ABSTRACT

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We study the relationship between lattice structures on finite Coxeter groups, which has implications for related lattice structures on analogous Artin groups.

In the first section, the complete results of a paper in progress (joint work with Barnard and Reading) centered around versions of noncrossing arc diagrams of types B and D are presented. The classical, type-A noncrossing arc diagrams are combinatorial objects in bijection with permutations and have been shown to provide insights into lattice congruences of the weak order. Two models of type-B noncrossing arc diagrams, one novel, are presented and used to characterize congruences of the weak order of signed permutations. Likewise, two original models of type-D noncrossing arc diagrams are presented and used to characterize congruences of the weak order of even-signed permutations.

In the second section, noncrossing arc diagrams of types A and B are used to realize the shard intersection orders of the same types. The shard intersection order on a Coxeter group arises from the hyperplane arrangement of the group; it is a weaker order than the weak order on the same group and contains the noncrossing partition lattice on the group as a sublattice. Two operations on noncrossing arc diagrams are presented in type A and shown to characterize the meet and join in the shard intersection order of type A. Next, the two models of type-B noncrossing arc diagrams presented in the first section combine to show that meet and join of the shard intersection order of type B can also be characterized using the type-A operations. Four surjective lattice homomorphisms from the weak order of type B to that of type A and the inverses of their restrictions to the lowest elements of their fibers are phrased in terms of noncrossing arc diagrams. Finally, the characterizations of meets and joins in the shard intersection orders of types A and B are used to prove whether each inverse embeds the type-A shard intersection order as a sublattice of the type-B shard intersection order.
Arches and Shards

by
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DEDICATION

To Nana.
BIOGRAPHY

The author was born and raised in Houston, Texas. After exploring several majors, she earned her bachelor’s degree in Mathematics with a Spanish minor from the University of Houston in 2018. She moved to Raleigh to pursue a doctorate in Mathematics from North Carolina State University, which she completed in Summer 2023, focusing primarily on algebraic combinatorics. Outside of her own studies, she passionately supported both her students and peers, including as the Mathematics Department’s first Graduate Support Resource TA. Among other recognition of her teaching and service, she was the first recipient of the Graduate Student Excellence Award given by the College of Sciences. She received the first Graduate Student Excellence Award given by the College of Sciences, along with honors from various institutions within NCSU.

At the time of writing, the author is excited to join the community of outstanding educators at North Carolina School of Science and Mathematics and to share strategies for self- and community care, in addition to mathematics, with the brilliant students at NCSSM.
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CHAPTER

1

INTRODUCTION

1.1 Context & Motivation

Permutations are perhaps the prototypical family of combinatorial objects, having a remarkably approachable introductory description, underlying structural depth, and connections to various fields of study. The family $S_n$ of permutations of $[n] = \{1, \ldots, n\}$ can be considered from combinatorial, algebraic, and geometric perspectives and can be generalized in various ways.

From a combinatorial point of view, a permutation $\pi \in S_n$ is a sequence $\pi_1 \pi_2 \cdots \pi_n$ of distinct elements of $[n]$. Each permutation may have some inversions, pairs of entries in which the earlier entry is greater than the later entry. The inversion sets of permutations in $S_n$ can be used to create a partial order on $S_n$, called the (right) weak order: $\pi$ is below $\sigma$ exactly when the inversion set of $\pi$ is contained in that of $\sigma$. In particular, $\sigma$ covers $\pi$ if there is exactly one inversion added to $\pi$’s inversion set to create $\sigma$’s. This happens precisely when two adjacent entries written in ascending order in $\pi$ are swapped to create a descent in $\sigma$ (meaning an inversion between adjacent entries). The weak order on the Coxeter group $S_3$, with vertices labeled by permutations and alternatively by their inversion sets, is pictured in Fig. 1.1. The Hasse diagram by which we can visualize the weak order on $S_n$ provides access to the geometric point of view: as an undirected graph, it coincides with the 1-skeleton of
the permutahedron, an \((n - 1)\)-dimensional simple polytope whose vertices are indexed by permutations. When realized in the “right” way, the permutahedron displays a great deal of symmetry which we discuss shortly.

The weak order on \(S_n\), denoted \((S_n, \leq)\), is a lattice (a particularly nice type of partially ordered set in which least upper bounds (joins) and greatest lower bounds (meets) exist for any subset of elements) [9, Theorem 8]. As a lattice, the weak order admits congruences, equivalence relations that respect meet and join operations in the usual algebraic sense. A congruence is said to contract an edge corresponding to the cover \(x < y\) when the congruence sets \(x\) and \(y\) equivalent (when this is true and \(y\) covers only \(x\), we sometimes say the congruence contracts \(y\)). In the corresponding lattice quotient, the result of modding out the original lattice by the congruence, we get a lattice structure on the set of congruence classes in the usual algebraic way. Many important combinatorial structures arise as lattice quotients of the weak order. For example, Cambrian lattices give a lot of insight into the combinatorics of cluster algebras: the Hasse diagram of the Cambrian lattice is the exchange graph of a finite cluster algebra of type A [32, 39]. Also, a certain simple lattice congruence gives a quotient whose elements are in bijection with certain rectangulations of a square (decompositions of the square into rectangles) [36, Example 10-7.25].

From a more algebraic point of view, \(S_n\) can be thought of as the Coxeter group of type \(A_{n-1}\). The group has simple reflections of the form \((i\ i+1)\) as generators. It is closely related to the braid group, which is the Artin group of type A. A Coxeter group is specified by its Coxeter diagram, a graph with vertices representing generators and edges (sometimes with a label \(\geq 4\)) signaling relationships between pairs of generators. Standard references on Coxeter groups include [10] (which has an explicitly combinatorial tilt), [12] (which works in an algebraic context), [21] (which approaches from a geometric angle). There is a complete classification of Coxeter groups of finite types, including three infinite families of irreducible (in the sense of direct product) finite Coxeter groups: type A, type B, and type D. The Coxeter diagrams for groups of types \(A_n\), \(B_n\), and \(D_n\) are pictured in Fig. 1.2. Coxeter groups

\[
\begin{array}{c}
321 \\
| \\
312 & 231 \\
| & |
\end{array}
\begin{array}{c}
\{(3, 2), (3, 1), (2, 1)\} \\
| \\
\{(3, 1), (3, 2)\} & \{(2, 1), (3, 1)\} \\
| & |
\end{array}
\begin{array}{c}
132 & 213 \\
| & |
\end{array}
\begin{array}{c}
\{(3, 2)\} & \{(2, 1)\} \\
| & |
\end{array}
\begin{array}{c}
123 \\
\end{array}
\end{array}
\]
of type B can be realized as groups of signed permutations, and those of type D can be realized as groups of even-signed permutations. There are weak orders of types B and D based on the containment of inversion sets, though the type-B and type-D analogues of inversions are slightly different than the original definition. As in type A, the Hasse diagrams of these weak orders coincide with the 1-skeletons of permutahedra of the appropriate type, which have a lot of symmetry baked in when they are realized in the “right” way.

As suggested by their Coxeter diagrams, there is an intimate and fairly straightforward relationship between Coxeter groups of types $A_n$ and $B_n$. In particular, we can get the diagram for $A_n$ by simply erasing the label 4 on the edge between $s_1$ and $s_2$ in the diagram for $B_n$. This is one instance (or infinitely many instances, depending on your mood) of a hierarchical relationship between Coxeter groups. A Coxeter group $W$ is said to dominate another Coxeter group $W'$ if we can get the diagram of $W'$ by lowering or erasing labels on edges or by erasing edges in the diagram of $W$ (and perhaps changing the indices on generators so that the labels on vertices agreed). The Coxeter groups of type $D_n$ do not fit nicely into either side of the dominance relationship with groups of type $A_n$ or $B_n$, because the vertex associated to $s_2$ has degree 3 and none of the generators adjacent to it are connected to one another by an edge.

The geometric point of view on $S_n$ and its signed relatives can be built from an understanding of the appropriate permutahedron. The type-A permutahedron in $\mathbb{R}^n$ has $n! (|S_n|)$ vertices, since its 1-skeleton is the weak order on $S_n$: elements of $S_n$ correspond to vertices, and covers in the weak order correspond to edges of the permutahedron. Similarly, the type-B and type-D permutahedra also have the same number of vertices as the size of their associated groups. The standard realization of the type-A permutahedron is highly symmetric: it has reflective symmetry across $n(n-1)/2$ distinct hyperplanes, one for each inversion which may occur in a permutation $\pi \in S_n$. In [33], a newer order called the shard intersection order on $S_n$ arises from this hyperplane arrangement. Informally, it is constructed as follows: After correctly identifying vertices with permutations, orient the permutahedron so that its center

![Coxeter diagrams for $A_n$, $B_n$, and $D_n$.](image)
is at the origin in $\mathbb{R}^n$ and the identity permutation $1 2 \cdots n$ is directly below the origin, then establish a hierarchy among reflecting hyperplanes in which the steepest hyperplanes (which correspond to simple reflections) are the most special. Next, allow steeper hyperplanes to separate less steep into pieces called shards, and finally order permutations by the reverse containment order on the intersections of the shards which most immediately separate them from the identity permutation. This order, called the shard intersection order and denoted $\Psi(S_n)$ or $(S_n, \preceq)$, is a finite lattice related to and weaker than the weak order (weaker in the sense that it has fewer order relations than the weak order). Analogous processes can be used to create shard intersection orders on signed permutations (type B) and even-signed permutations (type D).

The shard intersection order on $S_n$ contains the noncrossing partition lattice $NC(S_n)$ on $n$ elements as a sublattice. $NC(S_n)$ is also sometimes called the classical or type-A noncrossing partition lattice and was introduced by Kreweras in [25]. There is a noncrossing partition lattice $NC(W)$ for every finite Coxeter group $W$, and they have important connections to Artin groups and representation theory [3, 7, 8, 13, 14, 22, 24, 40]. The shard intersection order was introduced in [33] as a natural generalization of the noncrossing partition lattice and to give a new proof that noncrossing partitions form a lattice. More recently, the shard intersection order has emerged as an important object in its own right: Just as the noncrossing partition lattice is the lattice of wide subcategories for the corresponding quiver of type A, D, or E, the shard intersection order is the lattice of wide subcategories for the corresponding preprojective algebra [18, 20].

The fibers of the surjective lattice homomorphism from $(S_n, \leq)$ to the Tamari lattice, introduced in [43], form a congruence on the weak order whose equivalence classes are intervals, and the set of bottom elements of the intervals is closed under the meet and join operations of $\Psi(S_n)$ [34]. $(NC(S_n)$ is related to the Tamari lattice in the same way that $\Psi(S_n)$ is related to the weak order on $S_n$: they have the same underlying set, and the former is a weaker order than the latter.) For every finite Coxeter group $W$, there is an analogous sublattice relationship: $NC(W)$ is a sublattice of $\Psi(W)$ [33, Proposition 8.7].

We are finally in a position to articulate the initial motivation of this thesis, phrased as a question.

**Question 1.1.1.** If a finite Coxeter group $W$ dominates another finite Coxeter group $W'$, is $NC(W')$ a sublattice of $NC(W)$?

Our work focuses on this question within the dominance relationship of $B_n$ over $A_n$. Instead of answering the question directly, we approach it by considering a different but closely related question.
Question 1.1.2. If a finite Coxeter group \( W \) dominates another finite Coxeter group \( W' \), is \( \Psi(W') \) a sublattice of \( \Psi(W) \)?

One reason for posing and approaching this new question is that, since the noncrossing partition lattice of a Coxeter group is a sublattice of its shard intersection order (by [38, Theorem 8.3]), an affirmative answer to Question 1.1.2 will give the same answer to Question 1.1.1 “for free”. To provide a definite answer for the latter question, we need to consider surjective lattice homomorphisms \( \eta \) from \((B_n, \leq)\) to \((A_n, \leq)\). The set of lowest elements in fibers of \( \eta \) in \((B_n, \leq)\) must be closed under the join operation in \( \Psi(A_n) \) since in any shard intersection order, the join operation is simply intersection [35]. However, the set of lowest elements might fail to be closed under the meet operation in \( \Psi(A_n) \). Thus, we need to better understand the meet operation in \( \Psi(A_n) \), and we do this by presenting a new characterization of the meet. In order to make good sense of our characterization, it is beneficial to have a closely related and sensible characterization of the join operation in \( \Psi(A_n) \) as well as both operations in \( \Psi(B_n) \).

Another reason to deal primarily with Question 1.1.2 is that there are other combinatorial objects by which we can realize Coxeter groups of type A to better understand \( \Psi(S_n) \) without doing tedious bookkeeping of linear inequalities in \( \mathbb{R}^n \). Noncrossing arc diagrams on \( n \) points were introduced by Reading in [35] and shown to be in bijection with permutations of \([n]\). Each noncrossing arc diagram is a visual object consisting of points on a vertical line and some (or no) curves connecting points, with each curve following certain rules on its own and relative to the other curves present. Fig. 3.2 shows the noncrossing arc diagrams corresponding to the 24 permutations in \( S_4 \), and is included here as Fig. 1.3. Each allowed curve is called an arc, and arcs are shown to be in bijection with join-irreducible permutations (those with exactly one descent). Under this bijection, join-irreducible permutations correspond to shards. Moreover, in [4], Bancroft uses permutation pre-orders, which are in bijection with noncrossing arc diagrams (stated explicitly in Proposition 3.3.1), to find an EL-labeling of the shard intersection of \( S_n \). Since permutation pre-orders can be used to understand some of
the structure of the shard intersection order on $S_n$, it is reasonable to guess that noncrossing arc diagrams might also provide relevant information about $Ψ(S_n)$. Indeed, one of the main results of this thesis is the characterization of the meet and join operations in the shard intersection order of type A in terms of operations on noncrossing arc diagrams.

To characterize the operations underlying $Ψ(B_n)$ similarly, we first need to make sense of analogous versions noncrossing arc diagrams in type B. Chapter 2 is a version of a joint paper with Barnard and Reading, which will be submitted for publication in 2023. In it, we begin by summarizing existing results on the original, centrally symmetric construction of noncrossing arc diagrams in type B (first introduced in [6] and expanded upon in [27]). We then present a new construction which exploits the symmetry of the first construction. Using these constructions, we reach results analogous to those in [35], including results on how arcs can be used to convey the combinatorics of congruences on $(B_n, \leq)$. We also introduce two novel constructions of type-D noncrossing arc diagrams and present analogous results in type D.

The constructions and results in type B, which are set forth in Chapter 2, give us the necessary foundation to approach Question 1.1.2 head-on in Chapter 3. In it, we first state results characterizing $(A_n, \leq)$ using noncrossing arc diagrams, which is somewhat surprising but necessary since the shard intersection order is so closely related to the weak order. We then define two operations, each from the set of pairs of noncrossing arc diagrams to the set of noncrossing arc diagrams, and present two theorems, 3.3.15 and 3.3.19, stating that the operations characterize the meet and join in $Ψ(A_n)$. Then, combining the type-B noncrossing arc diagrams presented in Chapter 2 with work on type A in Section 3.3, we are able to define analogous operations in type B and state nearly immediate parallel theorems, 3.4.11 and 3.4.12, characterizing the meet and join in $Ψ(B_n)$. Finally, we consider four maps $η$ from $(B_n, \leq)$ to $(A_n, \leq)$ which are shown in [38] to be surjective lattice homomorphisms that effectively erase the label 4 on the Coxeter diagram for $B_n$. We reframe each map as an operation on type-B noncrossing arc diagrams and prove that, in all but one case, the inverse of the restriction of $η$ to the bottom elements of its fibers does in fact embed $Ψ(A_n)$ as a sublattice of $Ψ(B_n)$ (Theorems 3.5.8, 3.5.21 and 3.5.27).

1.2 Selected Results, Examples, & Figures

In this section we provide overviews of Chapters 2 and 3, including some major results as well as (sometimes abbreviated versions of) important definitions and results or figures that are worth at least a few paragraphs of prose. We follow the same general structure as the two chapters, but sometimes combine or skip sections when helpful.
1.2.1 Noncrossing arc diagrams beyond type A

Noncrossing arc diagrams, visual combinatorial objects in bijection with permutations, are presented and shown to be quite effective at characterizing congruences of the weak order of type A in [35]. The major results of Chapter 2, a version of a joint paper with Barnard and Reading which will be submitted in 2023, are type-B and type-D analogues of the major results of [35].

Type-B noncrossing arc diagrams, symmetric model

The first main result on the symmetric model of type-B noncrossing arc diagrams is Theorem 2.4.1, stated below as Theorem 1.2.1. In it, $\delta$ is the map from permutations of $[n]$ to noncrossing arc diagrams on $n$ points, originally described for type A in [35] and described in Section 2.3. The central symmetry mentioned is half-turn rotational symmetry, which sends the point $i$ to the point $-i$ (and vice versa) for each $i \in \{1, \ldots, n\}$.

**Theorem 1.2.1.** The map $\delta$ restricts to a bijection from $B_n$ to the set of centrally symmetric noncrossing arc diagrams on $2n$ points.

An example of $\delta$ applied to a signed permutation of $\pm[5]$ is pictured in Fig. 2.2 and included here as Fig. 1.4. The 8 centrally symmetric noncrossing arc diagrams for $B_2$ are included in Fig. 2.3 and here as Fig. 1.5.
Because of the symmetry of these diagrams, arcs occur in one of three forms in this model: as symmetric arcs, nonoverlapping symmetric pairs of arcs, or overlapping symmetric pairs of arcs. In Fig. 1.5, symmetric arcs are pictured in the innermost pair of diagrams, a nonoverlapping pair of arcs is pictured in the diagram immediately left of the symmetric arcs, and overlapping pairs of arcs are pictured in the two diagrams immediately right of them. Each symmetric arc or pair corresponds to a type-B inversion and thus to a join-irreducible signed permutation. The bijection from symmetric arcs and symmetric pairs to shards in the type-B Coxeter arrangement is made explicit in Propositions 2.4.3 and 2.4.4.

As in type A, we can characterize the canonical join representation (the minimal set, both element-wise in the weak order and set-wise, of join irreducible elements whose meet is whatever element of the Coxeter group we are considering) of a signed permutation in terms of the arcs in its symmetric noncrossing arc diagram. This result is Theorem 2.4.2 and is stated below as Theorem 1.2.2.

**Theorem 1.2.2.** Given $\pi \in B_n$, the canonical join representation of $\pi$ is the set of join-irreducible elements corresponding to the symmetric arcs and symmetric pairs of arcs in $\delta(\pi)$.

The concept of subarcs is introduced for type A in [35] as a partial pre-order on arcs in which the left/right data of a shorter arc matches the subset of the same data for a longer arc along their shared length. Definitions of subarcs/subarc pairs of subarcs/subarc pairs are stated in Definitions 2.4.19 and 2.4.20, but the idea is essentially the same in type B. Several examples of the symmetric subarc relation are pictured in Fig. 2.4 and included here as Fig. 1.6. There is one wrinkle as we translate from type A to type B, stated in Remark 2.4.8 and captured in the non-example of the subarc relation pictured in Fig. 2.5 and here as
Fig. 1.7. A non-example of a subarc pair.

See Remark 2.4.8

The last major result of Section 2.4.1 is the characterization of forcing of join-irreducible signed permutations (\(j_1\) forces \(j_2\) when every congruence on the weak order that contracts \(j_1\) must also contract \(j_2\)) in terms of subarcs/subarc pairs. It is stated in Theorem 2.4.9, also stated below as Theorem 1.2.3.

**Theorem 1.2.3.** Let \(j_1\) and \(j_2\) be join-irreducible signed permutations. Then \(j_1\) forces \(j_2\) if and only if the arc or pair of arcs corresponding to \(j_1\) is a subarc or subarc pair of the arc or pair of arcs corresponding to \(j_2\).

**Type-B noncrossing arc diagrams, orbifold model**

The first main result on the orbifold model of type-B noncrossing arc diagrams is Theorem 2.4.14, stated below as Theorem 1.2.4. An example of the application of \(\delta^\circ\) (the map in the statement) to a signed permutation of \(\pm[5]\) is pictured in Fig. 2.9 and included here as Fig. 1.8.

**Theorem 1.2.4.** The map \(\delta^\circ\) is a bijection from \(B_n\) to the set of type-B noncrossing arc diagrams on \(n\) points.

Because of the advantages of the orbifold model (including its compactness), we refer to these noncrossing arc diagrams simply as type-B noncrossing arc diagrams. All 8 type-B noncrossing arc diagrams for \(B_2\) are pictured in Fig. 3.22, included here as Fig. 1.9 (compare with Fig. 1.5), and all 48 diagrams for \(B_3\) are pictured in Fig. 3.23, included here as Fig. 1.10.

As in the symmetric model, type-B arcs in the orbifold model occur in one of three forms: ordinary arcs, orbifold arcs, and long arcs. In Fig. 1.10, there are only type-A arcs in the 6
Figure 1.8: The map $\delta^0$ applied to $(-4)352(-1)$.

Figure 1.9: Noncrossing arc diagrams for $B_2$.

Figure 1.10: Type-B noncrossing arc diagrams, $B_3$. 
diagrams on the left side of the bottom row. Type-A arcs are joined by symmetric arcs in the 6 diagrams on the right side of the bottom row and in all diagrams on the second row. In the top two rows of the figure, there is a single long arc in each diagram. As in the symmetric model, each type of arc corresponds to a type-B inversion and thus a join-irreducible signed permutation. The bijection from type-B arcs to shards in the type-B Coxeter arrangement is made explicit in Propositions 2.4.16 to 2.4.18.

Since we can think of each type-B noncrossing arc diagram as the result of taking symmetric noncrossing arc diagram and just “modding out” by its central symmetry, the rest of the main results in Section 2.4.2 are essentially direct translations of results in Section 2.4.1. This is certainly true of the following theorem, also stated as Theorem 2.4.15.

**Theorem 1.2.5.** Given \( \pi \in B_n \), the canonical join representation of \( \pi \) is the set of join-irreducible elements corresponding to the arcs in \( \delta^0(\pi) \).

Since every symmetric arc and symmetric pair in the original symmetric construction corresponds to exactly one arc in the orbifold construction, the subarc idea in the orbifold model is somewhat simpler and feels more similar to the original definition of subarcs in type A. Definitions of subarcs in type-B are stated in Definitions 2.4.19 and 2.4.20, with several examples of the type-B subarc relation pictured in Fig. 2.10 and included here as Fig. 1.11. The orbifold version of the wrinkle in the subarc idea from type A to type B is stated in Remark 2.4.21 and captured in Fig. 2.11 and here as Fig. 1.12. In this construction, the nuance in type-B is more easily stated and seen: a subarc of a long arc must not cross itself –
if the subarc not ordinary, it must be a valid long arc. (Compare Fig. 1.12 with Fig. 1.7.)

The last major result of Section 2.4.2, the characterization of forcing of join-irreducible signed permutations in terms of type-B subarcs, is Theorem 2.4.22, stated below as Theorem 1.2.6.

**Theorem 1.2.6.** Let $j_1$ and $j_2$ be join-irreducible signed permutations. Then $j_1$ forces $j_2$ if and only if the type-B arc corresponding to $j_1$ is a subarc of the type-B arc corresponding to $j_2$.

**Type-D noncrossing arc diagrams**

Because type D is much less relevant to Chapter 3 than type B, in this section we state only the most essential results and include pictures instead of definitions whenever possible.

We present two models of Type-D noncrossing arc diagrams. The first consists of equivalence classes of one or two collections of type-B arc that are mostly compatible in the type-B sense. We call this the equivalence class model. The second is the result of, in a way, collapsing pairs of certain long type-B arcs which are in the same equivalence class into a single arc with two endpoints, one endpoint in the type-B sense and one that somewhat challenges the “end” in endpoint. We informally call this model postmodern throughout this section. Because the two models are equivalent, we pass from one to the other in Section 2.5 and here when advantageous. As in type B, there are three types of type-D arcs: ordinary, partially doubled, and branched. All three types of arcs are defined in both the equivalence class and post modern constructions in Definition 2.5.1 and pictured in Fig. 2.12, included here as Fig. 1.13. We define a map $\delta^D$ from even-signed permutations to collections of type-D arcs in the equivalence class construction. An example of $\delta^D$ applied to an even-signed permutation of $\pm[5]$ is pictured in Fig. 2.16 and included here as Fig. 1.14.

The first main result on type-D noncrossing arc diagrams is Theorem 2.5.8, stated below as Theorem 1.2.7.

