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Development in Structural Systems Reliability Theory

Y. Murotsu
University of Osaka Prefecture, Dept. of Aeronautical Engineering, Sakai,
Osaka 591, Japan

Abstract

This paper is concerned with two topics on structural systems reliability theory. One covers automatic generation of failure mode equations, identification of stochastically dominant failure modes, and reliability assessment of redundant structures. Reduced stiffness matrixes and equivalent nodal forces representing the failed elements are introduced for expressing the safety margins of the elements, using a matrix method. Dominant failure modes are systematically selected by a branch-and-bound technique and heuristic operations. The other discusses the various optimum design problems based on reliability concept. Those problems are interpreted through a solution to a multi-objective optimization problem.

1. Introduction

Structural systems reliability theory is concerned with the assessment of structural safety in terms of probability, and it was initiated in the fields of civil engineering structures, aircraft, etc. [e.g., 1,2]. Its frame has been established and textbooks on the subject have been recently published [3-5], which encompass the following items :

1) Modelling the strengths of structural elements and structural systems in statistical terms
2) Modelling the applied loads in terms of random variables or random processes
3) Evaluating the reliability of structural elements and structural systems as interference phenomena of strengths and applied loads
4) Response analysis of structural systems to random excitation
5) Reliablity-based design of structural elements and structural systems

This paper reviews the topics among others on system reliability and reliability-based optimum design. In the former, the early studies were focused on estimation of reliability by evaluating its lower and upper bounds for given failure modes [6-10]. It is difficult in practice to specify the relevant modes of failure and their equations a priori for large structures with high degree of redundancy. Consequently, identification of stochastically significant failure modes is recognized to be an essential step to be done for reliability assessment of structural systems [11]. Researches for automatically generating mode equations have been developed [12-23]. The basic ideas for mode generation are briefly given in sections 2 and 3. A procedure for selecting dominant failure modes is provided in section 4. In the latter topics, progress is relatively slow [2,8,24-27] although the optimum design may be a goal of structural systems reliability theory. The various optimum design problems and their interpretation are given in section 5.

2. Failure in Structural System

A structural system is an assemblage of structural elements and functions to sustain the applied loads without excessive deformation from its original configuration. Consider a system shown in Fig. 1, where failure in any one element results in structural failure. This type of systems is called a weakest link system or a non-redundant one. On the other hand, such systems as shown in Figs. 2 and 3 do not fail even if one element fails, and they are called redundant systems. For example, failure in sets of elements (3,5), (5,6,7), etc., causes structural failure in the system of Fig. 2. A set of failed elements which causes structural failure is called a failure mode. It should be noted here that the different sequences of the failed elements lead to the same failure mode. For examples, 3 → 5 (this means that the sequential order of failure is from 3 to 5), 5 → 3 correspond to the failure mode (3,5), and 5 → 6 → 7, 5 → 7 → 6, 6 → 5 → 7, etc., do to the failure mode (5,6,7). A sequence of failed elements leading to structural failure is called a complete failure path. The maximum number (s+1) of the elements required for structural failure is specified for the structure, e.g., s=2 for the system of Fig. 2 and s=6 for that of Fig. 3, respectively. The number s is called degree of redundancy. However, the minimum number p_q of the required elements varies depending on the combination of the elements, which necessitates to determine structural failure for each combination of the failed elements. Further, the loads acting on the surviving elements are generally changed when some elements fail. Consequently, the stress analysis is needed at every failure stage to determine the element next to fail.

3. Automatic Generation of Failure Path

Some approaches have been initiated to generate automatically the failure modes of frame structures [14,19,20] The method given below is more general in the sense that it is applicable to a general class of element behaviors and various types of structural systems [12,13,15-18,21,22]. The procedure to find a complete failure path is as follows. The first element to have failed is determined by using a matrix method on the basis of linear elastic stress analysis. The stiffness matrix of the failed element is then replaced by a reduced one and the strength of the element is treated as an applied load. The modified structure is then reanalyzed and the element next to fail is selected. The procedure is repeated until structural failure is attained.

