2\frac{\partial^2}{\partial\psi\partial\theta} & k_4\frac{\partial^2}{\partial\psi^2}+k_5\frac{\partial^2}{\partial\theta^2} & \frac{\partial}{\partial\theta}-k_6\frac{\partial^3}{\partial\psi^2\partial\theta}-k\frac{\partial^3}{\partial\theta^3} \\ \nu\frac{\partial}{\partial\psi}+k_3\frac{\partial^3}{\partial\psi\partial\theta^2} & \frac{\partial}{\partial\theta}-k_6\frac{\partial^3}{\partial\psi^2\partial\theta}-k\frac{\partial^3}{\partial\theta^3} & 1+k(\frac{\partial^4}{\partial\psi^4}+2\frac{\partial^4}{\partial\psi^2\partial\theta^2}+\frac{\partial^4}{\partial\theta^4}) \end{bmatrix} \tag{7}$$

and the vector of displacements $\{U\}$ represents $\{u \quad v \quad w\}^t$. The geometric factor k is defined as $k=\frac{1}{12}(\frac{h}{r})^2$ and the constants k are given by



$$k_2=\frac{1+\nu}{2}-\frac{1-\nu}{2}\frac{3k}{4}; \quad k_3=\frac{1-\nu}{2}k$$

$$k_4=\frac{1-\nu}{2}(1+\frac{9k}{4}); \quad k_5=1+k; \quad k_6=\frac{3-\nu}{2} \tag{8}$$

The vibration parameter λ is related to the natural frequency of vibration ω (Hz), by $\lambda = r^2\rho(1-\nu^2)\omega^2/E$, where ρ is the mass density of the shell, ν the Poisson ratio, and E the Young's modulus.

For the application of the boundary conditions the relations between the stress resultants and displacement components are required. In the Budiansky theory these relations are given as

$$\begin{aligned} N_\theta &= \frac{K}{r}[\frac{\partial v}{\partial\theta}+w+\nu\frac{\partial u}{\partial\psi}] \\ N_y &= \frac{K}{r}[\frac{\partial u}{\partial\psi}+\nu(\frac{\partial v}{\partial\theta}+w)] \\ N_{\theta y} &= K\frac{1-\nu}{2r}[\frac{\partial v}{\partial\psi}+\frac{\partial u}{\partial\theta}] \\ M_\theta &= -\frac{D}{r^2}[\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}+\nu\frac{\partial^2 w}{\partial\psi^2}] \\ M_y &= -\frac{D}{r^2}[\frac{\partial^2 w}{\partial\psi^2}+\nu(\frac{\partial^2 w}{\partial\psi^2}-\frac{\partial v}{\partial\theta})] \\ M_{\theta y} &= -\frac{D(1-\nu)}{r^2}[\frac{\partial^2 w}{\partial\psi\partial\theta}-\frac{1}{4}\frac{\partial v}{\partial\psi}+\frac{1}{4}\frac{\partial u}{\partial\theta}] \end{aligned} \tag{9}$$

where $K = Eh/(1-\nu^2)$, $D = Kh^2/12$. The f_i are known operators in the parent coordinate system, and the h_i are known functions of the position variables, given in full by Hu [11].

Substitution of the coordinate and derivative transfer relations (1-2) into the boundary conditions (3-5), domain equations (6-7), and the expressions for the resultants (9) leads to a set of boundary-domain equations in the parent coordinate system that governs the problem. The full expressions were derived with the aid of the MAPLE software [12], and are given by Hu [11].

DIFFERENTIAL QUADRATURE METHOD

Following the DQM approach [13] a grid of sampling points is first created in the parent domain. The derivatives which appear in the domain and boundary equations are replaced by series involving the displacements at the sampling points of the grid. For a derivative in a given direction the series involves all the sampling points in that direction. Application of this procedure leads to a set of linear homogeneous equations in terms of the displacements at the sampling points, and the eigenvalue term λ. The solution of these equations yields the eigenvalues, and thus the natural frequencies for the shell.