**Theorem 1.2.7.** The map $\delta^D$ is a bijection from $D_n$ to the set of type-D noncrossing arc diagrams.
Figure 1.14: The map $\delta^D$ applied to $3(-4)35(-1)2$.

Figure 1.15: Type-D noncrossing arc diagrams for $D_3$. 
All 24 type-D noncrossing arc diagrams for $D_3$ are included in Fig. 2.15 and here as Fig. 1.15. The bottom two rows in the figure contain the type-D noncrossing arc diagrams corresponding to join-irreducible even-signed permutations, each of which is an equivalence class of either one or two equivalent type-B arcs. The postmodern versions of these single type-D arcs are pictured in Fig. 1.16.

The following theorem, another major result in type D, is Theorem 2.5.7.

**Theorem 1.2.8.** For any even-signed permutation $\pi$, the set of join-irreducible elements associated to $D^\pi(\pi)$ is the canonical join-representation of $\pi$.

The type-D version of subarcs is much less straightforward than the type-B version of subarcs in either construction. In particular, the ordinary and partially doubled subarcs of a partially doubled or branched arc do not need to have all (or even most of) the same left/right as the original arc. Definitions of subarcs of all three kinds of type-D arcs are given in Definitions 2.5.19 to 2.5.23. We include here only example figures in which left/right information changes from an arc to its subarc, included here as Figs. 1.17 and 1.18 and included as Figs. 2.21 and 2.22, with more subcases.

The last major result of Section 2.5, the characterization of forcing of join-irreducible even-signed permutations, is Theorem 2.5.24. It is stated below as Theorem 1.2.9.

**Theorem 1.2.9.** Let $j_1$ and $j_2$ be join-irreducible even-signed permutations. Then $j_1$ forces $j_2$ if and only if the type-D arc corresponding to $j_1$ is a subarc of the type-D arc corresponding to $j_2$. 
1.2.2 Arcs and Shards

Type A

Noncrossing arc diagrams and the weak order

The main result in Section 3.3.2 is Proposition 3.3.7, which characterizes going up in the weak order in terms of operations on noncrossing arc diagrams. All operations which send us up in the weak order are pictured in Figs. 3.5 and 3.6, which are also included here in Figs. 1.19 and 1.20.

Figure 1.18: Ordinary subarcs $\alpha_1$ of a partially doubled arc $\alpha_2$.

Figure 1.19: Simple ways to go up from $N$ in the weak order.
Cooperating and matting
The most important statements in Section 3.3.3 are the definition of cooperative and matted noncrossing arc diagrams, Definitions 3.3.8 and 3.3.10. Each operation takes two noncrossing arc diagrams as its input and gives a single noncrossing arc diagram as an output. (Lemmas 3.3.9 and 3.3.11 make explicit the fact that the outputs are, in fact, noncrossing arc diagrams.)

The cooperative noncrossing arc diagram of two noncrossing arc diagrams $N_1$ and $N_2$ is denoted $\text{cn}(N_1, N_2)$. In it, arcs are drawn only when two components, which we call blocks, in the two diagrams have arcs that share the same upper endpoint, the same lower endpoint, and the arcs in the two components pass weakly to the same side of each point between the shared endpoints. This operation is quite brittle, in the sense that the requirements to draw an arc can fail in many ways. The matted noncrossing arc diagram of $N_1$ and $N_2$ is denoted $\text{mn}(N_1, N_2)$, and it is much more predisposed toward making arcs than the cooperative process. In the matted noncrossing arc diagram, we “mat” each connected component of the union of the two original diagrams to create new blocks, by making each endpoint of a component an endpoint of the new block, passing to the same side of a point between two endpoints as arcs if they agree and adding any points of disagreement as new endpoints.

The definition of each operation is stated formally and then immediately followed by a less formal description of how we can draw the resulting noncrossing arc diagram once we have drawn the original pair of diagrams on the same set of points. These less formal descriptions are accompanied by example figures, including Figs. 3.12 to 3.15, which are consolidated and included here as Figs. 1.21 and 1.22.

Shard intersection order
The shard intersection order on $A_2$, realized both as the set of permutations on [3] and as the set of noncrossing arc diagrams on 3 points, is pictured in Fig. 3.18 and here as Fig. 1.23. The shard intersection order on $A_3$, realized as the set of noncrossing arc diagrams on 4 points, is pictured in Fig. 3.19 and here as Fig. 1.24.
Figure 1.21: Two examples of creating $cn(N_1, N_2)$.

Figure 1.22: Two examples of creating $mn(N_1, N_2)$.

Figure 1.23: The shard intersection order on $A_2$. 
We define two simple operations, link moves and merge moves on pairs of arcs in noncrossing arc diagrams, and examples of both operations are pictured in Fig. 3.20, also included as Fig. 1.25 here. A valid link move on a pair of blocks consists of adding an arc between them so that the result is a noncrossing arc diagram. A valid merge move consists of combining two blocks immediately next to one another to create a single block that has all endpoints of the two blocks and respects the left/right information of both blocks. In Proposition 3.3.14, also stated below as Proposition 1.2.10, we restate Proposition 3.3.13 in terms of noncrossing arc diagrams.

**Proposition 1.2.10.** Let $N$ and $N'$ be two noncrossing arc diagrams on $n$ points. $N'$ covers $N$ in $\Psi(A_{n-1})$ precisely when $N'$ is the result of doing a valid link move or a valid merge move on two blocks of $N$.

The two main results of Section 3.3.4 and indeed of Section 3.3 are Theorems 3.3.15
and 3.3.19, stated here as Theorems 1.2.11 and 1.2.12. In Section 3.3.4, we build up to Theorem 3.3.19 with a sequence of increasingly strong statements (Lemmas 3.3.16 and 3.3.18 and Proposition 3.3.17).

**Theorem 1.2.11.** Given two permutations $\sigma$ and $\tau$ in $S_n$ with noncrossing arc diagrams $N_1 = \delta(\sigma)$ and $N_2 = \delta(\tau)$, their meet in $\Psi(A_{n-1})$ is the permutation corresponding to $\text{cn}(N_1, N_2)$.

**Theorem 1.2.12.** Given two permutations $\sigma$ and $\tau$ in $S_n$ with noncrossing arc diagrams $N_1 = \delta(\sigma)$ and $N_2 = \delta(\tau)$, their join in $\Psi(A_{n-1})$ is the permutation corresponding to $\text{mn}(N_1, N_2)$.

**Type B**

**Signed permutations and noncrossing arc diagrams**

Symmetric noncrossing arc diagrams are type-B analogues of the original type-A noncrossing arc diagrams; their $2n$ points correspond to the entries of signed permutations, the type-B analogues of permutations. The only results worth noting in Section 3.4.1 on symmetric noncrossing arc diagrams are Lemmas 3.4.1 and 3.4.2, which state that when $N_1$ and $N_2$ are symmetric noncrossing arc diagrams, so are the cooperative and matted noncrossing arc diagrams of $N_1$ and $N_2$.

There are a few notable results in Section 3.4.1 on type-B noncrossing arc diagrams (in the orbifold construction). The first two such statements are Definitions 3.4.3 and 3.4.5, on type-B versions of the cooperative and matted noncrossing arc diagrams defined in Section 3.3.3. As in type A, the type-B cooperative noncrossing arc diagram $\text{cn}_B$ has quite restrictive criteria that must be met to draw an arc, and the type-B matted noncrossing arc diagram $\text{mn}_B$ “tangles” the connected components of the union of the two original diagrams. However, because of the subtleties inherent in type-B noncrossing arc diagrams, there criteria to draw arcs in both definition are broken out into separate conditions depending on the kinds of type-B blocks involved and how they relate to each other. Examples of arcs resulting from each condition of Definition 3.4.3 are included in Fig. 3.24 and here in Fig. 1.26. Similarly, examples of arcs resulting from each condition of Definition 3.4.5 are included in Figs. 3.25 and 3.26 and here in Figs. 1.27 and 1.28. The bilateral and unilateral woven blocks mentioned in the captions shorthand for whether a type-B block that does not only consist of type-A/ordinary arcs has two well-defined sides.

**Shard intersection order of type B**

The shard intersection order of type $B_2$, realized both as symmetric noncrossing arc diagrams on 4 points and as type-B noncrossing arc diagrams on 2 points, is pictured in Fig. 1.29. The
Figure 1.26: Type-B cooperative noncrossing arc diagrams.

Figure 1.27: Type-B matted noncrossing arc diagram, bilateral woven block.

Figure 1.28: Type-B matted noncrossing arc diagrams, unilateral woven blocks.
shard intersection order of type $B_3$, realized as type-B noncrossing arc diagrams on 3 points, is pictured in Fig. 3.28 and here as Fig. 1.30. In Fig. 1.29, the top and bottom element as well as the four center elements of the middle row look like type-A noncrossing arc diagrams on 3 points, except that the lowest point in the diagram is $\times$ instead of the numbered point 1. In Fig. 1.30, this phenomenon continues, with 11 elements in each of the two middle rows looking almost exactly like type-A noncrossing arc diagrams on 4 points. This foreshadows the first main result in Section 3.5.

An application of a well-known lattice theoretic result is Proposition 3.4.7, which states that $\Psi(B_n)$ is a sublattice of $\Psi(A_n)$. An immediate consequence of the proposition, in light of Lemmas 3.4.4 and 3.4.6 (which state that the cooperative and matting operations commute with modding out by the symmetry of symmetric noncrossing arc diagrams) is that the characterization of the meet and join in $\Psi(A_n)$ (using cooperative and matted noncrossing arc diagrams) also naturally applies to $\Psi(B_n)$. The two halves of this consequence are stated in Theorems 3.4.8 and 3.4.9 and here as Theorems 1.2.13 and 1.2.14.

**Theorem 1.2.13.** Given two signed permutations $\sigma$ and $\tau$ of $\pm[n]$ with centrally symmetric noncrossing arc diagrams $M_1 = \delta(\sigma)$ and $M_2 = \delta(\tau)$, their meet in $\Psi(B_n)$ is the signed permutation corresponding to $\text{cn}(M_1, M_2)$.

**Theorem 1.2.14.** Given two signed permutations $\sigma$ and $\tau$ of $\pm[n]$ with centrally symmetric noncrossing arc diagrams $M_1 = \delta(\sigma)$ and $M_2 = \delta(\tau)$, their join in $\Psi(B_n)$ is the signed permutation corresponding to $\text{mn}(M_1, M_2)$.

Finally, the two previous theorems can be easily translated into the orbifold construction to give the final two major results in type B. These results are stated in Theorems 3.4.11 and 3.4.12 and here as Theorems 1.2.15 and 1.2.16.
Figure 1.30: The shard intersection order on $B_3$.

**Theorem 1.2.15.** Given two signed permutations $\sigma$ and $\tau$ with type-B noncrossing arc diagrams $N_1 = \delta^\circ(\sigma)$ and $N_2 = \delta^\circ(\tau)$, their meet in $\Psi(B_n)$ is the signed permutation corresponding to $cn_B(N_1, N_2)$.

**Theorem 1.2.16.** Given two signed permutations $\sigma$ and $\tau$ with type-B noncrossing arc diagrams $N_1 = \delta^\circ(\sigma)$ and $N_2 = \delta^\circ(\tau)$, their join in $\Psi(B_n)$ is the signed permutation corresponding to $mn_B(N_1, N_2)$.

**Embeddings of type A into type B**

Each of the four subsections of Section 3.5 contains the same progression of results for a single surjective lattice homomorphism $\eta$ from $B_n$ to $A_n$. In this overview, we include a subset of the following for each $\eta$: an example illustrating how $\eta$ turns a signed permutation in $B_5$ into a permutation in $S_6 = A_5$, a characterization of the congruence defined by $\eta$ in terms of contracted and uncontracted type-B noncrossing arc diagrams, the outputs of $\zeta$ (the natural inclusion from $A_n$ to $B_n$ corresponding to $\eta$) for the same four permutations in $A_6 = S_7$, the type-A noncrossing arc diagrams corresponding to the four permutations and the type-B noncrossing arc diagrams of $\zeta$’s outputs, and a result that $\zeta$ either does (as in Theorems 3.5.8, 3.5.21 and 3.5.27) or does not (as in Example 3.5.15) embed $\Psi(A_n)$ as a sublattice of $\Psi(B_n)$.
Simion’s homomorphism

The first map we consider is $\eta_0$, which we call Simion’s homomorphism since it was originally presented by Simion in [41]. The following example illustrates how $\eta_0$ turns a signed permutation in $B_n$ into a permutation in $S_{n+1}$. It is also stated as Example 3.5.2.

**Example 1.2.17.** $\eta_0(2(-5)1(-4)(-3)) = 456132$

The arcs corresponding to the contracted join-irreducible signed permutations that generate the congruence defined by $\eta_0$ are pictured in Fig. 3.29, also included here as Fig. 1.31.

The four examples below illustrate how $\zeta_0$ turns a permutation in $A_n = S_{n+1}$ into a signed permutation in $B_n$. They are Examples 3.5.3 to 3.5.6.

**Example 1.2.18.** $\zeta_0(1452736) = 341625$

**Example 1.2.19.** $\zeta_0(4521736) = (-1)(-4)(-3)625$

**Example 1.2.20.** $\zeta_0(4251736) = (-4)(-1)(-3)625$

**Example 1.2.21.** $\zeta_0(4517326) = (-4)(-3)6215$
The following proposition is Proposition 3.5.7.

**Proposition 1.2.22.** The map \( \zeta_0 \) on permutations corresponds to the following operation on noncrossing arc diagrams: For any \( \pi \in A_n = S_{n+1} \), the type-B noncrossing arc diagram \( \delta^0(\zeta_0(\pi)) \) is identical to the type-A noncrossing arc diagram \( \delta(\pi) \), except that the numbered point 1 in \( \delta(\pi) \) is replaced by the orbifold point \( \times \) and the numbered points 2, \ldots, \( n+1 \) are renumbered as 1, \ldots, \( n \).

The operation on noncrossing arc diagrams corresponding to \( \zeta_0 \) for the set of examples above is pictured in Fig. 3.30 and here as Fig. 1.32.

The sublattice \( \zeta_0(\Psi(A_n)) \) of \( \Psi(B_n) \), is pictured in Fig. 3.31 and here as Fig. 1.33.

The following theorem is Theorem 3.5.8.

**Theorem 1.2.23.** \( \zeta_0 \) embeds the shard intersection order on the Coxeter group of type \( A_n \) as a sublattice of the shard intersection order on the Coxeter group of type \( B_n \).

A nonhomogeneous homomorphism

The following example illustrates how \( \eta_{-1} \), which is denoted \( \eta_{\nu} \) in [38], turns a signed permutation in \( B_n \) into a permutation in \( S_{n+1} \). It is also presented in Example 3.5.9.

**Example 1.2.24.** \( \eta_{-1}(2 (-5) 1 (-4) (-3)) = 451632 \)

\[ \begin{align*}
3 & 4 & -1 & 5 & -2 & & 2 & -5 & 1 & -4 & -3 \\
3 & 4 & -1 & 5 & & 2 & & 1 \\
3 & 4 & 0 & 5 & & 2 & & 1 \\
4 & 5 & 1 & 6 & & 3 & & 2
\end{align*} \]

read sequence \( \geq -1 \)

\(-1 \) becomes 0

add one

The arcs corresponding to the contracted join-irreducible signed permutations that generate the congruence defined by \( \eta_{-1} \) are pictured in Fig. 3.32, also included here as Fig. 1.34.

The four examples below illustrate how \( \zeta_{-1} \) turns a permutation in \( A_n = S_{n+1} \) into a signed permutation in \( B_n \). They are Examples 3.5.10 to 3.5.13.
Example 1.2.25. $\zeta_{-1}(1452736) = 341625$

\[
\begin{array}{cccccc}
1 & 4 & 5 & 2 & 7 & 3 & 6 \\
-1 & 3 & 4 & 1 & 6 & 2 & 5 \\
3 & 4 & 1 & 6 & 2 & 5
\end{array}
\]

\[j = 1, i = 4\]

Example 1.2.26. $\zeta_{-1}(4521736) = (-1)(-4)(-3)625$

\[
\begin{array}{cccccc}
4 & 5 & 2 & 1 & 7 & 3 & 6 \\
3 & 4 & 1 & -1 & 6 & 2 & 5 \\
-1 & -4 & -3 & 6 & 2 & 5
\end{array}
\]

\[j = 4, i = 3\]

---

Figure 1.33: The sublattice $\zeta_0(S_4)$ of $\Psi(B_3)$.

Figure 1.34: Contracted arcs that generate the congruence defined by $\eta_{-1}$.
Example 1.2.27. $\zeta_{-1}(4251736) = 4(-1)(-3)625$

\[
\begin{array}{cccccc}
4 & 2 & 5 & 1 & 7 & 3 & 6 \\
3 & 1 & 4 & -1 & 6 & 2 & 5 \\
4 & -1 & -3 & 6 & 2 & 5 \\
\end{array}
\]

$j = 4, i = 2$

Example 1.2.28. $\zeta_{-1}(4517326) = 621(-4)(-3)5$

\[
\begin{array}{cccccc}
4 & 5 & 1 & 7 & 3 & 2 & 6 \\
3 & 4 & -1 & 6 & 2 & 1 & 5 \\
6 & 2 & 1 & -4 & -3 & 5 \\
\end{array}
\]

$j = 3, i = 6$

The following proposition is Proposition 3.5.14.

**Proposition 1.2.29.** The map $\zeta_{-1}$ on permutations corresponds to the following operation on noncrossing arc diagrams: For any $\pi \in A_n = S_{n+1}$, the type-B noncrossing arc diagram $\delta^o(\zeta_{-1}(\pi))$ is identical to the type-A noncrossing arc diagram $\delta(\pi)$, except that the numbered point 1 is replaced by the orbifold point $\times$, the numbered points $2, \ldots, n+1$ are renumbered as 1, $\ldots$, $n$, and if $\delta(\pi)$ contains an arc $\alpha$ which passes left [resp. right] of 2 and has 1 as its lower endpoint, then this arc is replaced by a long arc whose left [resp. right] side agrees with $\alpha$ shifted down by one and whose right [resp. left] endpoint is 1.

The operation on noncrossing arc diagrams corresponding to $\zeta_{-1}$ for the set of examples above is pictured in Fig. 3.33 and here as Fig. 1.35.

The example below is Example 3.5.15, which demonstrates the fact that $\zeta_{-1}$ does not embed $\Psi(A_n)$ as a sublattice of $\Psi(B_n)$. The figure included here and mentioned in the theorem is also included as Fig. 3.35.

**Example 1.2.30.** Consider the permutations $\sigma = 3142$ and $\tau = 3241$. By Theorem 1.2.11, $\sigma \land \tau$ in $\Psi(A_3)$ is the permutation corresponding to the noncrossing arc diagram $\text{cn}(\delta(\sigma), \delta(\tau))$. 

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Figure 1.36: Example showing that $\zeta^{-1}(A_3)$ is not a sublattice of $\Psi(B_3)$.

As shown in the top of Fig. 1.36, the meet of $\sigma$ and $\tau$ is the permutation $1\ 2\ 3\ 4$, the identity in $A_3$. The signed permutation $\zeta^{-1}(\sigma \wedge \tau)$ is the identity in $B_3$, the signed permutation $1\ 2\ 3$.

The images of $\sigma$ and $\tau$ under $\zeta^{-1}$ are the signed permutations $\zeta^{-1}(\sigma) = 3\ 1\ (-2)$ and $\tau = 3\ (-1)\ (-2)$. By Theorem 1.2.15, $\zeta^{-1}(\sigma) \wedge \zeta^{-1}(\tau)$ in $\Psi(B_3)$ is the signed permutation corresponding to $\text{cn}_B(\delta^o(\zeta^{-1}(\sigma)), \delta^o(\zeta^{-1}(\tau)))$. As shown in the bottom of Fig. 1.36, the meet of $\zeta^{-1}(\sigma)$ and $\zeta^{-1}(\tau)$ is the signed permutation $1\ 3\ (-2)$, which is strictly above the identity in $\Psi(B_n)$. The signed permutation $1\ 3\ (-2)$ is not in the image of $\zeta^{-1}$, since both endpoints of the long arc in $\delta^o(1\ 3\ (-2))$ are above 1.

Two more homogeneous homomorphisms

The two remaining homomorphisms denoted $\eta_\delta$ and $\eta_\epsilon$ are hybrids of $\eta_0$ and $\eta_{-1}$. The difference between the two maps is that $\eta_\delta$ behaves as $\eta_0$ in exactly the cases where $\eta_\epsilon$ behaves as $\eta_{-1}$ and vice versa. This relationship is illustrated succinctly by the symmetry between Figs. 1.37 and 1.40, so we exclude several results for $\eta_\epsilon$ and $\zeta_\epsilon$ since the symmetry of the two figures can be safely extrapolated.

A hybrid map, $\eta_\delta$

The arcs corresponding to the contracted join-irreducible signed permutations that generate the congruence defined by $\zeta_\delta$ are pictured in Fig. 3.38, also included here as Fig. 1.39.

The four examples below illustrate how $\zeta_\delta$ turns a permutation in $A_n = S_{n+1}$ into a signed permutation in $B_n$. They are Examples 3.5.16 to 3.5.19.
Figure 1.37: Contracted arcs that generate the congruence defined by $\eta_\delta$.

Example 1.2.31. $\zeta_\delta(1452736) = 3 4 1 6 2 5$

```
1 4 5 2 7 3 6
0 3 4 1 6 2 5
```

Example 1.2.32. $\zeta_\delta(4521736) = (-1)(-4)(-3)625$

```
4 5 2 1 7 3 6
3 4 1 -1 6 2 5
-1 -4 -3 6 2 5
```

Example 1.2.33. $\zeta_\delta(4251736) = 4(-1)(-3)625$

```
4 2 5 1 7 3 6
3 1 4 -1 6 2 5
4 -1 -3 6 2 5
```

Example 1.2.34. $\zeta_\delta(4517326) = (-4)(-3)6215$

```
4 5 1 7 3 2 6
3 4 0 6 2 1 5
-4 -3 6 2 1 5
```

The following proposition is Proposition 3.5.20.

**Proposition 1.2.35.** The map $\zeta_\delta$ on permutations corresponds to the following operation on noncrossing arc diagrams: For any $\pi \in A_n = S_{n+1}$, the type-B noncrossing arc diagram $\delta^\circ(\zeta_\delta(\pi))$ is identical to the type-A noncrossing arc diagram $\delta(\pi)$, except that the numbered point 1 is replaced by the orbifold point $\times$, the numbered points 2, $\ldots$, $n+1$ are renumbered as 1, $\ldots$, $n$, and if $\delta(\pi)$ has an arc $\alpha$ which passes right of 2 and has 1 as its lower endpoint, then this arc is replaced by a long arc whose right side agrees with $\alpha$ shifted down by one and whose left endpoint is 1.
Figure 1.38: $\zeta_\delta$ on noncrossing arc diagrams.

Figure 1.39: The sublattice $\zeta_\delta(S_4)$ of $\Psi(B_3)$. 
Figure 1.40: Contracted arcs that generate the congruence defined by $\eta_\epsilon$.

The operation on noncrossing arc diagrams corresponding to $\zeta_\delta$ for the set of examples above is pictured in Fig. 3.37 and here as Fig. 1.38.

The sublattice $\zeta_\delta(\Psi(A_n))$ of $\Psi(B_n)$, is pictured in Fig. 3.38 and here as Fig. 1.39.

The following theorem is Theorem 3.5.21.

**Theorem 1.2.36.** $\zeta_\delta$ embeds the shard intersection order on the Coxeter group of type $A_n$ as a sublattice of the shard intersection order on the Coxeter group of type $B_n$.

A second hybrid map, $\eta_\epsilon$.

The arcs corresponding to the contracted join-irreducible signed permutations that generate the congruence defined by $\zeta_\epsilon$ are pictured in Fig. 3.39, also included here as Fig. 1.40.

The following theorem is Theorem 3.5.27.

**Theorem 1.2.37.** $\zeta_\epsilon$ embeds the shard intersection order on the Coxeter group of type $A_n$ as a sublattice of the shard intersection order on the Coxeter group of type $B_n$. 
2.1 Introduction

Noncrossing arc diagrams, introduced in [35], are combinatorial objects in bijection with permutations that bring to light lattice-theoretic information about the weak order on permutations (such as canonical join representations and lattice congruences) and related discrete-geometric information (such as shards and shard intersections). The results of [35] include:

- An explicit bijection from permutations to noncrossing arc diagrams and an explicit inverse;
- a bijection from arcs to join-irreducible elements of the weak order;
- a bijection from arcs to shards;
- a characterization of compatibility of join-irreducible elements (in the sense of canonical join complexes);
• a construction of the canonical join complex of the weak order in a way that proves that this complex is flag; and

• a characterization of lattice congruences on the weak order and their quotients.

