Abstract

INGRAM III, FRANK. On the Wreath Product of Schur Functions. (Under the direction of Naihuan Jing and Ernie Stitzinger.)

The characters of the wreath product $G \sim S_n$were first studied by Specht in [**4**]. Stanley later proposed the question of certain properties of the wreath product of Schur fucntions with respect to the classical theory of symmetric functions.

In this work a relationship is developed between a new power sum symmetric function and the classical power sum symmetric function. Properties for the wreath Schur function are developed from relationships with the classical Schur function and its properties. Finally, a character table is developed that links the new power sum symmetric function with the wreath Schur function.

On The Wreath Product of Schur Functions

by

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Biography

Frank Ingram III was born in Ayden, North Carolina on October 12, 1977 to Elsie and Frank Ingram Jr. He is married to Nakisha Ingram of Kinston, North Carolina. He has one son David Jaliyl Ingram. Frank has two sisters and three brothers. They are Angela L. Lloyd, Timothy C. Ingram, Felix H. Ingram, Ron S. Ingram, and Charity L. Ingram. Frank graduated from Ayden-Grifton high school in 1995. He then went on to attend North Carolina Agricultural and Technical State University in Greensboro, NC. He graduated from A&T in 2000, with a Bachelor of Science in Mathematics. Frank began his graduate education at North Carolina State University in the fall of 2000. Frank married his wife Nakisha on July 6, 2002 in Kinston, NC. He received the Master of Science in Mathematics in 2003 from N.C. State. Frank's son David was born on February 4, 2004. Frank's team Original Lace'em Up won the basketball graduate faculty championship in March 2005. Frank loves to play basketball, golf, and fish in his spare time.

Dedication

Dedicated to my beautiful and loving mother Elsie A. Ingram.

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First I would like to thank God, for putting me in all the right places at the right times. I thank Him for helping me climb mountains, that I could not have climbed without him.

To the love of my life, Nakisha Ingram, for her love, and those beautiful bright eyes that light up any room that she is in. Thank you baby! I love you with all I have and all I am.

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Chapter 1

Introduction

An interpretation of Schur functions in terms of the power sum symmetric functions and Young Tableaux is well known. For example, see Macdonald [**1**] and Stanley[**5**].The purpose of this work is to present an interpretation of the wreath product of Schur functions in terms of the power sum symmetric functions and Young Tableaux. We focus on the special case $\mathbb{Z}_r \sim S_n$. Let $P_{i,n}$ be the power sum symmetric function of degree n, where $i = 0, 1...r-1$ such that

$$P_{i,n} = x_{i,1}^n + x_{i,2}^n + \cdots$$

Let $\Lambda = \mathbb{C}\ [P_{i,n} \mid i = 0, 1...r-1, n \geq 1]$ be the ring of wreath product symmetric functions. We now define an inner product on Λ as follows. First of all we notice that the basis(power sum) of Λ is parameterized by colored partitions.
Let $\lambda = (\lambda_1, \lambda_{2,...})$be a partition, we define for each $\quad i = 0, 1...r-1$.

$$P_{i\lambda} = P_{i\lambda_1} P_{i\lambda_2} \ldots$$

For a colored partition $\underline{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots\right)$ where $\lambda^{(i)}$ are partitions with

$$|\underline{\lambda}| = \left|\lambda^{(0)}\right| + \left|\lambda^{(1)}\right| + \ldots = n.$$

we define

$$P_{\underline{\lambda}} = \prod_{i=1}^{r} P_{i,\lambda_{(i)}} \in \Lambda$$

the power sum associated with the colored partition $\underline{\lambda}$ or the partition valued function $\underline{\lambda}$ on $\{0, 1, \ldots\}$.

For any two colored partitions $\underline{\mu} = \left(\mu^{(0)}, \mu^{(1)}, \ldots\right)$ and $\underline{\sigma} = \left(\sigma^{(0)}, \sigma^{(1)}, \ldots\right)$ we define the inner product $\left\langle P_{\underline{\mu}}, P_{\underline{\sigma}} \right\rangle = \delta_{\underline{\mu}\underline{\sigma}}\ \mathbb{Z}_{\underline{\mu}}$

where $\mathbb{Z}_{\underline{\mu}} = \prod_{i=0}^{r-1} z_{\underline{\mu}(i)}.r^{l(\underline{\mu})}$ here for partition $\lambda = (\lambda_1, \lambda_{2,\cdots}) = (1^{m_1}, 1^{m_2}, \ldots)$ and $z_\lambda = 1^{m_1} m_1!, 2^{m_2} m_2! \ldots$ and $l(\ \underline{\mu})$ the number of partions of $\underline{\mu}$.

Chapter 2

Fundamental Concepts

Definition 2.0.1. *A **permutation** of a set A is a function $\phi : A \to A$ that is both one to one and onto.*

Example 2.0.1. *Suppose that $A = \{1, 2, 3, 4, 5\}$ and σ is the permutation given by* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}$ *so that $\sigma(1) = 4$, $\sigma(2) = 2$ and so on. Because the top line is fixed, we drop it to obtain the one line notation $\sigma = (1435)(2) = (1435)$.*

Definition 2.0.2. *Let $A = \{1, 2, \ldots n\}$ be a finite set . The group of all permutations of A is the **symmetric group** on n letters and is denoted by S_n.*

S_n has $n!$ elements where $n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$.

Example 2.0.2. *S_3 has the following form $S_3 = \{(1)(12), (13), (23), (123), (132)\}$.*

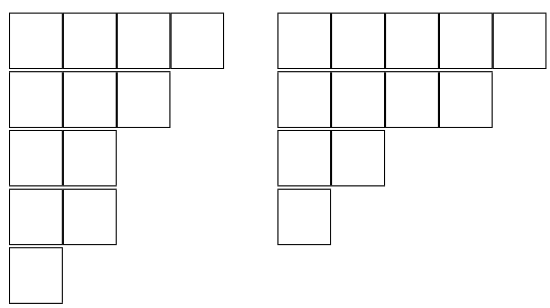
Definition 2.0.3. *A **partition** λ of n of length k, denoted by $\lambda \vdash n$, is a sequence $\lambda = (\lambda_1, \lambda_{2,\ldots}\lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ and $\sum_{i=1}^{k} = n$.The k summands are the parts of the partition, the length of λ is denoted $l(\lambda)$.*

Example 2.0.3. *The partitions of 3 are denoted by Par(3), where Par(3)= $\{(3), (2,1), (1,1,1)\}$. The partitions of 5 having 2 parts are (4,1) and (3,2).*

When l is repeated m_l times it is convenient to introduce another notation. The partition (6,6,5,5,5,5,3,2,2,2,1,1) is notated by $(6^2, 5^4, 3, 2^3, 1^2)$. Where the superscripts denote the multiplicities. In particular, $(6^2, 5^4, 3, 2^3, 1^2)$ is a 12 part partition of 43.

Suppose $\lambda = (\lambda_1, \lambda_{2, \dots} \lambda_k) \vdash n$.The corresponding Ferrers diagram or tableau of shape, λ consists of k left justified rows of boxes with λ_i boxes in row i for $1 \leq i \leq k$. The box in row i and column j has coordinates (i,j) as in a matrix. For example the partition $\lambda = (4, 3, 2, 2, 1)$ has shape. The *conjugate* partition of a partition λ is the partition ,λ',whose diagram is obtained by reflection in the main diagonal. The *hook length* h_x, of a tableau t at the point $x = (i, j)$ is defined to be, $h_x = \lambda_i + \lambda'_j - i - j + 1$. We call the product $h(\lambda) = \prod_{x \in T_\lambda} h_x$ the hook product of λ.

Table 1. $\lambda = (4, 3, 2, 2, 1)$ and $\lambda' = (5, 4, 2, 1)$



Definition 2.0.4. *A tableau T is* ***standard*** *if the numbered entries of T are such that the rows and columns of T are strictly increasing sequences. T is* ***semistandard*** *if the rows weakly increase and the columns strictly increase.*

Definition 2.0.5. *The* ***content*** *μ of a tableau T is the composition*

$\mu = (\mu_1, \mu_2, \dots, \mu_m)$, where $\mu_i$equals the number of i's in T.

