

Expected Number of Vertices of a Random Convex Polytope

I. Integral Formula and Asymptotic Bounds

by

D.G. Kelly*

Department of Statistics
Curriculum in Operations Research and Systems Analysis
Department of Mathematics

and

J.W. Tolle**

Curriculum in Operations Research and Systems Analysis
Department of Mathematics

University of North Carolina at Chapel Hill

Abstract

Given m points on the unit sphere in n -space, the hyperplanes tangent to the sphere at the given points bound a convex polytope with m facets. If the points are chosen independently at random from the uniform distribution on the sphere, the number V_{mn} of the vertices of the polytope is a random variable. We obtain an integral expression for EV_{mn} and asymptotic bounds of the form

$$\alpha \frac{n^{(n-6)/2}}{(m-n)} \leq EV_{mn} \leq \beta \frac{n^{(n-5)/2}}{m} .$$

*D.G. Kelly's research was supported by the Office of Naval Research under Contract N00014-76-C-0550.

**J.W. Tolle's research was partially supported by the Office of Naval Research under Contract N00014-76-C-0550 and partially by the Army Research Office under Grant DAAG29-79-G-0014.

I. Introduction

The main results of this working paper are: the integral expression (3.3) for the expected number of vertices of a random convex polytope; the upper bound (4.1) and consequent asymptotic upper bound (4.2) on the expected number; and the asymptotic lower bounds (4.7) and (4.8).

Briefly and informally, the bounds are these: Let p_1, \dots, p_m be independent random points each having the uniform distribution on the unit sphere in \mathbb{R}^n , and let V_{mn} be the number of vertices of the (random) polytope whose facets are the hyperplanes tangent to the sphere at p_1, \dots, p_m . Then there exist constants α and β independent of m and n such that asymptotically

$$\alpha n^{(n-6)/2} (m-n) \leq EV_{mn} \leq \beta n^{(n-5)/2} m.$$

The upper bound is valid for any $n \geq 2$, asymptotically as $m \rightarrow \infty$; the lower bound is valid for any large n , asymptotically as $m \rightarrow \infty$.

In a later paper we will present the results of numerical approximations of the integral (3.3) for moderate values of m and n .

II. Formulation

A. Given an n -vector \underline{a} and a real number b , the inequality $\underline{a}^T \underline{x} \leq b$ defines a half-space in \mathbb{R}^n with bounding hyperplane $H = \{x: \underline{a}^T x = b\}$. The non-empty intersection of a finite number, say k , of such half-spaces defines a polytope in \mathbb{R}^n with at most k facets. A *random polytope* is a polytope generated by choosing the components of the k n -vectors, $\underline{a}_1, \dots, \underline{a}_k$, and the k real numbers, b_1, \dots, b_k , from some probability distribution on the real line. Obviously the structure of random polytopes will depend on the choice of the probability distribution from which the data are drawn. For examples of certain distribution schemes the reader is referred to [2,4].

In this paper it is desired to consider only random polytopes in \mathbb{R}^n with a fixed number, m , of facets. The general scheme given above for generating polytopes will lead to redundant constraints especially in the case where k is large relative to n . This difficulty can be avoided if b_1, \dots, b_k are chosen as appropriate functions of the $\underline{a}_1, \dots, \underline{a}_k$. In particular, if for $j = 1, \dots, m$, each b_j satisfies

$$(2.1) \quad b_j = |\underline{a}_j| = \left(\sum_{i=1}^n a_{ji}^2 \right)^{1/2},$$

then the random polytope generated by the vectors $\underline{a}_1, \dots, \underline{a}_m$ is circumscribed about the unit sphere, S_{n-1} , and has exactly m facets. Each facet lies in one of the hyperplanes $H_j = \{x: \underline{a}_j^T x = |\underline{a}_j|\}$ and is tangent to S_{n-1} at $\underline{p}_j = \underline{a}_j / |\underline{a}_j|$. The polytope is thus completely determined by the unit vectors $\underline{p}_1, \dots, \underline{p}_m$. Consequently, an alternative generating scheme for this type of random polytope is to choose the vectors $\underline{p}_1, \dots, \underline{p}_m$ from some distribution on S_{n-1} .