The safety margin $Z_i^{(p)}$ of the surviving element i after the elements r_1, $r_2, \ldots,$ and r_{p-1} have failed is expressed in the form :

$$Z_i^{(p)} = R_i + \sum_{k=1}^{p-1} a_{ir_k} R_{r_k} - \sum_{j=1}^{3\ell} b_{ij} L_j \tag{1}$$

where R_j's and L_j's are the strengths of the elements and the applied loads, respectively.

Consequently, the failure criterion of element i is given by

$$Z_i^{(p)} \leqq 0 \tag{2}$$

On the other hand, structural failure is determined by the reduced total structure stiffness matrix $[\mathbb{K}^{(p_q)}]$, i.e., the structural system fails once the following condition is satisfied :

$$|[\mathbb{K}^{(p_q)}]| / |[\mathbb{K}^{(0)}]| \leqq \varepsilon \tag{3}$$

where superscripts (p_q) and (0) denote the p_q-th failure stage and the elastic condition, and ε is a constant specified for determining structural failure.

3.1 Truss Structure [12,13]

For a truss structure, the element stiffness matrixes for the failed elements are put to zero and their residual strengths are applied to the nodes as artificial loads, corresponding to the types of failure. When a brittle element fails, the residual strength is put to zero while in case of a ductile element the yield strength is taken as the residual strength.

3.2 Frame Structure Subjected to Bending Moment [13,15-17]

Elements are assumed to fail when the applied bending moments reach their fully plastic moments and plastic hinges form in them. The reduced stiffness matrixes $k_l^{(p)}$ and the equivalent nodal forces $\overline{X}_l^{(p)}$ are given in Table 1.

3.3 Frame Structure Subjected to Bending Moment and Axial Force [18]

Interaction of bending moment and axial force on yielding of the elements is taken into account. The behaviours of the elements are assumed to be perfectly elastic-plastic or elastic-brittle. The yield surface is approximated by a linearized function shown in Fig. 4. Consequently, the failure criterion of the element is given by

$$Z_k = R_k - \mathbb{C}_k^T \mathbb{X}_t = 0 \qquad (k=i,j) \tag{4}$$

where R_k : fully plastic moment of element k

$= \sigma_{yk} AZ_{pk}$ (AZ_{pk} : plastic section modulas of the element k, σ_{yk} : yield stress)

X_t : nodal force vector $=(F_{xi}, F_{yi}, M_{zi}, F_{xj}, F_{yj}, M_{zj})^T$

$C_i^T = (AZ_{pi}/A_{pi}\,\mathrm{sign}\,(F_{xi}), 0, \mathrm{sign}\,(M_{zi}), 0, 0, 0)$, $C_j^T = (0, 0, 0, AZ_{pj}/A_{pj}\,\mathrm{sign}\,(F_{xj}), 0, \mathrm{sign}\,(M_{zj}))$

(A_{pk} (k=i,j) : cross sectional area of the element end k, sign(·) : sign of (·))

The plasticity condition of a plane-frame structure subjected solely to bending moment mentioned above is obtained by setting the first term of $\mathbb{C}_i^T$ and the fourth of $\mathbb{C}_j^T$ equal to zero.

The reduced stiffness matrixes and the equivalent nodal forces are summarized in Table 1.

4. Selection of Dominant Failure Paths and Reliability Assessment

Consider a structural system which has s degree of redundancy. Elements are assumed to fail one by one up to some specific number p_q until structural failure results. The sequence of those elements to yield structural failure is symbolically denoted as $r_1, r_2, \ldots, r_p, \ldots,$ and r_{p_q} , which is called a complete failure path. On the other hand, the sequence of the elements which do not yield structural failure, e.g., the failure path $r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_p$ $(p<p_q)$ is called a partial failure path. Consider a case where structural failure still results even if some elements are removed from a complete failure path, for example, r_p in the complete failure path $r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_p \rightarrow \cdots \rightarrow r_{p_q}$. An element such as r_p is called a redundant one while those which can not be removed from a complete failure path to cause structural failure are called essential ones. The probability $P_{fp(q)}^{(p)}$ of a failure path $r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_p$ is calculated as

$$P_{fp(q)}^{(p)} = P\left[\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}\right] \tag{5}$$

where $F_{r_i(q)}^{(i)}$ is the failure event that element r_i fails at the i-th order of sequence, i.e., $F_{r_i(q)}^{(i)} = \left(Z_{r_i(q)}^{(i)} \leqq 0\right)$. Superscript p denotes the length of the failure path and q is used to denote a particular failure path. When $p<p_q$, $P_{fp(q)}^{(p)}$ is the probability of a partial failure path while it is the probability of a complete failure path for $p=p_q$.