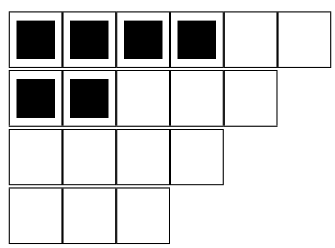
DEFINITION 2.0.6. *A **Semistandard Young Tableau**, denoted by $T_{\lambda,\mu}$, is a tableau T having shape λ and content μ*

TABLE 2. $T_{\lambda\mu}$ with $\lambda = (4, 3, 2, 2, 1)$ and $\mu = (3, 1, 3, 2, 2, 1)$

EXAMPLE 2.0.4.

1	*1*	*1*	*4*
2	*3*	*3*	
3	*4*		
5	*5*		
6			

Let λ and μ are partitions with shape$\mu \subseteq$ shapeλ. We extend the definition of a diagram of shape λ to one of skew shape, λ/μ. The set-theoretic difference $\theta = \lambda - \mu$ is called a *skew shape. Thus the diagram of shape λ/μ =(6,5,4,3)/(4,2) is given by*



The Kostka numbers, $K_{\lambda\mu}$, count the number of semistandard tableau.

DEFINITION 2.0.7. *The **Kostka numbers** are the number of semistandard tableau of shape λ and content μ.*

A generalized Young tableau, is a diagram whose boxes have been filled with positive integers satisfying no certain properties.

Definition 2.0.8. *Let λ and μ be partitions such that $\lambda \supseteq \mu$ (i.e. $\lambda_i \geq \mu_i$ for all i). A* ***semistandard Young tableau of skew shape*** *λ/μ, is a diagram of shape λ/μ whose boxes have been filled with positive integers that are weakly increasing in every row and strictly increasing in every column.*

For example, the generalized tableau of shape λ/μ =(6,5,4,3)/(4,2) is written as

■	■	■	■	*2*	*2*
■	■	*1*	*1*	*5*	
2	*3*	*3*	*3*		
5	*5*	*6*			

which we will also write as the array

$$\begin{array}{cccccc} & & & & 2 & 2 \\ & & 1 & 1 & 5 & \\ 2 & 3 & 3 & 3 & & \\ 4 & 5 & 6 & & & \end{array}$$

DEFINITION 2.0.9. [**3**] *Let G and H be permutation groups acting on sets S and T respectively. The wreath product, $G \sim H$, acts on $S \times T$ as follows. The elements of $G \sim H$ are all pairs of the form (g, h), where $h \in H$ and $g \in G^{|T|}$, so $g = (g_t : g_t \in G)$. The action is*

$$(g, h)(s, t) = (g_t s, ht).$$

It is easy to verify that $G \sim H$ is a group and that this actually a group action.

To illustrate, If $G = \mathcal{S}_3$ and $H = \mathcal{S}_2$ acting on {1,2,3} and {1,2}, respectively, then a typical element of $G \sim H$ is

$$(g, h) = (g, (1, 2)), \text{ where } g = (g_1, g_2) = ((1, 2, 3), (1, 2, 3)^2).$$

So an example of the action would be

$$(g, h)(3, 2) = (g_2(3), h(2)) = (2, 1).$$

Chapter 3

Symmetric Functions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ *and* $x = (x_1, x_2, \ldots)$ *be a set of indeterminates. A polynomial* $f(x_1, x_2, \ldots x_k)$ *is* ***symmetric*** *in* $x_1, x_2, \ldots x_k$ *if its value is unchanged by any permutation of the* k *indeterminates. i.e.*

$$f(x_1, x_2, \ldots x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots x_{\sigma(k)})$$

for every permutaton σ *of* $\{1, 2, \ldots k\}$*. The simplest functions fixed by this action are gotten by symmetrizing a monomial.*

The set of all symmetric functions is called the ***ring of symmetric functions*** *denoted by* $\Lambda = \oplus_{n \geq 0} \Lambda^n$*.* Λ *has five well known basis. Four are* $\mathbb{Z}-$*bases, The monomial symmetric functions,*m_λ*, the elementary symmetric functions,* e_λ *, the complete homogeneuos symmetric functions,* h_λ*, and the Schur functions* $s_{\lambda.}$ *One is a* $\mathbb{Q}-$*basis, the power sum symmetric functions,*p_λ *.*

Definition 3.0.10. *The* ***monomial symmetric function*** *corresponding to* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ *is* m_λ *where*

$$m_\lambda = \sum x_{i1}^{\lambda_1} x_{i2}^{\lambda_2} \ldots x_{ik}^{\lambda_k}$$

where the sum is over all distinct monomials having exponents $\lambda_1, \lambda_2, \ldots, \lambda_k$*.*

Definition 3.0.11. *The* n^{th} ***elementary symmetric function*** *is*

$$e_{(n)} = m_{(1^n)} = \sum_{i1 \geq \ldots \geq in} x_{i1} \ldots x_{in}$$

the sum of all square free monomials of degree n.

Let $\lambda = (\lambda_1, \lambda_{2,\ldots}\lambda_k)$*, then we have* $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_k}$

Definition 3.0.12. *The* n^{th} ***complete homogeneous symmetric function***

$$h_{(n)} = \sum_{\lambda \vdash n} m_\lambda = \sum_{i1 \geq \ldots \geq in} x_{i1} \ldots x_{in}$$

Let $\lambda = (\lambda_1, \lambda_{2,\ldots}\lambda_k)$*, then we have* $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_k}$

Definition 3.0.13. *The* n^{th} ***power sum symmetric function*** *is*

$$p_{(n)} \ = m_{(n)} = \sum_{i \geq 1} x_i^n$$

Let $\lambda = (\lambda_1, \lambda_{2,\ldots}\lambda_k)$*, then we have* $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}$

Example 3.0.5. *When n=4 we have*

$$\begin{aligned}
m_{22} &= x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + \ldots + x_2^2 x_3^2 + x_2^2 x_4^2 + \ldots \\
e_4 &= x_1 x_2 x_3 x_4 + x_1 x_2 x_4 x_5 + \ldots + x_2 x_3 x_4 x_5 + \ldots \\
h_4 &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + \ldots + x_1^3 x_2^1 + x_1 x_3^3 + x_1^3 x_4 + \ldots + x_1^2 x_2^2 + x_1^2 x_3^2 \\
&\quad + x_1^2 x_4^2 + \ldots + x_1 x_2 x_3 x_4 + \ldots + x_1 x_2 x_3 x_4 + \ldots \\
&= m_4 + m_{31} + m_{22} + m_{1111} \\
p_4 &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + \ldots = m_4
\end{aligned}$$

Given any composition $\mu = (\mu_1, \mu_{2,\cdots}\mu_l)$ *, there is a corresponding monomial weight in* $\mathbb{C}[[\mathbf{x}]]$:

$$\mathbf{x}^{\mu} := x_1^{\mu_1} x_2^{\mu_2} \dots x_m^{\mu_l}$$

Now consider a generalized tableau T *of shape* λ*,* T *has a weight, namely,* $\mathbf{x}^T := \mathbf{x}^{\mu}$ *where* μ *is the content of* T*.*
For example if

$$T = \begin{matrix} 2 & 5 & 3 \\ 3 & 1 & \end{matrix}$$

then $\boldsymbol{x}^T = x_1 x_2 x_3^2 x_5$

Definition 3.0.14. *Given a partition* λ*, the associated* ***Schur function*** *is*

$$s_{(\lambda)} = \sum_T \mathbf{x}^T$$

where the sum is over all semistandard λ*-tableaux* T*.*

Example 3.0.6. *If* $\lambda = (2, 1, 1)$ *then some possible tableau are*

$$T:\ \begin{matrix} 1 & 1 \\ 2 & \\ 3 & \end{matrix}\ ,\ \begin{matrix} 2 & 2 \\ 3 & \\ 4 & \end{matrix}\ ,\ \begin{matrix} 2 & 2 \\ 4 & \\ 5 & \end{matrix}\ , \dots \begin{matrix} 1 & 2 \\ 3 & \\ 4 & \end{matrix}\ ,\ \begin{matrix} 1 & 3 \\ 2 & \\ 4 & \end{matrix}\ ,\ \begin{matrix} 1 & 4 \\ 2 & \\ 3 & \end{matrix}\ \dots$$

thus,

$$\begin{aligned} s_{211}(\mathbf{x}) &= x_1^2 x_2 x_3 + x_2^2 x_3 x_4 + x_2^2 x_4 x_5 + \dots + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + \dots \\ &= m_{211}(\mathbf{x}) + 3m_{111}(\mathbf{x}) \end{aligned}$$

Remark 3.0.1. *By definition of the Schur functions and Kostka numbers,*

$$s_{(\lambda)} = \sum_{\mu} K_{\lambda\mu} x^{\mu}$$

where the sum is over all compositions μ of n.