The series used to replace derivatives in the governing equations have the form

$$\left.\frac{d^r f}{d\xi^r}\right|_{\xi=\xi_i} = \sum_{j=1}^{M} A_{ij}^{(r)} f(\xi_j) \tag{10}$$

where ξ is the generic variable of the generic function $f(\xi)$, r indicates the order of the derivative, i the number of the sampling point in the ξ direction, M the number of sampling points in the ξ direction, and the $A_{ij}^{(r)}$ are weighting coefficients. The $A_{ij}^{(r)}$ are set up a-priori for specified sampling point spacings, with the aid of chosen trial functions.

For the current study series for derivatives are required for both the ξ and η directions. Such a two-dimensional DQM approach was used previously by Redekop and Makhoul [14] for the DQM analysis of the buckling of cylindrical panels.

To determine the weighting coefficients the standard Legendre polynomial set of trial functions is used, together with the Chebyshev-Gauss-Lobatto spacing of sampling points [13]. For this system explicit formulas for the weighting coefficients are available. For the first order derivative the formulas are

$$A_{ij}^{(1)} = \frac{\pi(\xi_i)}{(\xi_i - \xi_j)\pi(\xi_j)}; i, j = 1,2..., M; i \neq j$$
$$\pi(\xi_i) = \prod_{j=1}^{M} (\xi_i - \xi_j); i \neq j \tag{11}$$

while for the higher order derivatives the formulas are

$$A_{ij}^{(r)} = r[A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{\xi_i - \xi_j}];$$
$$i, j = 1,2,..., M; i \neq j;\ \ 2 \leq r \leq (M-1)$$
$$A_{ij}^{(r)} = A_{ii}^{(r)} = -\sum_{k=1}^{M} A_{ik}^{(r)};$$
$$i, j = 1,2,..., M; i \neq k;\ \ 1 \leq r \leq (M-1) \tag{12}$$

In this system the coordinates of the sampling points for the generic variable are taken as $\xi_i = c_i \beta$ where the c_i are given by

$$c_1 = 0;\quad c_2 = \delta;\quad c_{M-1} = 1-\delta;\quad c_M = 1;$$
$$c_i = \frac{1 - \cos\dfrac{\pi(i-2)}{M-3}}{2};\quad 2 < i < (M-1) \tag{13}$$

where δ describes the position of 'δ' points, and β is a scaling factor.

At each sampling point either the DQM analogues of the domain or boundary equations are represented. For shells there are four boundary conditions, while there are only three governing equations. Along boundaries it is thus necessary to enforce one of the boundary conditions at an adjacent domain point. This point, a 'δ' point, is taken a short distance ($\delta \cong 10^{-5}$) from the boundary point.

Use of the quadrature rules (10-12) for the derivatives in the domain equations (6-8) leads to the transformed scalar DQM domain equations

$$\begin{aligned}
&\mu_{11a}\sum A_{ih}^{(2)}U_{hj} + \mu_{11b}\sum A_{ih}^{(1)}U_{hj} + \mu_{11c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} + \mu_{11d}\sum B_{jk}^{(2)}U_{ik} \\
&+ \mu_{12a}\sum A_{ih}^{(2)}V_{hj} + \mu_{12b}\sum A_{ih}^{(1)}V_{hj} + \mu_{12c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} \\
&+ \mu_{13a}\sum A_{ih}^{(3)}W_{hj} + \mu_{13b}\sum A_{ih}^{(2)}W_{hj} + \mu_{13c}\sum A_{ih}^{(1)}W_{hj} + \mu_{13d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}W_{hk} \\
&+ \mu_{13e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + \mu_{13f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}W_{hk} = \lambda U_{ij}
\end{aligned}$$