Of special interest here will be the case where $\underline{p}_1, \dots, \underline{p}_m$ are independently and uniformly distributed on S_{n-1} . As is well-known ([3]) this choice of the

tangent points is equivalent to choosing the components of the vectors \underline{a}_j , $j = 1, \dots, m$, independently from the standard normal distribution on the real line and the b_j according to (2.1).

It should be pointed out that requiring the polytopes to be circumscribed on the unit sphere as described above certainly limits their possible structure. Polytopes of certain combinatorial types cannot be realized with all of their facets tangent to a sphere. Therefore it may be argued that the more pathological polytopes are excluded by the method of generation.

B. Let p_1, \dots, p_m be independent and identically distributed points on S_{n-1} . As discussed above, these points determine a unique polytope in \mathbb{R}^n containing S_{n-1} and having exactly m facets. Let $V_{mn}(p_1, \dots, p_m)$ denote the number of vertices of this polytope. The function V_{mn} is a random variable on the space of m -tuples of n -vectors, with values in the nonnegative integers. The purpose of this paper is to investigate the expected value of V_{mn} , specifically when the p_j have the uniform distribution on S_{n-1} .

III. Integral Expression for EV_{mn} .

A. Fix integers m and n , $2 < n < m$, and let p_1, \dots, p_m be independent random points chosen from some distribution with density $g(p)$ on S_{n-1} . Let H_i be the hyperplane tangent to S_{n-1} at a_i , $i = 1, \dots, n$, and let $P(p_1, \dots, p_n)$ be the polytope bounded by H_1, \dots, H_n . Then with probability 1 any n of H_1, \dots, H_n intersect in a point of \mathbb{R}^n and there are $\binom{m}{n}$ such points of intersection, among which are the vertices of $P(p_1, \dots, p_n)$.

Denote by A_{mn} the family of all n -subsets of $1, 2, \dots, m$. For any $A = \{i_1, \dots, i_n\} \in A_{mn}$ let V_A be the event that the point of intersection of H_{i_1}, \dots, H_{i_n} is a vertex of $P(p_1, \dots, p_n)$. Then V is the sum of the indicator functions of the events V_A , and so

$$EV_{mn} = \sum_{A \in A_{mn}} \Pr(V_A) .$$

Because the p_i are identically distributed, we have by symmetry

$$EV_{mn} = \binom{m}{n} \Pr(V_n) ,$$

where V_n denotes $V_{\{1, \dots, n\}}$.

B. Now with probability 1, p_1, \dots, p_n lie on a unique small hypercircle on S_{n-1} , which divides S_{n-1} into two caps. Let $C(p_1, \dots, p_n)$ be the smaller of these two caps. Then V_n occurs if and only if none of p_{n+1}, \dots, p_m is in $C(p_1, \dots, p_n)$. Given p_1, \dots, p_n , therefore, the conditional probability of V_n is

$$\Pr(V_n | p_1, \dots, p_n) = \left(1 - \int_{C(p_1, \dots, p_n)} g(p) dp\right)^{m-n} .$$

So EV_{mn} is $\binom{m}{n}$ times the expected value of this function of p_1, \dots, p_n ; that is,

$$EV_{mn} = \binom{m}{n} E \Pr(V_n | p_1, \dots, p_n) = \int_{S_{n-1}} \cdots \int_{S_{n-1}} \left(1 - \int_{C(p_1, \dots, p_n)} g(p) dp\right)^{m-n} \cdot dp_1, \dots, dp_n . \quad (3.1)$$

C. Now we assume that the common distribution of p_1, \dots, p_n is the uniform distribution on S_{n-1} . In this case

$$\int_{C(p_1, \dots, p_n)} g(p) dp = \frac{\text{area of } C(p_1, \dots, p_n)}{\text{area of } S_{n-1}} = \frac{I_{n-2}(r)}{2I_{n-2}(\pi/2)},$$

where $r = r(p_1, \dots, p_n)$ is the angular radius of $C(p_1, \dots, p_n)$ ($0 < r < \pi/2$) and where

$$I_k(r) = \int_0^r \sin^k x \, dx \quad (0 < r < \pi/2, k = 0, 1, 2, \dots)$$

is the area of a cap of angular radius r on S_{k+1} . Thus (3.1) can be rewritten

$$EV_{mn} = \binom{m}{n} E \left(1 - \frac{I_{n-2}(r)}{2I_{n-2}(\pi/2)} \right)^{m-n}$$

where the random variable r is the radius of $C(p_1, \dots, p_n)$.