Λ has a well known scalar product. i.e. a bilinear form $\Lambda \times \Lambda \to \mathbb{Q}$,which is denoted by $\langle\ ,\ \rangle$. We have

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$$

thus the p_λ's form an orthogonal basis of λ

Definition 3.0.15. *A **horizontal strip** is a semistandard Young tableau of skew shape λ/μ with no two boxes in the same column. A **vertical strip** is a semistandard Young tableau of skew shape λ/μ with no two boxes in the same row.*

Definition 3.0.16. *A **lattice permutation** is a sequence of positive integers $A_n = a_1 a_2 \ldots a_n$, in which i occurs λ_i times ,such that for any subsequence $A_k = a_1 a_2 \ldots a_k$, the number of i's in A_k is at least as great as the number of $(i+1)'s$ in A_k.*

For instance, 11221 111212 1212121 are lattice permutations, where as 1221121 is not because the subsequence 122 has more twos than ones.

A central result in the theory of symmetric functions is the Littlewood-Richardson rule, which gives a combinatorial interpretation of the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$. We consider this rule along with a much easier special case when $\mu = (n)$, the partition with single part equal to n.

Theorem 3.0.1. ***Pieri's rule*** [**5**] *we have*

$$s_\lambda s_n = \sum_{\mu} s_\mu$$

summed over all partitions μ such that μ/λ is a horizontal strip of size n.

Let T be a tableau. We derive the word of T or $w(T)$ by reading the symbols in T from right to left in successive rows, starting with the top row. For instance, if T is the tableau

$$\begin{array}{cccccc} & & & & 5 & 5 \\ & & 1 & 1 & 6 & 7 \\ 2 & 3 & 3 & 3 & 7 & 8 \\ 4 & 4 & 6 & 7 & 8 & 9 \end{array}$$

$w(T)$ is the word 557611873332987644.

The integer $\langle s_\lambda, s_\mu s_\nu \rangle = \left\langle s_{\lambda/\nu}, s_\mu \right\rangle = \left\langle s_{\lambda/\mu}, s_\nu \right\rangle$ is denoted $c^\lambda_{\mu\nu}$ and is called the Littlewood- Richardson coefficient.

THEOREM 3.0.2. ***Littlewood-Richardson rule***. *Let λ, μ, ν be partitions. Then $c^\lambda_{\mu\nu}$ is equal to the number of tableau T of shape λ/μ and content ν such that $w(T)$ is a lattice permutation. The product $s_\mu s_\nu$ is an integral linear combination of Schur functions:*

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

or equivalently

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$$

Chapter 4

Wreath Product Schur Functions

In this section we develop the wreath product Schur function. Recall that given a partition λ, *the associated* ***Schur function*** *is*

$$s_{(\lambda)} = \sum_T \mathbf{x}^T$$

where the sum is over all semistandard λ*-tableaux* T.

For $\underline{\lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$ *and content* $\underline{\nu} = (\nu^1, \nu^2, \ldots, \nu^{r-1})$ *we define* $\widetilde{T}$ *to be a colored array of semistandard Young tableau such that*

$$\widetilde{T} = T^0, T^1, \ldots, T^{r-1}$$

where T^i *has shape* $\lambda^{(i)}$ *and content* $\nu^{(i)}$ *i.e.*

$$\widetilde{T}_{\underline{\lambda},\underline{\nu}} = T^0_{\lambda^0,\nu^0}, T^1_{\lambda^1,\nu^1}, \ldots, T^{r-1}_{\lambda^{r-1},\nu^{r-1}}$$

Let $\underline{\lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$ *and* $\underline{\mu} = (\mu^1, \mu^2, \ldots, \mu^{r-1})$. *The shape* $\underline{\lambda}/\underline{\mu}$ *is defined to be an array of colored diagrams. Where the i-th colored diagram has shape* $\lambda^{(1)}/\mu^{(i)}$ *i.e.*

$$\underline{\lambda}/\underline{\mu} = \lambda^0/\mu^0, \lambda^1/\mu^1, \ldots,$$

We define $\widetilde{T}$ *of shape* $\underline{\lambda}/\underline{\mu}$ *in a similiar way.*

$$\widetilde{T}_{\underline{\lambda}/\underline{\mu}} = T^0_{\lambda^0/\mu^0}, T^1_{\lambda^1/\mu^1}, \ldots, T^{r-1}_{\lambda^{r-1}/\mu^{r-1}}$$

DEFINITION 4.0.17. *Let $\underline{\lambda}\backslash\underline{\mu}$ be a skew shape. The **Wreath Schur function**,*

$S_{\underline{\lambda}\backslash\underline{\mu}}(\mathbf{Y})$

of shape $\underline{\lambda}\backslash\underline{\mu}$ in the variables $\mathbf{Y} = (Y^0, Y^1, \ldots, Y^{r-1})$ is the power series

$$S_{\underline{\lambda}\backslash\underline{\mu}}(\mathbf{Y}) = \sum_{\widetilde{T}} \mathbf{Y}^{\widetilde{T}}$$

where the sum is over all semistandard tableaux $\widetilde{T}$ of shape $\underline{\lambda}\backslash\underline{\mu}$ and
$\mathbf{Y}^{\widetilde{T}} = (Y^{T^{(0)}}, \ldots, Y^{T^{(r-1)}})$.
If $\mu = \emptyset$, then $\underline{\lambda}\backslash\underline{\mu}$=$\underline{\lambda}$ and we say that $S_{\underline{\lambda}}$ is the wreath Schur function of shape $\underline{\lambda}$

Let ξ be an r^{th} of unity. $\xi = e^{\frac{2\pi i}{r}}$. Introduce another set of power sum symmetric functions inside Λ.
For each $i \in \{0, 1...r-1.\}$ we define

$$P_n^{(i)} = \frac{1}{r}\sum_{j=0}^{r-1} \xi^{ij} P_{j,n}$$

In particular, $P_n^{(0)} = \frac{1}{r}(P_{0,n} + P_{1,n} + \ldots + P_{(r-1),n})$.

EXAMPLE 4.0.7. *For r=2 we have*
$P_n^{(0)} = \frac{1}{2}(P_{0,n} + P_{1,n})$
$P_n^{(1)} = \frac{1}{2}(P_{0,n} - P_{1,n})$

EXAMPLE 4.0.8. *For r=3 we have*
$P_n^{(0)} = \frac{1}{3}(P_{0,n} + P_{1,n} + P_{2,n})$
$P_n^{(1)} = \frac{1}{3}(P_{0,n} + \xi P_{1,n} + \xi^2 P_{2,n})$
$P_n^{(2)} = \frac{1}{3}(P_{0,n} + \xi^2 P_{1,n} + \xi P_{2,n})$
thus we have $P_{i,n} = \sum_{j=0}^{r-1} P_n^{(j)}$

REMARK 4.0.2. *The set of power sum $P_{\underline{\lambda}}^{(i)}$ $i = 0, 1...r-1, |\underline{\lambda}| = n$ also form a basis for Λ.*

Definition 4.0.18. *Let $S_{\lambda^{(i)}}$ be the Schur function associated with the partition $\lambda^{(i)}$, then the* ***Wreath Schur Symmetric Function*** *is defined to be*

$$S_{\underline{\lambda}}(\mathbf{Y}) = S_{\lambda^{(0)}}(Y^{(0)})S_{\lambda^{(1)}}(Y^{(1)})\dots$$

for $\mathbf{Y} = (Y^{(0)}, Y^{(1)}, \dots)$ *where* $Y_n^{(i)} = (y_{i,n})_{n\geq 1}$ *are variables.*

We now introduce another notation. We denote by φ, the unique colored partition $(0, 0, 0\dots)$ of 0, where $S_\varphi(Y^{(i)}) = 1$

Lemma 4.0.1. $S_{\lambda,\varphi}(\mathbf{Y}) \cdot S_{\varphi,\mu}(\mathbf{Y}) = S_{\lambda,\mu}(\mathbf{Y})$

Proof.

$$\begin{aligned} S_{\lambda,\varphi}(\mathbf{Y}) \cdot S_{\varphi,\lambda}(\mathbf{Y}) &= \big(S_\lambda(Y^0)S_\varphi(Y^1)\big) \cdot \big(S_\varphi(Y^0)S_\mu(Y^1)\big) \\ &= S_\lambda(Y^0) \cdot S_\mu(Y^1) \\ &= S_{\lambda,\mu}(\mathbf{Y}) \end{aligned}$$