$$\begin{aligned}
&\mu_{21a}\sum A_{ih}^{(2)}U_{hj} + \mu_{21b}\sum A_{ih}^{(1)}U_{hj} + \mu_{21c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} \\
&+ \mu_{22a}\sum A_{ih}^{(2)}V_{hj} + \mu_{22b}\sum A_{ih}^{(1)}V_{hj} + \mu_{22c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} + \mu_{22d}\sum B_{jk}^{(1)}V_{ik} \\
&+ \mu_{23a}\sum A_{ih}^{(3)}W_{hj} + \mu_{23b}\sum A_{ih}^{(2)}W_{hj} + \mu_{23c}\sum A_{ih}^{(1)}W_{hj} + \mu_{23d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}W_{hk} \\
&+ \mu_{23e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + \mu_{23f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}W_{hk} + \mu_{23g}\sum B_{jk}^{(3)}W_{ik} \\
&+ \mu_{23h}\sum B_{jk}^{(1)}W_{hk} = \lambda V_{ij}
\end{aligned} \tag{14}$$

$$\mu_{31a}\sum A_{ih}^{(3)}U_{hj} + \mu_{31b}\sum A_{ih}^{(2)}U_{hj} + \mu_{31c}\sum A_{ih}^{(1)}U_{hj} + \mu_{31d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}U_{hk}$$
$$+\mu_{31e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} + \mu_{31f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}U_{hk}$$
$$+\mu_{32a}\sum A_{ih}^{(3)}V_{hj} + \mu_{32b}\sum A_{ih}^{(2)}V_{hj} + \mu_{32c}\sum A_{ih}^{(1)}V_{hj} + \mu_{32d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}V_{hk}$$
$$+\mu_{32e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} + \mu_{32f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}V_{hk} + \mu_{32g}\sum B_{jk}^{(3)}V_{ik} + \mu_{32h}\sum B_{jk}^{(1)}V_{ik}$$
$$+\mu_{33a}\sum A_{ih}^{(4)}W_{hj} + \mu_{33b}\sum A_{ih}^{(3)}W_{hj} + \mu_{33c}\sum A_{ih}^{(2)}W_{hj} + \mu_{33d}\sum A_{ih}^{(1)}W_{hj}$$
$$+\mu_{33e}\sum A_{ih}^{(3)}\sum B_{jk}^{(1)}W_{hk} + \mu_{33f}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}W_{hk} + \mu_{33g}\sum A_{ih}^{(2)}\sum B_{jk}^{(2)}W_{hk}$$
$$+\mu_{33h}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + \mu_{33i}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}W_{hk} + \mu_{33j}\sum A_{ih}^{(1)}\sum B_{jk}^{(3)}W_{hk}$$
$$+\mu_{33k}\sum B_{jk}^{(2)}W_{ik} + W_{ij} = \lambda W_{ij}$$

where the μ_{lmn} signify known functions of the geometric coordinates and the material properties, given in full by Hu [11]. The $A_{lm}^{(r)}$ and $B_{lm}^{(r)}$ respectively are known weighting coefficients for the ϕ and η variables. The U_{ij}, V_{ij}, W_{ij}, are the unknown displacements at the sampling points. Similar DQM equations may be derived for the boundary conditions.

The assembly of the domain and boundary equations ultimately yields a matrix equation of the form

$$\left([K_{dd}]-[K_{db}]\,[K_{bb}]^{-1}[K_{bd}]\right)\{\Delta_d\} = \lambda\{\Delta_d\} \tag{15}$$

The sub-matrices $[K_{bb}],[K_{bd}]$ stem from the boundary conditions, while the sub-matrices $[K_{db}],[K_{dd}]$ stem from the domain equations. The vector $\{\Delta_d\}$ contains the displacements at the domain sampling points.

The equation (15) may be used to solve the eigenvalue problem. The vibration parameters λ are thus determined, from which the natural frequencies are found. The theory for the vibration of the clamped-clamped and clamped-symmetry oblique shells was coded in the Matlab^{TM} programs obcycc.m and obcysc.m. Results from these programs are given in the following.

FINITE ELEMENT METHOD

The commercial FEM program ADINA [15] was used to provide an alternate solution for the problem. A flat four-noded twenty-four degree-of-freedom shell element is available in this program for the solution of shell vibration problems. The problem was solved in the physical space, with no account made of any symmetry in the geometry.