The probability $P_{fp(q)}^{(p)}$ is estimated by evaluating its lower and upper bounds, $P_{fp(q)(L)}^{(p)}$ and $P_{fp(q)(U)}^{(p)}$ [23]:

$$P_{fp(q)(L)}^{(p)} \leqq P_{fp(q)}^{(p)} = P\left[\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}\right] \leqq P_{fp(q)(U)}^{(p)} \tag{6}$$

$$P_{fp(q)(U)}^{(p)} = \min_{j \in \{2,\cdots,p\}} P\left[F_{r_1(q)}^{(1)} \cap F_{r_j(q)}^{(j)}\right] \tag{7}$$

$$P_{fp(q)(L)}^{(p)} = 1 - P\left[F_{r_1(q)}^{(1)}\right] - \sum_{i=2}^{p} \min_{j \in \{1,2,\ldots,i-1\}} \left\{P\left[F_{r_j(q)}^{(j)} \cap \overline{F}_{r_i(q)}^{(i)}\right]\right\} \tag{8}$$

$$P_{fp(q)(L)}^{(p)} = \max \{ 0, \ P[F_{r_1(q)}^{(1)}] - P[F_{r_1(q)}^{(1)} \cap \overline{F}_{r_2(q)}^{(2)}] - \sum_{j=3}^{p} \min (P_{f_p(q)(U)}^{(j-1)}, \ P[F_{r_1(q)}^{(1)} \cap \overline{F}_{r_j(q)}^{(j)}]) \} \tag{9}$$

Eq. (8) needs the safety margins at all the failure stages, while Eq. (9) uses only the safety margins at the first and last stages and the upper bounds of the preceding failure path probabilities.

There are too many failure paths in a redundant structure to generate all of them, which neccssitates a procedure for selecting only the probabilistically significant failure paths. Efficient methods by using a branch-and-bounding technique have been proposed as described below [13,15-17,23].

4.1 Branching Operations

These operations are to select the elements such that stochastically dominant failure paths may be obtained. An element is selected as the element to fail at the p-th failure stage based on the criterion that the joint probability to fail is to be the largest. By repeating the selecting process, a sequence of elements to cause structural failure , e.g., r_1, r_2,..., and r_{p_q} is found. Then, the lower bound of the complete failure path is evaluated.

The maximum P_{fpM} of the lower bounds of the selected complete failure path probability is calculated :

$$P_{fpM} = \max_{q} P_{fp(q)(L)}^{(p_q)} \tag{10}$$

P_{fpM} is updated when a new complete failure path is found and its failure probability is larger than the previous P_{fpM}. The branching operations are terminated when no elements are left for selection.

4.2 Bounding Operations

These operations are to select the elements to be discarded. The elements deleted at the p-th failure stage are those :

$$P[F_{i_1}^{(1)}] / P_{fpM} < 10^{-\gamma} \quad \text{for } p=1 \qquad P[F_{r_1(q)}^{(1)} \cap F_{i_p}^{(p)}] / P_{fpM} < 10^{-\gamma} \quad \text{for } p \geqq 2 \tag{11}$$

where γ is a constant.

From this, it is concluded that the neglected failure paths are those which have the failure probabilities smaller than $10^{-\gamma} \cdot P_{fpM}$[23].

4.3 Heuristic Operations

The number of branchings becomes enormous for a large structure with high degree of redundancy, even though the branch-and-bound method is applied. To reduce the computational efforts, the following heuristic operations may be effective. First, the reliability assessment is performed of the complete failure paths which are presumed to be critical, and the estimated values of the failure probabilities are used as the reference value P_{fpM} of the bounding operations. Second, the set of elements for branching is restricted to the elements belonging to the neighbouring nodes, the same story, the same port, etc. or those which satisfy the monotonicity conditions of the failure probabilities [17]. Third, The number of branchings from one failure stage is limited to a specified number. Fourth, when the lower bounds of the complete failure path probabilities evaluated by Eq. (8) or Eq. (9) are consecutively zero up to the certain number of branchings, the reference probability P_{fpM} for the bounding operations is replaced by the maximum of upper bounds [13,15-17].