Now it follows from results of R.A. Miles ([5], theorem 4, p. 368) that $t = \sin^2 r$ has the beta $((n-1)^2/2, 1/2)$ distribution, whose density is proportional to $t^{((n-1)^2/2)-1} (1-t)^{-1/2}$ on $(0, 1)$. From this it follows that the density of r is $f_{n(n-2)}(r)$, where

$$f_k(r) = C_k \sin^k r, \quad 0 < r < \pi/2$$

and

$$C_k = \frac{2\Gamma(\frac{k+2}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{k+1}{2})}, \quad k = 0, 1, 2, \dots \quad (3.2)$$

Moreover, the distribution function of this density is

$$F_k(r) = \frac{I_k(r)}{I_k(\pi/2)}, \quad 0 < r < \pi/2.$$

Using the above notation we obtain the expression

$$EV_{mn} = \binom{m}{n} \int_0^{\pi/2} \left(1 - \frac{1}{Z} F_{n-2}(r)\right)^{m-n} f_{n(n-2)}(r) dr . \quad (3.3)$$

Although we will not use it, the following alternative expression may be of interest. If B_k and b_k denote the distribution function and density of the beta($k/2$, $1/2$) distribution, then the substitution $t = \sin^2 r$ in (3.3) gives

$$EV_{mn} = \binom{m}{n} \int_0^1 \left(1 - \frac{1}{Z} B_{n-1}(t)\right)^{m-n} b_{(n-1)2}(t) dt .$$

A. The evaluation of (3.3) is not difficult in case n is 2 or 3:

$$EV_{m2} = \binom{m}{2} \int_0^{\pi/2} \left(1 - \frac{r}{\pi}\right)^{m-2} \cdot \frac{2}{\pi} dr = m \left(1 - \left(\frac{1}{2}\right)^{m-1}\right),$$

and

$$\begin{aligned} EV_{m3} &= \binom{m}{3} \int_0^{\pi/2} \left(1 - \frac{1}{2}(1 - \cos r)\right)^{m-3} \cdot \frac{3}{2} \sin^3 r dr \\ &= 2m-4 - \left(\frac{1}{2}\right)^{m-1} (m+1)(m-2). \end{aligned}$$

(We note in passing that m and $m-1$ are the only possible values of V_{m2} , the latter occurring only when p_1, \dots, p_m are all in one semicircle. The value of $E_{m2}V$ given above gives the well-known value $m\left(\frac{1}{2}\right)^{m-1}$ for the probability of this event.)

The above calculations suggest that for fixed n , EV_{mn} is asymptotically of the form K_n . This section is devoted to finding upper and lower bounds on EV_{mn} which establish this asymptotic growth rate and bound the growth of K_n for large n . (Unfortunately, the tempting conjecture that $K_n = n-1$ is false: K_n is approximately of order $(\beta n)^{n/2}$ as we have noted.)

B. First we will establish an upper bound by showing that for $2 \leq n < m$,

$$EV_{mn} \leq A_n m (1 - B_{mn}) \quad (4.1)$$

where

$$A_n = C_{n(n-2)} \left(\frac{2}{C_{n-2}}\right)^{n-1} \frac{(n-1)^{n-3}}{n},$$

$$B_{mn} = \left(\frac{1}{2}\right)^{m-1} \sum_{j=0}^{n-2} \binom{m-1}{j} \left(\frac{C_{n-2}}{n-1}\right)^j.$$

Before proceeding, we remark that B_{mn} is negligible:

$$B_{mn} < \left(\frac{1}{2}\right)^{m-1} \left(1 + \frac{C_{n-2}}{n-1}\right)^{m-1},$$

and since $\frac{C_{n-2}}{n-1} < \frac{3}{4}$ when $n \geq 3$, $B_{mn} \leq \left(\frac{7}{8}\right)^{m-1}$.