CONVERGENCE, VALIDATION AND RESULTS

DQM solutions are mesh dependent just as are FEM solutions, and it is necessary for a given problem to establish the necessary mesh density for convergence. Sample convergence study results are given herein for the case of a clamped-clamped (CC) cylinder with geometric and material properties as follows; $L' = 0.1m$, $r = 0.1m$, $h = 0.001m$, $\alpha = 20^0$, $E = 0.183e12Pa$, $\nu = 0.3$, and $\rho = 7492kg/m^3$.

Results are given in Table 1 for the first six natural frequencies of vibration of this cylinder. Six different mesh densities are represented ranging from 14 x 14 to 24 x 24. Convergence is steady with all eigenvalues represented in each mesh. Five figure accuracy is obtained for the fundamental frequency with the 22 x 22 mesh. This density was therefore used in all subsequent analyses of this study.

Results from theoretical studies for oblique cylindrical shells were not found in the literature, and thus the validation was restricted to cylindrical shells with perpendicular ends ($\alpha = 0^0$). Validation here is for the case of a CC cylinder with geometric and material properties as follows; $L' = 15.65in$, $r = 1.924in$, $h = 0.101in$, $E = 30{,}000kpi$, $\nu = 0.3$, and $\rho = 0.7324 \times 10^{-3}\ lb\ sec^2 / in^4$.

Results for the first five natural frequencies of this cylinder are given in Table 2 together with the mode numbers *m,n*. These two integer values indicate respectively the wave numbers in the axial and circumferential directions. The DQM results are compared with results from experiment, analytical solutions based on the Fluegge and Sanders theory [16], and an FEM solution. The maximum difference in frequencies between the FEM and DQM is less than 2%. There is also close agreement with the experimental and analytical results.

Brief parametric studies were conducted for the clamped-clamped (CC) boundary conditions and clamped-symmetry (CSym) conditions discussed earlier. The material properties for the cylinders of the parametric study were as follows $E = 0.183e12Pa$, $\nu = 0.3$, and $\rho = 7492kg/m^3$. In the studies the length ratio L/h was varied from 100 to 300, the radius ratio r/h from 25 to 100, and the obliquity angle α from 0^0 to 30^0.

Comparisons of the fundamental frequencies for the CC case are given in Table 3. Both FEM and DQM results are given for each case considered. Frequencies are seen to decrease significantly with increases in the length ratio, and the radius ratio.0 Frequencies also generally decrease with increase in the obliquity angle, but the decrease is smaller. There is close agreement in the results between the two methods with a maximum difference in values of less than 2%.

Comparisons of the fundamental frequencies for the CSym case are given in Table 4. The parametric study now does not cover the *r/h* = 25 case. The dependence of the frequencies on the parameters is similar to that of the CC case. There is again close agreement in the results between the two methods. Results for the fundamental frequencies for the CSym case are all significantly lower than those for the CC case.

CONCLUSIONS

The vibration problem of cylindrical shells with oblique ends has been solved with the aid of domain mapping, and the new differential quadrature method. Results obtained from this method agree well with finite element results. The paper marks a significant advance in the use of the differential quadrature method, presenting a general procedure that may be applied for the analysis of shells of irregular shape.