4.4 Reliability Assessment

Let X_c and X_t respectively denote the set of the selected complete failure paths and that of the discarded failure paths. Then, the system failure probability P_f is estimated by [23]

$$P\left[\bigcup_{q\in X_c}\left(\bigcap_{i=1}^{p_q} F_{r_i(q)}^{(i)}\right)\right] \leq P_f \leq P\left[\bigcup_{q\in X_c}\left(\bigcap_{i=1}^{p_q} F_{r_i(q)}^{(i)}\right) \bigcup_{q\in X_t}\left(\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}\right)\right] \tag{12}$$

Although the lower and upper bounds of Eq. (12) are in principle evaluated by using the well developed approximation methods [7-11], computational difficulties may arise for large structures with high degree of redundancy. An alternative upper bound is obtained by selecting the specified number of the events in the intersections, e.g., k :

$$P_{fU}^{(k)} = P\left[\bigcup_{q\in X_c}\left(\bigcap_{i=s_1}^{s_k} F_{r_i(q)}^{(i)}\right) \bigcup_{q\in X_t}\left(\bigcap_{i=s_1}^{s_k} F_{r_i(q)}^{(i)}\right)\right] \tag{13}$$

5. Reliability-Based Optimum Design

There are various optimum design problems of frame structures based on reliability concept [24], which are listed in Table 2. The most familiar is to determine a structure minimizing the structural cost or weight under the constraint on the allowable failure probability of the structure. The optimum design problem is formulated as follows [26]:

Problem 1

Find $\mathbf{X}$

such that $C_S(\mathbf{X}) \longrightarrow$ minimize

subject to $P_f(\mathbf{X}) \leqq P_{fa}$

Table 2 Various Optimum Design Problems

Problem No.	*Objective function*	*Constraints*
1	$C_S(X)$	$P_F(X) \leqq P_{Fa}$
2	$C_S(X)$	$P_{Fj}(X) \leqq P_{Faj}$
3	$P_F(X)$	$C_S(X) \leqq C_{Sa}$
4	$C_{ET}(X)$	———
5	$P_F(X)$	$C_{ET}(X) \leqq C_{ETa}$
6	$C_S(X)$	$C_{ET}(X) \leqq C_{ETa}$

C_S : structural cost

P_F : structural failure prob.

P_{Fj} : failure prob. of j-th critical section ($j = 1,2,\ldots,m$)

C_{ET} : expected total cost

Subscript a denotes the allowable value.

where $\mathbf{X}$ is the design variable vector, C_S the structural cost (initial cost) or weight, P_f the failure probability of the structure, and P_{fa} the allowable failure probability of the structure.

In case of a large structure, it takes much time to evaluate the system's failure probability. Moreover, the evaluation is repeated many times in the optimum design processes, and the total computation time will become too enormous to be practical. Thus introduced are the constraints on the failure probabilities of the critical elements [27] whose failure triggers structural failure or cause a catastrophic damage to the performance of the system, instead of the constraint on the structural failure probability. The resulting design problem is Problem 2 as given in Table 2.

Problem 3 is to determine a structure to minimize the structural failure probability under the constraint on the allowable structural cost.

The expected total cost is defined as a sum of the structural cost and the expected failure cost $C_{ET}(\mathbf{X})$ due to structural failure or partial failure :

$$C_{ET}(\mathbf{X}) = C_S(\mathbf{X}) + C_{EF}(\mathbf{X}) \tag{14}$$

Problem 4 is to minimize the expected total cost. Problems 5 and 6 are respectively to minimize the structural failure probability and the structural cost under the constraint on the allowable expected total cost.

