A SIMPLE CONSTITUTIVE LAW FOR ARTIFICIAL GRAPHITE-LIKE MATERIALS

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SUMMARY

A plasticity theory capable of realistically describing non-linearly hardening material behavior for multiaxial loadings with stress reversals, finds a particularly simple representation in the case of artificial graphite-like materials.

The basic idea of the general theory is to include in the plastic strain rate constitutive relation parameters associated with the history of the most recent unloading-reloading in different directions. This is achieved indirectly by introducing in stress space the concept of the "bounding surface" which encloses the yield surface. The proximity of these two surfaces in the course of their simultaneous translation and possible deformation determines the value of the generalized plastic modulus. The special feature of the model in the case of artificial graphite-like materials is that, due to the absence of pure elastic response, the yield surface "shrinks" to a single point, the point representing the stress state itself. This implies many other simplifications of the general theory.

The concept of the bounding surface is generated by generalizing to multidimensional stress space, observations made on uniaxial experiments. Without further reference to these observations, here, the multidimensional aspect of the model can be described as follows. In stress space, the stress state is represented by point a with coordinates σ_{ij}, lying inside the bounding surface; the latter can be represented by an ellipse (for two stress components) of center k with coordinates β_{ij}. In the course of plastic loading, both the point a and the bounding surface translate simultaneously in stress space at different rates. The motion of point a is given directly by the stress rate $\dot{\sigma}_{ij}$, while an appropriate relation for $\dot{\beta}_{ij}$ yields the translation of the bounding surface. The point a may reach the bounding surface but never cross it. The plastic strain rate is given by

$$\dot{\varepsilon}^p_{ij} = \frac{1}{K}(\dot{\sigma}_{kl} n_{kl}) n_{ij}$$

where n_{ij} is the unit normal to the bounding surface at point b which corresponds to point a according to an appropriate rule. The value of the generalized plastic modulus K depends on the proximity of point a to the bounding surface, measured by the distance δ between points a and b. K depends also on δ_{in}, δ_{in} being the valueof δ at initiation of a loading process. Thus, δ_{in} changes at each stress reversal and is precisely the parameter associated with the history of the most recent event of unloading reloading.

Numerical examples show a good agreement with experimental results in a uniaxial case. The model describes highly non-linear material response under stress reversal, and because of its simplicity is especially tractable for numerical applications.

1. Introduction

The material behavior in the plastic range under loading, unloading and reloading conditions, has become an increasingly important factor in structural analysis. Numerous works are devoted to this area, but because of the restrictive nature of this presentation, no attempt will be made to cover the extensive literature on the subject. The paper will focus on presenting a recently developed constitutive law for plastic material response under stress reversals [1,2,3] for metals in general, and in particular for artificial graphite-like materials.

By artificial graphite-like materials is meant a material which exhibits no purely elastic response, thus the yield surface in stress space degenerates into a point, which represents the stress state itself. In classical plasticity theory loading-unloading criteria are established in terms of a yield surface (or a loading surface). Therefore materials not having such a surface, cannot be appropriately described in a multi-dimensional stress space under stress reversals. By contrast, in one-dimensional stress space, loading and unloading is easily defined by the direction of the stress rate, therefore the vanishing yield surface does not cause any difficulty. In fact, most of the related works on the subject are confined to uniaxial loading and for example one of the simplest representations of material behavior - the Ramberg-Osgood model - implies zero elastic range, although it cannot very successfully describe the material response under a cyclic loading of an irregular kind. It is under multiaxial loading where the difficulty arises, and it is the purpose of this presentation to show how such a case can be treated as a special case of a more general model developed for materials with or without a yield surface. Besides overcoming the difficulty of a vanishing yield surface, the presented model can describe plastic material response under irregular cyclic loading. This feature of the model may be especially important.