REFERENCES

1. Hsieh, M.F., Moffat, D.G. and Mistry, J., "Nozzles in the knuckle region of a torispherical head", *Proc. of 9th International Conference on Pressure Vessel Technology*, Vol. 1, pp. 79-90, Sydney, 2000.
2. Kitching, R. and Hose, D.R., "Experimental and theoretical stress analysis of multi-mitred lined glass reinforced plastic pipe bends", *International Journal of Mechanical Science*, Vol. 37, 1995, pp. 97-119.
3. Williams, D.K. and Lewis, G. D., "Development of B1 and C1 stress indices for trunnion elbow supports", *Journal of Pressure Vessel Technology*, Vol. 106, 1984, pp. 166-171.
4. Sobieszczanski, J., "Strength of a pipe mitred bend", *Journal of Engineering for Industry*, Vol. 92, 1970, pp. 767-773.
5. Gill, S.S., *The Stress Analysis of Pressure Vessels and Pressure Vessel Components*, Pergamon, New York, 1970.
6. Jones, N. and Kitching, R., "A theoretical study of in-plane bending of a single unreinforced mitred-bend", *Journal of Strain Analysis,* Vol. 1, 1966, pp. 264-276.
7. Leissa, A.W., *Vibration of Shells*, NASA SP-288, Scientific and Technical Information Office, Washington, 1973.
8. Malik, M. and Bert, C.W., "Vibration analysis of plates with curvilinear quadrilateral planforms by DQM using blending functions", *Journal of Sound & Vibration,* Vol. 230, 1996, pp. 949-954.
9. Sanders, J.L. Jr., "Cutouts in shallow shells", *Journal of Applied Mechanics*, Vol. 42, 1970, pp. 374 -383.
10. Budiansky, B., "Notes on nonlinear shell theory", *Journal of Applied Mechanics*, Vol. 40, 1968, pp. 393-401.
11. Hu, X. J. *Vibration of Some Cylindrical Shells*, M. A. Sc. Thesis, U. of Ottawa, in progress.
12. Char, W. C.*, MAPLE User's Guide*, Watcom Pub. Ltd., Waterloo, 1985.
13. Shu, C., *Differential Quadrature and its Application in Engineering*, Springer, Berlin, 2001.
14. Redekop D. and Makhoul E., "Use of the DQM for the buckling analysis of cylindrical shell panels", Structural Engineering & Mechanics, Vol. 10, 2000, pp. 451-462.
15. *ADINA Verification Manual,* ARD 97-9, ADINA R & D Inc., Watertown, 1998.
16. Bert, C. W. and Malik, M., "Free vibration analysis of thin cylindrical shells by the Differential Quadrature Method", *Journal of Pressure Vessels Technology,* Vol. 118, 1996, pp. 1-12.