In this chapter, we construct models analogous to noncrossing arc diagrams for Coxeter groups of type B (signed permutations) and type D (even-signed permutations) and prove the analogous results. (The theorem on flagness of the canonical join complex was already vastly generalized by Barnard in [5], so we don’t state it again here for Coxeter groups of types B and D.)

The first challenge in constructing models in types B and D is to choose the correct combinatorial object. After that, bijections to elements of the Coxeter groups, bijections to join-irreducible elements and to shards, and characterization of canonical join complexes are all relatively straightforward. Significantly more challenging is the combinatorial characterization of lattice congruences and quotients, particularly in type D.

The resulting characterization of congruences in type B is a great improvement over the earlier state of the art [31, Theorem 7.3], which is usable (e.g. in [32, 38]) but clunky. The improvement is in two directions. First, we give a direct one-step characterization of forcing relations among join-irreducible congruences, rather than only characterizing the arrows whose transitive closure is the forcing relation. Second, the description of the entire forcing relation is much simpler even than the earlier description of single arrows.

The characterization of congruences in type D is the first of its kind. Although the proof of the type-D characterization is quite complicated, the payoff is that much of the complication can be left behind in the proof, so that the resulting tool is again a one-step characterization, and so is not prohibitively complicated.

The type-B model comes in two equivalent versions, analogous to the description of signed permutations as sequences $\pi_{-n} \pi_{-n+1} \cdots \pi_{-1} \pi_1 \pi_2 \cdots \pi_n$ or as sequences $\pi_1 \pi_2 \cdots \pi_n$. In the first version, signed permutations of $\{\pm 1, \ldots, \pm n\}$ are in bijection with centrally symmetric noncrossing arc diagrams on $2n$ points. (See Fig. 2.3.) This version has the advantage that the comparison with type-A noncrossing arc diagrams is transparent. Parts of this model have appeared in [6, 27]. Notably, the latter paper includes the characterization of lattice congruences and a construction of quotientopes (polytopes realizing lattice quotients of the weak order in the same way the permutohedron realizes the weak order [29]). The second version of the type-B model passes to the quotient of centrally symmetric noncrossing diagrams, modulo the symmetry, to obtain an orbifold model. (See Fig. 2.8.) In addition to the obvious advantage of spatial compactness, the orbifold version brings to light some more subtle and surprising connections with type-A diagrams: In Chapter 3, we characterize
the meet and join operations in the shard intersection orders of types A and B in terms of the usual noncrossing arc diagrams and in terms of the orbifold model and uses these characterizations to show that the shard intersection order of type $A_n$ (permutations in $S_{n+1}$) embeds as a sublattice of the shard intersection order of type $B_n$.

The type-D model also has two versions. In one version, type-D arcs are equivalence classes of arcs in the type-B orbifold model. Most classes are singletons, but some have two elements. In the other version, the type-D arcs look less like arcs of type A or B, and come in three types: ordinary (like type A), partially doubled, and branched. (See Fig. 2.12.) It would be interesting to make the connection between type-D arcs and the characterization of join-irreducible elements of the type-D weak order produced in a representation-theory context in [23, Section 6.2] and [2]. It also seems possible that the existence of a combinatorial model for lattice congruences of the weak order in type D will allow the construction of quotientopes [29, 27] to be extended to type D.

2.2 Preliminaries

In this section, we establish background on lattice theory, finite Coxeter groups, posets of regions, and shards. We assume the most basic notions, particularly for lattice theory and Coxeter groups. In Section 2.2.3, we prove a new characterization of shard arrows, which provides a convenient tool to study forcing among congruences in the case of lattices of regions of a simplicial arrangement.

2.2.1 Lattices and congruences

An element $j$ of a finite lattice $L$ is called join-irreducible if and only if it covers exactly one element. We write $j_\ast$ for the unique element covered by $j$.

The canonical join representation (CJR) of an element $x$ in a finite lattice $L$ is the unique antichain $X$ in $L$ such that $x = \bigvee X$ and such that, if $X'$ is any subset of $L$ with $x = \bigvee X'$, the order ideal generated by $X'$ contains the order ideal generated by $X$. In particular, $X$ consists of join-irreducible elements called the canonical joinands of $x$. An element $x$ may or may not have a canonical join representation.

The property that every element of a finite lattice $L$ has a canonical join representation is equivalent to a property called join-semidistributivity [19, Theorem 2.24]. We will not need the usual definition of join-semidistributivity here, but for finite lattices, one can profitably take the existence of canonical join representations as the definition. If both join-semidistributivity and the dual condition hold, then $L$ is called semidistributive.
A natural question, given a finite join-semidistributive lattice \( L \), is to characterize which subsets of \( L \) are canonical join representations of elements of \( L \). Thus the question is to characterize the collection \( \{ X \subseteq L : X \text{ is the CJR of } \bigvee X \} \). This collection is called the canonical join complex, because it is a simplicial complex (with vertex set the set of join-irreducible elements).

A simplicial complex is flag if each minimal non-face is a 2-element set. (Equivalently, a set \( X \) of vertices forms a face if and only if every 2-element subset of \( X \) is an edge.) The following theorem is part of [5, Theorem 2].

**Theorem 2.2.1.** Suppose \( L \) is a finite join-semidistributive lattice. The canonical join complex of \( L \) is flag if and only if \( L \) is semidistributive.

Theorem 2.2.1 is extremely important for understanding the combinatorics of finite semidistributive lattices. Given a finite semidistributive lattice \( L \), we say that two join-irreducible elements \( j \) and \( j' \) of \( L \) are compatible if and only if \( \{ j, j' \} \) is a face of the canonical join complex. (That is, if and only if there is an element whose canonical join representation is \( \{ j, j' \} \).) Equivalently, if and only if there is an element whose canonical join representation contains \( \{ j, j' \} \). Since, in particular, \( L \) is join-semidistributive (so that every element has a canonical join representation), the elements of \( L \) are in bijection with the faces of the canonical join complex. But then since \( L \) is semidistributive, Theorem 2.2.1 implies that the faces of the canonical join complex are precisely the pairwise compatible sets of join-irreducible elements of \( L \). In later sections, we will give a bijection between join-irreducible elements of the weak order and various kinds of “arcs”, define notions of compatibility of arcs that correspond to compatibility of join-irreducible elements, and thus show that elements of the Coxeter group are in bijection with pairwise compatible sets of arcs.

A congruence on a lattice \( L \) is an equivalence relation \( \Theta \) such that if \( x_1 \equiv x_2 \) and \( y_1 \equiv y_2 \) modulo \( \Theta \), then \( (x_1 \land y_1) \equiv (x_2 \land y_2) \) and \( (x_1 \lor y_1) \equiv (x_2 \lor y_2) \). The quotient \( L/\Theta \) of \( L \) modulo \( \Theta \) is the lattice whose elements are the \( \Theta \)-classes \([x]_\Theta \) and whose meet and join are given by \([x]_\Theta \land [y]_\Theta = [x \land y]_\Theta \) and \([x]_\Theta \lor [y]_\Theta = [x \lor y]_\Theta \).

Congruences and quotients have a nice order-theoretic description as well. An equivalence relation \( \Theta \) on a finite lattice \( L \) is a lattice congruence if and only if (1) each equivalence class is an interval, (2) the map sending an element to the bottom of its equivalence class is order-preserving, and (3) the map sending an element to the top of its equivalence class is order-preserving. The quotient \( L/\Theta \) is isomorphic to the subposet induced by the set of bottom elements of congruence classes. Said another way: An element \( x \in L \) is said to be contracted by \( \Theta \) if \( x \) is congruent to some element strictly less than \( x \). The quotient \( L/\Theta \) is isomorphic to the subposet of \( L \) induced by uncontracted elements.
A join-irreducible element $j$ is contracted by $\Theta$ if and only if $j \equiv j_*$ modulo $\Theta$. It is well known that a congruence is completely determined by the set of join-irreducible elements it contracts. (For a precise statement, see [37, Theorem 9-5.12].) Thus, to completely characterize congruences on a given lattice, it suffices to characterize which sets of join-irreducible elements can be contracted by congruences. In this chapter, we carry out this characterization in the case of the weak order on Coxeter groups of types B and D.

Canonical join representations behave well when passing to quotients.

**Proposition 2.2.2.** [37, Proposition 10-5.29] Suppose $L$ is a finite join-semidistributive lattice and $\Theta$ is a congruence on $L$. Then an element $x \in L$ is contracted by $\Theta$ if and only if one or more of its canonical joinands is contracted by $\Theta$. If $x$ is not contracted by $\Theta$, then its canonical join representation in the quotient $L/\Theta$ coincides with its canonical join representation in $L$.

In the proposition, the quotient $L/\Theta$ is realized, as before, as the subposet of $L$ induced by the elements of $L$ that are not contracted by $\Theta$. The second assertion of the proposition implies, in particular, that the join-irreducible elements of the quotient are precisely the join-irreducible elements of $L$ that are not contracted by $\Theta$.

The following corollary is an immediate consequence of Proposition 2.2.2. (Recall that the vertices of the canonical join complex of $L$ are the join-irreducible elements of $L$.)

**Corollary 2.2.3.** Suppose $L$ is a finite join-semidistributive lattice and $\Theta$ is a congruence on $L$. Then the canonical join complex of $L/\Theta$ is the subcomplex of the canonical join complex of $L$ induced by the join-irreducible elements of $L$ not contracted by $\Theta$.

Join-irreducible elements cannot be contracted independently. Instead, there is a partial (pre-)order that mediates which sets of join-irreducibles can be contracted by a congruence. We say that a join-irreducible element $j_1$ forces another join-irreducible element $j_2$ if every congruence that contracts $j_1$ also contracts $j_2$. In general, the forcing relation is a pre-order (a reflexive, transitive, but not necessarily antisymmetric relation) on the set of join-irreducible elements.

### 2.2.2 Coxeter groups and the weak order

Let $(W, S)$ be a Coxeter system. We write $\ell(w)$ for the usual length function, the length of $w$ in the alphabet $S$. A (left) inversion of an element $w \in W$ is a reflection $t$ such that $\ell(tw) < \ell(w)$. Let $^{-1}(w)$ denote the set of inversions of $w$. We write “$\leq$” for the (right) weak order on $W$, which is the partial order on $W$ with $u \leq w$ if and only if $\text{inv}(u) \subseteq \text{inv}(w)$. The
cover relations in the weak order are precisely the relations \(ws < w\) such that \(w \in W, s \in S\), and \(\ell(ws) < \ell(w)\).

Given an element \(w\) of a Coxeter group \(W\), a \textit{cover reflection} of \(w\) is a reflection \(t\) such that \(tw < w\). Cover reflections of \(w\) are in bijection with elements covered by \(w\): If \(ws < w\), then \(ws w^{-1}\) is a cover reflection. Let \(\text{cov}(w)\) denote the set of cover reflections of \(w\).

The weak order on a finite Coxeter group is a semidistributive lattice [26, Lemme 9].

**Theorem 2.2.4.** [36, Theorem 10-3.9] Suppose \(W\) is a finite Coxeter group and \(w \in W\). For each \(t \in \text{cov}(w)\), there is a unique minimal element \(j_t\) in \(\{v : v \leq w, t \in t^{-1}(v)\}\). The canonical join representation of \(w\) is \(w = \bigvee \{j_t : t \in \text{cov}(w)\}\).

The weak order on a finite Coxeter group has a property called congruence uniformity [15], which in particular implies that the forcing relation is a partial order. In later sections, we describe this forcing order on join-irreducible elements of the weak order of types A, B, and D in terms of “subarc” relationships on arcs.

### 2.2.3 Shards, congruences, and forcing

We now briefly recall some discrete-geometric notions that can be used to understand lattice congruences of the weak order, and prove a characterization of forcing that is useful in describing the forcing order on join-irreducible elements. In fact, we prove the characterization in the more general setting of lattices of regions of simplicial arrangements. For more details on the background material, see [37, 36].

A \textit{(real, central) hyperplane arrangement} is a finite collection \(\mathcal{A}\) of linear hyperplanes in \(\mathbb{R}^d\). (We us the adjective “real” because hyperplane arrangements in vector spaces over other fields are often of interest. The adjective “central” emphasizes that our hyperplanes are linear, as opposed to affine hyperplanes which might not contain the origin.) The \textit{regions} of \(\mathcal{A}\) are the closures of the connected components of \(\mathbb{R}^d \setminus \bigcup \mathcal{A}\). Choosing a \textit{base region} \(B\), we define the \textit{separating set} \(S(R)\) of a region \(R\) to be the set of hyperplanes in \(\mathcal{A}\) that separate the interior of \(R\) from the interior of \(B\). The \textit{poset of regions} \(\mathcal{P}(\mathcal{A}, B)\) is the set of regions, partially ordered by setting \(Q \leq R\) if and only if \(S(Q) \subseteq S(R)\).

A hyperplane arrangement is \textit{simplicial} if every region is a simplicial cone (the nonnegative linear span of a linearly independent set of vectors). Semidistributivity in the following theorem is [30, Theorem 3]. The lattice property was proven earlier in [11, Theorem 3.4].

**Theorem 2.2.5.** If \(\mathcal{A}\) is a simplicial hyperplane arrangement, then \(\mathcal{P}(\mathcal{A}, B)\) is a semidistributive lattice.
The set \( A_W \) of reflecting hyperplanes of a Coxeter group \( W \) (in the usual reflection representation) constitute the Coxeter arrangement associated to \( W \). Choosing \( B \) to be a region bounded by the reflecting hyperplanes for the simple reflections \( S \), the map \( w \mapsto wB \) is an isomorphism from the weak order on \( W \) to the poset of regions \( \mathcal{P}(A_W, B) \). It is well known that the Coxeter arrangements are simplicial. (See [36, Theorem 10.2.1] and the citation notes at the end of that chapter.) Thus Theorem 2.2.5 in particular implies the semidistributivity of the weak order, already mentioned in Section 2.2.2.

A rank-two subarrangement of an arrangement \( A \) is a subset \( A' \) of \( A \), with \( |A'| > 1 \) that can be described as \( A' = \{ H \in A : H \supset U \} \) for some codimension-2 linear subspace \( U \). Any two distinct hyperplanes \( H_1, H_2 \in A \) are contained in a unique rank-two subarrangement, namely \( \{ H \in A : H \supset (H_1 \cap H_2) \} \).

Suppose that we have fixed a base region \( B \) as in the definition of \( \mathcal{P}(A, B) \). Then a rank-two subarrangement has two distinguished hyperplanes called its basic hyperplanes. The subarrangement cuts \( \mathbb{R}^d \) into regions, and \( B \) is contained in one of these regions, call it \( B' \). The basic hyperplanes of \( A' \) are the two hyperplanes that bound \( B' \).

A hyperplane \( H_1 \in A \) cuts a hyperplane \( H_2 \in A \) if \( H_1 \) is basic in the rank-two subarrangement \( A' \) containing \( H_1 \) and \( H_2 \), and \( H_2 \) is not basic in \( A' \). Given \( H \in A \), the shards in \( H \) are the closures of connected components of \( \{ H \setminus H' : H' \text{ cuts } H \} \). The set of shards of \( A \) is the set of all shards in all hyperplanes in \( A \). Thus to make the shards of \( A \), we “slice” each hyperplane along all of its intersections with hyperplanes that cut it. We emphasize that the construction of shards depends on the choice of base region \( B \).

Given a shard \( \Sigma \), we write \( H_\Sigma \) for the hyperplane containing \( \Sigma \). Suppose \( R \) is a region of \( A \) and \( \Sigma \) is a shard of \( A \). If \( R \cap \Sigma \) is a facet of \( R \) and \( H_\Sigma \in S(R) \), then \( R \) is an upper region of \( \Sigma \) and \( \Sigma \) is a lower shard of \( R \).

**Proposition 2.2.6.** [31, Propositions 3.2, 3.5] If \( A \) is a simplicial arrangement, then each shard \( \Sigma \) has a unique minimal upper region \( J_\Sigma \). The region \( J_\Sigma \) is join-irreducible in \( \mathcal{P}(A, B) \) and is the unique join-irreducible upper region of \( \Sigma \). The map \( \Sigma \mapsto J_\Sigma \) is a bijection from the set of shards of \( A \) to the set of join-irreducible elements of \( \mathcal{P}(A, B) \). The inverse map takes a join-irreducible region \( J \) to its unique lower shard.

**Theorem 2.2.7.** [33, Theorem 3.6] If \( A \) is a simplicial arrangement and \( R \) is a region, then the canonical join representation of \( R \) in \( \mathcal{P}(A, B) \) is \( \{ J_\Sigma : \Sigma \text{ is a lower shard of } R \} \).

Recall from Section 2.2.1 that two join-irreducible elements \( j_1, j_2 \) of a semidistributive lattice \( L \) are compatible if and only if there exists an element of \( L \) whose canonical join representation is \( \{ j_1, j_2 \} \) if and only if there exists an element of \( L \) (not necessarily the same
element) whose canonical join representation contains \( \{j_1, j_2\} \). We will say that two shards are \textit{compatible} if and only if the intersection of their relative interiors is nonempty.

**Proposition 2.2.8.** Suppose \( \mathcal{A} \) is a simplicial arrangement. Two shards are compatible if and only if the two corresponding join-irreducible regions in \( \mathcal{P}(\mathcal{A}, B) \) are compatible.

**Proof.** By Theorem 2.2.7, two join-irreducible regions \( J_1 \) and \( J_2 \) in a simplicial poset of regions are compatible if and only if \( \Sigma_{J_1} \) and \( \Sigma_{J_2} \) are the two lower shards of some region \( R \). In that case, the hyperplanes containing \( \Sigma_{J_1} \) and \( \Sigma_{J_2} \) are the two basic hyperplanes in the rank-two subarrangement containing them, so they don’t cut each other. Since their intersection has codimension 2 (because it contains the intersection of two facets of the simplicial region \( R \)), we see that the relative interiors of \( \Sigma_{J_1} \) and \( \Sigma_{J_2} \) have nonempty intersection. Conversely, if there are two shards \( \Sigma_1 \) and \( \Sigma_2 \) whose relative interiors have nonempty intersection, then there is a region \( Q \) having both as lower shards. (Such a region \( Q \) can be found by starting at a generic point in the intersection of the two relative interiors and moving a small distance in the direction away from the interior of \( B \).) Then Theorem 2.2.7 says that the corresponding join-irreducible regions are in the canonical join representation of \( Q \), and thus are compatible. \( \square \)

We have seen that shards and their incidences encode the canonical join representations of regions in \( \mathcal{P}(\mathcal{A}, B) \). We will also see that they encode the forcing (pre-)order on join-irreducible elements in \( \mathcal{P}(\mathcal{A}, B) \). Define the \textit{shard digraph} to be the directed graph whose vertices are the shards, with \( \Sigma_1 \rightarrow \Sigma_2 \) if and only if \( H_{\Sigma_1} \) cuts \( H_{\Sigma_2} \) and \( \Sigma_1 \cap \Sigma_2 \) has codimension 2.

**Theorem 2.2.9.** [37, Theorem 9-7.17] If \( \mathcal{A} \) is simplicial and \( \Sigma_1 \) and \( \Sigma_2 \) are shards, then \( J_{\Sigma_1} \) forces \( J_{\Sigma_2} \) if and only if there is a directed path in the shard digraph from \( \Sigma_1 \) to \( \Sigma_2 \).

Said another way, the map \( \Sigma \mapsto J_\Sigma \) is an isomorphism from the reflexive-transitive closure of the shard digraph to the forcing (pre-)order defined in Section 2.2.1. We emphasize that, even when the forcing pre-order is a partial order, shard arrows are not necessarily cover relations in the forcing order. Instead, there may be pairs of shards that are related by a shard arrow and also by a longer path in the shard digraph.

We now state and prove the main theorem of this section, a technical result that rephrases the definition of the shard digraph in a way that is useful (in later sections) for describing forcing in terms of “subarc” relationships between arcs.

**Theorem 2.2.10.** Suppose \( \mathcal{A} \) is a simplicial hyperplane arrangement and suppose \( \Sigma_1 \) and \( \Sigma_2 \) are shards. Then \( \Sigma_1 \rightarrow \Sigma_2 \) in the shard digraph if and only if there exists a shard \( \Sigma'_1 \) satisfying the following conditions:
(i) $\Sigma_1$ and $\Sigma'_1$ are compatible,

(ii) $H_{\Sigma_2}$ is in the rank-two subarrangement containing $H_{\Sigma_1}$ and $H_{\Sigma'_1}$ but is not basic in that subarrangement, and

(iii) $\Sigma_1 \cap \Sigma'_1 \subseteq \Sigma_2$.

Note that the compatibility of $\Sigma_1$ and $\Sigma'_1$ in (i) implies that there is a region having both $\Sigma_1$ and $\Sigma'_1$ as lower shards. In particular, $H_{\Sigma_1}$ and $H_{\Sigma'_1}$ are basic in the rank-two subarrangement containing them.

A shard intersection is an intersection of a set of shards. The shard intersection order is the reverse containment order on the set of all shard intersections. The proof of Theorem 2.2.10 uses the fact that the reverse containment order on the set of all intersections of shards is graded by codimension [33, Proposition 5.1].

Proof of Theorem 2.2.10. Suppose $\Sigma_1 \to \Sigma_2$. Write $H_1$ for $H_{\Sigma_1}$ and $H_2$ for $H_{\Sigma_2}$. Then in particular $H_1$ is basic in the rank-two subarrangement $\mathcal{A}'$ containing $H_1$ and $H_2$, but $H_2$ is not basic in $\mathcal{A}'$. Let $H'_1$ be the other basic hyperplane in $\mathcal{A}'$. Since $\Sigma_1 \cap \Sigma_2$ has codimension 2 and is contained in the intersection of the hyperplanes of $\mathcal{A}'$, we can find a point $x \in \Sigma_1 \cap \Sigma_2$ that is not contained in any hyperplane in $\mathcal{A} \setminus \mathcal{A}'$. Thus, since $H_2$ does not cut $H_1$, the point $x$ is in the relative interior of $\Sigma_1$. Since $H_2$ also does not cut $H'_1$, $x$ is also in the relative interior of some shard $\Sigma'_1$ in $H'_1$. By definition, $\Sigma_1$ and $\Sigma'_1$ are compatible. Since $x$ is in the relative interior of $\Sigma'_1$, there is some ball about $x$ whose intersection with $\Sigma'_1$ is the same as its intersection with $H'_1$. Since $H'_1$ contains $\Sigma_1 \cap \Sigma_2$, we see that $\Sigma_1 \cap \Sigma'_1 \cap \Sigma_2$ also has codimension 2. Since $H_1$ and $H'_1$ are not the same hyperplane, $\Sigma_1 \cap \Sigma'_1$ has codimension 2 as well. Since the shard intersection order is graded by codimension and $\Sigma_1 \cap \Sigma'_1 \cap \Sigma_2 \subseteq \Sigma_1 \cap \Sigma'_1$, these two intersections must in fact be the same. Thus $\Sigma_1 \cap \Sigma'_1 \subseteq \Sigma_2$.

Conversely, suppose there exists $\Sigma'_1$ such that the three conditions of the theorem hold. Then certainly $H_{\Sigma_1}$ cuts $H_{\Sigma_2}$, so we must show that $\Sigma_1 \cap \Sigma_2$ has codimension 2. Since $H_{\Sigma_1} \neq H_{\Sigma_2}$, certainly the codimension of $\Sigma_1 \cap \Sigma_2$ is at least 2. But the fact that $\Sigma_1$ and $\Sigma'_1$ are compatible means that they intersect in their relative interiors, so $\Sigma_1 \cap \Sigma'_1$ has codimension 2. But since $\Sigma_1 \cap \Sigma'_1 \subseteq \Sigma_2$, we have $\Sigma_1 \cap \Sigma'_1 = \Sigma_1 \cap \Sigma'_1 \cap \Sigma_2$. In particular, the codimension of $\Sigma_1 \cap \Sigma_2$ cannot be more than 2.

The third condition in Theorem 2.2.10 can be restated as the existence of a certain order relation in the shard intersection order (in the sense of [33]). In this paper, we will use the third condition directly, without explicitly working with the shard intersection order.
2.3 Noncrossing arc diagrams of type A

In this section, we review results of [35] on noncrossing arc diagrams for permutations in \( S_n \), and describe how Theorem 2.2.10 applies to the proof of one result from [35].