□

Example 4.0.9. *Consider when n=2 and r=2, we then have the following :*

$$\begin{aligned} P_2^{(0)} &= \frac{1}{2}p_{0,2} + \frac{1}{2}p_{1,2} = \frac{1}{2}p_{2,\varphi} + \frac{1}{2}p_{\varphi,2} \\ P_2^{(1)} &= \frac{1}{2}p_{0,2} - \frac{1}{2}p_{1,2} = \frac{1}{2}p_{2,\varphi} - \frac{1}{2}p_{\varphi,2} \\ P_{11}^{(0)} &= P_0^1 \cdot P_0^1 = \frac{1}{4}p_{0,11} + \frac{1}{4}p_{1,11} + \frac{1}{2}p_{0,1}p_{1,1} = \frac{1}{4}p_{11,\varphi} + \frac{1}{4}p_{\phi,11} + \frac{1}{2}p_{1,1} \\ P_{11}^{(1)} &= P_1^1 \cdot P_1^1 = \frac{1}{4}p_{0,11} + \frac{1}{4}p_{1,11} - \frac{1}{2}p_{0,1}p_{1,1} = \frac{1}{4}p_{11,\varphi} + \frac{1}{4}p_{\phi,11} - \frac{1}{2}p_{1,1} \\ P_1^{(0)} \cdot P_1^{(1)} &= \frac{1}{4}p_{0,11} - \frac{1}{4}p_{1,11} = \frac{1}{4}p_{11,\varphi} - \frac{1}{4}p_{\varphi,11} \end{aligned}$$

which is represented by the following coefficient matrix $[P]_{p'}$ where p has the ordered bases $p = \{p_{2,\varphi}, p_{11,\varphi}, p_{\varphi,2}, p_{\varphi,11}, p_{1,1}\}$ $\mathcal{P} = \{P_2^0, P_{11}^0, P_2^1, P_{11}^1, P_1^0 P_1^1\}$

$$[P]_{p'} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \end{bmatrix}$$

The Schurs in terms of the power sum symmetric functions is well known. We have

$s_2 = \frac{1}{2}p_2 + \frac{1}{2}p_{11}$

$s_{11} = -\frac{1}{2}p_2 + \frac{1}{2}p_{11}$

thus, we have

$S_{2,\varphi} = \frac{1}{2}(P_1^{(0)})^2 + \frac{1}{2}(P_2^{(0)})$

$S_{11,\varphi} = \frac{1}{2}(P_1^{(0)})^2 - \frac{1}{2}(P_2^{(0)})$

$S_{\varphi,2} = \frac{1}{2}(P_1^{(1)})^2 + \frac{1}{2}(P_2^{(1)})$

$S_{\varphi,11} = \frac{1}{2}(P_1^{(1)})^2 - \frac{1}{2}(P_2^{(1)})$

$S_{1,1} = P_1^{(0)}P_1^{(1)}$

which is represented by the following matix $[S]_{\mathcal{P}}$*, with respect to the ordered basis* $\mathcal{P} = \left\{P_2^{(0)}, P_{11}^{(0)}, P_2^{(1)}, P_{11}^{(1)}, P_1^{(0)}P_1^{(1)}\right\}$ *and* $S = \{S_{2,\varphi}, S_{11,\varphi}, S_{\varphi,2}, S_{\varphi,11}, S_{1,1}\}$

$$[S]_{\mathcal{P}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} [S_{\mathbf{2}}(Y^{(0)})] & & \\ & [S_{\mathbf{2}}(Y^{(1)})] & \\ & & [S_{\mathbf{1}}(Y^{(0)}) \otimes S_{\mathbf{1}}(Y^{(1)})] \end{bmatrix}$$

where $[S_{\mathbf{n}}(x)]$ *is the matrix representation all the Schur functions with respect to* $Par(n)$.

The Wreath Schur matrix $S_{p'}$ *is as follows*

$$[S]_{p'} = [S]_{\mathcal{P}} \cdot [P]_{p'} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{4} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \end{bmatrix}$$

hence,

$S_{2,\varphi} = \frac{1}{4}P_{2,\varphi} + \frac{1}{8}P_{11,\varphi} + \frac{1}{4}P_{\varphi,2} + \frac{1}{8}P_{\varphi,11} + \frac{1}{4}P_{1,1}$

$S_{\varphi,2} = \frac{1}{4}P_{2,\varphi} + \frac{1}{8}P_{11,\varphi} - \frac{1}{4}P_{\varphi,2} + \frac{1}{8}P_{\varphi,11} - \frac{1}{4}P_{1,1}$

We now calculate $S_{2,\varphi}$ explicitly. Recall that

$$\begin{aligned} P_1^{(0)} &= \frac{1}{2}p_{0,1} + \frac{1}{2}p_{1,1} = \frac{1}{2}p_{1,\varphi} + \frac{1}{2}p_{\varphi,1} \\ P_2^{(0)} &= \frac{1}{2}p_{0,2} + \frac{1}{2}p_{1,2} = \frac{1}{2}p_{2,\varphi} + \frac{1}{2}p_{\varphi,2} \end{aligned}$$

and $S_2 = \frac{1}{2}p_2 + \frac{1}{2}p_{11} = \frac{1}{2}p_2 + \left(\frac{1}{2}p_1\right)^2$ *so we have*

$$\begin{aligned} S_{2,\varphi} &= \frac{1}{2}(P_1^{(0)})^2 + \frac{1}{2}(P_2^{(0)}) \\ &= \frac{1}{2}[\frac{1}{2}p_{1,\varphi} + \frac{1}{2}p_{\varphi,1}]^2 + \frac{1}{2}[\frac{1}{2}p_{2,\varphi} + \frac{1}{2}p_{\varphi,2}] \\ &= \frac{1}{4}P_{2,\varphi} + \frac{1}{8}P_{11,\varphi} + \frac{1}{4}P_{\varphi,2} + \frac{1}{8}P_{\varphi,11} + \frac{1}{4}P_{1,1} \end{aligned}$$

Example 4.0.10. *When n=3 and r=2 ,with respect to the ordered bases*

$p = \{p_{3,\varphi}; p_{21,\varphi}; p_{111,\varphi}; p_{\varphi,3;}p_{\varphi,21;}p_{\varphi,111}; p_{2,1}, p_{11,1}; p_{\varphi,2}; p_{1,2}; p_{1,11}\}$

$\mathcal{P} = \left\{P_3^{(0)}; P_{21}^{(0)}; P_{111}^{(0)}; P_3^{(1)}; P_{21}^{(1)}; P_{111}^{(1)}; P_2^{(0)}P_1^{(1)}; P_{11}^{(0)}P_1^{(1)}; P_1^{(0)}P_2^{(1)}; P_1^{(0)}P_{11}^{(1)}\right\}$

$\mathcal{S} = \{S_{3,\varphi}; S_{21,\varphi}; S_{111,\varphi}; S_{\varphi,3;}S_{\varphi,21;}S_{\varphi,111}; S_{2,1}, S_{11,1}; S_{\varphi,2}; S_{1,2}; S_{1,11}\}$

we have

$$[P]_{p'} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & -\frac{1}{8} & 0 & -\frac{3}{8} & 0 & -\frac{3}{8} \\ 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & 0 & -\frac{1}{8} \\ 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} \end{bmatrix}$$

$$[S]_{\mathcal{P}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \left[S_{\mathbf{3}}(Y^{(0)})\right] & & & \\ & \left[S_{\mathbf{3}}(Y^{(1)})\right] & & \\ & & \left[S_{\mathbf{2}}(Y^{(0)}) \otimes S_{\mathbf{1}}(Y^{(1)})\right] & \\ & & & \left[S_{\mathbf{1}}(Y^{0}) \otimes S_{\mathbf{2}}(Y^{(1)})\right] \end{bmatrix}$$

$$[S]_{p'} = [S]_{\mathcal{P}} \cdot [P]_{p'}$$

$$= \begin{bmatrix}
\frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix}$$