In an analogous manner Phillips and Greenstreet [4,5] have developed a theory of an elastic-plastic continuum with emphasis on artificial graphite, which is mainly an extension of a theory by the former author introducing the concept of loading surfaces in stress space. A continuous nucleation of such surfaces in stress space upon stress reversal is the key point of their theory, which renders their model somewhat cumbersome in use for large scale computer programs. The present model makes use of only one surface, the bounding surface, and it is simple enough to be especially tractable for use in numerical schemes.

Artificial graphite exhibits no pure elastic response, thus it falls in the above class of materials. In addition, it is an anisotropic material with an axis of symmetry and transverse isotropy, but such a complication will not be considered here, focusing only on the property of nonexistance of pure elastic behavior.

2. The General Model

In this section a brief discussion of the general theory developed in [1,2,3] and other forthcoming publications is given, emphasizing on the important features and end results.

The basic idea underlying the model is the introduction of the bounding surface in stress space which encloses the yield surface and where the proximity of these two surfaces determines the plastic deformation rate. The main feature of the model is the establishment of a rule according to which the plastic modulus changes in multidimensional stress space, by generalizing conclusions derived from uniaxial experiments. A typical schematic representation of such an experiment is shown in Fig. 1. Here, every stress-strain curve approaches asymptotically or coincides with the bounds represented by the lines XX', YY'. If

$$\dot{\sigma}_1 = E^p \dot{\varepsilon}_1^p \tag{1}$$

where E^p is the plastic modulus and $\dot{\sigma}_1$, $\dot{\varepsilon}_1^p$ the stress and plastic strain rates, the material behavior can be easily described by considering E^p a function of the distance $\delta = AB$ of the stress state A from the corresponding bound, and also a function of the value of δ at the initiation of yielding for each loading process, denoted by δ_{in}. The parameter δ measures the proximity of each point to the bound, and the parameter δ_{in} the proximity of the nonlinear part of the stress-strain curve as a whole to this bound. Thus, a relation $E^p = \tilde{E}^p(\delta,\delta_{in})$ is suggested, with $E_o^p = \tilde{E}^p(o,\delta_{in})$, the value of E^p on the bounds, and $\tilde{E}^p(\delta_{in},\delta_{in}) = \infty$ at the initiation of yielding for a smooth transition from the elastic to the elastoplastic range. $\tilde{E}^p$ is an absolutely increasing function of δ. The parameter δ_{in} changes at each stress reversal and therefore can be thought of as a parameter associated with the history of the most recent event of unloading-reloading. In addition δ_{in} is a measure of the plastic deformation which took place in a direction opposite to the current one since, for example, the larger the value of δ_{in} the more the material has been workhardened before unloading-reloading.

The line segment $A\bar{A}$ in Fig. 1 represents the elastic region which lies within the larger segment $B\bar{B}$, defined by the two bounds XX', YY'. The relative position of these two segments, which translate simultaneously during loading, determines δ. By projecting on the σ-axis and then generalizing in multidimensional stress space, the end points $A,\bar{A}$ become the yield surface and the end points $B,\bar{B}$ a second surface enclosing the yield surface, called the bounding surface, Fig. 2. The two surfaces translate simultaneously in stress space in a coupled way and possibly deform. They may come in contact but do not intersect. Throughout this course, the continually changing distance δ in stress space, measured by the usual Euclidian metric, between the stress state a on the yield surface and the corresponding point b on the bounding surface, determines the value of the generalized plastic modulus K, in a manner analogous to the uniaxial case as

$$K = \tilde{K}(\delta,\delta_{in}) \tag{2}$$

with $K_o = \tilde{K}(0,\delta_{in})$ the value of K for points on the bounding surface after contact $(\delta = 0)$. The constitutive relation for the plastic strain rate $\dot{\varepsilon}_{ij}^p$ is given by

$$\dot{\varepsilon}_{ij}^p = \frac{1}{K}\dot{\sigma}\,\rho_{ij} \tag{3}$$

where σ_{ij} is the stress tensor, $\dot{\sigma} = \dot{\sigma}_{ij}\,n_{ij}$ is the projection of the stress rate on the unit normal n_{ij} to the yield surface at point a and ρ_{ij} is a unit vector indicating the direction of $\dot{\varepsilon}_{ij}^p$ (normality condition is obtained as the special case $\rho_{ij} = n_{ij}$). If the equation of the yield surface is given by $f(\sigma_{ij} - \alpha_{ij}, q_n) = 0$, with α_{ij} the coordinates of its center, and q_n internal variables, it is found that