As noticed from the optimum design problems mentioned above, a major goal of structural design is to achieve harmony between maximum safety and minimum cost. Thus, optimum design is regarded as a multi-objective optimization problem such as

Problem 7

Find $\mathbf{X}$

such that $C_S(\mathbf{X}) \longrightarrow$ minimize

$P_f(\mathbf{X}) \longrightarrow$ minimize

(or $P_{fj}(\mathbf{X}) \longrightarrow$ minimize ($j=1,2,\ldots,m$))

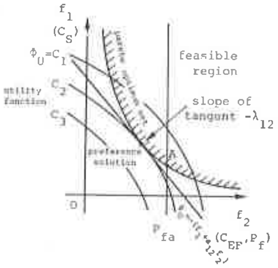

Fig. 5 Concept of Multi-Objective Optimization

In the multi-objective optimization problem, preference solution, e.g., to maximize the decision maker's utility, is chosen from the so-called pareto optimum set. The relation among the pareto optimum set, utility function and preference solution on a C_S-P_f plane is illustrated in Fig. 5. It is noted that the optimum solution of Problem 1 corresponds to the point A of Fig. 5 which is also a pareto optimum solution. Consequently, it is suggested that the allowable failure probability P_{fa} or P_{fj} in Problem 1 or 2 should be specified to the value of P_f or P_j which corresponds to the preference solution of Problem 7. In the multi-objective problem, the preference solution can be obtained if the utility function is determined. However, the utility function generally differs from one decision maker to another, and hence the function could not be known explicitly. Therefore, it is required to assume a reasonable utility function to obtain an impartial preference solution.

Now, suppose the case where the utility function Φ_U is expressed as a linear combination of two objective functions f_1 and f_2. That is

$$\Phi_U = - (f_1 + a_{12} f_{12}) \tag{15}$$

The coefficient a_{12} coincides with λ_{12} whose value is given by $-(\partial f_1 / \partial f_2)$ at the preference solution point as shown in Fig. 5. The coefficient λ_{12} is called a trade-off ratio between f_1 and f_2.

Consider a case when the expected failure cost is expressed as $C_{EF}(\mathbf{X}) = C_F P_f(\mathbf{X})$ in Problem 3 [25]. The expected total cost C_T could be interpreted as a linear utility function for Problem 7, and then the trade-off ratio is 1 between C_S and C_{EF} or C_F between C_S and P_f.

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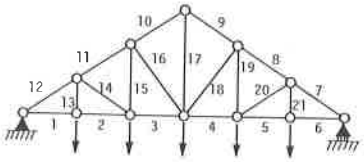


Fig. 1 Non-redundant Truss

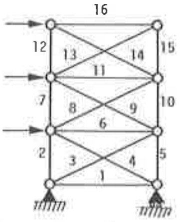
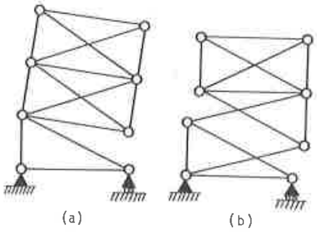


Fig. 2 Redundant Truss



(a) failure in member 3 and 5

(b) failure in member 5, 6 and 7

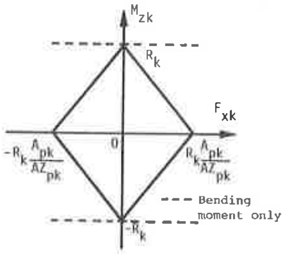


Fig. 4 Linearized Plasticity Condition

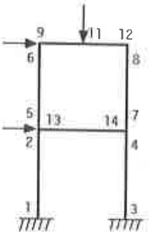


Fig. 3 Redundant Frame Structure

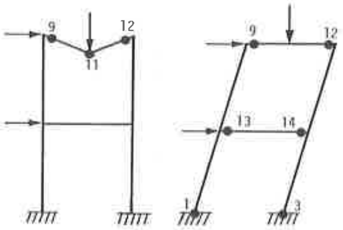
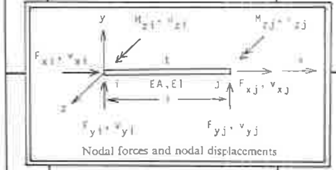


Table 1 Reduced element stiffness matrix $k_i^{(p)}$, and equivalent nodal force vector $\overline{X}_i^{(p)}$