$$\cdot \begin{bmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & -\frac{1}{8} & 0 & -\frac{3}{8} & 0 & -\frac{3}{8} \\
0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & 0 & -\frac{1}{8} \\
0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & -\frac{1}{8} & 0 & -\frac{1}{8}
\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{1}{8} & \frac{1}{48} & \frac{1}{6} & \frac{1}{8} & \frac{1}{48} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ -\frac{1}{6} & 0 & \frac{1}{24} & -\frac{1}{6} & 0 & \frac{1}{24} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{8} & \frac{1}{48} & \frac{1}{6} & -\frac{1}{8} & \frac{1}{48} & -\frac{1}{8} & \frac{1}{16} & -\frac{1}{8} & \frac{1}{16} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{48} & -\frac{1}{6} & \frac{1}{8} & -\frac{1}{48} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} \\ -\frac{1}{6} & 0 & \frac{1}{24} & \frac{1}{6} & 0 & -\frac{1}{24} & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} \\ \frac{1}{6} & -\frac{1}{8} & \frac{1}{48} & -\frac{1}{6} & -\frac{1}{8} & -\frac{1}{48} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{8} & -\frac{1}{16} \\ 0 & \frac{1}{8} & \frac{1}{16} & 0 & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{8} & \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} \\ 0 & -\frac{1}{8} & \frac{1}{16} & 0 & \frac{1}{8} & -\frac{1}{16} & \frac{1}{8} & \frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} \\ 0 & \frac{1}{8} & \frac{1}{16} & 0 & -\frac{1}{8} & \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} \\ 0 & -\frac{1}{8} & \frac{1}{16} & 0 & \frac{1}{8} & \frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} & \frac{1}{8} & -\frac{1}{16} \end{bmatrix}$$

THEOREM 4.0.3. *(Specht-Macdonald) For $\underline{\lambda}$ colored partitions of n the Wreath Product Schur functions $S_{\underline{\lambda}}$ form an orthonormal basis of Λ* $\left\langle S_{\underline{\lambda}}, S_{\underline{\mu}} \right\rangle = \delta_{\underline{\lambda}\underline{\mu}}$.

Chapter 5

Pieri's Rule

Theorem 5.0.4. ***(Pieri's Rule)*** *Let* $\mathbf{m} = (m_0, m_1, \dots m_{r-1})$ *then we have*
$S_{\underline{\lambda}}(Y)\, S_{\mathbf{m}}(Y) = \sum_{\underline{\mu}} S_{\underline{\mu}}(Y)$
summed over all partitions $\underline{\mu}$ *such that* $\mu^{(i)}/\lambda^{(i)}$ *is a horizontal* m_i *strip.*

Proof.

$$
\begin{aligned}
&S_{\underline{\lambda}}(Y)\, S_{\mathbf{m}}(Y) \\
&\quad = S_{(\lambda^{(0)},\lambda^{(1)},\dots\lambda^{(r-1)})}(Y) \cdot S_{(m_0,m_1\dots m_{r-1},)}(Y) \\
&\quad = (S_{\lambda^{(0)}}(Y^{(0)}) S_{\lambda^{(1)}}(Y^{(1)}) \dots S_{\lambda^{(r-1)}}(Y^{(r-1)}) \cdot (S_{m^{(0)}}(Y^{(0)}) \dots S_{m^{(r-1)}}(Y^{(r-1)})) \\
&\quad = \left(\sum_{\mu^{(0)}} S_{\mu^{(0)}}\left(Y^{(0)}\right)\right) \cdot \left(\sum_{\mu^{(1)}} S_{\mu^{(1)}}\left(Y^{(1)}\right)\right) \dots \left(\sum_{\mu^{(r-1)}} S_{\mu^{(r-1)}}\left(Y^{(r-1)}\right)\right) \\
&\quad = \sum_{(\mu^{(0)},\mu^{(1)},\dots\mu^{(r-1)})} S_{(\mu^{(0)},\mu^{(1)},\dots\mu^{(r-1)})}(Y) \\
&\quad = \sum_{\underline{\mu}} S_{\underline{\mu}}(Y)
\end{aligned}
$$

□

Example 5.0.11. $S_{(32,21)}\left(Y\right)S_{(3,2)}\left(Y\right)=\left(S_{32}\left(Y^{(1)}\right)S_{3}\left(Y^{(1)}\right)\right)\cdot\left(S_{21}\left(Y^{(2)}\right)S_{2}\left(Y^{(2)}\right)\right)$
The ways of adding a horizontal strip of size 3 to the shape 32 and horizontal strip of size 2 to shape 21 are given by

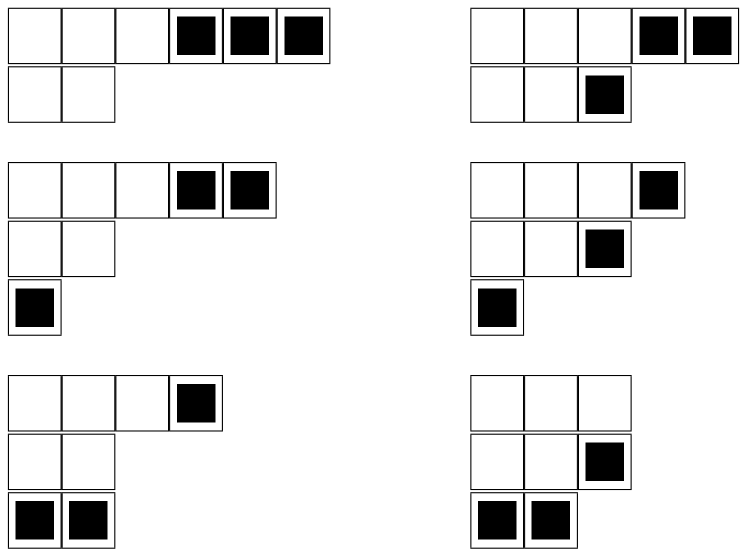


Figure 1. thus, $S_{32}S_3 = S_{62} + S_{53} + S_{521} + S_{431} + S_{422} + S_{332}$

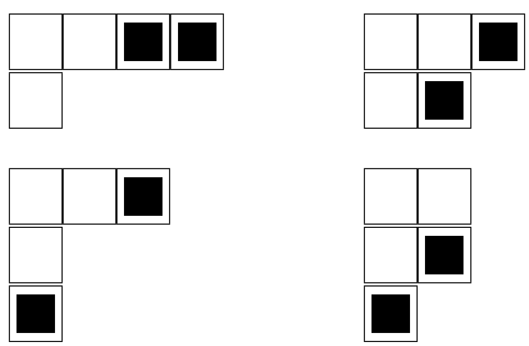


Figure 2. thus, $S_{21}S_2 = S_{41} + S_{32} + S_{311} + S_{221}$

therefore,

$$\begin{aligned} S_{(32,21)}\left(Y\right)S_{(3,2)}\left(Y\right) &= S_{32}(Y^0)S_3(Y^0)\cdot S_{21}(Y^1)S_2(Y^1) \\ &= S_{62,41}+S_{62,32}+\ldots+S_{62,221}+\ldots+S_{53,41}+\ldots+S_{332,221} \end{aligned}$$

COROLLARY 5.0.1. *Let* $1^{\mathbf{m}} = (1^{m_0}, 1^{m_1}, \dots 1^{m_{r-1}})$ *then we have* $S_{\underline{\lambda}}(Y)\, S_{1^{\mathbf{m}}}(Y) = \sum_{\underline{\mu}} S_{\underline{\mu}}(Y)$

summed over all partitions $\underline{\mu}$ *such that* $\mu^{(i)}/\lambda^{(i)}$ *is a vertical* m_i *strip.*

PROOF.

$$
\begin{aligned}
S_{\underline{\lambda}}(Y)\, S_{1^{\mathbf{m}}}(Y) & \\
&= S_{(\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{(r-1)})}(Y) \cdot S_{(1^{m_0},1^{m_1},\dots,1^{m_{r-1}})}(Y) \\
&= (S_{\lambda^{(0)}}(Y^{(0)}) S_{\lambda^{(1)}}(Y^{(1)}) \dots S_{\lambda^{(r-1)}}(Y^{(r-1)}) \cdot (S_{1^{m_0}}(Y^{(0)}) \dots S_{1^{m_{(r-1)}}}(Y^{(r-1)})) \\
&= \left(\sum_{\mu^{(0)}} S_{\mu^{(0)}}\left(Y^{(0)}\right)\right) \cdot \left(\sum_{\mu^{(1)}} S_{\mu^{(1)}}\left(Y^{(1)}\right)\right) \dots \left(\sum_{\mu^{(r-1)}} S_{\mu^{(r-1)}}\left(Y^{(r-1)}\right)\right) \\
&= \sum_{(\mu^{(0)},\mu^{(1)},\dots\mu^{(r-1)})} S_{(\mu^{(0)},\mu^{(1)},\dots\mu^{(r-1)})}(Y) \\
&= \sum_{\underline{\mu}} S_{\underline{\mu}}(Y)
\end{aligned}
$$

such that $\mu^{(i)}/\lambda^{(i)}$ is a vertical m_i strip.