The Coxeter group of type \( A_{n-1} \) can be realized as the group of permutations of \( \{1, \ldots, n\} \). The one-line notation of \( \pi \in S_n \) is the sequence \( \pi_1 \pi_2 \cdots \pi_n \) where \( \pi_i = \pi(i) \). One can also realize \( S_n \) as a reflection group in \( \mathbb{R}^n \) in the usual way, with each simple reflection \( s_i = (i \ i+1) \) acting as a reflection orthogonal to \( e_{i+1} - e_i \). (The \( e_i \) are the standard basis vectors.) The reflections in \( S_n \) are the transpositions \( (i \ j) \) for \( 1 \leq i < j \leq n \).

The weak order on \( S_n \) has cover relations given by \( \pi \preceq \sigma \) if \( \sigma \) is obtained from \( \pi \) by exchanging the entries \( \pi_i \) and \( \pi_{i+1} \) for some \( i \in \{1, \ldots, n-1\} \) such that \( \pi_i > \pi_{i+1} \). The cover reflection of \( \sigma \) associated to this cover is \( (\pi_i \ \pi_{i+1}) \), and multiplying \( \pi \) on the left by the reflection swaps the entries \( \pi_i \) and \( \pi_{i+1} \). A join-irreducible element of \( S_n \) is a permutation whose one-line notation has exactly one descent \( \pi_i > \pi_{i+1} \).

We now define noncrossing arc diagrams. We place \( n \) distinct points on a vertical line, identified with the numbers \( 1, \ldots, n \) from bottom to top. An arc is a curve connecting a point \( q \in \{1, \ldots, n\} \) to a strictly lower point \( p \in \{1, \ldots, n\} \), moving monotone downwards from \( q \) to \( p \) without touching any other numbered point, but rather passing to the left of some points and to the right of others. A noncrossing arc diagram is a collection of arcs that don’t intersect, except possibly at their endpoints, such that no two arcs share the same upper endpoint or the same lower endpoint. The combinatorial data determining an arc consists of which pair of points it connects and which points in between are left or right of the arc. Two arcs are combinatorially equivalent if they have the same combinatorial data. We consider arcs and noncrossing arc diagrams up to combinatorial equivalence. When we need to distinguish these diagrams from the objects defined later (for Coxeter groups of types B and D), we will refer to them as type-A arcs and type-A noncrossing arc diagrams.

Noncrossing arc diagrams can also be understood in terms of a compatibility relation on arcs. We say two arcs are compatible if they don’t intersect except possibly at one common endpoint, and if they don’t share the same upper endpoint or the same lower endpoint. A noncrossing arc diagram is the same thing as a set of pairwise compatible arcs. (Certainly the arcs in a noncrossing arc diagram are pairwise compatible, and [35, Proposition 3.2] verifies that any collection of pairwise compatible arcs is combinatorially equivalent to some noncrossing arc diagram.)

We now describe the bijection \( \delta \) from \( S_n \) to the set of noncrossing arc diagrams on \( n \) points. Given \( \pi = \pi_1 \cdots \pi_n \in S_n \), write each entry \( \pi_i \) at the point \( (i, \pi_i) \) in the plane. For every \( i \) such that \( \pi_i > \pi_{i+1} \), draw a straight line segment from \( \pi_i \) to \( \pi_{i+1} \). These line segments
become arcs: We move the numbers 1, . . . , n horizontally to put them into a single vertical line, allowing the line segments to curve, so that they avoid passing through any numbers and one another. We define \( \delta(\pi) \) to be the resulting noncrossing arc diagram.

We can alternatively describe \( \delta(\pi) \) by listing its arcs: For every descent \( \pi_i > \pi_{i+1} \), there is an arc with endpoints \( \pi_i \) and \( \pi_{i+1} \). This arc goes right of every entry \( \pi_j \) with \( \pi_{i+1} < \pi_j < \pi_i \) and \( j < i \) and left of every entry \( \pi_j \) with \( \pi_{i+1} < \pi_j < \pi_i \) and \( j > i \).

**Theorem 2.3.1.** [35, Theorem 3.1] The map \( \delta \) is a bijection from \( S_n \) to the set of noncrossing arc diagrams on \( n \) points.

The proof of [35, Theorem 3.1] includes an explicit description of the inverse map, which we quote here. (An example is shown in Table 2.1.) A noncrossing arc diagram has one or more **components** (the components of the diagram, viewed as an embedded graph). Each component is a single numbered point or a sequence of arcs sharing endpoints. Reading each component from top to bottom, we recover the maximal descending runs of the permutation. The noncrossing arc diagram has at least one **left component**, meaning a component “with nothing to its left”. More formally, no other arc or numbered point can be reached from that component by moving horizontally to the left. The left components can be totally ordered from lowest to highest. One can obtain \( \pi \) from \( \delta(\pi) \) recursively by taking the lowest left component, writing its numbered points in decreasing order and then deleting it from the noncrossing arc diagram. Recursively, we then remove the lowest left component of what remains and continue writing the one-line notation for \( \pi \) from left to right.

Since \( \delta \) maps a permutation in \( S_n \) with \( k \) descents to a noncrossing arc diagram with \( k \) arcs, in particular \( \delta \) restricts to a bijection from permutations that are join-irreducible in the weak order to single arcs. Suppose \( \alpha \) is an arc connecting point \( q \) and \( p \) with \( q > p \). Write \( L(\alpha) \) for the set of **left points of** \( \alpha \) (points in the interval \((p, q)\) that are to the left of \( \alpha \)) and write \( R(\alpha) \) for the set of **right points of** \( \alpha \) (points in the interval \((p, q)\) that are to the right of \( \alpha \)). Then the join-irreducible element corresponding to \( \alpha \) is the permutation that starts with \( 1 \cdots (p-1) \), followed by the elements of \( L(\alpha) \) in increasing order, then \( q \), then \( p \), then \( \pi \).
Table 2.1: The map from type-A noncrossing arc diagrams to permutations.

<table>
<thead>
<tr>
<th>Step</th>
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<td>386752</td>
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</tbody>
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Diagram remaining

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then the elements of $R(\alpha)$ in increasing order, and finally $(q + 1) \cdots n$.

Join-irreducible elements of the weak order are in bijection with shards, and thus shards are in bijection with arcs. Combining [36, Proposition 10-5.8] with the bijection described above between arcs and join-irreducible permutations, we can write the bijection between arcs and shards as follows.

**Proposition 2.3.2.** Suppose $\alpha$ is an arc with endpoints $q > p$. The shard associated to $\alpha$ is

$$\{ x \in \mathbb{R}^n : x_p = x_q \text{ and } x_p \leq x_i \forall i \in R(\alpha) \text{ and } x_p \geq x_i \forall i \in L(\alpha) \}.$$ 

Noncrossing arc diagrams record canonical join representations of permutations, as explained in [35, Section 3] and summarized in the following theorem.

**Theorem 2.3.3.** Given a permutation $\pi \in S_n$, the canonical join representation of $\pi$ is the set of join-irreducible elements corresponding to the set of arcs in $\delta(\pi)$.

Recall that two join-irreducible elements in a semidistributive lattice are called compatible if and only if they can appear together in a canonical join representation. As a consequence of Theorem 2.3.3, two join-irreducible elements of $S_n$ are compatible if and only if the corresponding arcs are compatible.

We now use Theorem 2.2.10 to recover a characterization of the forcing relation on join-irreducible elements in terms of subarcs, which we now define. As an aid to defining subarcs, we define a function $h$ from the plane to $\mathbb{R}$ that returns the vertical height of a point, calibrated so that each numbered point $i$ has height $i$. 

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An arc $\alpha'$ with endpoints $q' > p'$ is a **subarc** of an arc $\alpha$ with endpoints $q > p$ if and only if $q \geq q' > p' \geq p$ and $R(\alpha') = R(\alpha) \cap (p', q')$.

**Remark 2.3.4.** It will be convenient in later generalizations to construct subarcs explicitly. Given an arc $\alpha$, parametrized as a function from the interval $[0, 1]$ into the plane, we obtain a subarc $\alpha'$ of $\alpha$ as follows: Choose $t_1$ and $t_2$ with $0 \leq t_1 < t_2 \leq 1$ such that $h(\alpha(t_1)) \in \{1, \ldots, n\}$ and $h(\alpha(t_2)) \in \{1, \ldots, n\}$. Also choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $|h(\alpha(t_1 + \epsilon_1)) - h(\alpha(t_1))| < 1, |h(\alpha(t_2 - \epsilon_2)) - h(\alpha(t_2))| < 1$, and $t_1 + \epsilon_1 < t_2 - \epsilon_2$. Define $\alpha'$ to be the arc obtained by concatenating three curves: First, the straight line segment from the point numbered $h(\alpha(t_1))$ to the point $\alpha(t_1 + \epsilon_1)$; second, the restriction of $\alpha$ to the interval $[t_1 + \epsilon_1, t_2 - \epsilon_2]$; and third, the straight line segment from $\alpha(t_2 - \epsilon_2)$ to the point numbered $h(\alpha(t_2))$.

If $\alpha'$ is a subarc of $\alpha$, then we say $\alpha$ is a **superarc** of $\alpha'$. Thus, given $\alpha'$, a superarc $\alpha$ is obtained by pushing $\alpha'$ right or left of the top and/or bottom endpoint, independently, and then extending upward and/or downward to make a longer arc.

The following theorem is [35, Theorem 4.4], but we prove it again here because we will reuse the argument in type D and to give an introduction to how we use Theorem 2.2.10 in type B. We also prove some corollaries from [35].

**Theorem 2.3.5.** Let $j_1$ and $j_2$ be join-irreducible permutations, corresponding to arcs $\alpha_1$ and $\alpha_2$ respectively. Then $j_1$ forces $j_2$ if and only if $\alpha_1$ is a subarc of $\alpha_2$.

Before proving the theorem, we state two important corollaries. The first corollary (which is part of [35, Corollary 4.5]), is simply a rephrasing of Theorem 2.3.5.

**Corollary 2.3.6.** A set $U$ of arcs corresponds to the set of uncontracted join-irreducible permutations of some congruence $\Theta$ on $S_n$ if and only if $U$ is closed under passing to subarcs.

Combining Theorems 2.3.1 and 2.3.3 with Corollary 2.2.3, we obtain the following result, which is also part of [35, Corollary 4.5].

**Corollary 2.3.7.** If $\Theta$ is a congruence on $S_n$ and $U$ is the set of arcs corresponding to join-irreducible permutations not contracted by $\Theta$, then $\delta$ restricts to a bijection from the quotient $S_n/\Theta$ (the set of permutations not contracted by $\Theta$) to the set of noncrossing arc diagrams consisting only of arcs in $U$.

We now turn to the proof of Theorem 2.3.5. Recall that the correspondence between join-irreducible elements and shards is an isomorphism from the forcing order to the reflexive-transitive closure of the shard digraph. Recall also that, while every shard arrow is a relation
in the forcing order, a shard arrow can fail to be a cover relation. Since arcs are also in
bijection with shards, we will characterize the shard digraph, and then the forcing order,
in terms of arcs. We write $\alpha_1 \rightarrow \alpha_2$ if the corresponding shards have $\Sigma_1 \rightarrow \Sigma_2$. The key to
Theorem 2.3.5 is the following proposition, which interprets Theorem 2.2.10 in terms of arcs.

**Proposition 2.3.8.** Two arcs $\alpha_1$ and $\alpha_2$ have $\alpha_1 \rightarrow \alpha_2$ if and only if $\alpha_1$ is a subarc of $\alpha_2$
and the two arcs have exactly one endpoint in common.

In other words, $\alpha_1 \rightarrow \alpha_2$ if and only if $\alpha_2$ is a superarc of $\alpha_1$ obtained by extending
$\alpha_1$ up or down, but not both.

**Proof.** Let $\Sigma_1$ and $\Sigma_2$ be the shards corresponding to $\alpha_1$ and $\alpha_2$. We use Proposition 2.2.8
implicitly throughout the proof.

By Theorem 2.2.10, to have $\Sigma_1 \rightarrow \Sigma_2$, there must exist $\Sigma'_1$ satisfying certain conditions. Write $\alpha'_1$ for the arc corresponding to the shard $\Sigma'_1$. Theorem 2.2.10 requires in particular
that $\alpha_1$ and $\alpha'_1$ are compatible and that the rank-two subarrangement containing $H_{\Sigma_1}$ and $H_{\Sigma'_1}$
must contain more than two hyperplanes. If $\alpha_1$ has endpoints $p$ and $q$, then $H_{\Sigma_1}$ is given
by $x_p = x_q$, and $H_{\Sigma'_1}$ is similarly described in terms of the endpoints of $\alpha'_1$. We see that $\Sigma_1$, $\Sigma'_1$ and $\Sigma_2$ satisfy conditions (i) and (ii) of Theorem 2.2.10 if and only if (up to switching $\Sigma_1$ and $\Sigma'_1$), there exist $p < q < r$ such that $\alpha_1$ has endpoints $p$ and $q$, $\alpha'_1$ has endpoints $q$ and $r$, and $\alpha_2$ has endpoints $p$ and $r$.

Supposing there exist such $p$, $q$, and $r$, we claim that $\Sigma_1 \cap \Sigma'_1 \subseteq \Sigma_2$ if and only if $\alpha_1$ and $\alpha'_1$ are both subarcs of $\alpha_2$.

On the one hand, suppose $\alpha_1$ and $\alpha'_1$ are both subarcs of $\alpha_2$. Then Proposition 2.3.2
implies that

$$\Sigma_1 \cap \Sigma'_1 = \{ x \in \mathbb{R}^n : x_p = x_q = x_r \text{ and } x_p \leq x_i \forall i \in R(\alpha_2) \text{ and } x_p \geq x_i \forall i \in L(\alpha_2) \} = \{ x \in \Sigma_2 : x_p = x_q \}.$$

On the other hand, suppose one of $\alpha_1$ and $\alpha'_1$ is not a subarc of $\alpha_2$. Then there exists a
numbered point $i \in (p, q) \cup (q, r)$ such that $i$ is left of $\alpha_1$ or $\alpha'_1$ and right of $\alpha_2$ or vice versa.
In the case where $i$ is right of $\alpha_2$, by Proposition 2.3.2, the only constraint on $x_i$ in $\Sigma_1 \cap \Sigma'_1$
is that $x_p \geq x_i$, while the only constraint on $x_i$ in $\Sigma_2$ is that $x_p \leq x_i$. Thus $\Sigma_1 \cap \Sigma'_1$ contains
points not in $\Sigma_2$. In the case where $i$ is left of $\alpha_2$, the same argument works, with inequalities
reversed. We have proved the claim.

Now, if $\alpha_1 \rightarrow \alpha_2$, then there exist $p$, $q$, and $r$ as above and, by Theorem 2.2.10 and
the claim, $\alpha_1$ is a subarc of $\alpha_2$, necessarily sharing an endpoint with $\alpha_2$. Conversely, if $\alpha_1$ is a
subarc of $\alpha_2$, sharing an endpoint, let $\alpha'_1$ be the subarc of $\alpha_2$ obtained by, essentially, deleting $\alpha_1$ from $\alpha_2$. Then there exist $p$, $q$, and $r$ as above, and the claim says that $\alpha_1 \rightarrow \alpha_2$. □

Proof of Theorem 2.3.5. One direction of the theorem follows from Proposition 2.3.8 because the subarc relation is transitive and forcing is the transitive closure of the $\rightarrow$ relation. The other direction of the theorem follows from Proposition 2.3.8 by passing from $\alpha_1$ to $\alpha_2$ by a sequence of one or two arrows. One arrow (if necessary) lengthens $\alpha_1$ by moving its lower endpoint down to agree with the lower endpoint of $\alpha_2$, and the other arrow (if necessary) moves the upper endpoint of $\alpha_1$ up to the upper endpoint of $\alpha_2$. □

We conclude this section with a result that will be helpful in Section 2.4. A finite Coxeter group has an element $w_0$ that is longer than every other element (in the usual sense of length in terms of number of letters in a reduced word or number of inversions). The element $w_0$ is an involution.

In $S_n$, this element $w_0$ is $n(n-1)\cdots321$. Given $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, $\pi w_0$ is the permutation with one-line notation $\pi_n\pi_{n-1}\cdots\pi_1$ and $w_0\pi$ is the permutation with one-line notation $(n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$. The one-line notation of $w_0\pi w_0$ is $(n+1-\pi_n)(n+1-\pi_{n-1})\cdots(n+1-\pi_1)$.

Proposition 2.3.9. If $\pi \in S_n$, then the arc diagrams $\delta(\pi)$ and $\delta(w_0\pi w_0)$ are related by a half turn that sends the point labeled $i$ to the point labeled $n+1-i$.

For this proposition to make sense, we must assume that the initial placement of points had this half-turn symmetry.

Proof. There is a descent $\pi_i > \pi_{i+1}$ in $\pi$ if and only if there is a descent $(n+1-\pi_{i+1}) > (n+1-\pi_i)$ in $w_0\pi w_0$, and an entry $a$ with $\pi_{i+1} < a < \pi_{i+1}$ occurs left of $\pi_i$ in $\pi$ if and only if $(n+1-a)$ is right of $(n+1-\pi_i)$ in $w_0\pi w_0$. Thus $\delta(w_0\pi w_0)$ is obtained from $\delta(\pi)$ by a reflection taking each point labeled $i$ to the point labeled $n+1-i$, followed by a reflection in the vertical line containing the points. The composition of these two reflections is a half turn that sends each point labeled $i$ to the point labeled $n+1-i$. □

2.4 Noncrossing arc diagrams of type B

In this section, we establish a notion of noncrossing arc diagrams for Coxeter groups of type B. In fact, we establish two notions that, while equivalent, look quite different: a centrally symmetric model and an orbifold model. Before we explain these models, we recall some background about Coxeter groups of type $B_n$ and establish notation.
The Coxeter group of type $B_n$ can be realized as the group of signed permutations of $\{\pm 1, \ldots, \pm n\}$. These are the permutations $\pi$ of $\{\pm 1, \ldots, \pm n\}$ with the property that $\pi(-i) = -\pi(i)$ for $i = 1, \ldots, n$. The long one-line notation of $\pi \in B_n$ is the sequence $\pi_1 \pi_2 \cdots \pi_n$. But $\pi \in B_n$ is completely determined by its short one-line notation (or simply one-line notation) $\pi_1 \pi_2 \cdots \pi_n$.

We realize $B_n$ as usual as a reflection group in $\mathbb{R}^n$, with $s_0$ acting as a reflection orthogonal to the standard basis vector $e_1$ and with each $s_i$ acting as a reflection orthogonal to $e_{i+1} - e_i$ for $i = 1, \ldots, n-1$. For convenience, we write $e_{-i}$ to mean $-e_i$ for each $i = 1, \ldots, n$. Given a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $x_i$ to mean $-x_i$.

The reflections in $B_n$ are the permutations with cycle notation $(i \rightarrow -i)$ for $i \in \{\pm 1, \ldots, \pm n\}$ or $(i j)(-i \rightarrow -j)$ for $i, j \in \{\pm 1, \ldots, \pm n\}$, and $|j| \neq |i|$.

The weak order on $B_n$ has cover relations given by $\pi < \sigma$ if $\sigma$ is obtained in one of two ways from $\pi$. One way is that $\pi_{-1} < \pi_1$ and $\sigma$ is obtained by exchanging $\pi_{-1}$ and $\pi_1$. The cover reflection of $\sigma$ associated to this cover is the involution $(\pi_1 \pi_{-1})$. The other way is that $\pi_i < \pi_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ (and equivalently $\pi_{i-1} < \pi_{-i}$) and $\sigma$ is obtained from $\pi$ by exchanging $\pi_i$ and $\pi_{i+1}$ and exchanging $\pi_{i-1}$ and $\pi_{-i}$. The cover reflection of $\sigma$ associated to this cover is $(\pi_i \pi_{i+1})(\pi_{-i} \pi_{-i-1})$.

A join-irreducible element of $B_n$ is a signed permutation whose long one-line notation either has exactly one descent $\pi_{-1} > \pi_1$ or has exactly two descents that are symmetric to each other: $\pi_i > \pi_{i+1}$ and $\pi_{i-1} > \pi_{-i}$ for some $i \in \{1, \ldots, n-1\}$. Equivalently, either $\pi_1 < 0$ and the (short) one-line notation is increasing or $\pi_1 > 0$ and there is exactly one $i \in \{1, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.

### 2.4.1 A centrally symmetric model

In the centrally symmetric model, we begin with $2n$ distinct points on a vertical line, and the points identified with the numbers $-n, \ldots, -2, -1, 1, 2, \ldots, n$, in order, with $-n$ at the bottom. The points are placed so that the antipodal map $x \mapsto -x$ (a half turn about the origin) takes $i$ to $-i$ for all $i \in \{\pm 1, \ldots, \pm n\}$. A centrally symmetric noncrossing arc diagram is a collection of arcs on these points, satisfying the same requirements as in type A, with the additional requirement that the entire diagram is symmetric with respect to the half-turn symmetry.

A centrally symmetric noncrossing arc diagram consists of symmetric arcs (arcs that are fixed by the central symmetry) and symmetric pairs of arcs (pairs of arcs that are compatible in the type-A sense of Section 2.3 and are mapped to each other by the central symmetry). Symmetric pairs of arcs come in two types: A non-overlapping symmetric...
Figure 2.2: The map $\delta$ applied to $\pi = (-4)352(-1)$

Figure 2.3: Centrally symmetric noncrossing arc diagrams for $B_2$.

pair is a symmetric pair $\{\alpha, -\alpha\}$ such that $\alpha$ connects positive points and $-\alpha$ connects negative points, so that neither arc is left or right of the other. An overlapping symmetric pair is a symmetric pair $\{\alpha, -\alpha\}$ in which each arc connects a positive point to a negative point. In an overlapping pair, since the two arcs are compatible, one is to the right of the other.

For the purposes of this section, we can harmlessly recast the results of Section 2.3 in terms of a group $S_{2n}$ of permutations of $\{\pm 1, \ldots, \pm n\}$ and in terms of noncrossing arc diagrams on points labeled $-n, \ldots, -1, 1, \ldots, n$. As an immediate consequence of Theorem 2.3.1 and Proposition 2.3.9, we have the following theorem.

**Theorem 2.4.1.** The map $\delta$ restricts to a bijection from $B_n$ to the set of centrally symmetric noncrossing arc diagrams on $2n$ points.

An example of $\delta$ applied to a signed permutation in $B_5$ is shown in Fig. 2.2. All centrally symmetric noncrossing arc diagrams for $B_2$ are included in Fig. 2.3.

A join-irreducible signed permutation is mapped by $\delta$ to an arc diagram with only one arc (a symmetric arc), or only two arcs (a symmetric pair), and this is a bijection from join-irreducible signed permutations to symmetrics arcs/pairs. We describe this bijection explicitly in the table below as a map from symmetric arcs/pairs to join-irreducible signed
permutations in (short) one-line notation. In this description, when we write a set as part of the one-line notation, we mean the elements of that set in increasing order. Also in this description, \( i \cdots j \) will always stand for the sequence of elements increasing by 1 from \( i \) to \( j \), or the empty sequence if \( j = i - 1 \).

Recall from Section 2.3 that the set \( L(\alpha) \) of \textit{left points} of an arc \( \alpha \) is the set of numbers that are left of \( \alpha \), and the set \( R(\alpha) \) of \textit{right points} of \( \alpha \) is the set of numbers that are right of \( \alpha \). (Applying this definition to arc diagrams on points \{\( \pm 1, \ldots, \pm n \} \), of course 0 is never a right point or a left point.) We write \(-L(\alpha)\) for \(\{ -i : i \in L(\alpha) \}\) and similarly \(-R(\alpha)\).