We observe also that Wallis's formula ([1], p. 258), viz.

$$\frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} \sim \frac{1}{\sqrt{\alpha}} \left(1 - \frac{1}{8\alpha} + O\left(\frac{1}{\alpha^2}\right)\right) \text{ as } \alpha \rightarrow \infty,$$

implies

$$C_{n(n-2)} \sim \sqrt{2/\pi} n$$

and

$$\frac{2}{C_{n-2}} \sim \sqrt{2\pi/(n-2)} \left(1 - \frac{1}{4(n-2)}\right).$$

Thus

$$A_n \sim \frac{(2\pi)^{n/2}}{\pi e^{1/4}} n^{(n-5)/2}$$

and so if $m \geq n$, then asymptotically as $n \rightarrow \infty$

$$EV_{mn} \leq \frac{(2\pi)^{n/2}}{\pi e^{1/4}} n^{(n-5)/2} m. \quad (4.2)$$

The proof of (4.1) begins with (3.3), which says that

$$EV_{mn} = \binom{m}{n} C_{n(n-2)} S_{n-2, m-n, n(n-2)} \quad (4.3)$$

where

$$S_{j k \ell} = \int_0^{\pi/2} \left(1 - \frac{1}{2} F_j(r)\right)^k \sin^\ell r \, dr. \quad (4.4)$$

If $\ell \geq j+1$, an integration by parts using

$$u = -\frac{2}{C_j} \sin^{\ell-j} r, \quad dv = \left(1 - \frac{1}{2} F_j(r)\right)^k \left(-\frac{1}{2} C_j \sin^j r\right) dr$$

yields

$$S_{j k \ell} = \frac{-2}{C_j} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1} + \frac{2}{C_j} \frac{\ell-j}{k+1} \int_0^{\pi/2} \left(1 - \frac{1}{2} F_j(r)\right)^{k+1} \sin^{\ell-j-1} r \cos r \, dr$$

and hence

$$S_{jkl} \leq -\frac{2}{C_j} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1} + \frac{2}{C_j} \frac{\ell-j}{k+1} S_{j,k+1,\ell-j-1} \quad \text{for } \ell \geq j+1. \quad (4.5)$$

Iterating (4.5) gives, for $\ell \geq v(j+1)$,

$$S_{jkl} \leq -\frac{2}{C_j} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1} - \sum_{i=2}^v \left(\frac{2}{C_j}\right)^i \frac{(\ell-j)(\ell-2j-1)\dots(\ell-(i-1)j-(i-2))}{(k+1)(k+2)\dots(k+i)} \left(\frac{1}{2}\right)^{k+i} \\ + \left(\frac{2}{C_j}\right)^v \frac{(\ell-j)(\ell-2j-1)\dots(\ell-vj-(v-1))}{(k+1)(k+2)\dots(k+v)} S_{j,k+v,\ell-v(j+1)}. \quad (4.6)$$

In particular this is valid for $j = n-2$, $k = m-n$, $\ell = n(n-2)$, and $v = n-2$.

In this case for $i = 2, 3, \dots, n-2$ we have

$$(\ell-j)(\ell-2j-1)\dots(\ell-(i-1)j-(i-2)) \\ = [(n-1)(n-2)][(n-1)(n-3)]\dots[(n-1)(n-i)] = (n-1)^{i-1} \frac{(n-2)!}{(n-i-1)!}$$

and also by a simple integration

$$S_{j,k+v,\ell-v(j+1)} = S_{n-2,m-2,n-2} = \frac{2}{C_{n-2}} \frac{1}{m-1} \left(1 - \left(\frac{1}{2}\right)^{m-1}\right).$$