□

Chapter 6

Littlewood-Richardson

Theorem 6.0.5. ***(Littlewood-Richardson)***
Let $\underline{\lambda}, \underline{\mu}, \underline{\nu}$ *be colored partitions. Then* $c^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}}$ *is equal to the number of colored tableau* $\widetilde{T}$ *of shape* $\underline{\lambda}/\underline{\mu}$ *and content* $\underline{\nu}$ *such that* $w(T)$ *is a lattice permutation. The product* $S_{\underline{\mu}}S_{\underline{\nu}}$ *is an integral linear combination of Schur functions:*

$$S_{\underline{\mu}}S_{\underline{\nu}} = \sum_{\underline{\lambda}} c^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}} S_{\underline{\lambda}}$$

or equivalently

$$S_{\underline{\lambda}/\underline{\mu}} = \sum_{\underline{\nu}} c^{\lambda}_{\underline{\mu}\underline{\nu}} S_{\nu}$$

Proof.

$$\begin{aligned}
S_{\underline{\lambda}/\underline{\mu}}(\mathbf{Y}) &= S_{(\lambda^{(0)}/\mu^{(0)},\lambda^{(1)}/\mu^{(1)},\ldots\lambda^{(r-1)}/\mu^{(r-1)})}(\mathbf{Y}) \\
&= \left(S_{\lambda^{(0)}/\mu^{(0)}}(Y^{(0)})S_{\lambda^{(1)}/\mu^{(1)}}(Y^{(1)})\ldots S_{\lambda^{(r-1)}/\mu^{(r-1)}}(Y^{(r-1)})\right) \\
&= \left(\sum_{\nu^{(0)}} c^{\lambda^{(0)}}_{\mu^{(0)}\nu^{(0)}} S_{\nu^{(0)}}\left(Y^{0}\right)\right)\ldots\left(\sum_{\nu^{(r-1)}} c^{\lambda^{(r-1)}}_{\mu^{(r-1)}\nu^{(r-1)}} S_{\nu^{(r-1)}}\left(Y^{(r-1)}\right)\right) \\
&= \left(\sum_{\nu^{(0)},\nu^{(1)},\ldots,\nu^{(r-1)}} c^{\lambda^{(0)}\lambda^{(1)}\ldots\lambda^{(r-1)}}_{\left(\mu^{(0)}\ldots\mu^{(r-1)}\right)\left(\nu^{(0)}\ldots\nu^{(r-1)}\right)} S_{\nu^{(0)}\ldots\nu^{(r-1)}}(\mathbf{Y})\right) \\
&= \sum_{\underline{\nu}} c^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}} S_{\underline{\nu}}(\mathbf{Y})
\end{aligned}$$

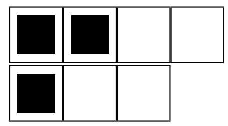
□

Example 6.0.12. *Consider*

$$S_{(21,543)}\left(Y\right)S_{(211,111)}\left(Y\right)=\left(S_{21}\left(Y^{(0)}\right)S_{211}\left(Y^{(0)}\right)\right)\cdot\left(S_{543}\left(Y^{(1)}\right)S_{111}\left(Y^{(1)}\right)\right).$$

For $S_{21}\left(Y^{(0)}\right)S_{211}\left(Y^{(0)}\right)$, *first note that* $|\mu|+|\nu|=|\lambda|=7$. *We fix* $\mu=(2,1)$*and* $\nu=(2,1,1)$ *and sum over all possible* λ*-tableau. The possible tableaux are as follows:*

$\lambda=(4,3)$

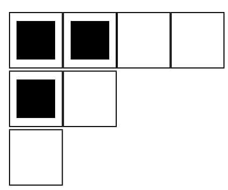


where the possible tableaux are:

$$\begin{array}{cccc} & & 1 & 1\\ & 2 & 3 & \end{array},\quad \begin{array}{cccc} & & 1 & 2\\ & 1 & 3 & \end{array},\quad \begin{array}{cccc} & & 1 & 3\\ & 1 & 2 & \end{array}$$

which have words 1132, 2131, 3121; none of which are lattice permutations, thus $c_{\mu\nu}^{\lambda}=0$. *Only the tableaux where* $c_{\mu\nu}^{\lambda}\neq 0$. *will now be listed.*

for $\lambda=(4,2,1)$ *we have shape*

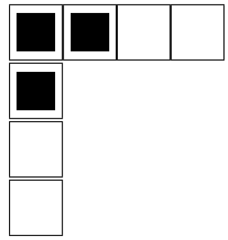


where the possible tableaux are:

$$\begin{array}{ccc} & 1 & 1\\ 2 & & \\ 3 & & \end{array},\quad \begin{array}{ccc} & 1 & 1\\ 3 & & \\ 2 & & \end{array},\quad \begin{array}{ccc} & 1 & 2\\ 3 & & \\ 1 & & \end{array},\quad \begin{array}{ccc} & 1 & 2\\ 1 & & \\ 3 & & \end{array}\quad \begin{array}{ccc} & 1 & 3\\ 1 & & \\ 2 & & \end{array}\quad \cdots$$

which have words 1123,1132, 2131, 2113, 3112...; where 1123 is a lattice permutation, thus $c_{\mu\nu}^{(421)}=1$.

for $\lambda = (4, 1, 1, 1)$ *we have shape*

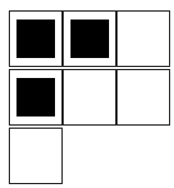


where the only possible tableau is:

		1	1
2			
3			

which has word 1123, which is a lattice permutation, thus $c_{\mu\nu}^{(4111)} = 1$.

for $\lambda = (3, 3, 1)$ *we have shape*

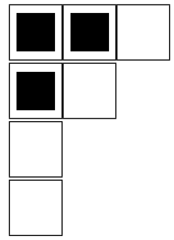


where the possible tableaux are:

		1			1			1			2
	2	3 ,		1	2 ,		1	3 ,		1	3
1			3			2			1		

which have words 1321, 1213, 1312, 2311; where 1213 is a lattice permutation, thus $c_{\mu\nu}^{(331)} = 1$.

for $\lambda = (3,2,1,1)$ *we have shape*

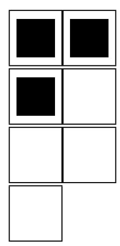


where the possible tableaux are:

$$\begin{array}{ccc} & & 1 \\ & 1 & \\ 2 & & \\ 3 & & \end{array} , \quad \begin{array}{ccc} & & 1 \\ & 2 & \\ 1 & & \\ 3 & & \end{array} , \quad \begin{array}{ccc} & & 1 \\ & 3 & \\ 1 & & \\ 2 & & \end{array}$$

which have words 1123, 1213, 1312; where 1123 and 1213 are lattice permutations, thus $c_{\mu\nu}^{(3211)} = 2$.

for $\lambda = (2,2,2,1)$ *we have shape*

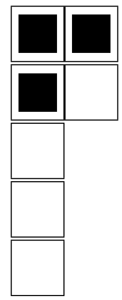


where the possible tableaux are:

$$\begin{array}{cc} & 1 \\ 1 & 2 \\ 3 & \end{array} , \quad \begin{array}{cc} & 1 \\ 1 & 3 \\ 2 & \end{array} ,$$

which have words 1213, 1312 ; where 1213 is a lattice permutation, thus $c_{\mu\nu}^{\lambda} = 1$.

for $\lambda = (2,2,1,1,1)$ *we have shape*



where the only possible tableau is:

$$\begin{array}{cc} & \\ & 1 \\ 1 & \\ 2 & \\ 3 & \end{array}$$

which has word 1123, which is a lattice permutation, thus $c_{\mu\nu}^{(22111)} = 1$*.*

hence we have:

$$\begin{aligned} S_{21}\left(Y^{(0)}\right)S_{211}\left(Y^{(0)}\right) &= S_{421}\left(Y^{(0)}\right) + S_{41^3}\left(Y^{(0)}\right) + S_{3^21}\left(Y^{(0)}\right) + 2S_{321^2}\left(Y^{(0)}\right) \\ &\quad + S_{2^31}\left(Y^{(0)}\right) + S_{2^21^3}\left(Y^{(0)}\right) \end{aligned}$$