$$\dot{\alpha}_{ij} = \frac{1}{n_{k\ell}\,\nu_{k\ell}}\,\frac{K_\alpha}{K}\,\dot{\sigma}\,\nu_{ij} \tag{4}$$

with $K_\alpha/K = 1 + (1/g\dot{\sigma})(\partial f/\partial q_n)\dot{q}_n$, where ν_{ij} is the unit vector in the direction of $\dot{\alpha}_{ij}$, K_α is a modulus for $\dot{\alpha}_{ij}$, $g = [(\partial f/\partial\sigma_{ij})(\partial f/\partial\sigma_{ij})]^{\frac{1}{2}}$ and where use of the consistency condition $\dot{f} = 0$ was made. Eq. (4) applies to any kind of kinematic hardening for which ν_{ij} is specified. The q_n are usually taken to be the plastic strain ε^p_{ij} and a parameter k controlling the size of the yield surface.

Similarly, if the equation of the bounding surface is given by $F(\sigma_{ij} - \beta_{ij}, q_n) = 0$, with β_{ij} the coordinates of its center, and q_n internal variables, the equation coupling its translation to the translation of the yield surface is found to be

$$\dot{\beta}_{ij} = \dot{\alpha}_{ij} - [1 - \frac{K_0}{K}] \frac{\dot{\sigma}}{n_{k\ell}\mu_{k\ell}} \mu_{ij} - \frac{(\partial f/\partial q_n)\dot{q}_n}{g\, n_{k\ell}\mu_{k\ell}} \mu_{ij} + \frac{(\partial F/\partial q_n)\dot{q}_n}{\bar{g}\, n_{k\ell}\mu_{k\ell}} \mu_{ij} \qquad (5)$$

where $\bar{g} = [(\partial F/\partial\sigma_{ij})(\partial F/\partial\sigma_{ij})]^{\frac{1}{2}}$ and μ_{ij} is a unit vector emanating from point a toward the corresponding point b , Fig. 2. Point b is defined such that the normal at points a and b on the two surfaces has the same direction. The form (5) guarantees that the point of possible contact is also the stress state. For no deformation of the yield and bounding surfaces, $(\partial f/\partial q_n) = (\partial F/\partial q_n) = 0$, and eqs. (4) and (5) simplify accordingly. In addition, when contact of the two surfaces occurs, i.e., $\delta = 0$, eq. (2) yields $K = K_0$ and eq. (5) yields $\dot{\beta}_{ij} = \dot{\alpha}_{ij}$, i.e., the two surfaces move together remaining in contact as long as loading continues. This, in the uniaxial case, corresponds to the event that the stress-strain curve has merged into the corresponding bound.

3. Specialization for Artificial Graphite-Like Materials

The previously described general model can be applied to any specific hardening rule by appropriately determining the corresponding parameters, for example the q_n for isotropic hardening, or the ν_{ij} for kinematic hardening, etc. Along these lines, this section will show how the model specializes for artificial graphite-like materials.

For graphite-like materials, due to the nonexistance of purely elastic response, the yield surface degenerates into a single point a as shown in Fig. 3. This point represents the stress state. The motion of point a is given directly by the stress rate $\dot{\sigma}_{ij}$, while an appropriate relation for $\dot{\beta}_{ij}$, determined in the following, yields the translation of the bounding surface. The point a may reach the bounding surface but never crosses it.