	General	Truss $C_i^T=(AZ_{pi}/A_{pi}\mathrm{sign}(F_{xi}),0,0,0,0,0)$ $C_j^T=(0,0,0,AZ_{pj}/A_{pj}\mathrm{sign}(F_{xj}),0,0)$	Simple Frame $C_i^T=(0,0,\mathrm{sign}(M_{zi}),0,0,0)$ $C_j^T=(0,0,0,0,0,\mathrm{sign}(M_{zj}))$	Frame with Combined Load Effect $C_i^T=(AZ_{pi}/A_{pi}\,\mathrm{sign}(F_{xi}),0,\mathrm{sign}(M_{zi}),0,0,0)$ $C_j^T=(0,0,0,AZ_{pj}/A_{pj}\,\mathrm{sign}(F_{xj}),0,\mathrm{sign}(M_{zj}))$
In case of failure at the left-hand end	$k_i^{(p)}(=k_i^L)=k_i-k_iC_iC_i^Tk_i/(C_i^Tk_iC_i)$ $\overline{X}_i^{(p)}(=\overline{X}_i^L)=R_ik_iC_i/(C_i^Tk_iC_i)$ k_i: elastic element stiffness matrix  Nodal forces and nodal displacements	$k_i^{(p)}=0$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}\sigma_{yi}A_{pi}\\0\\0\\-\sigma_{yi}A_{pi}\\0\\0\end{bmatrix}$	$k_i^{(p)}$ $\begin{bmatrix}EA/l&0&0&-EA/l&0&0\\&3EI/l^3&0&0&-3EI/l^3&3EI/l^2\\&&0&0&0&0\\&&&EA/l&0&0\\&\text{SYM}&&&3EI/l^3&-3EI/l^2\\&&&&&3EI/l\end{bmatrix}$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}0\\\frac{3}{2\ell}R_i'\\R_i'\\0\\-\frac{3}{2\ell}R_i'\\R_i'/2\end{bmatrix}$	$k_i^{(p)}$ $\begin{bmatrix}\frac{EA}{\ell}\frac{1}{1+\square}&-\frac{EA}{\ell}\frac{3C^L}{2(1+\square)}&-EA\frac{C^L}{1+\square}&-\frac{EA}{\ell}\frac{1}{1+\square}&\frac{EA}{\ell}\frac{3C^L}{2(1+\square)}&-EA\frac{C^L}{2(1+\square)}\\&\frac{3EI}{\ell^3}\frac{1+4\square}{1+\square}&\frac{6EI}{\ell^2}\frac{\square}{1+\square}&\frac{EA}{\ell}\frac{3C^L}{2(1+\square)}&-\frac{3EI}{\ell^3}\frac{1+4\square}{1+\square}&\frac{3EI}{\ell^2}\frac{1+2\square}{1+\square}\\&&\frac{4EI}{\ell}\frac{\square}{1+\square}&EA\frac{C^L}{1+\square}&-\frac{6EI}{\ell^2}\frac{\square}{1+\square}&\frac{2EI}{\ell}\frac{\square}{1+\square}\\&\text{SYM}&&\frac{EA}{\ell}\frac{\square}{1+\square}&-\frac{EA}{\ell}\frac{3C^L}{2(1+\square)}&EA\frac{C^L}{2(1+\square)}\\&&&&\frac{3EI}{\ell^3}\frac{1+4\square}{1+\square}&-\frac{3EI}{\ell^2}\frac{1+2\square}{1+\square}\\&&&&&\frac{3EI}{\ell}\frac{3+4\square}{3(1+\square)}\end{bmatrix}$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}\frac{\square/C^L}{1+\square}\frac{R_i'}{\ell}\\\frac{1}{1+\square}\frac{3}{2\ell}R_i'\\\frac{1}{1+\square}R_i'\\-\frac{\square/C^L}{1+\square}\frac{R_i'}{\ell}\\-\frac{1}{1+\square}\frac{3}{2\ell}R_i'\\\frac{1}{1+\square}\frac{R_i'}{2}\end{bmatrix}$ where $R_i'=\mathrm{sign}(M_{zi})R_i$, $C^L=\mathrm{sign}(M_{zi})AZ_{pi}/\mathrm{sign}(F_{xi})A_{pi}\ell$, $\square=(EA\ell^2/4EI)(C^L)^2$