For $S_{543}\left(Y^{(1)}\right)S_{111}\left(Y^{(1)}\right)$ *we use the same process to obtain*

$$
\begin{aligned}
S_{543}\left(Y^{(1)}\right) S_{111}\left(Y^{(1)}\right) \quad &= \quad S_{654}\left(Y^{(1)}\right) + S_{6431^2}\left(Y^{(1)}\right) + S_{5^2 31^2}\left(Y^{(1)}\right) + S_{543^3}\left(Y^{(1)}\right) \\
&\quad + S_{54^2 1^2}\left(Y^{(1)}\right)
\end{aligned}
$$

therefore,

$$
\begin{aligned}
S_{(21,543)}(Y) S_{(211,111)}(Y) \quad &= \quad \big(S_{21}(Y^0) S_{211}(Y^0)\big) \cdot \big(S_{543}(Y^1) S_{111}(Y^1)\big) \\
&= \quad \big(S_{421}(Y^0) + \ldots + S_{2^2 1^3}(Y^0)\big) \cdot \big(S_{654}(Y^1) + \ldots + S_{54^2 1^2}(Y^1)\big) \\
&= \quad S_{421}(Y^0) \cdot S_{654}(Y^1) + \ldots + S_{2^2 1^3}(Y^0) \cdot S_{54^2 1^2}(Y^1) \\
&= \quad S_{(421,654)}(Y) + S_{(421,6431^2)}(Y) + \ldots + S_{(2^2 1^3, 54^2 1^2)}(Y)
\end{aligned}
$$

Chapter 7

The Characters of $\mathbb{Z}_r \sim S_n$

It is well known that the irreducible characters of the symmetric group S_n are indexed by partitions of (n). The last desciption of $S_{\underline{\lambda}}$ will involve the irreducible characters of $\mathbb{Z}_r \sim S_n$ Let G be a group. Let $A(g), g \in G$ be a matrix representation. The character table, χ of A is $A(g) = trA(g)$, where tr is the trace of $A(g)$.

Definition 7.0.19. [**3**] *A class function on a group G is a mapping $f : G \to \mathbb{C}$ such that $f(g) = f(h)$ whenever g and h are in the same conjugacy class. The set of all class functions on G is denoted by $R(G)$.*

If k is a conjugacy class and χ is a character, we define χ_k to be the value of the given character on the given class. i.e.

$$\chi_k = A(g)$$

for any $g \in k$.

DEFINITION 7.0.20. [**3**] *Let G be a group, the character table of G is an array with rows indexed by the inequivalent irreducible characters of G and columns indexed by the conjugacy classes. The table entry in row χ and column k is χ_k*

	$\cdots$	k	$\cdots$
		$\vdots$	
χ	$\cdots$	χ_k	

EXAMPLE 7.0.13. [**3**] *Recall that a conjugacy class in $G = S_n$ consists of all permutations of a given cycle type. In particular for S_3*

$$k_1 = \{\in\} \ k_2 = \{(1,2),(1,3),(2,3)\} \text{ and } k_3 = \{(1,2,3),(1,3,2)\}.$$

Thus, there of three irreducible representations of S_3. We have the character table for S_3

	k_1	k_2	k_3	
$\chi^{(1)}$	1	1	1	$\leftarrow$ *trivial representation*
$\chi^{(2)}$	1	-1	1	$\leftarrow$ *sign representation*
$\chi^{(3)}$	2	0	-1	$\leftarrow$ *defining representation*

Next we turn to the irreducible characters of $\mathbb{Z}_r \sim S_n$. Let $r-1$ be the number of irreducible characters of $\mathbb{Z}_r$. Then the irreducible characters of $\mathbb{Z}_r \sim S_n$ are indexed by the $(r-1)$-tuples $(\lambda^0, \ldots, \lambda^{r-1})$ of partitions λ^i such that $\sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. Let $(Y^{(r-1)})_n$ be the set of such $(r-1)$-tuples of partitions. We denote by $\chi^{\underline{\lambda}}$ the irreducible character corresponding to $\underline{\lambda} \in \left(Y^{(r-1)}\right)_n$

7.1. Conjugacy Classes

We now use [**2**] *to give a brief review of conjugacy classes and the irreducible representations of a finite group* Γ.

Let $x = (\gamma(1), \ldots, \gamma(n); \sigma)$ *be an element of* $\Gamma \sim S_n$. *We write* σ *as a product of disjoint cycles:*

$$\sigma = \sigma_1 \cdots \sigma_l \qquad \text{where} \;\; \sigma_s = (i_{s,1}, \ldots, i_{s,\mu_s}) \qquad \text{and} \;\; \sum_{s=1}^{l} \mu_s = n.$$

Then the product $\gamma(i_{s,\mu(s)})\gamma(i_{s,\mu(s-1)}) \cdots \gamma(i_{s,1})$ *is called the cycle product associated with a cycle* σ_s. *For a Conjugacy class* C *of* Γ, *let* $\mu_{s_1}, \ldots, \mu_{s_m}$ $(s_1 < \cdots < s_m)$ *be the lengths of cycles* μ_s *such that the associated cycle product is contained on* C *and let* $\mu^C(x)$ *be the partition obtained by rearranging* $\mu_{s_1}, \ldots, \mu_{s_m}$ *in descending order.*

Theorem 7.1.1. [**2**] *Two elements* x *and* y *of* $\Gamma \sim S_n$ *are conjugate if and only if* $\mu^C(x) = \mu^C(y)$ *for all conjugacy classes* C *of* Γ.

So we call $\mu^C(x)$ *the* C*-type of* x. *If* Γ *has exactly* r *conjugacy classes , the number of conjugancy classes of* $\Gamma \sim S_n$ *is equal to the cardinality of* $(Y^r)_n$.

7.1.1. Irreducible Representations[2]. *For a partition λ of n, let χ^λ be the irreducible character corresponding to λ and V^λ be an irreducible $\mathbb{C}S_n$ module affording χ^λ. Then the degree $\chi^\lambda(1) = dimV^\lambda$ is equal to $n!\backslash h(\lambda)$. Let $\{W_1, \ldots, W_k)$ be the complete set of representatives of isomorphism classes of irreducible $\mathbb{C}\Gamma$-modules. We fix the numbering of $W_1, \ldots, W_k$ and put the $d_i = dimW_i$. Then the irreducible $\mathbb{C}\,(\Gamma \sim S_n)$-modules can be constructed as follows. For $\underline{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(k)}\right) \in (Y^k)_n$, we put*

$$K^{\underline{\lambda}} = \Gamma^n \left(S_{n_1} \times \cdots \times S_{n_k}\right) \ ,$$

$$T^{\underline{\lambda}} = T^{n_1}(W_1) \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} T^{n_k}(W_k),$$

and

$$V^{\underline{\lambda}} = V^{\lambda^1} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} V^{\lambda^k}$$

where $n_i = |\lambda^i|$ and $T^i(V)$ denotes the i-fold tensor product of a vector space V. We regard $T^{\underline{\lambda}}$ and $V^{\underline{\lambda}}$ as $\mathbb{C}K^{\underline{\lambda}}$-modules by defining

$$(\gamma(1), \ldots, \gamma(n); \sigma)(t_1 \otimes \cdots \otimes t_n) = \gamma(1)t_{\sigma^{-1}(1)} \otimes \cdots \otimes \gamma(n)t_{\sigma^{-1}(n)},$$
$$(\gamma(1), \ldots, \gamma(n); \sigma)(v_1 \otimes \cdots \otimes v_k) = \sigma_1 v_1 \otimes \cdots \otimes \sigma_k v_k,$$

where $\gamma(i) \in \Gamma$, $\sigma = \sigma_1 \cdots \sigma_k \in S_{n_1} \times \cdots \times S_{n_k}$, $t_1 \otimes \cdots \otimes t_n \in T^{\underline{\lambda}}$, and $v_1 \otimes \cdots \otimes v_k \in V^{\underline{\lambda}}$:

$$W^{\underline{\lambda}} = \mathbb{C}(\Gamma \sim S_n) \otimes_{\mathbb{C}K^{\underline{\lambda}}} (T^{\underline{\lambda}} \otimes_{\mathbb{C}} V^{\underline{\lambda}}).$$

Then

$$dimW^{\underline{\lambda}} = n! \prod_{i=1}^{k} \frac{d_i^{|\lambda^{(i)}|}}{h(\lambda^{(i)})}.$$