The basic question now is how loading is defined in the above case. To begin with, there is no such phenomenon as unloading. Whichever the direction of $\dot{\sigma}_{ij}$, plastic deformation follows. The rate of such a deformation, however, depends on the direction of $\dot{\sigma}_{ij}$ and the easiest manner for determining this dependence is to consider that point a is obtained in the limit as the yield surface shrinks. This allows one to locate the corresponding point b on the bounding surface, thus the unit normal n_{ij} , essential for the set of eqs. (3)-(5), is defined; recall that at point b , the unit normal n_{ij} is the same as the normal at the corresponding point a on the yield surface. Since the yield surface degenerates into the stress state point, the direction of $\dot{\alpha}_{ij}$ coincides with the direction of stress rate, that is if

$$\dot{\sigma}_{ij} = \|\dot{\tilde{\sigma}}\| \, \ell_{ij} \tag{6}$$

with $\|\dot{\sigma}\|$ the norm of $\dot{\sigma}_{ij}$ and ℓ_{ij} the unit vector along the $\dot{\sigma}_{ij}$, then

$$\ell_{ij} = \nu_{ij} \tag{7}$$

Then, using eqs. (6) and (7), since $(\partial f/\partial q_n) = 0$, $(K_\alpha/K) = 1$ and eq. (4) yields:

$$\dot{\alpha}_{ij} = \frac{1}{n_{k\ell}\nu_{k\ell}} (\dot{\sigma}_{pq} n_{pq}) \ell_{ij} = \frac{1}{n_{k\ell}\nu_{k\ell}} (\ell_{pq} n_{pq}) \|\dot{\tilde{\sigma}}\| \, \ell_{ij} = \dot{\sigma}_{ij} \tag{8}$$

which demonstrates the expected result.

Eq. (3) remains the same; in addition, for simplicity it may be assumed that $\rho_{ij} = n_{ij}$, and the important point now is to locate point b on the bounding surface in order to define n_{ij} . The location of b however, depends on the kinematic rule assumed, and possible approaches are demonstrated on the following for three major kinematic rules.

(i) Ziegler's Kinematic Rule

The motion of the yield surface takes place along the radius connecting its center to the stress state. Here, since according to eq. (7) $\nu_{ij} = \ell_{ij}$, the radius of the shrinking yield surface is along the $\dot{\sigma}_{ij}$. Thus, assuming that the yield surface is similar to the bounding surface (which is not restrictive since the yield surface degenerates anyway into a point), point b is obtained as the intersection of the radius kb with the bounding surface, where kb is parallel to $\dot{\sigma}_{ij}$; this case is shown in Fig. 3. Then, using eq. (8) and assuming for simplicity that $(\partial F/\partial q_n) = 0$, eq. (5) yields

$$\dot{\beta}_{ij} = \dot{\sigma}_{ij} - [1 - \frac{K_o}{K}] \frac{\dot{\sigma}}{n_{k\ell}\mu_{k\ell}} \mu_{ij} \tag{9}$$

where n_{ij} is the normal to the bounding surface at point b .

(ii) Prager's Kinematic Rule

Assuming that Prager's Kinematic Rule (motion of the yield surface along the normal) holds for the subspace of Fig. 3, following the same argument as before, it can be concluded that the normal direction n_{ij} is along $\dot{\sigma}_{ij}$. Thus point b is defined as the point on the bounding surface where $n_{ij} = \ell_{ij}$. The rest remains the same. This assumption does not necessarily guarantee the incompressibility condition $\dot{\varepsilon}^p_{ii} = 0$.

(iii) Mróz's Kinematic Rule

According to Mróz's rule [6,7], the motion of the yield surface takes place along μ_{ij} . Thus from eq. (7), $\nu_{ij} = \ell_{ij} = \mu_{ij}$, and point b is obtained as the intersection of $\dot{\sigma}_{ij}$ with the bounding surface. In addition, using eq. (6), eq. (5) yields

$$\dot{\beta}_{ij} = \dot{\sigma}_{ij} - [1 - \frac{K_o}{K}] \frac{1}{n_{k\ell}\ell_{k\ell}} \|\dot{\tilde{\sigma}}\| (\ell_{pq} n_{pq}) \ell_{ij} = \frac{K_o}{K} \dot{\sigma}_{ij} \tag{10}$$