In case of failure at the right-hand end	$k_i^{(p)}(=k_i^R)=k_i-k_iC_jC_j^Tk_i/(C_j^Tk_iC_j)$ $\overline{X}_i^{(p)}(=\overline{X}_i^R)=R_jk_iC_j/(C_j^Tk_iC_j)$	$k_i^{(p)}=0$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}-\sigma_{yj}A_{pj}\\0\\0\\\sigma_{yj}A_{pj}\\0\\0\end{bmatrix}$	$k_i^{(p)}$ $\begin{bmatrix}EA/l&0&0&-EA/l&0&0\\&3EI/l^3&3EI/l^2&0&-3EI/l^3&0\\&&3EI/l&0&-3EI/l^2&0\\&&&EA/l&0&0\\&\text{SYM}&&&3EI/l^3&0\\&&&&&0\end{bmatrix}$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}0\\\frac{3}{2\ell}R_j'\\R_j'/2\\0\\-\frac{3}{2\ell}R_j'\\R_j'\end{bmatrix}$	$k_i^{(p)}$ $\begin{bmatrix}\frac{EA}{\ell}\frac{1}{1+\boxed{/}}&\frac{EA}{\ell}\frac{3C^R}{2(1+\boxed{/})}&EA\frac{C^R}{2(1+\boxed{/})}&-\frac{EA}{\ell}\frac{1}{1+\boxed{/}}&-\frac{EA}{\ell}\frac{3C^R}{2(1+\boxed{/})}&EA\frac{C^R}{1+\boxed{/}}\\&\frac{3EI}{\ell^3}\frac{1+4\boxed{/}}{1+\boxed{/}}&\frac{3EI}{\ell^2}\frac{1+2\boxed{/}}{1+\boxed{/}}&-\frac{EA}{\ell}\frac{3C^R}{2(1+\boxed{/})}&-\frac{3EI}{\ell^3}\frac{1+4\boxed{/}}{1+\boxed{/}}&\frac{6EI}{\ell^2}\frac{\boxed{/}}{1+\boxed{/}}\\&&\frac{3EI}{\ell}\frac{3+4\boxed{/}}{3(1+\boxed{/})}&-EA\frac{C^R}{2(1+\boxed{/})}&-\frac{3EI}{\ell^2}\frac{1+2\boxed{/}}{1+\boxed{/}}&\frac{2EI}{\ell}\frac{\boxed{/}}{1+\boxed{/}}\\&\text{SYM}&&\frac{EA}{\ell}\frac{1}{1+\boxed{/}}&\frac{EA}{\ell}\frac{3C^R}{2(1+\boxed{/})}&-EA\frac{C^R}{1+\boxed{/}}\\&&&&\frac{3EI}{\ell^3}\frac{1+4\boxed{/}}{1+\boxed{/}}&-\frac{6EI}{\ell^2}\frac{\boxed{/}}{1+\boxed{/}}\\&&&&&\frac{4EI}{\ell}\frac{\boxed{/}}{1+\boxed{/}}\end{bmatrix}$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}-\frac{\boxed{/}/C^R}{1+\boxed{/}}\frac{R_j'}{\ell}\\\frac{1}{1+\boxed{/}}\frac{3}{2\ell}R_j'\\\frac{1}{1+\boxed{/}}\frac{R_j'}{2}\\\frac{\boxed{/}/C^R}{1+\boxed{/}}\frac{R_j'}{\ell}\\-\frac{1}{1+\boxed{/}}\frac{3}{2\ell}R_j'\\\frac{1}{1+\boxed{/}}R_j'\end{bmatrix}$ where $R_j'=\mathrm{sign}(M_{zj})R_j$, $C^R=\mathrm{sign}(M_{zj})AZ_{pj}/\mathrm{sign}(F_{xj})A_{pj}\ell$, $\boxed{/}=(EA\ell^2/4EI)(C^R)^2$