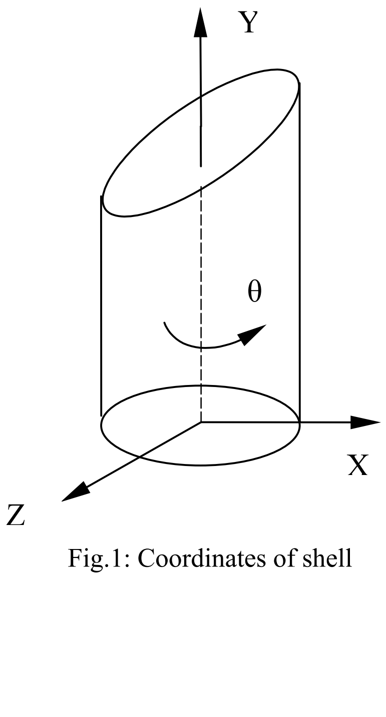


Fig.1: Coordinates of shell

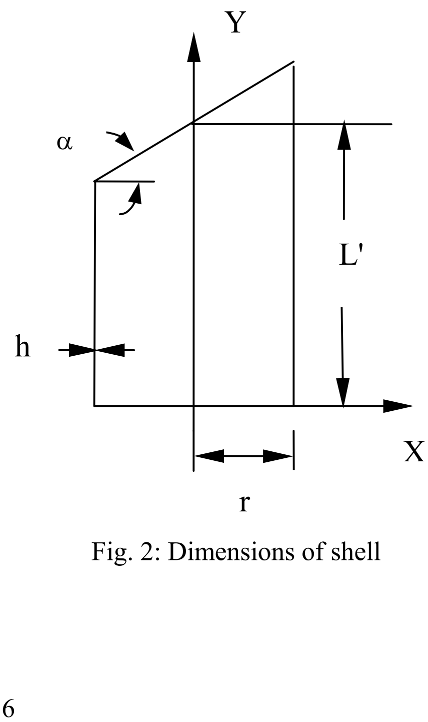


Fig. 2: Dimensions of shell

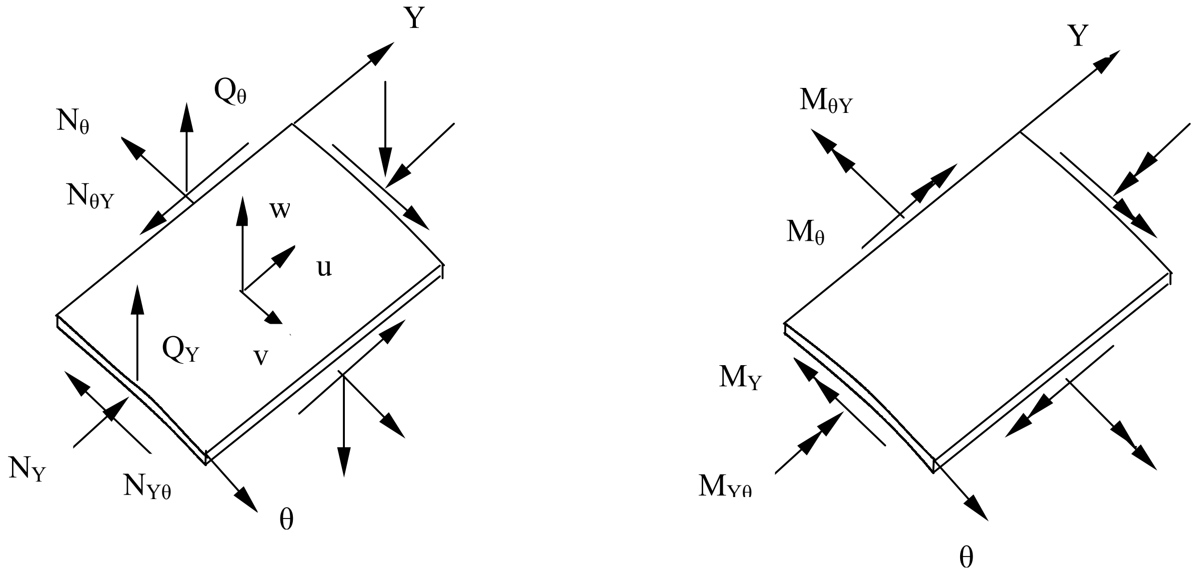


Fig.3 Displacements and stress resultants

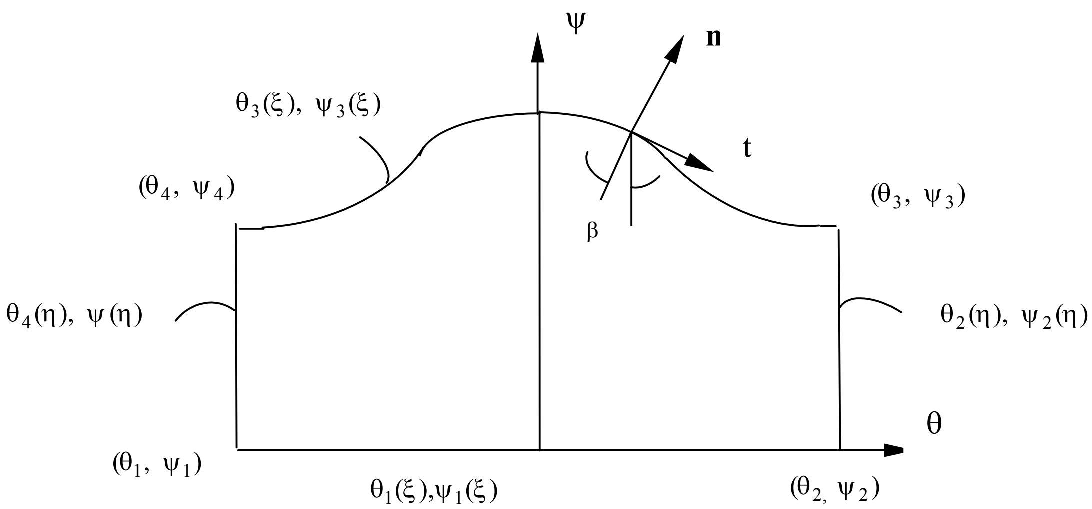


Fig. 4 Mapped domain

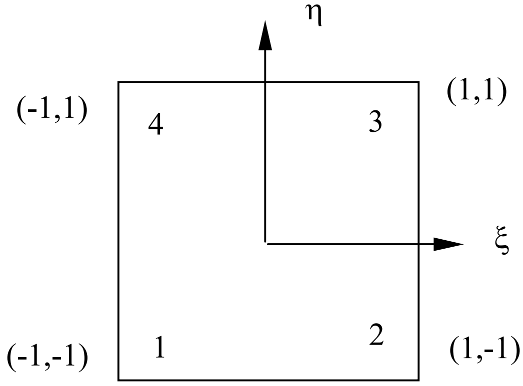


Fig. 5 Parent domain

Table 1: DQM Convergence – frequencies (x 10^3 Hz) – CC supports, $\alpha = 20^0$

	Mesh size					
Mode	14 x 14	16 x 16	18 x 18	20 x 20	22 x 22	24 x 24
1	14.383	12.969	12.392	12.293	12.288	12.288
2	14.421	13.010	12.425	12.293	12.288	12.288
3	18.782	16.306	14.822	14.248	14.181	14.179
4	18.844	16.357	14.918	14.297	14.181	14.179
5	23.352	19.932	17.677	16.406	16.016	15.966
6	23.372	19.978	17.803	16.531	16.051	15.966

Table 2: Validation – frequencies (x 10^3 Hz) – CC supports, $\alpha = 0^0$

Root	m, n	Expt.	Fluegge	Sanders	FEM	DQM
1	1,2	7.791	8.108	8.100	8.155	8.100
2	1,3	13.509	13.381	13.370	13.602	13.370
3	1,1	-	-	-	14.654	14.645
4	2,3	15.771	16.177	16.152	16.419	16.151
5	2,2	15.331	16.400	16.345	16.498	16.383

Table 3: Fundamental - frequencies (x 10^3 Hz) – CC supports