Now the importance of the parameter δ_{in} in the proposed theory can be recognized. As is apparent from the preceeding development, no unloading ever takes place and loading is defined entirely in terms of the unit normal to the bounding surface. The important question is, then, how δ_{in} changes in the multi-directional case. In the uniaxial case no such difficulty arises since a change of the loading direction is uniquely defined. For the multi-directional case, when a loading process begins, δ_{in} assumes a certain value; during the process δ may decrease until it becomes zero; or alternatively, due to a continuous or discontinuous change in the direction of loading, δ may become greater than δ_{in}. Here is the crucial point: It is assumed that if δ becomes greater than δ_{in}, then δ_{in} assumes this new greater value, signaling the initiation of a new process. This condition is necessary not only for the present case, but also for the cases where the yield surface has not degenerated into a single point. Indeed, consider a plastic process where loading takes place almost tangentially to the yield surface, i.e., almost neutral loading. In this way, with almost nonexisting plastic deformation, loading in the reverse direction may develop, which is equivalent for all practical purposes to another process with unloading and reverse reloading. But in the former case no change in δ_{in} occurs, unless it is assumed that δ_{in} will change when δ eventually becomes greater. Therefore δ_{in} plays the important role of differentiating one loading process from another with or without a yield surface, and it can be recognized as a discrete memory parameter of the most recent event of unloading-reloading.

Lastly it should be noted that the bounding surface can expand or contract or even deform as a function of the total plastic work W^p, the strain rate, and the temperature. This corresponds to a parallel translation of the bounds XX' and YY' in the uniaxial case, which is sometimes observed as a result of a change of the above quantities.

4. Numerical Examples

So far no specific functional form of equation (2) for $\tilde{K}$, or equivalently for $\tilde{E}^p$ has been given. For the Mises yield criterion and radial loading, it can be shown [2] that $K = \frac{2}{3} E^p$, so that in the remainder of this section E^p will be studied and K will be assumed to be 2/3 of E^p

An equation for E^p used successfully in different numerical examples [1,2] is

$$E^p = E_o^p + h \left[\frac{\delta}{\delta_{in}-\delta}\right] \tag{11}$$

where h is a hardening shape parameter. Since $E^p = (d\sigma/d\varepsilon^p)$, eq. (11) is the differential equation of the stress-plastic strain curve. Integrating eq. (11), it is easily obtained

$$h = E_o^p - \frac{\sigma_1}{\varepsilon_1^p} - \frac{\delta_{in}}{\varepsilon_1^p} \ln\left[1 + \frac{E_o^p \varepsilon_1^p - \sigma_1}{\delta_{in}}\right] \tag{12}$$

where σ_1 and ε_1^p are the coordinates of any point on the nonlinear part of the stress-plastic strain curve, using as origin the point where yielding initiates.

The expressions (11) and (12) have been used to obtain the comparison of experiment with theory shown in Fig. 4 for artificial graphite. The experimental data were taken from [8]. Writing $h = E_o^p \tilde{h}$, it was found that $\tilde{h} = 7.5$. The bounds XX' and YY' do not exhibit themselves in the experiment, but they were assumed to be straight lines towards which the curves converge asymptotically. The agreement between theory and experiment is good, and it should be mentioned that for other materials, where the bounds exhibit themselves in the experiment, the results are much better [2].

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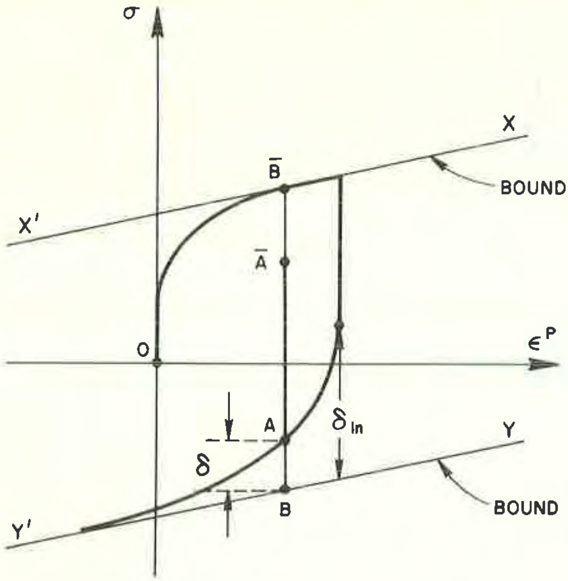

Figure 1. Schematic representation of uniaxial loading indicating the parameters δ and δ_{in} .