A join-irreducible signed permutation \( \pi \) with one descent \( \pi_{-1} \succ \pi_1 \) corresponds to a symmetric arc \( \alpha \). If \( \alpha \) has endpoints \(-p \prec p \), then \( \pi \) is

\[
(-p) R(\alpha) (p + 1) \cdots n.
\]

A join-irreducible signed permutation \( \pi \) that has two descents \( \pi_i > \pi_{i+1} \) and \( \pi_{-i-1} > \pi_{-i} \) corresponds to a symmetric pair \( \{ \alpha, -\alpha \} \) of arcs. Suppose \( \alpha \) has endpoints \( p < q \). If \( \{ \alpha, -\alpha \} \) is nonoverlapping, then we assume that \( \alpha \) is above \(-\alpha \), so that \( 0 < p < q \). Then \( \pi \) is

\[
1 \cdots (p - 1) L(\alpha) q p R(\alpha) (q + 1) \cdots n.
\]

If \( \{ \alpha, -\alpha \} \) is overlapping, then we assume that \( \alpha \) is right of \(-\alpha \). If \( p < 0 < -p < q \), then \( \pi \) is

\[
\left[ \begin{array}{c}
\{(0, -p) \cap L(\alpha) \cap -L(\alpha)\} \\
\{(0, q) \cap L(\alpha) \cap -L(\alpha)\}
\end{array} \right] \pi p R(\alpha) (q + 1) \cdots n
\]

If \( p < 0 < q < -p \), then \( \pi \) is

\[
\left[ \begin{array}{c}
\{(0, q) \cap L(\alpha) \cap -L(\alpha)\} \\
\{(q, -p) \cap L(\alpha) \cap -L(\alpha)\}
\end{array} \right] (q - p + 1) \cdots n.
\]

Given a signed permutation \( \pi \in B_n \), each symmetric arc or symmetric pair of arcs in \( \delta(\pi) \) is associated to a cover reflection \( t \) of \( \pi \). Each symmetric arc or symmetric pair also specifies a join-irreducible element \( j \) as described above. By inspection of the three cases described above, we see that \( j \leq \pi \) and that \( j \) is minimal with respect to the property that \( t \in \pi^{-1} (j) \). Thus Theorem 2.2.4 implies the following theorem.

**Theorem 2.4.2.** Given \( \pi \in B_n \), the canonical join representation of \( \pi \) is the set of join-irreducible elements corresponding to the symmetric arcs and symmetric pairs of arcs in \( \delta(\pi) \).

Recall from Section 2.3 that two arcs are called compatible if and only if they don’t
intersect, except possibly at endpoints, and don’t share the same top endpoint or the same bottom endpoint. We now define one symmetric arc/pair $A$ to be compatible with another symmetric arc pair $A'$ if and only if every arc in $A$ is compatible with every arc in $A'$. As a consequence of Theorem 2.4.2, two symmetric arcs/pairs are compatible if and only if the corresponding join-irreducible elements are compatible.

The following two propositions are obtained by combining the correspondence between arcs and join-irreducible signed permutations with simple observations in [31, Sections 3,5]. In those sections, join-irreducible elements are shown to be in bijection with certain “signed subsets”, and the signed subsets are used to write inequalities for shards.

**Proposition 2.4.3.** Suppose $\alpha$ is a symmetric arc having endpoints $-p$ and $p$, with $p > 0$. The shard associated to $\alpha$ is

$$\{x \in \mathbb{R}^n : x_p = 0 \text{ and } 0 \leq x_i \forall i \in R(\alpha)\}.$$

**Proposition 2.4.4.** Suppose $\{\alpha, -\alpha\}$ is a symmetric pair of arcs such that $\alpha$ has endpoints $p$ and $q$ with $p < q$ and $q > 0$. If $p < 0$ (that is, if the pair is overlapping), then assume that $\alpha$ is to the right of $-\alpha$. The shard associated to $\{\alpha, -\alpha\}$ is

$$\{x \in \mathbb{R}^n : x_p = x_q \text{ and } x_p \leq x_i \forall i \in R(\alpha) \text{ and } x_p \geq x_i \forall i \in L(\alpha)\}.$$

**Remark 2.4.5.** There is a global change in the direction of inequalities between Propositions 2.4.3 and 2.4.4 and [31], because in [31], hyperplanes were identified with their normal vectors pointing away from a “base region” $B$. A standard convention for root systems has the positive roots pointing towards a “fundamental chamber”. This convention was used in [36], from which we quoted Proposition 2.3.2, so to keep conventions consistent in this paper, we apply the antipodal map relative to [31].

We now use Theorem 2.2.10 to characterize forcing of join-irreducible signed permutations in terms of a centrally symmetric version of subarcs, which we now define. Since join-irreducible signed permutations correspond either to centrally symmetric arcs or to centrally symmetric pairs of arcs, we will use the terminology of a “subarc pair” of a symmetric arc or symmetric pair, or a “subarc” of a symmetric arc or symmetric pair. Fig. 2.4 shows subarcs and subarc pairs of a symmetric arc and an overlapping symmetric pair.

**Definition 2.4.6** (Subarcs/subarc pairs of a symmetric arc). Suppose $\alpha$ is a symmetric arc with endpoints $-p < p$. A subarc of $\alpha$ is a symmetric arc $\alpha'$ with endpoints $-p' < p'$ with $p' \leq p$ and $R(\alpha') = R(\alpha) \cap (-p', p')$. A subarc pair of $\alpha$ is a non-overlapping pair $\{\alpha', -\alpha'\}$ of arcs such that $\alpha'$ has endpoints $p'$ and $q'$ with $0 < p' < q' \leq p$ and $R(\alpha') = R(\alpha) \cap (p', q')$. 49
Definition 2.4.6 (Subarcs/subarc pairs of a symmetric pair of arcs). Suppose \(\{\alpha, -\alpha\}\) is a symmetric pair of arcs and \(\alpha\) has endpoints \(p < q\). A **subarc** of \(\{\alpha, -\alpha\}\) is a symmetric arc \(\alpha'\) with endpoints \(-p' < p' \leq q\) and \(R(\alpha') = R(\alpha) \cap (-p', p') = R(-\alpha') \cap (-p', p')\). This can only happen if \(\alpha\) and \(-\alpha\) are overlapping and agree (in the sense of going left/right of points) on the interval \((-p', p')\). Now, if \(\{\alpha, -\alpha\}\) is overlapping, suppose further that \(\alpha\) is right of \(-\alpha\). A **subarc pair** of \(\{\alpha, -\alpha\}\) is a symmetric pair \(\{\alpha', -\alpha'\}\) with endpoints \(p' < q' \leq q\) and \(R(\alpha') = R(\alpha) \cap (p', q')\), satisfying an additional requirement: If \(\{\alpha', -\alpha'\}\) is overlapping then \(\alpha'\) is also right of \(-\alpha'\). The left two pictures of Fig. 2.5 show a failure of this additional requirement.

**Remark 2.4.8.** To understand this additional requirement, it is useful to construct subarcs
of \{\alpha, -\alpha\} explicitly. Similarly to the type-A construction, we take \(h\) to be a function from the plane to \(\mathbb{R}\) that returns the vertical height of a point, calibrated so that each numbered point \(i \in \{\pm 1, \ldots, \pm n\}\) has height \(i\). Choose \(t_1\) and \(t_2\) with \(0 \leq t_1 < t_2 \leq 1\) such that \(h(\alpha(t_1)), h(\alpha(t_2)) \in \{\pm 1, \ldots, \pm n\}\) and choose \(\epsilon_1 > 0\) and \(\epsilon_2 > 0\) with \(|h(\alpha(t_1 + \epsilon_1)) - h(\alpha(t_1))| < 1, |h(\alpha(t_2 - \epsilon_2)) - h(\alpha(t_2))| < 1, \) and \(t_1 + \epsilon_1 < t_2 - \epsilon_2\). Then \(\alpha'\) is obtained by concatenating three curves: the segment from the point numbered \(h(\alpha(t_1))\) to the point \(\alpha(t_1 + \epsilon_1)\), the restriction of \(\alpha\) to \([t_1 + \epsilon_1, t_2 - \epsilon_2]\) and the segment from \(\alpha(t_2 - \epsilon_2)\) to the point numbered \(h(\alpha(t_2))\). The additional requirement on \(\{\alpha', -\alpha'\}\) is that (up to choosing another symmetric arc pair combinatorially equivalent to \(\{\alpha, -\alpha\}\)) the specific curves \(\alpha'\) and \(-\alpha'\) constructed here do not intersect each other. The right picture of Fig. 2.5 shows an example of \(\alpha'\) and \(-\alpha'\) crossing.

We will prove the following theorem.

**Theorem 2.4.9.** Let \(j_1\) and \(j_2\) be join-irreducible signed permutations. Then \(j_1\) forces \(j_2\) if and only if the arc or pair of arcs corresponding to \(j_1\) is a subarc or subarc pair of the arc or pair of arcs corresponding to \(j_2\).

Theorem 2.4.9 lets us prove the type-B analogues of Corollaries 2.3.6 and 2.3.7. The following result is a rephrasing of Theorem 2.4.9.

**Corollary 2.4.10.** A set \(U\) of symmetric arcs/symmetric pairs corresponds to the set of uncontracted join-irreducible signed permutations of some congruence \(\Theta\) on \(B_n\) if and only if \(U\) is closed under passing to subarcs/subarc pairs.

Combining Corollary 2.4.10 with Theorems 2.4.1 and 2.4.2 and Corollary 2.2.3, we obtain the following result.

**Corollary 2.4.11.** If \(\Theta\) is a congruence on \(B_n\) and \(U\) is the set of symmetric arcs/pairs corresponding to join-irreducible signed permutations not contracted by \(\Theta\), then \(\delta\) restricts to a bijection from the quotient \(B_n/\Theta\) (the set of signed permutations not contracted by \(\Theta\)) to the set of centrally symmetric noncrossing arc diagrams consisting only of arcs in \(U\).

It is convenient to understand the opposite of the subarc relation. We say that a symmetric arc/pair \(A_2\) is a **superarc** or **superarc pair** of a symmetric arc/pair \(A_1\) if and only if \(A_1\) is a subarc/subarc pair of \(A_2\). We now describe how to construct a superarc/superarc pair of a given symmetric arc/pair. The construction is a direct rephrasing of Definitions 2.4.6 and 2.4.7.

**Superarcs/superarc pairs of a symmetric arc \(\alpha\).** We construct a superarc by pushing \(\alpha\) right or left of its top endpoint, symmetrically pushing it left or right of its bottom endpoint,
and extending $\alpha$ symmetrically upwards and downwards. We construct a superarc pair by first replacing $\alpha$ with two arcs that are combinatorially equivalent to $\alpha$, antipodal images of each other, and disjoint except at their endpoints. We then push one or both copies right or left of the top endpoint, independently but without making the curves cross each other, make the symmetric change at the bottom endpoint, and extend the curves upward and downward symmetrically.

Superarcs/superarc pairs of a non-overlapping symmetric pair of arcs $\{\alpha, -\alpha\}$. To construct a superarc or superarc pair, we first push $\alpha$ and $-\alpha$ left or right of their inner endpoints and/or outer endpoints in a way that preserves the symmetry. We construct a superarc by extending the curves inward and connecting them, and also possibly extending both outwards. We construct a superarc pair by extending the curves symmetrically to create a symmetric pair of compatible arcs, either overlapping or not.

Superarc pairs of an overlapping symmetric pair of arcs $\{\alpha, -\alpha\}$. (There are no superarcs of $\{\alpha, -\alpha\}$ and no non-overlapping superarc pairs, only overlapping superarc pairs.) We construct a superarc pair by pushing $\alpha$ right or left of one or both endpoints independently, making the symmetric change to $-\alpha$, and then extending the curves symmetrically to create a symmetric pair of compatible overlapping arcs. The additional requirement in Definition 2.4.7 is implied by this description of superarcs: If $\{\alpha_1, -\alpha_1\}$ fails to be a subarc pair of $\{\alpha_2, -\alpha_2\}$ because it fails the additional requirement, then the attempt to extend $\alpha_1$ and $-\alpha_1$ to obtain $\alpha_2$ and $-\alpha_2$ will result in a pair of arcs that cross each other. (See Fig. 2.5.)

A first step towards the proof of Theorem 2.4.9 is the following observation.

**Proposition 2.4.12.** The subarc/subarc pair relation on symmetric arcs/pairs is transitive.

The proposition is verified by checking the various cases (e.g. a nonoverlapping subarc pair of a symmetric subarc of an overlapping arc pair). There are many cases to check, but each individual case is easy.

As a next step towards proving Theorem 2.4.9, we use Theorem 2.2.10 to characterize arrows in the shard digraph in terms of arcs and arc pairs. Given symmetric arcs/pairs $A_1$ and $A_2$, we write $A_1 \to A_2$ if and only if the corresponding shards $\Sigma_1$ and $\Sigma_2$ have $\Sigma_1 \to \Sigma_2$ in the shard digraph.

**Proposition 2.4.13.** Suppose $A_1$ is a symmetric arc or pair and $A_2$ is another symmetric arc or pair, then $A_1 \to A_2$ if and only if $A_1$ is a subarc/subarc pair of $A_2$ and one of the following conditions holds.

(i) $A_2$ is a symmetric pair $\{\alpha_2, -\alpha_2\}$ such that $\alpha_2$ has endpoints $p$ and $q$ with $p < q$ and $A_1$ is a symmetric pair $\{\alpha_1, -\alpha_1\}$ such that $\alpha_1$ has endpoints $p'$ and $q$ with $p' \neq -p$ or $\alpha_1$ has endpoints $p$ and $q'$ with $q' \neq -q$.
Proposition 2.4.13.(i)

(ii) $A_1$ and $A_2$ are both symmetric arcs with no endpoints in common.

(iii) $A_2$ is a symmetric arc $\alpha_2$ with endpoints $p$ and $-p$ and $A_1$ is a non-overlapping symmetric pair $\{\alpha_1, -\alpha_1\}$ such that $\alpha_1$ has an endpoint $p$.

(iv) $A_2$ is a symmetric pair $\{\alpha_2, -\alpha_2\}$ such that $\alpha_2$ has endpoints $p$ and $q$ with $-q < p < 0 < -p < q$, having the property that each point in $\{\pm 1, \ldots, \pm (p - 1)\}$ is left of $\alpha_2$ if and only if it is left of $-\alpha_2$, and $A_1$ is a symmetric arc with endpoints $p$ and $-p$.

(v) $A_2$ is a symmetric pair $\{\alpha_2, -\alpha_2\}$ such that $\alpha_2$ has endpoints $p$ and $q$ with $-q < p < 0 < -p < q$, having the property that each point in $\{\pm 1, \ldots, \pm (p - 1)\}$ is left of $\alpha_2$ if and only if it is left of $-\alpha_2$, and $A_1$ is a non-overlapping symmetric pair $\{\alpha_1, -\alpha_1\}$ such that $\alpha_1$ has endpoints $-p$ and $q$.

Examples of arrows $A_1 \to A_2$ are shown in Fig. 2.6. We emphasize that the conditions for an arrow in Proposition 2.4.13 include the condition that $A_1$ is a subarc/subarc pair of $A_2$. This requirement includes the additional requirement in Definition 2.4.7, which is crucial for the arrows satisfying condition (i) in Proposition 2.4.13, as explained in Case 1 of the proof below.

Proof. Let $\Sigma_1$ and $\Sigma_2$ be the shards corresponding to $A_1$ and $A_2$. Theorem 2.2.10 says that $A_1 \to A_2$ if and only if there exists a symmetric arc/pair $A_1'$ (corresponding to a shard $\Sigma_1'$) satisfying certain properties. In particular, $A_1$ and $A_1'$ must be compatible and there needs to exist a non-basic hyperplane in the rank-two subarrangement containing $H_{\Sigma_1}$ and $H_{\Sigma_1'}$. In other words, that subarrangement must have more than two hyperplanes. Each rank-two subarrangement looks like a rank-two parabolic subgroup, so in type B it can have 2, 3, or 4 hyperplanes.
A symmetric arc with endpoints $\pm p$ determines a shard whose hyperplane is orthogonal to $e_p$. A symmetric pair such that one arc has endpoints $p$ and $q$ determines a shard whose hyperplane is orthogonal to $e_q - e_p$ (recalling the convention that $e_{-i}$ means $-e_i$). If $A_1$ and $A_1'$ have no endpoints in common, then $H_{\Sigma_1}$ and $H_{\Sigma_1'}$ are orthogonal, and thus, since these hyperplanes are basic in the rank-two subarrangement they determine, that subarrangement has only two hyperplanes. Thus we need only consider cases where $A_1$ and $A_1'$ have endpoints in common. We break into two cases, where $A_1$ and $A_1'$ determine a rank-two subarrangement with 3 or 4 hyperplanes.

Case 1. $H_{\Sigma_1}$ and $H_{\Sigma_1'}$ are the basic hyperplanes in a rank-two subarrangement with 3 hyperplanes. Then $A_1$ is a symmetric pair $\{\alpha_1, -\alpha_1\}$ such that $\alpha_1$ has endpoints $p < q$ and $A_1'$ is a symmetric pair $\{\alpha_1', -\alpha_1'\}$ such that $\alpha_1'$ has endpoints $q < r$ with $q \not\in \{-p, -r\}$. At most one of the pairs $A_1$ and $A_1'$ is overlapping, and if one is, we can assume (up to renaming arcs and points) that $\alpha_1$ or $\alpha_1'$ is to the right of its negative. The basic hyperplane $H_{\Sigma_1}$ is orthogonal to $e_q - e_p$ and the other basic hyperplane $H_{\Sigma_1'}$ is orthogonal to $e_r - e_q$. The unique non-basic hyperplane in the subarrangement is orthogonal to $e_r - e_p$. Thus an arc/arc pair specifies a non-basic hyperplane in the subarrangement if and only if it is a symmetric arc pair $\{\alpha_2, -\alpha_2\}$ such that $\alpha_2$ has endpoints $p$ and $r$. Proposition 2.4.4 implies that

$$
\Sigma_1 \cap \Sigma_1' = \{x \in \mathbb{R}^n : x_p = x_q = x_r, x_p \leq x_i \forall i \in (R(\alpha_1) \cup R(\alpha_1')), x_p \geq x_i \forall i \in (L(\alpha_1) \cup L(\alpha_1'))\}.
$$

On the other hand,

$$
\Sigma_2 = \{x \in \mathbb{R}^n : x_p = x_r, x_p \leq x_i \forall i \in R(\alpha_2), x_p \geq x_i \forall i \in L(\alpha_2)\}.
$$

Since $\Sigma_1 \cap \Sigma_1'$ is in the subspace where $x_p = x_q$, it is contained in $\Sigma_2$ if and only if it is contained in $\{x \in \Sigma_2 : x_p = x_q\}$, which equals

$$
\{x \in \mathbb{R}^n : x_p = x_q = x_r, x_p \leq x_i \forall i \in R(\alpha_2) \setminus \{q\}, x_p \geq x_i \forall i \in L(\alpha_2) \setminus \{q\}\}.
$$

Thus $\Sigma_1 \cap \Sigma_1' \subseteq \Sigma_2$ if and only if $R(\alpha_2) \setminus \{q\} = R(\alpha_1) \cup R(\alpha_1')$ (equivalently, if and only if $L(\alpha_2) \setminus \{q\} = L(\alpha_1) \cup L(\alpha_1')$). This is in turn equivalent to the condition that $\alpha_2$ is obtained by pushing the curve $\alpha_1 \cup \alpha_1'$ left or right of the shared endpoint $q$.

Now Theorem 2.2.10 implies that every arrow among shards that determine rank-two subarrangements with 3 hyperplanes are of the form $A_1 \to A_2$ or $A_1' \to A_2$ for $A_1, A_1'$, and
$A_2$ as above such that $\alpha_2$ is obtained by pushing the curve $\alpha_1 \cup \alpha'_1$ left or right of $q$. Given $A_1$ and $A_2$, the existence of $A'_1$ such that $\alpha_2$ is obtained by pushing $\alpha_1 \cup \alpha'_1$ left or right of $q$ is exactly the condition that $A_1$ is a subarc pair of $A_2$, except for the additional requirement on subarc pairs of symmetric arc pairs in Definition 2.4.7. Similarly, given $A'_1$ and $A_2$, the existence of an appropriate $A_1$ is exactly that $A'_1$ is a subarc pair, without the additional requirement.

Furthermore, Theorem 2.2.10 says that if $\alpha_2$ is obtained by pushing $\alpha_1 \cup \alpha'_1$ left or right of $q$, then these arrows $A_1 \rightarrow A_2$ or $A'_1 \rightarrow A_2$ exist if and only if $A_1$ and $A'_1$ are compatible. We will show that, under the condition that $\alpha_2$ is obtained by pushing $\alpha_1 \cup \alpha'_1$ left or right of $q$, the compatibility of $A_1$ and $A'_1$ is equivalent to the additional requirement.

Compatibility of $A_1$ and $A'_1$ means that all four arcs $\alpha_1$, $-\alpha_1$, $\alpha'_1$, and $-\alpha'_1$ can be in the same noncrossing arc diagram. Recall also that if one of the pairs $\{\alpha_1, -\alpha_1\}$ or $\{\alpha'_1, -\alpha'_1\}$ is overlapping, then $\alpha_1$ or $\alpha'_1$ is on the right. For each pair which is nonoverlapping, $\alpha_1$ or $\alpha'_1$ is above its opposite.

Suppose $A_1$ and $A'_1$ are compatible. Then the union $\alpha_1 \cup \alpha'_1$ is either above or right of its opposite $(\alpha_1)$ or $(\alpha'_1)$. Since $\alpha_2$ is obtained by pushing $\alpha_1 \cup \alpha'_1$ left or right of $q$, then either $\{\alpha_2, -\alpha_2\}$ is nonoverlapping and $\alpha_2$ is above $-\alpha_2$ or $\{\alpha_2, -\alpha_2\}$ is overlapping and $\alpha_2$ is right of $-\alpha_2$.

Now, suppose $A_1$ and $A'_1$ are not compatible. The individual arcs $\alpha_1$ and $\alpha'_1$ are compatible in any case, so the failure of compatibility of $A_1$ and $A'_1$ implies that $\alpha_1$ and $-\alpha'_1$ cross each other. Thus the embedding of $\alpha_2$ obtained by pushing $\alpha_1 \cup \alpha'_1$ left or right of $q$ also crosses its antipodal opposite. Since by supposition $\{\alpha_2, -\alpha_2\}$ is a symmetric arc pair, there is an embedding of $\alpha_2$ that does not cross its opposite. This embedding must therefore have $\alpha_2$ left of $-\alpha_2$, as in Fig. 2.5.

We have shown that arrows of the form $A_1 \rightarrow A_2$ and $A'_1 \rightarrow A_2$ arising from Case 1 are precisely the arrows described in condition (i).

**Case 2.** $H_{\Sigma_1}$ and $H_{\Sigma'_1}$ are the basic hyperplanes in a rank-two subarrangement with 4 hyperplanes. Then (up to swapping $A_1$ and $A'_1$) $A_1$ is a symmetric arc $\alpha_1$ with endpoints $\pm p$ for $p > 0$ and $A'_1$ is a non-overlapping arc pair $\{\alpha'_1, -\alpha'_1\}$ such that $\alpha'_1$ has endpoints $p$ and $q$ with $p < q$. Any such $A_1$ and $A'_1$ are compatible, with no additional requirements needed.

The basic hyperplanes in the subarrangement are orthogonal to $e_p$ and $e_q - e_p$. The two non-basic hyperplanes are $e_q$ and $e_q + e_p$. Thus an arc/arc-pair $A_2$ specifies a non-basic hyperplane in the subarrangement if and only if it is a symmetric arc with endpoints $\pm q$ or an overlapping symmetric pair one of whose arcs has endpoints $-p$ and $q$. To determine all possible arrows arising from this compatible pair, it remains to determine necessary and sufficient conditions on $A_2$ so that the corresponding shard has $\Sigma_1 \cap \Sigma'_1 \subseteq \Sigma_2$. 

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Propositions 2.4.3 and 2.4.4 combine to say that

\[ \Sigma_1 \cap \Sigma'_1 = \{ x \in \mathbb{R}^n : x_p = x_q = 0, 0 \leq x_i \, \forall i \in (R(\alpha_1) \cup R(\alpha'_1)), \, 0 \geq x_i \, \forall i \in L(\alpha'_1) \}. \]

Since \( x_{-i} = -x_i \), we can rewrite this as

\[ \Sigma_1 \cap \Sigma'_1 = \{ x \in \mathbb{R}^n : x_p = x_q = 0, 0 \leq x_i \, \forall i \in (R(\alpha_1) \cup R(\alpha'_1) \cup (-L(\alpha'_1))) \}. \]  \hspace{1cm} (2.1)

Since \( \Sigma_1 \cap \Sigma'_1 \) is in the subspace where \( x_p = 0 \), it is contained in \( \Sigma_2 \) if and only if it is contained in \( \{ x \in \Sigma_2 : x_p = 0 \} \).