Putting the two expressions above into (4.6) yields

$$S_{n-2,m-n,n(n-2)} \leq \left(\frac{2}{C_{n-2}}\right)^{n-1} \frac{(n-2)!(m-n)!}{(m-1)!} (n-1)^{n-2} \\ - \sum_{i=1}^{n-1} \left(\frac{2}{C_{n-2}}\right)^i \frac{(n-2)!(m-n)!}{(n-i-1)!(m-n+i)!} (n-1)^{i-1} \left(\frac{1}{2}\right)^{m-n+i}.$$

Multiplying by $C_{n(n-2)} \binom{m}{n}$ gives an upper bound on EV_{mn} ; simplifying and letting the sum run over $j = n-1-i$ then gives (4.1).

C. Next we show that for any $n \geq 2$ and any $\epsilon \in (0,1)$,

$$EV_{mn} \geq \frac{d_{(n-1)}^2}{d_{n-1}^{n-1}} \frac{2^{n-1} \epsilon^n}{n! e^{n-1}} (n-1)^{n-1} (m-n), \quad \text{asymptotically as } m \rightarrow \infty. \quad (4.7)$$

where

$$d_j = \frac{\Gamma(\frac{j+1}{2})}{\Gamma(\frac{j}{2})\Gamma(\frac{j+2}{2})} .$$

Before proceeding we notice that Wallis's formula implies

$$d_j \sim \sqrt{2/\pi j} \text{ as } j \rightarrow \infty .$$

This and Stirling's formula provide that

$$\frac{d_{(n-1)^2}}{d_{n-1}^{n-1}} \frac{2^{n-1} \epsilon^n}{n! e^{n-1}} (n-1)^{n-1} \sim \frac{(2\pi)^{\frac{n-1}{2}}}{\pi e^{1/2}} \epsilon^n n^{\frac{n-6}{2}} \text{ as } n \rightarrow \infty .$$

Therefore, for fixed large n , (4.7) implies that for any $\epsilon \in (0,1)$, we have asymptotically as $m \rightarrow \infty$

$$E_{mn} V \geq \frac{(2\pi)^{\frac{n-1}{2}}}{\pi e^{1/2}} \epsilon^n n^{\frac{n-6}{2}} (m-n) . \quad (4.8)$$

To prove (4.7) we begin again with (3.3). For any $\alpha \in (0, \frac{\pi}{2})$

$$\begin{aligned} EV_{mn} &\geq \binom{m}{n} \int_0^\alpha (1 - \frac{1}{2} F_{n-2}(\alpha))^{m-n} f_{n(n-2)}(r) dr \\ &= \binom{m}{n} (1 - \frac{1}{2} F_{n-2}(\alpha))^{m-n} F_{n(n-2)}(\alpha) . \end{aligned} \quad (4.9)$$

It can be checked that for $0 \leq \alpha < \frac{\pi}{2}$

$$F_k(\alpha) = \cos \alpha (d_{k+1} \sin^{k+1} \alpha + d_{k+3} \sin^{k+3} \alpha + \dots)$$

by differentiating and using elementary properties of the gamma function.

Moreover, $d_0 \geq d_1 \geq d_2 \geq \dots$. Therefore

$$1 - \frac{1}{2} F_{n-2}(\alpha) \geq 1 - \frac{\cos \alpha}{2} (d_{n-1} (\sin^{n-1} \alpha + \sin^{n+1} \alpha + \dots))$$

$$= 1 - \frac{d_{n-1} \sin^{n-1} \alpha}{2 \cos \alpha}$$

and

$$F_{n(n-2)}(\alpha) \geq d_{(n-1)^2} \cos \alpha \sin^{(n-1)^2} \alpha.$$

So, writing τ for $\sin \alpha$, we have

$$E_{mn} V \geq \binom{m}{n} d_{(n-1)^2} \tau^{(n-1)^2} \sqrt{1-\tau^2} \left(1 - \frac{d_{n-1}}{2} \frac{\tau^{n-1}}{\sqrt{1-\tau^2}} \right)^{m-n} \quad \text{for any } \tau \in (0,1).$$