		r / h = 25		r / h = 50		r / h = 100	
L / h	α	FEM	DQM	FEM	DQM	FEM	DQM
100	0^0	29.524	29.304	21.941	21.688	15.293	15.099
	10^0	29.504	29.297	21.263	21.063	13.769	13.669
	20^0	29.484	29.279	20.375	20.191	12.369	12.288
	30^0	29.459	29.252	19.468	19.292	11.137	11.064
200	0^0	14.495	14.408	11.638	11.459	8.098	7.940
	10^0	14.501	14.415	11.510	11.358	7.832	7.710
	20^0	14.531	14.442	11.318	11.178	7.464	7.353
	30^0	14.593	14.500	11.118	10.983	7.080	6.976
300	0^0	9.143	9.053	7.359	7.269	5.487	5.384
	10^0	9.146	9.057	7.356	7.269	5.428	5.335
	20^0	9.159	9.068	7.358	7.270	5.307	5.218
	30^0	9.186	9.093	7.365	7.273	5.152	5.066

Table 4: Table 3: Fundamental - frequencies (x 10^3 Hz) – CSym supports

		r / h = 50		r / h = 100	
L / h	α	FEM	DQM	FEM	DQM
100	0^0	11.546	11.459	8.011	7.940
	10^0	11.228	11.160	7.388	7.352
	20^0	10.809	10.745	6.702	6.669
	30^0	10.385	10.320	6.076	6.042
200	0^0	5.747	5.698	4.178	4.143
	10^0	5.734	5.687	4.077	4.044
	20^0	5.699	5.653	3.915	3.881
	30^0	5.645	5.599	3.737	3.702
300	0^0	3.539	3.524	2.829	2.795
	10^0	3.539	3.524	2.800	2.769
	20^0	3.541	3.526	2.739	2.711
	30^0	3.548	3.530	2.666	2.639