**Case 2a.** If \( A_2 \) is a symmetric arc \( \alpha_2 \) with endpoints \( \pm q \), then Proposition 2.4.3 says that

\[ \{ x \in \Sigma_2 : x_p = 0 \} = \{ x \in \mathbb{R}^n : x_p = x_q = 0, 0 \leq x_i \, \forall i \in R(\alpha_2) \}. \]

If \( A_1 \) is a subarc of \( A_2 \) as in condition (ii) and \( A'_1 \) is a subarc pair of \( A_2 \) as in condition (iii), then in particular \( R(\alpha_2) \setminus \{ \pm p \} = R(\alpha_1) \cup R(\alpha'_1) \cup (-L(\alpha'_1)) \), so \( \Sigma_1 \cap \Sigma'_1 = \{ x \in \Sigma_2 : x_p = 0 \} \).

Conversely, if \( \Sigma_1 \cap \Sigma'_1 \subseteq \Sigma_2 \), then for every \( i \in R(\alpha_2) \), the inequality \( 0 \leq x_i \) holds in \( \Sigma_1 \cap \Sigma'_1 \). Thus \( R(\alpha_2) \setminus \{ \pm p \} = R(\alpha_1) \cup R(\alpha'_1) \cup (-L(\alpha'_1)) \), so \( A_1 \) is a subarc of \( A_2 \) and \( A'_1 \) is a subarc pair of \( A_2 \).

We have shown that arrows of the form \( A_1 \rightarrow A_2 \) and \( A'_1 \rightarrow A_2 \) in Subcase 2a are precisely the arrows described in conditions (ii) and (iii).

**Case 2b.** If \( A_2 \) is an overlapping symmetric pair \( \{ \alpha_2, -\alpha_2 \} \) such that \( \alpha_2 \) has endpoints \( -p \) and \( q \), then we consider Proposition 2.4.4 in two further cases, given by whether \( \alpha_2 \) is to the right or left of \( -\alpha_2 \). If \( \alpha_2 \) is to the right of \( -\alpha_2 \), then Proposition 2.4.4 implies that

\[ \{ x \in \Sigma_2 : x_p = 0 \} = \{ x \in \mathbb{R}^n : x_p = x_q = 0, 0 \leq x_i \, \forall i \in R(\alpha_2), \, 0 \geq x_i \, \forall i \in L(\alpha_2) \}. \]

If \( \alpha_2 \) is to the left of \( -\alpha_2 \), then since \( -L(\alpha_2) \) is the set of points to the right of \( -\alpha_2 \) and \( -R(\alpha_2) \) is the set of points to the left of \( -\alpha_2 \), Proposition 2.4.4 implies that

\[ \{ x \in \Sigma_2 : x_p = 0 \} = \{ x \in \mathbb{R}^n : x_p = x_q = 0, 0 \leq x_i \, \forall i \in -L(\alpha_2), \, 0 \geq x_i \, \forall i \in -R(\alpha_2) \}. \]

In either case,

\[ \{ x \in \Sigma_2 : x_p = 0 \} = \{ x \in \mathbb{R}^n : x_p = x_q = 0, 0 \leq x_i \, \forall i \in (R(\alpha_2) \cup (-L(\alpha_2))) \} \]  \hspace{1cm} (2.2)

If \( A_1 \) is a subarc of \( A_2 \) as in condition (iv) and \( A'_1 \) is a subarc pair of \( A_2 \) as in condition (v), then no points are between \( \alpha_2 \) and \( -\alpha_2 \), so \( R(\alpha_2) \cup (-L(\alpha_2)) \) contains no pairs \( \pm i \).
Furthermore, $R(\alpha_2) \cap (-p, p) = R(\alpha_1)$, and $R(\alpha_2) \cap (p, q) = R(\alpha'_1)$, and $L(\alpha_2) \cap (p, q) = L(\alpha'_1)$. So comparing (2.2) with (2.1), we see that $\Sigma_1 \cap \Sigma'_1 = \{ x \in \Sigma_2 : x_p = 0 \}$.

Conversely, if $\Sigma_1 \cap \Sigma'_1 \subseteq \{ x \in \Sigma_2 : x_p = 0 \}$, then by (2.1) and (2.2), we know that $(R(\alpha_2) \cup (-L(\alpha_2))) \subseteq (R(\alpha_1) \cup R(\alpha'_1) \cup (-L(\alpha'_1)))$. In this case, no point can be between $\alpha_2$ and $-\alpha_2$, because $R(\alpha_1)$ contains no pairs $\pm i$. So, $R(\alpha_2)$ and $-L(\alpha_2)$ coincide along the interval $(-p, p)$. Therefore $R(\alpha_2) \setminus \{ p \} \subseteq R(\alpha_1) \cup R(\alpha'_1)$ and $L(\alpha_2) \setminus \{ p \} \subseteq (-R(\alpha_1)) \cup L(\alpha'_1) = L(\alpha_1) \cup L(\alpha'_1)$, so $A_1$ is a subarc of $A_2$ and $A'_1$ is a subarc pair of $A_2$.

We have shown that arrows of the form $A_1 \to A_2$ and $A'_1 \to A_2$ in Subcase 2b are precisely the arrows described in conditions (iv) and (v).

**Proof of Theorem 2.4.9.** Both directions of the theorem use Proposition 2.4.13. Let $A_1$ be the symmetric arc/pair corresponding to $j_1$ and $A_2$ be the symmetric arc/pair corresponding to $j_2$. We continue to use the notation $A_1 \to A_2$ to mean $j_1 \to j_2$.

If $j_i$ forces $j_2$, then there is a sequence of arrows from $j_1$ to $j_2$. Thus Propositions 2.4.12 and 2.4.13 imply that $A_1$ is a subarc/subarc pair of $A_2$.

Conversely, suppose that $A_1$ is a subarc/subarc pair of $A_2$, with $A_1 \neq A_2$. We will show that, in every case, there is a sequence of one, two, or three arrows from $j_1$ to $j_2$. For the purposes of this proof, we will refer to the kinds of arrows described in Proposition 2.4.13 as “type (i)”, etc.

**Case 1.** $A_2$ is a symmetric arc.

**Case 1a.** $A_1$ is also a symmetric arc. In this case, $A_1 \to A_2$ by an arrow of type (ii)

**Case 1b.** $A_1$ is a non-overlapping symmetric pair. In this case, suppose that the endpoints of $A_2$ are $\pm p$ with $p > 0$ and the endpoints of one arc of $A_1$ are $p'$ and $q'$ with $0 < p' < q' \leq p$. Then there is an arrow of type (iii) from $A_1$ to a symmetric arc $A'$ with endpoints $\pm q'$ that is a subarc of $A_2$, and (unless $A' = A_2$) an arrow of type (ii) from $A'$ to $A_2$, as illustrated in the left picture of Fig. 2.7.
Case 2. $A_2$ is a nonoverlapping symmetric pair. In this case, $A_1$ is also a nonoverlapping symmetric pair. If the outer endpoints of $A_1$ and $A_2$ are not the same, then there is an arrow of type (i) from $A_1$ to an arc pair $A'$ whose outer endpoints agree with those of $A_2$; otherwise, let $A' = A_1$. If the inner endpoints of $A'$ and $A_2$ agree then $A' = A_2$. If not, then there is another arrow of type (i) from $A'$ to $A_2$, as illustrated in the center picture of Fig. 2.7.

Case 3. $A_2 = \{\alpha_2, -\alpha_2\}$ is an overlapping symmetric pair. Write $p$ and $q$ with $p < q$ for the endpoints of $\alpha_2$.

Case 3a. $A_1 = \{\alpha_1, -\alpha_1\}$ is also an overlapping symmetric pair. In this case, assume that $\alpha_2$ is right of $-\alpha_2$, that $\alpha_1$ is right of $-\alpha_1$, and that $\alpha_1$ has endpoints $p'$ and $q'$ with $p \leq p' < 0 < q' \leq q$. If $p < p' < 0 < q' < q$, then we find a sequence of two arrows of type (i) from $A_1$ to $A_2$. If $q' = q$ and/or $p = p'$, then one or both of these arrows is replaced by equality. If $-q > p$, then also $-q' > p$, so there is a type (i) arrow from $A_1$ to a subarc pair $A' = \{\alpha', -\alpha'\}$ of $A_2$ such that $\alpha'$ has endpoints $p$ and $q'$ and an arrow of type (i) from $A'$ to $A_2$, as illustrated in the right picture of Fig. 2.7. If $-p > q$, then also $-p' > q$, so there is a type (i) arrow from $A_1$ to a subarc pair $A' = \{\alpha', -\alpha'\}$ of $A_2$ such that $\alpha'$ has endpoints $p'$ and $q$ and an arrow of type (i) from $A'$ to $A_2$. (Separating into two cases $-q < p$ and $-p < q$ is necessary. For example, when $-q > p$, it is possible that $p' = -q$, so that there is no subarc pair $A' = \{\alpha', -\alpha'\}$ of $A_2$ such that $\alpha'$ has endpoints $p'$ and $q$. This is the case illustrated in the right picture of Fig. 2.7.)

Case 3b. $A_1 = \{\alpha_1, -\alpha_1\}$ is a non-overlapping symmetric pair. In this case, assume that $\alpha_1$ is a subarc of $\alpha_2$ (making no assumption about which of $\alpha_2$ or $-\alpha_2$ is to the right). Write $p'$ and $q'$ for the endpoints of $\alpha_1$, this time with $p < 0 < p' < q' \leq q$. If $p' \neq -p$, then there is a subarc pair $A'$ of $A_2$ with endpoints $p'$ and $q$ and an arrow $A' \rightarrow A_2$ of type (i). If $p' = -p$, then there exists a subarc pair $A' = \{\alpha', -\alpha'\}$ of $A_2$ such that $\alpha'$ has endpoints 1 and $q$ and an arrow from $A'$ to $A_2$. The arrow is of type (i) if $p < -1$ or (v) if $p = -1$. Either way, $A_1$ is a subarc of $A'$ and by Case 2, there is a (possibly empty) sequence of arrows from $A_1$ to $A'$.

Case 3c. $A_1$ is a symmetric arc $\alpha_1$. Write $p'$ and $-p'$ with $p \leq p' < 0 < -p' \leq q$ for the endpoints of $\alpha_1$. If $p' = p$ or $-p' = q$, then there is an arrow of type (iv) from $A_1$ to $A_2$, so it remains to consider the case where $p < p' < 0 < -p' < q$. Without loss of generality (up to swapping $\pm \alpha_2$), we may as well assume that $p > -q$. Therefore also $p' > -q$, so there is a type (iv) arrow from $A_1$ to a subarc pair $A' = \{\alpha', -\alpha'\}$ of $A_2$ such that $\alpha'$ has endpoints $p'$ and $q$ and an arrow from $A'$ to $A_2$ of type (i).

\[\square\]
2.4.2 An orbifold model

We now take the centrally symmetric model for Coxeter groups of type $B_n$ and pass to a quotient modulo the central symmetry, obtaining what we call an orbifold model. Most simply—but not very “drawably”—the orbifold model lives in the quotient space where each point of the plane is identified with its antipodal opposite. The quotient map takes a point $x$ in the plane to $\{\pm x\}$.

To make a more “drawable” model, we consider the same space as a different quotient of the plane: We cut the plane in half with a horizontal line through the origin. Points strictly above the horizontal line are not identified with any other points. Each point on or below the horizontal line is identified with all other points on or below the line at the same distance from the origin. Thus, each “point” in the quotient is either a point strictly above the horizontal line (a \textit{point in the upper halfplane}), the origin, or a semicircle below the line with endpoints on the line (or a degenerate semicircle consisting only of the origin, called a \textit{semicircle point}). We will refer to this quotient as the \textit{orbifold plane}.

We are not interested in the natural quotient map associated to the orbifold plane. Rather, we are interested in the map $\phi$ defined as follows: If $x$ is not on the horizontal line, then $\phi(x)$ is the point in $\{\pm x\}$ that is above the horizontal line. If $x$ is on the horizontal line, then $\phi(x)$ is the semicircle of points of length $|x|$ on or below the horizontal line (or $\phi(0)$ is the degenerate semicircle at the origin).

A symmetric arc $\alpha$ or pair $\{\alpha, -\alpha\}$ in the centrally symmetric model can be uniquely recovered from $\phi(\alpha)$. Thus also symmetric arc diagrams and the various results and constructions in Section 2.4.1 can be recovered from their images under $\phi$. The goal now is to define arcs, arc diagrams, a bijection to $B_n$, etc. in the orbifold plane so that the results of Section 2.4.1 translate to results in the orbifold plane via the map $\phi$. With the right definitions, these new results are simply “translations” into a new setting, and will not require new proofs. Collectively, we will refer to these constructions and results as the \textit{orbifold model} for $B_n$.

The orbifold model for $B_n$ starts with $n$ distinct points on a vertical line containing the origin, with each point strictly above the origin. The origin itself is called the \textit{orbifold point}, and is marked with an “$\times$”. (We will sometimes also refer, in prose, to the orbifold point as “$\times$”.) The $n$ points above the origin are identified with the numbers $1, 2, \ldots, n,$ in order, with $n$ at the top, and are called \textit{numbered points}.

A \textit{type-B arc} (or in context simply an \textit{arc}) is a curve in the orbifold plane with each endpoint at a numbered point or at $\times$ (the origin), satisfying one of the following three descriptions:
An **ordinary arc** is an arc on the points 1, \ldots, n satisfying the same rules as an arc in type A (Section 2.3).

An **orbifold arc** is an arc with one endpoint at \( \times \) and the other at a numbered point \( p \), moving monotone downwards from \( p \) to \( \times \) without touching any other numbered point, passing to the left or right of any numbered points below \( p \).

A **long arc** is an arc \( \alpha \) containing exactly one semicircle point. The **left piece** of \( \alpha \) moves monotone downward from the **left endpoint** of \( \alpha \), passing left or right of numbered points between, and hits the semicircle point left of \( \times \). The **right piece** moves monotone downward from the **right endpoint**, passing left or right of numbered points, hitting the semicircle point right of \( \times \). The right and left pieces are disjoint, so in particular the left and right endpoints do not coincide.

An ordinary arc or orbifold arc \( \alpha \) is specified combinatorially by its endpoints and the set \( R(\alpha) \) of numbered points to its right, or equivalently by its endpoints and the set \( L(\alpha) \) of numbered points to its left. A long arc \( \alpha \) is specified combinatorially by its endpoints (right and left), the set \( R(\alpha) \) of points right of its right piece and the set \( L(\alpha) \) of points left of its left piece. We emphasize that for long arcs (in contrast to the situation for ordinary and orbifold arcs), the sets \( R(\alpha) \) and \( L(\alpha) \) do not completely determine each other, though they must be disjoint, and both sets are necessary to determine \( \alpha \) combinatorially. We also emphasize that to determine \( \alpha \), it is necessary to specify which endpoint is right and which is left. (For example, there are many long arcs \( \alpha \) with \( R(\alpha) = \emptyset \) and \( L(\alpha) = \emptyset \), one for each choice of a left endpoint and right endpoint.)

A (type-B) **noncrossing arc diagram (on \( n \) points)** is a collection of type-B arcs on points \( \times \) and 1, \ldots n that don’t intersect, except possibly at their endpoints, with no two arcs sharing an endpoint from which they both go down or both go up. Again, we consider arcs and noncrossing arc diagrams up to combinatorial equivalence.

All noncrossing arc diagrams for \( B_2 \) are included in Fig. 2.8.

The map \( \phi \) is a bijection from the set of centrally symmetric arcs and centrally symmetric pairs of arcs to the set of type-B arcs, and also induces a bijection on combinato-

![Figure 2.8: Noncrossing arc diagrams for \( B_2 \).](image)
rial equivalence classes. Nonoverlapping symmetric pairs, symmetric arcs, and overlapping symmetric pairs map respectively to ordinary, orbifold, and long arcs. Furthermore, \( \phi \) induces a bijection from centrally symmetric noncrossing arc diagrams to type-B noncrossing arc diagrams. We write \( \delta^o \) for the map \( \phi \circ \delta \), where \( \delta \) is understood as in Section 2.4.1 to map signed permutations to centrally symmetric noncrossing arc diagrams. The following theorem is a restatement of Theorem 2.4.1.

**Theorem 2.4.14.** The map \( \delta^o \) is a bijection from \( B_n \) to the set of type-B noncrossing arc diagrams on \( n \) points.

We now give a direct description of the bijection \( \delta^o \) from \( B_n \) to the set of type-B noncrossing arc diagrams on \( n \) points, as illustrated in Fig. 2.9. Given \( \pi \in B_n \) with one-line notation \( \pi_1 \cdots \pi_n \), write each entry \( \pi_i \) at the point \((i, \pi_i)\) in the plane, for \( i = 1, \ldots, n \). For every \( i \) such that \( \pi_i > \pi_{i+1} \), draw a straight line segment from \( \pi_i \) to \( \pi_{i+1} \). Additionally, if \( \pi_1 < 0 \), draw a line segment from the origin to \( \pi_1 \). These line segments become arcs: First, we move the numbers \( \pi_1, \ldots, \pi_n \) horizontally to put them into a single vertical line, allowing the line segments to curve, so that they don’t pass through any of the numbers or one another. Then, we rotate the negative numbers 180 degrees clockwise, allowing the (already curved) line segments connecting positive numbers to negative numbers to stretch around the origin. We remove the negative signs on the numbers. Since \( \{|\pi_1|, \ldots, |\pi_n|\} = \{1, \ldots, n\} \), the numbers are now \( 1, \ldots, n \) from bottom to top. The resulting type-B noncrossing arc diagram is \( \delta^o(\pi) \).

We continue to translate the results of Section 2.4.1 into the language of the orbifold model. We next describe the bijection from arcs to join-irreducible elements of \( B_n \). We continue the conventions from Section 2.4.1 for describing join-irreducible signed permutations.

A join-irreducible signed permutation \( \pi \) with a single descent \( \pi_{-1} > \pi_1 \) corresponds to
an orbifold arc $\alpha$. If $\alpha$ has upper endpoint $p$, then $\pi$ is

$$(-p) (-L(\alpha)) R(\alpha) (p + 1) \cdots n.$$ 

A join-irreducible signed permutation $\pi$ with two symmetric descents $\pi_i > \pi_{i+1}$ and $\pi_{-i-1} > \pi_{-i}$ such that $\pi_i$ and $\pi_{i+1}$ have the same sign corresponds to an ordinary arc $\alpha$. (If $\pi_i$ and $\pi_{i+1}$ are both negative, there must be an additional inversion between $\pi_i$ and the entry before it, either $\pi_{i-1}$ or $\pi_{-1}$.) If $\alpha$ has endpoints $p$ and $q$ with $0 < p < q$, then $\pi$ is

$$1 \cdots (p - 1) L(\alpha) q p R(\alpha) (q + 1) \cdots n.$$ 

A join-irreducible element $\pi$ with two symmetric descents each involving a positive and a negative number corresponds to a long arc $\alpha$. Suppose $\alpha$ has left endpoint $p$ and right endpoint $q$ (noting that as we translate this description from Section 2.4.1, the $p$ here is the $-p$ there). Necessarily, $p \neq q$. Write $B(\alpha)$ (suggesting “between”) to denote the set of points that are left of the right piece of $\alpha$ and right of the left piece. Thus $B$ is a subset of $(0, \min(p, q))$. If $p < q$, then $\pi$ is

$$B(\alpha) [(p, q) \setminus R(\alpha)] q (-p) (-L(\alpha)) R(\alpha) (q + 1) \cdots n.$$ 

If $q < p$, then $\pi$ is

$$B(\alpha) q (-p) (-L(\alpha)) R(\alpha) [(q, p) \setminus L(\alpha)] (p + 1) \cdots n.$$ 

The following is a restatement of Theorem 2.4.2 in the orbifold model.

**Theorem 2.4.15.** Given $\pi \in B_n$, the canonical join representation of $\pi$ is the set of join-irreducible elements corresponding to the arcs in $\delta^o(\pi)$.

Two type-B arcs are **compatible** if and only if they do not intersect, except possibly at numbered endpoints, and don’t share the same top endpoint or the same bottom endpoint. In other words, they are compatible if and only if they can appear together in a type-B noncrossing arc diagram. As a consequence of Theorems 2.4.14 and 2.4.15, two join-irreducible elements of $B_n$ are compatible (can appear together in a canonical join representation) if and only if the corresponding type-B arcs are compatible.

The following three propositions are restatements of Propositions 2.4.3 and 2.4.4. (Propositions 2.4.17 and 2.4.18 break Proposition 2.4.4 into two cases.)
Proposition 2.4.16. Suppose $\alpha$ is an orbifold arc upper endpoint $p$. The shard associated to $\alpha$ is

$$\{ x \in \mathbb{R}^n : x_p = 0 \text{ and } 0 \leq x_i \forall i \in R(\alpha) \text{ and } 0 \geq x_i \forall i \in L(\alpha) \}. $$

Proposition 2.4.17. Suppose $\alpha$ is an ordinary arc with endpoints $p$ and $q$. The shard associated to $\alpha$ is

$$\{ x \in \mathbb{R}^n : x_p = x_q \text{ and } x_p \leq x_i \forall i \in R(\alpha) \text{ and } x_p \geq x_i \forall i \in L(\alpha) \}. $$

Proposition 2.4.18. Suppose $\alpha$ is a long arc with left endpoint $p$ and right endpoint $q$. The shard associated to $\alpha$ is

$$\{ x \in \mathbb{R}^n : x_q = -x_p \text{ and } x_q \leq x_i \forall i \in (R(\alpha) \cup (-L(\alpha))) \text{ and } x_q \geq x_i \forall i \in ((-p, q) \setminus (R(\alpha) \cup (-L(\alpha))) \}. $$

We now turn to translating the definitions of subarcs and superarcs, as well as Theorem 2.4.9, to the orbifold model, using the map $\phi$. There is a case-free way to define subarcs of a type-B arc $\alpha$: Informally, we can cut $\alpha$ shorter at one or both of its endpoints and reattach the shortened curve to a numbered point. If the result is a type-B arc (or can be easily modified to become an orbifold arc as made precise below), then it is a subarc of $\alpha$. Together, Definitions 2.4.19 and 2.4.20, below, correspond to Definitions 2.4.6 and 2.4.7, but the split into two definitions is not the same here as in Section 2.4.1.

Definition 2.4.19 (Subarcs of ordinary and orbifold arcs). Suppose $\alpha$ is an ordinary or orbifold arc with endpoints $p < q$, with $\times = 0$ for the purposes of this definition. A subarc of $\alpha$ is an arc $\alpha'$ with endpoints $p'$ and $q'$ having $p \leq p' \leq q' \leq q$ and $R(\alpha') = R(\alpha) \cap (p', q')$. The arc $\alpha'$ is ordinary unless $p' = 0$, in which case $\alpha'$ is an orbifold arc.
Definition 2.4.20 (Subarcs of long arcs). Suppose $\alpha$ is a long arc with left endpoint $p$ and right endpoint $q$. A subarc of $\alpha$ can be ordinary, orbifold, or long.

- An ordinary arc $\alpha'$ with endpoints $p' < q'$ is a subarc of $\alpha$ if and only if one of the following occurs: either $q' \leq p$ and $L(\alpha') = L(\alpha) \cap (p', q')$ or $q' \leq q$ and $R(\alpha') = R(\alpha) \cap (p', q')$.

- An orbifold arc $\alpha'$ with upper endpoint $p'$ is a subarc of $\alpha$ if and only if $p' \leq \min(p, q)$, $L(\alpha') = L(\alpha) \cap (0, p')$, and $R(\alpha') = R(\alpha) \cap (0, p')$.

- A long arc $\alpha'$ with left endpoint $p'$ and right endpoint $q'$ is a subarc of $\alpha$ if and only if $p' \leq p$, $q' \leq q$, $L(\alpha') = L(\alpha) \cap (0, p')$, $R(\alpha') = R(\alpha) \cap (0, q')$, and the following additional conditions hold: $p' \not\in R(\alpha)$ and $q' \not\in L(\alpha)$.

Remark 2.4.21. To better understand long subarcs of long arcs in Definition 2.4.20, we discuss an explicit construction of these subarcs. Compare similar constructions in Remarks 2.3.4 and 2.4.8. Take $h$ to be a height function on the orbifold plane with each numbered point $i$ having height $i$ and with every semicircle point in the closed lower halfplane having height 0.