Let ε be an arbitrary number in $(0,1)$; then we have

$$E_{mn} V \geq \binom{m}{n} d_{(n-1)^2} \tau^{(n-1)^2} \varepsilon \left(1 - \frac{d_{n-1}}{2\varepsilon} \tau^{n-1} \right)^{m-n} \quad \text{if } 0 < \varepsilon < 1$$

$$\text{and } 0 < \tau < \sqrt{1-\varepsilon^2}.$$

If we now write σ for $\frac{d_{n-1}}{2\varepsilon} \tau^{n-1}$, then this inequality becomes

$$E_{mn} V \geq \binom{m}{n} d_{(n-1)^2} \left(\frac{2\varepsilon\sigma}{d_{n-1}} \right)^{n-1} \varepsilon (1-\sigma)^{m-n} \quad \text{if } 0 < \varepsilon < 1$$

$$\text{and } 0 < \sigma < \frac{d_{n-1}(1-\varepsilon^2)^{\frac{n-1}{2}}}{2\varepsilon}.$$

Notice that the value of σ maximizing the right side of the above inequality is $\frac{n-1}{m-1}$. So if $\frac{n-1}{m-1} < \frac{d_{n-1}(1-\varepsilon^2)^{\frac{n-1}{2}}}{2\varepsilon}$, then

$$E_{mn} V \geq \binom{m}{n} \frac{d_{(n-1)^2}}{d_{n-1}^{n-1}} 2^{n-1} \varepsilon^n \left(\frac{n-1}{m-1} \right)^{n-1} \left(1 - \frac{n-1}{m-1} \right)^{m-n}.$$

Since $\binom{m}{n} \geq \frac{(m-n)^n}{n!}$,

$$E_{mn} V \geq (m-n) \frac{d_{(n-1)^2}}{d_{n-1}^{n-1}} \frac{2^{n-1} \varepsilon^n}{n!} \left(1 - \frac{n-1}{m-1} \right)^{m-n} (n-1)^{n-1},$$

and (4.7) follows from this.

References

1. Abramowitz, M., Stegun, I. *Handbook of Mathematical Functions*. Dover Publications, New York, 1965.
2. Dunham, J., Kelly, D., Tolle, J. Some experimental results concerning the expected number of pivots for solving randomly generated linear programs. Technical Report 77-16, ORSA Curriculum, University of North Carolina, Chapel Hill, 1977.
3. Feller, W. *An Introduction to Probability Theory and its Applications*, vol. 2, 2nd edition. John Wiley & Sons, Inc., New York, 1971.
4. Kelly, D., Tolle, J. Expected simplex algorithm behaviour for random linear programs. *Operations Research Verfahren* 31 (1979), 361-367.
5. Miles, R. Random points, sets, and tessellations on the surface of a sphere. *Sankhyā, Series A*, 33 (1971), 145-174.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Expected Number of Vertices of a Random Convex Polytope: I. Integral Formula and Asymptotic Bounds		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
7. AUTHOR(s) D.G. Kelly and J.W. Tolle		6. PERFORMING ORG. REPORT NUMBER Mimeo Series No. 1261
9. PERFORMING ORGANIZATION NAME AND ADDRESS		8. CONTRACT OR GRANT NUMBER(s) Contract N00014-76-C-0550 Grant DAAG29-79-G-0014
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics and Probability Program (Code 436) Arlington, VA 22217		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE January 1980
		13. NUMBER OF PAGES 13
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
16. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release: Distribution Unlimited		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) convex polytope, random polytope, vertices		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Given m points on the unit sphere in n -space, the hyperplanes tangent to the sphere at the given points bound a convex polytope with m facets. If the points are chosen independently at random from the uniform distribution on the sphere, the number V_{mn} of the vertices of the polytope is a random variable. We obtain an integral expression for EV_{mn} and asymptotic bounds of the form $\alpha \binom{n}{m} \binom{n-6}{m-n} / 2 \leq EV_{mn} \leq \beta \binom{n}{m} \binom{n-5}{m} / 2$		