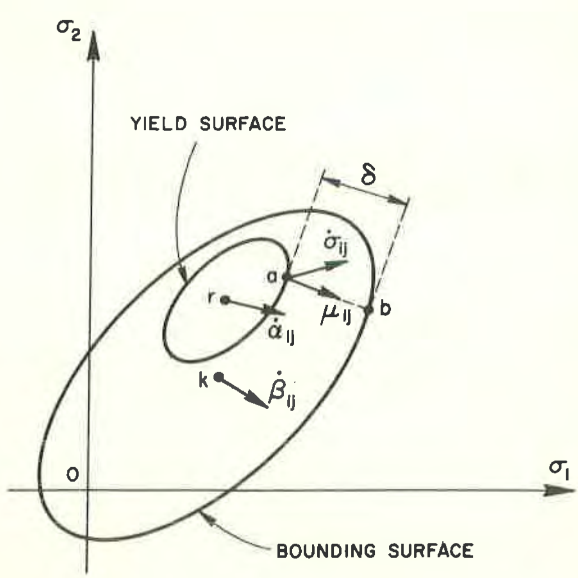

Figure 2. Schematic representation of the Yield and Bounding Surfaces and indication of their motions.

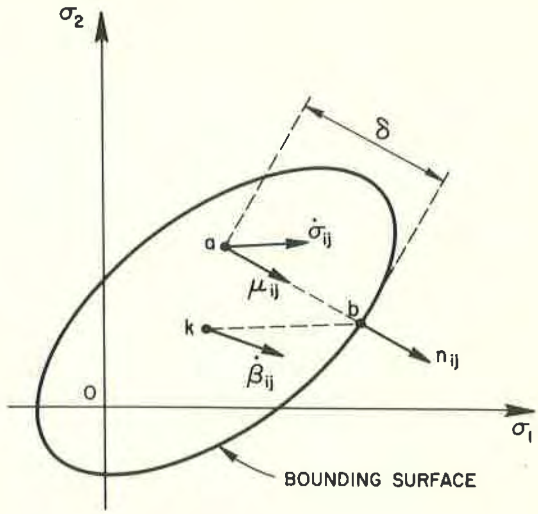

Figure 3. Schematic representation of the stress state and the Bounding Surface for nonexisting Yield Surface.

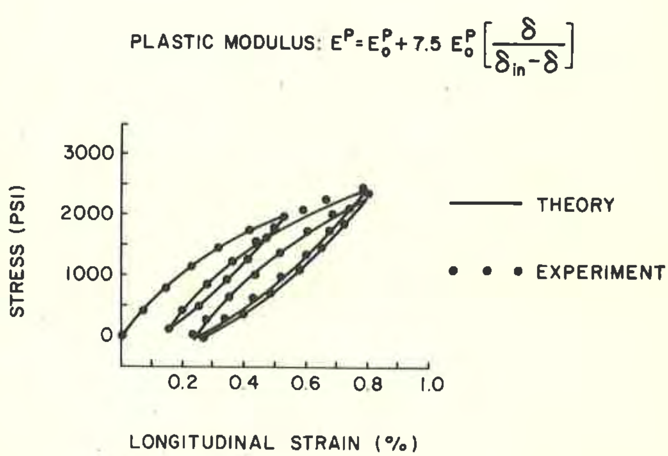

Figure 4. Random cyclic loading on across-grain EGCR-type AGOT specimen.