Suppose $\alpha$ is a long type-B arc, parametrized as a function from the interval $[0, 1]$ into the orbifold plane. The construction of a long subarc $\alpha'$ begins as follows: Choose $t_1$ and $t_2$ with $0 \leq t_1 < t_2 \leq 1$ such that $h(\alpha(t_1))$ and $h(\alpha(t_2))$ are distinct elements of $\{1, \ldots, n\}$. Choose $\epsilon_1 > 0$ such that $|h(\alpha(t_1 + \epsilon_1)) - h(\alpha(t_1))| < 1$ and such that $h(\alpha(t)) > 0$ for all $t$ in the interval $(t_1, t_1 + \epsilon_1)$. Similarly, choose $\epsilon_2 > 0$ such that $t_1 + \epsilon_1 < t_2 - \epsilon_2$, such that $|h(\alpha(t_2 - \epsilon_2)) - h(\alpha(t_2))| < 1$, and such that $h(\alpha(t)) > 0$ for all $t$ in the interval $(t_2 - \epsilon_2, t_2)$. Define $\alpha'$ to be the curve obtained by concatenating three curves: First, the straight line segment from the point numbered $h(\alpha(t_1))$ to the point $\alpha(t_1 + \epsilon_1)$; second, the restriction of $\alpha$ to the interval $[t_1 + \epsilon_1, t_2 - \epsilon_2]$; and third, the straight line segment from $\alpha(t_2 - \epsilon_2)$ to the point numbered $h(\alpha(t_2))$.

If $\alpha'$ is a valid long arc, then it is a subarc of $\alpha$. It fails to be a type-B arc if and only if it crosses itself, in which case, this choice of $t_1$ and $t_2$ does not produce a subarc of $\alpha$. See Fig. 2.11.

The map $\phi$ translates the subarc relation on symmetric arcs/pairs exactly to the subarc relation on type-B arcs. The following results are direct translations of Theorem 2.4.9 and Corollaries 2.4.10 and 2.4.11.

Theorem 2.4.22. Let $j_1$ and $j_2$ be join-irreducible signed permutations. Then $j_1$ forces $j_2$ if and only if the type-B arc corresponding to $j_1$ is a subarc of the type-B arc corresponding to $j_2$. 

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Figure 2.11: A failed construction of a subarc, per Definition 2.4.20 and Remark 2.4.21.

**Corollary 2.4.23.** A set $U$ of type-B arcs corresponds to the set of uncontracted join-irreducible signed permutations of some congruence $\Theta$ on $B_n$ if and only if $U$ is closed under passing to subarcs.

**Corollary 2.4.24.** If $\Theta$ is a congruence on $B_n$ and $U$ is the set of type-B arcs corresponding to join-irreducible permutations not contracted by $\Theta$, then $\delta^r$ restricts to a bijection from the quotient $B_n/\Theta$ (the set of signed permutations not contracted by $\Theta$) to the set of type-B noncrossing arc diagrams consisting only of arcs in $U$.

A type-B arc $\alpha'$ is a **superarc** of a type-B arc $\alpha$ if and only if $\alpha$ is a subarc of $\alpha'$. We now describe how to construct superarcs.

**Superarcs of an orbifold arc $\alpha$.** These can be orbifold arcs or long arcs. We construct an orbifold superarc by pushing $\alpha$ left or right of its top endpoint and extending it upwards to a new upper endpoint. We construct a long superarc by first replacing $\alpha$ with a curve that goes around $\times$, with both endpoints at the original endpoint of $\alpha$, and with its left piece and its right piece having exactly the same right points as $\alpha$. We then push one or both ends off of the endpoint (one left and one staying, one right and one staying, both left, one on each side, or both right, but not creating a self-intersection), and extend upwards without creating self-intersections, to make a long superarc.

**Superarcs of an ordinary arc $\alpha$.** These can be of any of the three kinds. To construct a superarc, we push the top or bottom of $\alpha$ independently to the left or right. We then extend upwards and/or downwards. The downward extension can end before reaching $\times$ (to make an ordinary superarc), can have an endpoint at $\times$ (to make an orbifold superarc), or can go around $\times$ before ending (to make a long superarc), provided that no self-intersections are created.

**Superarcs of a long arc $\alpha$.** Every superarc of $\alpha$ is long. We construct a superarc by pushing one or both endpoints of $\alpha$ left or right and extending upwards, without creating self-intersections.
We close this section with a translation of Proposition 2.4.13, which will be useful in later sections. As before, we write arrows $\to$ between type-B arcs to indicate arrows between the corresponding shards.

**Proposition 2.4.25.** Suppose $\alpha_1$ and $\alpha_2$ are type-B arcs. Then $\alpha_1 \to \alpha_2$ if and only if $\alpha_1$ is a subarc of $\alpha_2$ and one of the following conditions holds.

(i) Neither $\alpha_1$ nor $\alpha_2$ is an orbifold arc, and $\alpha_1$ and $\alpha_2$ have exactly one endpoint in common.

(ii) $\alpha_1$ and $\alpha_2$ are orbifold arcs with different upper endpoints.

(iii) $\alpha_2$ is an orbifold arc and $\alpha_1$ is ordinary, with the same upper endpoint.

(iv) $\alpha_2$ is a long arc with endpoints $p < q$ and no numbered points between its left and right pieces, and $\alpha_1$ is an orbifold arc with upper endpoint $p$.

(v) $\alpha_2$ is a long arc with endpoints $p < q$ and no numbered points between its left and right pieces, and $\alpha_1$ is an ordinary arc with endpoints $p$ and $q$.

We emphasize that condition (i) disallows arrows from a long arc to an ordinary arc with the same endpoints, but condition (v) provides an exception when the long arc has no numbered points between its left and right pieces. Also, in (iv) and (v), it does not matter whether $p$ or $q$ is the left endpoint.

**2.5 Noncrossing arc diagrams of type D**

We now construct a model of noncrossing arc diagrams for finite Coxeter groups of type D. One early step in the construction, namely the treatment of shards in type D, draws on a similar treatment in [28], but with some differences of conventions.

**2.5.1 Shards, arcs, and join-irreducible elements in type D**

An **even-signed permutation** is a signed permutation $\pi$ with one-line notation $\pi_1 \cdots \pi_n$ such that the set $\{i \in [n] : \pi_i < 0\}$ has an even number of elements. The Coxeter group of type $D_n$ is the group of even-signed permutations of $\{\pm 1, \ldots, \pm n\}$. It has simple reflections $s_0 = (-21)(-12)$ and $s_i = (i \ i + 1)(-i - 1 \ -i)$ for $i = 1, \ldots, n - 1$. The reflections in $D_n$ are the elements $(i \ j)(-j \ -i)$ for $i, j \in \pm [n]$ with $i \neq -j$.

The Coxeter arrangement $A(D_n)$ consists of hyperplanes in $\mathbb{R}^n$ with normal vectors $\{e_i - e_j : i, j \in \pm [n], i > j \neq -i\}$, where again we use the shorthand $e_{-i}$ for $-e_i$. Similarly,
we continue the shorthand $x_{-i}$ for $-x_i$ when $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector in $\mathbb{R}^n$. We will refer to the hyperplanes by their normal vectors, but each hyperplane can be named in two ways, as $e_i - e_j$ or as $e_{-j} - e_{-i}$. The base region $B$ is the region containing the point $(1, 2, \ldots, n)$. This region is bounded by the reflecting hyperplanes $e_2 + e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}$ associated to the simple reflections.

The inversions of a even-signed permutation $\pi$ are the reflections $(i \ j)(-j \ -i)$ with $i > j \neq -i$ such that $i$ precedes $j$ in the long one-line notation for $\pi$. (This description mentions each inversion exactly twice.) Each even-signed permutation $\pi$ is associated to the region whose separating set consists of the reflecting hyperplanes of all inversions of $\pi$. Concretely, for $i > j \neq -i$, the reflection $(i \ j)(-j \ -i)$ is an inversion of $\pi$ if and only if points in the region for $\pi$ have nonpositive dot product with $e_i - e_j$. Thus a vector $\mathbf{x}$ in the region for $\pi$ has $x_i \leq x_j$ if $i$ precedes $j$ in the long one-line notation for $\pi$ and has $x_i \geq x_j$ if $i$ follows $j$.

Given a even-signed permutation $\pi$, the **descents** of $\pi$ are the pairs $(\pi_i, \pi_j)$ with $\pi_i > \pi_j$ for $i \in \{-n, \ldots, -2\} \cup \{1, \ldots, n-1\}$ and $j = i+1$ or for $i \in \{-2, -1\}$ and $j = i+3$. Descents come in symmetric pairs $(\pi_i, \pi_j), (\pi_{-j}, \pi_{-i})$, and these pairs correspond to the cover reflections of $\pi$, with a descent pair $(\pi_i, \pi_j), (\pi_{-j}, \pi_{-i})$ specifying the cover reflection $(\pi_i \ \pi_j)(\pi_{-j} \ \pi_{-i})$.

The rank-two subarrangements of size $> 2$ have the form

$$\left\{e_i - e_j, e_i - e_k, e_j - e_k\right\} \text{ for } i > j > k \text{ with } j \neq -i, k \neq -i, k \neq -j,$$

where the underlined vectors correspond to the basic hyperplanes.

Thus, for $i > k$ and $k \neq -i$ the shards in the hyperplane orthogonal to $e_i - e_k$ are defined by inequalities of the form $x_i \leq x_j$ or $x_i \geq x_j$ for $i > j > k$ and $j \notin \{-i, -k\}$. (Since $x_i = x_k$ in that hyperplane, the inequalities $x_k \leq x_j$ or $x_k \geq x_j$ are redundant.) Moreover, for each $j$ with $i > j > k$ and $j \notin \{-i, -k\}$, exactly one of the inequalities $x_i \leq x_j$ and $x_i \geq x_j$ must hold in the shard. A choice of one inequality from each pair defines a type-D shard if and only if it defines an $(n-1)$-dimensional subset of the hyperplane.

We now construct a type-D noncrossing arc diagram model. The model for $D_n$ begins like the orbifold model in type B, with a point $\times$ at the origin and $n$ points in a vertical line above the origin, numbered $1, \ldots, n$ from bottom to top. We begin by defining type-D arcs and showing that they are in bijection with shards.

**Definition 2.5.1.** We will represent type-D arcs in two ways: as equivalence classes of certain type-B arcs, or as single “arcs” with more complicated rules. These two different representations are useful for different aspects of the model: The equivalence class representation is
convenient for compatibility and noncrossing arc diagrams, and the single-arc model is more convenient for forcing. Thus we will pass freely between the two representations.

The type-B arcs that we consider are the ordinary and long arcs (i.e. orbifold arcs are excluded), and we define an equivalence relation on long arcs. If $\alpha$ is a long type-B arc with endpoints $a < b$ that has no numbered points between its left piece and its right piece, then $\alpha$ is equivalent to an arc $\alpha'$ that is nearly identical to $\alpha$ in the following sense: the left piece of $\alpha'$ is combinatorially the same as the right piece of $\alpha$ and vice versa, except that the longer piece of $\alpha'$ passes to the opposite side of $a$ as the longer piece of $\alpha$. (An example of this equivalence appears below in Fig. 2.12.) Aside from these 2-element equivalence classes, all other equivalence classes are singletons.

The type-D arcs are of three types, as described here and pictured in Fig. 2.12.

- An ordinary arc is an arc on the points $1, \ldots, n$ satisfying the same rules as an arc in type A (Section 2.3). It can be represented as an ordinary type-B arc (a singleton equivalence class under the equivalence relation on type-B arcs). If $\alpha$ is an ordinary arc, then we write $R(\alpha)$ for the set of numbered points right of $\alpha$ and $L(\alpha)$ for the set of numbered points left of $\alpha$.

- A partially doubled arc with and internal endpoint a external endpoint b (with $a < b$) is a curve connecting $b$ to $\times$ and also passing through $a$. Between $\times$ and $a$ and between $a$ and $b$, the curve satisfies the same rules as an arc in type A. A partially doubled arc $\alpha$ can be represented as a 2-element equivalence class consisting of two long type-B arcs with endpoints $a$ and $b$ that both pass to the same side of each numbered point (except necessarily the point $a$). If $\alpha$ is a partially doubled arc with internal endpoint $a$ and external endpoint $b$, then we write $R(\alpha)$ for the subset of $(0,a) \cup (a,b)$ consisting of points right of $\alpha$ and write $L(\alpha)$ for the subset of $(0,a)$ consisting of points left of $\alpha$. This definition breaks the natural symmetry of a partially doubled arc, but proves useful later.

- A branched arc is a union of two curves, one connecting $\times$ to a left endpoint $b_L$ and the other connecting $\times$ to a right endpoint $b_R \neq b_L$, satisfying the following requirements: each curve follows the rules for an arc in type A; there is a numbered point $a < \min(b_L,b_R)$ and a number $0 < \epsilon < 1$ such that the two curves coincide from height 0 to height $a - \epsilon$ and are disjoint above height $a - \epsilon$; and the curve to $b_L$ passes left of $a$ while the curve to $b_R$ passes right of $a$. For precision, we have defined this arc as a union of two curves, but we think of it as a single “branched” arc, with branch point $a$ and a left branch (to $b_L$) and a right branch (to $b_R$). A branched arc $\alpha$ can be represented as a long type-B arc that has at least one point between its left and
right pieces. The lowest point between the two pieces is \( a \). (This is a single-element equivalence class.) Given a branched arc \( \alpha \), we write \( R(\alpha) \) for the set of points right of its right branch and write \( L(\alpha) \) for the set of points left of its left branch.

As in types A and B, the combinatorial data of a type-D arc \( \alpha \) consists of its endpoints and the sets \( L(\alpha) \) and \( R(\alpha) \). There are many embeddings of \( \alpha \) with the same combinatorial data, and we consider arcs and noncrossing arc diagrams up to combinatorial equivalence.

The main reason for representing type-D arcs as equivalence classes of type-B arcs is to allow us to re-use results from type B. The first such re-use is the following proposition.

**Proposition 2.5.2.** Given a type-D arc, represented as an equivalence class of type-B arcs, the union, over the equivalence class, of the corresponding type-B shards is a type-D shard. This correspondence is a bijection between type-D arcs and type-D shards.

**Proof.** We first argue that each union, over equivalence classes of arcs, of type-B shards is a type-D shard. Let \( \alpha \) be a type-D arc.

First, suppose \( \alpha \) is an ordinary arc. In this case, the equivalence class has only one type-B arc, and Proposition 2.4.17 describes the type-B shard using inequalities that define a type-D shard.

Next, suppose \( \alpha \) is a branched arc with endpoints \( b_L \) and \( b_R \). Again, the equivalence class is a singleton. Proposition 2.4.18 describes the type-B arc using inequalities appropriate for a type-D arc, but with one extra inequality not appropriate for a type-D arc: If \( b_R < b_L \), then the extra inequality is \( x_{b_R} \geq -x_{b_L} \). If \( b_L < b_R \), then the extra inequality is \( x_{b_R} \geq x_{b_L} \), but since \( x_{b_R} = -x_{b_L} \), this says \( x_{b_L} \geq -x_{b_R} \). Thus, in either case, the extra inequality is \( x_{b_R} \geq 0 \). But this extra inequality is implied by the other inequalities: The point \( a \) is neither right of the right piece of the type-B arc nor left of the left piece. Thus, according to Proposition 2.4.18, two of the defining equations of the type-B shard are \( x_{b_R} \geq x_a \) and \( x_{b_R} \geq -x_a \), which implies that \( x_{b_R} \geq 0 \). We conclude that the type-B shard is also a type-D shard.
Finally, suppose \( \alpha \) is a partially doubled arc with internal endpoint \( a \) and external endpoint \( b \). Of the two equivalent type-B arcs, let \( \beta \) be the one with \( a \) as its left endpoint and \( b \) as its right endpoint. The type-B shard corresponding to \( \beta \) is described in Proposition 2.4.18 as

\[
\{ x \in \mathbb{R}^n : x_b = -x_a \text{ and } x_b \leq x_i \forall i \in (R(\beta) \cup (-L(\beta))) \\
\text{and } x_b \geq x_i \forall i \in ((-a, b) \setminus R(\beta) \cup (-L(\beta))) \}.
\]

Let \( \beta' \) be the other type-B arc in the class, having \( a \) as its right endpoint and \( b \) its left endpoint. Then \( L(\beta') = (0, b) \setminus (R(\beta) \cup \{a\}) \) and \( R(\beta') = (0, a) \setminus L(\beta) \). Therefore \( R(\beta') \cup (-L(\beta')) \) is \((-b, a) \setminus (L(\beta) \cup (-R(\beta)) \cup \{-a\})\). Thus the type-B shard corresponding to \( \beta' \) is described in Proposition 2.4.18 as

\[
\{ x \in \mathbb{R}^n : x_a = -x_b \text{ and } x_a \leq x_i \forall i \in ((-b, a) \setminus L(\beta) \cup (-R(\beta)) \cup \{-a\}) \\
\text{and } x_a \geq x_i \forall i \in (L(\beta) \cup (-R(\beta)) \cup \{-a\}) \}.
\]

Comparing these sets of inequalities in light of the equality \( x_b = -x_a \) that holds in both cases, we see that they differ only in that one set of inequalities requires \(-x_a \geq x_a\) (i.e. \( x_a \leq 0 \)) and the other requires \( x_a \geq -x_a\) (i.e. \( x_a \geq 0 \)). The union of these two type-B shards is obtained by eliminating the either requirement, \( x_a \geq 0 \) or \( x_a \leq 0 \). Thus this union is described by a collection of inequalities appropriate to describe a type-D shard.

Having shown that each union of type-B shards is a type-D shard, we show that all type-D shards are of this form. The type-B shards whose arcs (in the orbifold model) are not orbifold arcs are precisely the type-B shards in hyperplanes whose normal vectors are \( e_i - e_j \) for \( i, j \in \pm[n] \) \( i > j \), \( j \neq -i \), or in other words, the reflecting hyperplanes for \( D_n \). Each such hyperplane is a union of type-B shards, so the hyperplane is also a union of the type-D shards that are formed as unions over equivalence classes of type-B shards. Thus every type-D shard is such a union.

Since ordinary and branched type-D arcs can be represented by singleton equivalence classes of type-B arcs, Proposition 2.4.18 gives explicit inequalities for the corresponding shards. The proof of Proposition 2.5.2 shows that one inequality is redundant in the branched case and also shows how to write down inequalities in the partially doubled case. We summarize the association between type-D shards and the various kinds of type-D arcs in the following propositions.

**Proposition 2.5.3.** Suppose \( \alpha \) is an ordinary type-D arc with endpoints \( a \) and \( b \). The shard
Proposition 2.5.4. Suppose $\alpha$ is a partially doubled arc with internal endpoint $a$ and external endpoint $b$. The shard associated to $\alpha$ is

$$\{ x \in \mathbb{R}^n : x_b = -x_a \text{ and } x_b \leq x_i \forall i \in (R(\alpha) \cup (-L(\alpha))) \text{ and } x_b \geq x_i \forall i \in ((-a, b) \setminus (R(\alpha) \cup (-L(\alpha)))) \}. $$

Proposition 2.5.5. Suppose $\alpha$ is a branched type-D arc with branch point $a$, left endpoint $b_L$, and right endpoint $b_R$. The shard associated to $\alpha$ is

$$\{ x \in \mathbb{R}^n : x_{b_R} = -x_{b_L} \text{ and } x_{b_R} \leq x_i \forall i \in (R(\alpha) \cup (-L(\alpha))) \text{ and } x_{b_R} \geq x_i \forall i \in ((-b_L, b_R) \setminus (R(\alpha) \cup (-L(\alpha))) \cup \{b_L, -b_R\}) \}. $$

2.5.2 Type-D noncrossing arc diagrams

In this section, we define type-D noncrossing arc diagrams and a bijection from $D_n$ to type-D noncrossing arc diagrams.

To define type-D noncrossing arc diagrams, we need a notion of compatibility of type-D arcs. Figs. 2.13 and 2.14 illustrate compatibility of type-D arcs.

Definition 2.5.6. To define compatibility of type-D arcs in terms of equivalence classes of type-B arcs, we need a notion of type-D compatibility of ordinary and long type-B arcs (i.e. type-B arcs that are not orbifold arcs). Two such arcs are type-D compatible if either they

![Figure 2.13: Compatibility between type-D arcs.](image)

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Figure 2.14: Compatibility between type-D arcs.

are compatible as type-B arcs in the sense of Section 2.4 or one is a long arc with endpoints \( a < b \) and no numbered points between its left piece and right piece, while the other is an ordinary arc with the same endpoints \( a < b \) and the two arcs agree between \( a \) and \( b \). We say that two type-D arcs are **compatible** if it is possible to choose an equivalence-class representative of each such that the two representatives are type-D compatible.

A **type-D noncrossing arc diagram** is a collection of pairwise compatible type-D arcs. Examples of type-D noncrossing arc diagrams, represented both in the single-arc model and as equivalence classes, are shown in Fig. 2.15.

We now define a map \( \delta^D \) from even-signed permutations to collections of type-D arcs, which eventually will be the bijection between \( D_n \) and type-D noncrossing arc diagrams (Theorem 2.5.8). The map is illustrated in Fig. 2.16. The map is like the map from \( B_n \) to type-B noncrossing arc diagrams, except that we draw line segments that reflect the natural type-D notion of descents (and at the end we pass to equivalence classes).

Given an even-signed permutation \( \pi \in D_n \), for \( i = -n, \ldots, -2, 2, \ldots, n \), write the entry \( \pi_i \) at the point \((i, \pi_i)\). Also write the entry \( \pi_{-1} \) at the point \((0, \pi_{-1})\) and the point \( \pi_1 \) at the point \((0, \pi_1)\). For \( i = 1, \ldots, n - 1 \), if \( \pi_i > \pi_{i+1} \), draw a line segment from \((i, \pi_i)\) to \((i + 1, \pi_{i+1})\) and from \((-i - 1, \pi_{-i-1})\) to \((-i, \pi_{-i})\). Also, if \( \pi_{-1} > \pi_2 \) draw a line segment from \((-1, \pi_{-1})\) to \((2, \pi_2)\) and from \((-2, \pi_{-2})\) to \((1, \pi_1)\). These line segments do not intersect, and no two of them share a top endpoint or a bottom endpoint, except that we may have drawn a parallelogram with vertices \((0, \pi_1)\), \((2, \pi_2)\), \((-2, \pi_{-2})\), and \((0, \pi_{-1})\). When we move all the points to a vertical line, the result is a collection of symmetric pairs of arcs in the centrally symmetric model, which determine a set of type-B arcs in the orbifold model. None of the arcs in the orbifold model are equivalent to each other, and the set of their equivalence classes
Figure 2.15: Type-D noncrossing arc diagrams for $D_3$. 
is a set $\delta^D(\pi)$ of type-D arcs.

One might naturally wonder why we begin $\delta^D$ with a symmetric picture and then pass to the orbifold model at the end, instead of beginning with a map more like $\delta^o$ from the start. The reason to begin in the symmetric case is that we can think of even-signed permutations as signed permutations which are ambivalent about whether $\pi_1$ or $\pi_{-1}$ is negative: if we know $\pi_2 \cdots \pi_n$, and specifically how many entries in the sequence are negative, we can determine the sign of $\pi_1$ based on that information. Thus, the type-B inversion from $\pi_{-1}$ to $\pi_1$ does not really come into play in type D. Rather, we need to account for inversions of the form $\pi_{-1} > \pi_2$ and/or $\pi_1 > \pi_2$. Since we need more than just the short one-line notation, we may as well start by plotting a complete, symmetric set of points for $\pi$ and then pass to the orbifold model from the symmetric model at the end.

Recall from Proposition 2.2.6 that there is a bijection between shards and join-irreducible elements of the weak order. In particular, Proposition 2.5.2 implies that type-D arcs are in bijection with join-irreducible even-signed permutations.

**Theorem 2.5.7.** For any even-signed permutation $\pi$, the set of join-irreducible elements associated to $\delta^D(\pi)$ is the canonical join-representation of $\pi$.

**Proof.** By Theorem 2.2.7, we need to show that the set of shards associated to the type-D arcs in $\delta^D(\pi)$ is the set of lower shards for the region associated to $\pi$. By construction, there is one arc in $\delta^D(\pi)$ for each symmetric pair of descents of $\pi$, and the corresponding shard $\Sigma$ is in the hyperplane $H$ that is crossed when those descents are reversed. We will check that $\Sigma$ is the lower shard for $\pi$ that lives in $H$ by checking that, for any hyperplane $H'$ that cuts $H$, the shard $\Sigma$ and the region for $\pi$ are on the same side of $H'$. Recall that for $i > j \neq -i$, the region for $\pi$ satisfies $x_i \leq x_j$ if $i$ precedes $j$ in the long one-line notation for $\pi$ and satisfies
$x_i \geq x_j$ if $i$ follows $j$.