In case of failure at the both ends	$k_i^{(p)}(=k_i^{LR})=k_i-[H]^T[G^{-1}][H]$ $\overline{X}_i^{(p)}(=\overline{X}_i^{LR})=[H]^T[G^{-1}]\begin{Bmatrix}R_i\\R_j\end{Bmatrix}$ $[G^{-1}]=\begin{bmatrix}C_i^Tk_iC_i^T&C_i^Tk_iC_j\\C_j^Tk_iC_i&C_j^Tk_iC_j\end{bmatrix}^{-1}$ $[H]=\begin{bmatrix}C_i^Tk_i\\C_j^Tk_i\end{bmatrix}$		$k_i^{(p)}$ $\begin{bmatrix}EA/l&0&0&-EA/l&0&0\\&0&0&0&0&0\\&&0&0&0&0\\&&&EA/l&0&0\\&\text{SYM}&&&0&0\\&&&&&0\end{bmatrix}$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}0\\(R_i'+R_j')/\ell\\R_i'\\0\\-(R_i'+R_j')/\ell\\R_j'\end{bmatrix}$	$k_i^{(p)}$ $\begin{bmatrix}\frac{EA}{\ell}\frac{3}{\boxed{}}&-\frac{EA}{\ell}\frac{3(C^L-C^R)}{\boxed{}}&-EA\frac{3C^L}{\boxed{}}&-\frac{EA}{\ell}\frac{3}{\boxed{}}&\frac{EA}{\ell}\frac{3(C^L-C^R)}{\boxed{}}&EA\frac{3C^R}{\boxed{}}\\&\frac{12EI}{\ell^3}\frac{\square+\boxed{/}-2\boxtimes}{\boxed{}}&\frac{12EI}{\ell^2}\frac{\square-\boxtimes}{\boxed{}}&\frac{EA}{\ell}\frac{3(C^L-C^R)}{\boxed{}}&-\frac{12EI}{\ell^3}\frac{\square+\boxed{/}-2\boxtimes}{\boxed{}}&\frac{12EI}{\ell^2}\frac{\boxed{/}-\boxtimes}{\boxed{}}\\&&\frac{12EI}{\ell}\frac{\square}{\boxed{}}&EA\frac{3C^L}{\boxed{}}&-\frac{12EI}{\ell^2}\frac{\square-\boxtimes}{\boxed{}}&\frac{12EI}{\ell}\frac{\boxed{/}}{\boxed{}}\\&&&\frac{EA}{\ell}\frac{3}{\boxed{}}&-\frac{EA}{\ell}\frac{3(C^L-C^R)}{\boxed{}}&-EA\frac{3C^R}{\boxed{}}\\&\text{SYM}&&&\frac{12EI}{\ell^3}\frac{\square+\boxed{/}-2\boxtimes}{\boxed{}}&-\frac{12EI}{\ell^2}\frac{\boxed{/}-\boxtimes}{\boxed{}}\\&&&&&\frac{12EI}{\ell}\frac{\boxed{/}}{\boxed{}}\end{bmatrix}$ $\overline{X}_i^{(p)}$ $\begin{bmatrix}\frac{4\square+2\boxtimes}{\boxed{}\,\ell}\frac{R_i'}{C^L}-\frac{4\boxed{/}+2\boxtimes}{\boxed{}\,\ell}\frac{R_j'}{C^R}\\\frac{3(1-2\boxed{/}+2\boxtimes)}{\boxed{}}\frac{R_i'}{\ell}+\frac{3(1+2\square+2\boxtimes)}{\boxed{}}\frac{R_j'}{\ell}\\\frac{3-4\boxed{/}+2\boxtimes}{\boxed{}}R_i'+\frac{2(\square+2\boxtimes)}{\boxed{}}R_j'\\-\frac{4\square+2\boxtimes}{\boxed{}\,\ell}\frac{R_i'}{C^L}+\frac{4\boxed{/}+2\boxtimes}{\boxed{}\,\ell}\frac{R_j'}{C^R}\\-\frac{3(1+2\boxed{/}+2\boxtimes)}{\boxed{}}\frac{R_i'}{\ell}-\frac{3(1+2\square+2\boxtimes)}{\boxed{}}\frac{R_j'}{\ell}\\\frac{2(\boxed{/}+2\boxtimes)}{\boxed{}}R_i'+\frac{3+4\square+2\boxtimes}{\boxed{}}R_j'\end{bmatrix}$ where $\boxtimes=(EA\ell^2/4EI)(C^LC^R)$, $\boxed{}=3-4\square-4\boxed{/}-4\boxtimes$