THEOREM 7.1.2. [**2**] *The $\mathbb{C}(\Gamma \sim S_n)$-module $W^{\underline{\lambda}}$ constructed above is irreducible. These $W^{\underline{\lambda}}$ ($\underline{\lambda} \in (Y^{(r-1)})_n$)are pairwise non-isomorphic and exhast the isomorphism classes of irreducible $\mathbb{C}(\Gamma \sim S_n)$ modules.*

DEFINITION 7.1.1. *For $\underline{\lambda} \in (Y^{(r-1)})_n$, let $\chi^{\underline{\lambda}}_{\mathbb{Z}_r \sim S_n}$(or $\chi^{\underline{\lambda}}$) denote the character of irreducible $\mathbb{C}(\mathbb{Z}_r \sim S_n) -$ module.*

Let $\underline{\rho} = (\rho^{(0)}, \rho^{(1)}, \ldots, \rho^{(r-1)})$ we define

$$Z_{\underline{\rho}} = \left(\prod_{i=0}^{r-1} z_{\rho^{(i)}}\right) r^{||\rho||}$$

where $z_\rho = \prod_{j \geq 1} j^{m_j} m_j!$ with $\rho = (1^{m_1} 2^{m_2} \cdots)$

THEOREM 7.1.3. [**1**] *For a family of partitions* $\underline{\rho} = (\rho^{(0)}, \rho^{(1)}, \ldots)$ *we have*

$$S_{\underline{\lambda}} = \sum_{\underline{\rho}} Z_{\underline{\rho}}^{-1} \chi_{\underline{\rho}} P_{\underline{\rho}}$$

or equivalently

$$\chi_{\underline{\rho}}^{\underline{\lambda}} = \left\langle S_{\underline{\lambda}}, P_{\underline{\rho}} \right\rangle$$

The simplest example of this theory is when $r = 2$. *We then have the hyperoctahedral group. The hyperoctahedral group is the semi-direct product* $H_n = \mathbb{Z}_2^n \rtimes S_n$, *where* $\mathbb{Z}_2 = \{-1, 1\}$ *is considered as the multipicative group of two elements. We will now extract the irreducible characters from* $S_{\underline{\lambda}}$ *to form the character table for* $\mathbb{Z}_2 \sim S_2$. *Recall that:*

$$\begin{aligned}
S_{2,\varphi} &= \frac{1}{4}p_{2,\varphi} + \frac{1}{8}p_{11,\varphi} + \frac{1}{4}p_{\varphi,2} + \frac{1}{8}p_{\varphi,11} + \frac{1}{4}p_{1,1} \\
S_{11,\varphi} &= -\frac{1}{4}p_{2,\varphi} + \frac{1}{8}p_{11,\varphi} - \frac{1}{4}p_{\varphi,2} + \frac{1}{8}p_{\varphi,11} + \frac{1}{4}p_{1,1} \\
S_{\varphi,2} &= \frac{1}{4}p_{2,\varphi} + \frac{1}{8}p_{11,\varphi} - \frac{1}{4}p_{\varphi,2} + \frac{1}{8}p_{\varphi,11} - \frac{1}{4}p_{1,1} \\
S_{\varphi,11} &= -\frac{1}{4}p_{2,\varphi} + \frac{1}{8}p_{11,\varphi} + \frac{1}{4}p_{\varphi,2} + \frac{1}{8}p_{\varphi,11} - \frac{1}{4}p_{1,1} \\
S_{1,1} &= \frac{1}{4}p_{11,\varphi} - \frac{1}{4}p_{\varphi,11}
\end{aligned}$$

For $\underline{\rho} = (2, \varphi)$ *or* $\underline{\rho} = (\varphi, 2)$, *we have* $Z_{\underline{\rho}} = (Z_{\rho^{(0)}} \cdot Z_{\rho^{(1)}}) = (2 \cdot 1)2^1 = 4$.
For $\underline{\rho} = (11, \varphi)$ *or* $\underline{\rho} = (\varphi, 11)$, *we have* $Z_{\underline{\rho}} = (Z_{\rho^{(0)}} \cdot Z_{\rho^{(1)}}) = ((1^2 \cdot 2!) \cdot 1)2^2 = 8$.
For $\underline{\rho} = (1, 1)$,$Z_{\underline{\rho}} = (Z_{\rho^{(0)}} \cdot Z_{\rho^{(1)}}) = (1 \cdot 1)2^2 = 4$.

Factoring $\frac{1}{Z_{\underline{\rho}}}$ *from each* $S_{\underline{\lambda}}$ *we obtain the following:*

$$\begin{aligned}
S_{2,\varphi} &= \frac{1}{4}(1)p_{2,\varphi} + \frac{1}{8}(1)p_{11,\varphi} + \frac{1}{4}(1)p_{\varphi,2} + \frac{1}{8}(1)p_{\varphi,11} + \frac{1}{4}(1)p_{1,1} \\
S_{11,\varphi} &= -\frac{1}{4}(-1)p_{2,\varphi} + \frac{1}{8}(1)p_{11,\varphi} - \frac{1}{4}(-1)p_{\varphi,2} + \frac{1}{8}(1)p_{\varphi,11} + \frac{1}{4}(1)p_{1,1} \\
S_{\varphi,2} &= \frac{1}{4}(1)p_{2,\varphi} + \frac{1}{8}(1)p_{11,\varphi} - \frac{1}{4}(-1)p_{\varphi,2} + \frac{1}{8}(1)p_{\varphi,11} - \frac{1}{4}(-1)p_{1,1} \\
S_{\varphi,11} &= -\frac{1}{4}(-1)p_{2,\varphi} + \frac{1}{8}(1)p_{11,\varphi} + \frac{1}{4}(1)p_{\varphi,2} + \frac{1}{8}(1)p_{\varphi,11} - \frac{1}{4}(-1)p_{1,1} \\
S_{1,1} &= \frac{1}{4}(2)p_{11,\varphi} - \frac{1}{4}(-2)p_{\varphi,11}
\end{aligned}$$

thus we have the character table for $\mathbb{Z}_2 \sim S_2$

TABLE 1. Characters of $\mathbb{Z}_2 \sim S_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 & 0 \end{bmatrix}$$

The Characters for $\mathbb{Z}_2 \sim S_3$ are as follows The wreath product $\mathbb{Z}_2 \sim S_4$ can also be

TABLE 2. Characters of $\mathbb{Z}_2 \sim S_3$

$$\begin{bmatrix} 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 & -1 & 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 2 & 0 & -1 & -2 & 0 & 1 & -2 & 0 & 2 & 0 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 3 & -1 & 0 & -3 & 1 & 0 & 1 & 1 & -1 & -1 \\ 3 & 1 & 0 & -3 & -1 & 0 & 1 & -1 & -1 & 1 \\ 3 & -1 & 0 & 3 & 1 & 0 & -1 & -1 & -1 & 1 \\ 3 & 1 & 0 & 3 & -1 & 0 & -1 & 1 & -1 & -1 \end{bmatrix}$$

represented by the symmetry group W_4 of the for dimminsional cube, consisting of pure rotations and rotations combined with reflections, is a group of order 384. It has the following character table.

Table 3. Characters of $\mathbb{Z}_2 \sim S_4$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
2 & 2 & 0 & 0 & 2 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 2 & 2 & 0 & -1 & -1 & 2 & 0 & 2 \\
2 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & -2 & -1 & 0 & 0 & 2 & -2 & 0 & -1 & 1 & 2 & 0 & 2 \\
3 & 3 & 1 & 1 & 3 & 0 & 1 & 1 & 3 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 3 \\
3 & -3 & -1 & 1 & 3 & 0 & 1 & -1 & -3 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & 3 \\
3 & 3 & -1 & -1 & 3 & 0 & -1 & -1 & 3 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 3 \\
3 & -3 & 1 & -1 & 3 & 0 & -1 & 1 & -3 & 0 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 3 \\
4 & 2 & 2 & 2 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & -2 & -1 & -1 & 0 & -2 & -4 \\
4 & -2 & -2 & 2 & 0 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & 1 & 0 & -2 & 4 \\
4 & 2 & -2 & -2 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 2 & -4 \\
4 & -2 & 2 & -2 & 0 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & -2 & -1 & 1 & 0 & 2 & -4 \\
6 & 0 & 2 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & -2 & 0 & 6 \\
6 & 0 & -2 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & -2 & 0 & 6 \\
6 & 0 & 0 & 2 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 2 & 6 \\
6 & 0 & 0 & -2 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & -2 & 6 \\
8 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -8 \\
8 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -8
\end{bmatrix}
$$

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