The hyperplane $H$ is normal to $e_i - e_k$ for some $i > k \neq -i$, and $H'$ is normal to either $e_i - e_j$ or $e_j - e_k$ for some $j$ with $i > j > k$ with $j \not\in \{-i, -k\}$. Either the entry $i$ immediately precedes $k$ in the long one-line notation for $\pi$, or $i$ and $k$ are in positions $-1$ and $2$ or $-2$ and $1$. If $j$ is left of $i$ and $k$, then the region for $\pi$ satisfies $x_i \geq x_j$ and $x_j \leq x_k$. In this case, inspection of Propositions 2.5.3, 2.5.4 and 2.5.5 shows that $\Sigma$ satisfies the same conditions.

If $j$ is right of $k$, then the region for $\pi$ satisfies $x_i \leq x_j$ and $x_j \geq x_k$, and again $\Sigma$ satisfies the same conditions.

**Theorem 2.5.8.** The map $\delta^D$ is a bijection from $D_n$ to the set of type-D noncrossing arc diagrams.

**Proof.** It is immediate from the definition of $\delta^D$ that, for any $\pi \in D_n$, $\delta^D(\pi)$ is a type-D noncrossing arc diagram. Furthermore, the construction of $\delta^D(\pi)$ produces a representation of $\delta^D(\pi)$ by as type-D compatible type-B arcs and a specific embedding of these arcs in the plane such that no two arcs intersect except possibly at endpoints. Since an element is uniquely determined by its canonical join-representation, Theorem 2.5.7 implies that $\delta^D$ is one-to-one.

We now show that any set of pairwise compatible type-D arcs is $\delta^D(\pi)$ for some $\pi \in D_n$. As a start, we show that, for any two compatible type-D arcs $\alpha_1$ and $\alpha_2$, there exists $\pi$ such that $\alpha_1, \alpha_2 \in \delta^D(\pi)$.

If $\alpha_1$ and $\alpha_2$ are compatible because there is an equivalence-class representative of each that make a type-B compatible pair, then by Theorem 2.4.1, there is a signed permutation $\pi$ such that $\delta(\pi)$ contains the two representatives. Since neither of the representatives is an orbifold arc, they do not arise in $\delta(\pi)$ from the descent in positions $\pm 1$. We can swap $\pi_1$ and $\pi_{-1}$ if necessary to turn $\pi$ into an even-signed permutation $\pi'$, and $\delta^D(\pi')$ contains $\alpha_1$ and $\alpha_2$. (If the representative of $\alpha_1$ or $\alpha_2$ comes from the descent in positions $1, 2$ and $-2, -1$ in $\delta^D(\pi)$, then the same type-B arc will arise in $\delta^D(\pi')$ from positions $1, -2$ and $-1, 2$ and vice versa.)

If $\alpha_1$ is a partially doubled arc with endpoints $0 < a < b$ and $\alpha_2$ is an ordinary arc with the same endpoints, then we construct a type-D permutation $\pi$ by placing $b, a, -a, -b$ in positions $-2, -1, 1, 2$ respectively and then place the remaining elements of $\{\pm 1, \ldots, \pm n\}$ in the long one-line notation for $\pi$, subject to sign-symmetry and placing every element of $R(\alpha_1)$ on the right of the long one-line notation, every element of $L(\alpha_1) \cup L(\alpha_2)$ on the left of the long one-line notation, and otherwise arbitrarily. If the result is not even-signed, then we swap $\pm a$. 

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We have shown that, for any two compatible type-D arcs, there is an even-signed permutation $\pi$ such that $\delta^D(\pi)$ contains them. Thus by Theorem 2.5.7 (and the fact that the canonical join complex is a simplicial complex), any two compatible type-D arcs encode two join-irreducible elements that form a face of the canonical join complex. Since the canonical join complex of the weak order is flag (by Theorem 2.2.1, because the weak order is semidistributive), we conclude that any pairwise compatible set $U$ of type-D arcs encodes a set of join-irreducible elements that forms a face in the canonical join complex. This face is the canonical join-representation of some element $\pi$, and Theorem 2.5.7 says that $U = \delta^D(\pi)$. \qed

The proof of Theorem 2.5.8 does not provide an explicit inverse to $\delta^D$, but we describe the inverse map below in Theorem 2.5.13. For emphasis, we gather some important equivalences in the following theorem, which is an immediate consequence of Theorems 2.5.7 and 2.5.8 and Proposition 2.2.8.

**Theorem 2.5.9.** Two join-irreducible even-signed permutations are compatible (i.e. form a face in the canonical join complex) if and only if the corresponding type-D arcs are compatible if and only if the corresponding shards are compatible.

There are some potential subtleties in the notion of a type-D noncrossing arc diagram that we now clarify. Clarifying these subtleties allows definite answers to some questions that are crucial in working out examples, such as whether a given drawing actually represents a type-D noncrossing arc diagrams, whether a given drawing proves that some collection of type-D arcs is not a type-D noncrossing arc diagram, and how many ways the same type-D noncrossing arc diagram can be represented by type-B arcs. Clarifying these subtleties also clears the way for a description of the inverse to $\delta^D$.

The first subtlety is that we can check the compatibility of a pair $\alpha_1, \alpha_2$ of type-D arcs by choosing one type-B arc representing $\alpha_1$ and we can check the compatibility of a pair $\alpha_1, \alpha_3$ by choosing a different type-B arc representing $\alpha_1$. Thus, it is conceivable that a set $\{\alpha_1, \ldots, \alpha_k\}$ of type-D arcs is pairwise compatible, but that there is no set $\{\alpha'_1, \ldots, \alpha'_k\}$ of pairwise type-D compatible type-B arcs such that each $\alpha'_i$ represents $\alpha_i$. The following corollary shows such a set $\{\alpha'_1, \ldots, \alpha'_k\}$ always exists.

**Corollary 2.5.10.**

1. Every type-D noncrossing arc diagram can be represented as a collection of ordinary and/or long type-B arcs that are pairwise type-D compatible.

2. Every pairwise type-D compatible collection of ordinary and/or long type-B arcs represents a type-D noncrossing arc diagram.
Both representatives of $\alpha_1$ are compatible with $\alpha_2$.

$\alpha'_2$ is not compatible with either representative of $\alpha_1$.

Figure 2.17: Replaceable and irreplaceable type-B arcs, as in Proposition 2.5.11.

Proof. Assertion 1 follows from Theorem 2.5.8 and the definition of $\delta^D$. Assertion 2 is part of the definition of a type-D noncrossing arc diagram, and is included here for emphasis.

Related to the first subtlety is the question of how many ways a given type-D noncrossing arc diagram can be represented by type-B arcs. The following proposition answers that question.

**Proposition 2.5.11.** A noncrossing arc diagram can be represented in at most two ways by a collection of type-B arcs that are pairwise type-D compatible. If there are two ways, they are related by replacing exactly one type-B arc in the collection by an equivalent type-B arc. (In this case, the one type-B arc that can be replaced represents the partially doubled arc with the highest internal endpoint.)

**Proof.** First consider two type-D compatible type-B arcs $\alpha_1$ and $\alpha_2$, both long and having nothing between them. Collectively, these two arcs have 4 distinct endpoints. Let $p$ be the smallest of the 4 and suppose without loss of generality that $\alpha_2$ has $p$ as an endpoint. Then $p$ is either left or right of both sides of $\alpha_1$. Thus if $\alpha'_2$ is the type-B arc that is equivalent (but not equal) to $\alpha_2$, then $\alpha'_2$ is not type-D compatible with $\alpha_1$. See Fig. 2.17.

Now suppose a type-D noncrossing arc diagram has more than one partially doubled arc. Since these are pairwise compatible, one can choose a representative type-B arc for each partially doubled arc to obtain pairwise type-D compatible type-B arcs, all long and having nothing between them. The previous paragraph shows that at most one of these arcs can be exchanged for its equivalent partner without destroying compatibility with the other arcs.

The second subtlety is that we can check the type-D compatibility of a pair $\alpha_1, \alpha_2$ of type-B arcs by choosing one topological-equivalence representative of $\alpha_1$, and we can check the type-D compatibility of a pair $\alpha_1, \alpha_3$ by choosing a different topological-equivalence representative of $\alpha_1$. The following theorem says that we can see the pairwise type-D
compatibility of a collection of type-B arcs while choosing topological representatives of the arcs once and for all.

**Theorem 2.5.12.** Every pairwise type-D compatible collection of ordinary and/or long type-B arcs has a set of topological-equivalence representatives that are pairwise non-intersecting except perhaps at endpoints.

**Proof.** Suppose $A$ is a pairwise type-D compatible collection of ordinary and/or long type-B arcs. This set represents a set $D$ of pairwise compatible type-D arcs. Corollary 2.5.10.1 says that there is a set $A'$ of pairwise type-D compatible type-B arcs that represents $D$. Furthermore, the map $\delta^D$ constructs a set of topological-equivalence representatives of $A'$ that are pairwise non-intersecting except perhaps at endpoints. Proposition 2.5.11 says that either $A = A'$ or $A$ and $A'$ differ by swapping exactly one type-B arc $\alpha$ with its equivalent partner $\alpha'$. Thus $\alpha$ and $\alpha'$ are both compatible with every arc in $A' \setminus \{\alpha'\}$. In particular, no arc in $A' \setminus \{\alpha'\}$ passes between the left piece and right piece of $\alpha'$. Thus the set of pairwise non-intersecting topological equivalence representatives for $A'$ can be turned into a set of pairwise non-intersecting topological equivalence representatives for $A$ by simply replacing the representative of $\alpha'$ with a suitable representative of $\alpha$.

Proposition 2.5.11 states that a type-D noncrossing arc diagram has one or two representations as a set of pairwise type-D compatible type-B arcs, and that any two representations are related by exchanging exactly one type-B arc for its equivalent partner. That fact is helpful in describing the inverse map to $\delta^D$.

**Theorem 2.5.13.** Suppose $N$ is a type-D noncrossing arc diagram. To construct the even-signed permutation $(\delta^D)^{-1}(N)$, first, construct a type-B noncrossing arc diagram as follows:

- If $N$ has only one representation as a collection of pairwise type-D compatible type-B arcs, then these arcs constitute a type-B noncrossing arc diagram.

- If $N$ has two representations and each is a type-B noncrossing arc diagram, then for the partially doubled arc in $N$ with the highest internal endpoint, choose the representation with the internal endpoint on the left.

- If $N$ has two representations and neither is a type-B noncrossing arc diagram, then each has a unique pair of type-D compatible arcs that is not type-B compatible (an ordinary arc and a partially doubled arc in $N$ with the same pair of endpoints), then choose the representation for the partially doubled arc in that pair with the internal endpoint on the right side and delete the ordinary arc.
Apply \((\delta^o)^{-1}\) to that type-B noncrossing arc diagram to obtain a signed permutation. If the result is even-signed, then it is \((\delta^D)^{-1}(N)\). Otherwise, swap the entries in positions \(-1\) and \(1\) to obtain an even-signed permutation, which is \((\delta^D)^{-1}(N)\).

**Proof.** Let \(\pi\) be the signed permutation described in the theorem (obtained by applying \((\delta^o)^{-1}\) to the chosen type-B noncrossing arc diagram). The definition of \(\delta^D\) does not require its input to be even-signed, so \(\delta^D(\pi)\) is not changed by swapping the entries \(\pi_{-1}\) and \(\pi_1\). Thus, to prove the theorem, we need only prove that \(\delta^D(\pi) = N\).

First, suppose \(N\) has only one representation as a collection of pairwise type-D compatible type-B arcs. These type-B arcs are in fact compatible as type-B arcs. Otherwise, they include a pair of arcs with the same endpoints \(a < b\), one partially doubled and one ordinary. Either type-B arc in the equivalence class for this partially doubled arc is compatible with the other type-B arcs since the two inversions that lead to the sharing must be “innermost” in the symmetric plotting of \(\pi\) and thus all other long arcs must pass to both sides of \(a\) or have their internal endpoint below \(a\), so there are two representations of the type-D noncrossing arc diagram. Now it follows from the definitions that \(\delta^D(\pi) = N\).

Next, suppose \(N\) has two representations as a collection of pairwise type-D compatible type-B arcs, both of which are type-B noncrossing arc diagrams. Then \(N\) does not contain a partially doubled arc and an ordinary arc with the same endpoints. Take \(\pi' = (\delta^D)^{-1}(N)\). Because we take the representative with the internal endpoint on the left side, \(|\pi'_1| > |\pi'_2|\) and \(\pi'_2\) is negative. So either \(\pi'_{-1} > \pi'_2\) (if the entries in positions \(-1\) and \(1\) were swapped) or \(\pi'_1 > \pi'_2\) (if they were not swapped), but not both. The equivalent type-B arcs \(\alpha\) and \(\alpha'\) that appear in a representation of \(N\) are associated to the inequality that holds. For this permutation, the construction of \(\delta^D(\pi')\) starts by constructing a set of type-D compatible (and in fact type-B compatible) type-B arcs, including whichever of \(\alpha\) or \(\alpha'\) has its left endpoint at \(|\pi'_2|\), lower than its right endpoint at \(|\pi'_1|\).

Finally, suppose \(N\) has two representations as a collection of pairwise type-D compatible type-B arcs and neither is a type-B noncrossing arc diagram. Then \(N\) contains a partially doubled arc and an ordinary arc with the same endpoints. Again, let \(\pi' = (\delta^D)^{-1}(N)\). Because we take the representative with the internal endpoint on the right side, \(|\pi'_1| < |\pi'_2|\) and \(\pi'_2\) is negative, so \(\pi'_{-2} > \pi'_1 > \pi'_2\) and equivalently \(\pi'_{-2} > \pi'_{-1} > \pi'_2\). Thus, in the process of applying \(\delta^D\) to \(\pi'\), we construct two type-B arcs with endpoints \(|\pi'_2|\) and \(|\pi'_1|\), one ordinary and one long. The long arc has right endpoint at \(|\pi'_1|\), lower than its left endpoint at \(|\pi'_2|\). Taking this representation of \(N\), but deleting the ordinary arc with endpoints \(|\pi_2|\) and \(|\pi_1|\) and applying \((\delta^o)^{-1}\), we obtain either \(\pi'\) or the permutation obtained from \(\pi'\) by swapping the entries \(\pi'_{-1}\) and \(\pi'_1\). \(\square\)
2.5.3 Shard arrows

As a first step towards characterizing lattice congruences of the weak order of type D, in this section we describe the shard arrows.

If \( \alpha_1 \) and \( \alpha_2 \) are type-D arcs, we write \( \alpha_1 \rightarrow \alpha_2 \) to indicate that the corresponding type-D shards have \( \Sigma_1 \rightarrow \Sigma_2 \). We begin by characterizing arrows in terms of equivalence classes.

**Proposition 2.5.14.** Suppose \( \alpha_1 \) and \( \alpha_2 \) are type-D arcs. Then \( \alpha_1 \rightarrow \alpha_2 \) in the type-D shard digraph if and only if there is a type-B arc \( \gamma_1 \) in the equivalence class representing \( \alpha_1 \) and a type-B arc \( \gamma_2 \) in the equivalence class representing \( \alpha_2 \) such that \( \gamma_1 \) is a subarc of \( \gamma_2 \) (in the type-B sense) and \( \gamma_1 \) and \( \gamma_2 \) share exactly one endpoint.

Proposition 2.5.14 follows immediately from Proposition 2.4.25 and the following lemma.

**Lemma 2.5.15.** Suppose \( \alpha_1 \) and \( \alpha_2 \) are type-D arcs. Then \( \alpha_1 \rightarrow \alpha_2 \) in the type-D shard digraph if and only if there is a type-B arc \( \gamma_1 \) in the equivalence class representing \( \alpha_1 \) and a type-B arc \( \gamma_2 \) in the equivalence class representing \( \alpha_2 \) such that \( \gamma_1 \rightarrow \gamma_2 \) in the type-B shard digraph and \( \gamma_1 \) and \( \gamma_2 \) have exactly one endpoint in common.

The requirement that \( \gamma_1 \) and \( \gamma_2 \) have exactly one endpoint in common rules out one of the two kinds of arrows between long/ordinary arcs in type B, namely the case where \( \gamma_1 \) and \( \gamma_2 \) have the same pair of endpoints and \( \gamma_2 \) is long while \( \gamma_1 \) is ordinary (the arrow of type (v) in Proposition 2.4.25). Thus all arrows \( \gamma_1 \rightarrow \gamma_2 \) in Proposition 2.5.14 and Lemma 2.5.15 are of type (i) in Proposition 2.4.25.

**Proof of Lemma 2.5.15.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be the shards associated with \( \alpha_1 \) and \( \alpha_2 \). We use the definition of the shard digraph, which says that \( \Sigma_1 \rightarrow \Sigma_2 \) if and only if \( H_{\Sigma_1} \) cuts \( H_{\Sigma_2} \) and \( \Sigma_1 \cap \Sigma_2 \) has codimension 2.

Suppose \( \Sigma_1 \rightarrow \Sigma_2 \) in the type-D shard digraph. Since \( H_{\Sigma_1} \) cuts \( H_{\Sigma_2} \), the rank-two subarrangement \( \mathcal{A}' \) containing them contains three hyperplanes, and thus is a rank-two subarrangement in type B as well. Since the type-D base region contains the type-B base region, \( H_{\Sigma_1} \) cuts \( H_{\Sigma_2} \) in the type-B arrangement as well. The intersection \( \Sigma_1 \cap \Sigma_2 \) is the union of the intersections \( T_1 \cap T_2 \) over all type-B shards \( T_1 \) contained in \( \Sigma_1 \) and \( T_2 \) contained in \( \Sigma_2 \). If all intersections \( T_1 \cap T_2 \) have codimension greater than 2, then \( \Sigma_1 \cap \Sigma_2 \) has codimension greater than 2. Thus there is some \( T_1 \) and \( T_2 \) such that \( T_1 \cap T_2 \) has codimension 2, and we see that \( T_1 \rightarrow T_2 \) in the type-B shard digraph. The type-B arcs \( \gamma_1 \) and \( \gamma_2 \) corresponding to \( T_1 \) and \( T_2 \) have \( \gamma_1 \rightarrow \gamma_2 \). The rank-two subarrangement \( \mathcal{A}' \) has hyperplanes \( e_i - e_j, e_i - e_k \)
Figure 2.18: Arcs $\alpha_1$ that arrow an ordinary arc $\alpha_2$ (Proposition 2.5.16).

and $e_j - e_k$ for some $i > j > k$ with $j \neq -i$, $k \neq -i$, and $k \neq -j$. In particular, $\gamma_1$ and $\gamma_2$ do not have the same pair of endpoints.

Conversely, suppose $\gamma_1$ is in the equivalence class representing $\alpha_1$, $\gamma_2$ is in the equivalence class representing $\alpha_2$, suppose $\gamma_1 \rightarrow \gamma_2$, and suppose $\gamma_1$ and $\gamma_2$ do not have the same pair of endpoints. Let $T_1$ and $T_2$ be the corresponding type-B shards, so that $T_1 \subseteq \Sigma_1$ and $T_2 \subseteq \Sigma_2$. Then $T_1 \cap T_2$ has codimension 2, and thus $\Sigma_1 \cap \Sigma_2$ has codimension 2, since it contains $T_1 \cap T_2$. Since $\gamma_1$ and $\gamma_2$ do not have the same pair of endpoints, their hyperplanes $H_{T_1}$ and $H_{T_2}$ are in a rank-two subarrangement with three hyperplanes, and therefore these hyperplanes are also in the type-D arrangement. Since $H_{T_1}$ cuts $H_{T_2}$ in the type-B arrangement, it also cuts $H_{T_2}$ in the type-D arrangement. But $H_{T_1} = H_{\Sigma_1}$ and $H_{T_2} = H_{\Sigma_2}$, and we conclude that $\Sigma_1 \rightarrow \Sigma_2$.

In order to describe forcing on type-D arcs with anything approaching conciseness, it will be necessary to pass from the equivalence-class description of type-D arcs to the single-arc description. We describe the arrows $\alpha_1 \rightarrow \alpha_2$ in pictures, breaking into cases according to what kind of arc $\alpha_2$ is. In the pictures, we leave out as many numbered points as we can. First, the arrows to an ordinary arc $\alpha_2$ are described in Proposition 2.5.16 and illustrated in Fig. 2.18. The proposition says that the arrows among ordinary type-D arcs are exactly the same as the arrows in type A. The proposition is an immediate consequence of Proposition 2.5.14.

**Proposition 2.5.16.** Suppose that $\alpha_2$ is an ordinary arc with upper endpoint $b$ and lower endpoint endpoint $a < b$. Then $\alpha_1 \rightarrow \alpha_2$ if and only if $\alpha_1$ is an ordinary arc, shares exactly one endpoint with $\alpha_2$, and has one new endpoint $c$ fulfilling one of the following criteria:

1. $\alpha_1$ has endpoints $b$ and $c$ with $b > c > a$ and $R(\alpha_1) = R(\alpha_2) \cap (c, b)$, or equivalently $L(\alpha_1) = L(\alpha_2) \cap (c, b)$.

2. $\alpha_1$ has endpoints $c$ and $a$ with $b > c > a$ and $R(\alpha_1) = R(\alpha_2) \cap (a, c)$, or equivalently $L(\alpha_1) = L(\alpha_2) \cap (a, c)$.

Next, the arrows to a partially doubled arc $\alpha_2$ are described in Proposition 2.5.17 and illustrated in Fig. 2.19. The arrows on the right pictures represent the relative sides of a
Figure 2.19: Arcs $\alpha_1$ that arrow a partially doubled arc $\alpha_2$ (Proposition 2.5.17).

and $c$; there are also arrows where the diagrams shown in the figure are reflected through a vertical line.

**Proposition 2.5.17.** Suppose that $\alpha_2$ is a partially doubled arc with internal endpoint $a$ and external endpoint $b > a$. Then $\alpha_1 \rightarrow \alpha_2$ if and only if $\alpha_1$ shares exactly one endpoint of $\alpha_2$, has one new endpoint $c < b$, and fulfills one of the following criteria:

1. $\alpha_1$ is a partially doubled arc such that
   (a) $\alpha_1$ has internal endpoint $a$ and external endpoint $c$ with $b > c > a$, $L(\alpha_1) = L(\alpha_2)$, and $R(\alpha_1) = R(\alpha_2) \cap (0, c)$.
   (b) $\alpha_1$ has internal endpoint $c$ and external endpoint $b$ with $b > a > c$, $L(\alpha_1) = L(\alpha_2) \cap (0, c)$, and either $R(\alpha_1) = R(\alpha_2)$ if $c \in L(\alpha_2)$ or $R(\alpha_1) = R(\alpha_2) \cup \{a\} \setminus \{c\}$ if $c \in R(\alpha_2)$.
   (c) $\alpha_1$ has internal endpoint $c$ and external endpoint $a$ with $b > a > c$, $L(\alpha_1) = L(\alpha_2) \cap (0, c)$, and $R(\alpha_1) = R(\alpha_2) \cap (0, a) \setminus \{c\}$.

2. $\alpha_1$ is an ordinary arc such that
   (a) $\alpha_1$ has upper endpoint $b$ and lower endpoint $c$ with $b > c > a$, and $R(\alpha_1) = R(\alpha_2) \cap (c, b)$.
   (b) $\alpha_1$ has upper endpoint $a$ and lower endpoint $c$ with $b > a > c$, and $R(\alpha_1) = R(\alpha_2) \cap (c, a)$.
   (c) $\alpha_1$ has upper endpoint $b$ and lower endpoint $c$ with $b > a > c$, and either $R(\alpha_1) = R(\alpha_2) \cap (c, b) \cup \{a\}$ if $c \in L(\alpha_2)$ or $R(\alpha_1) = R(\alpha_2) \cap (c, b)$ if $c \in R(\alpha_2)$.

**Proof.** The criteria record the possible cases of Proposition 2.5.14 when $\alpha_2$ is partially doubled. Suppose first that $\gamma_2$ has $a$ as its right endpoint and $b$ as its left endpoint. Choosing a $\gamma_1$ corresponds to cutting $\gamma_2$ into two pieces at some point $c$ in $(0, a)$ or $(a, b)$ and choosing
one of the pieces. More specifically, we have the following choices. Choosing $c \in (a, b)$ the two pieces correspond to the criteria (1a) and (2a). Choosing $c \in (0, a)$ and cutting the right piece of $\gamma_2$ makes two pieces that correspond to the criteria (1b) and (2b). Since $a$ is the right endpoint of $\gamma_2$, a cut can only be made at $c$ if $c \in R(\gamma_2)$, and thus $c \in R(\alpha_2)$. For the same reason, if we cut at $c$ and choose the resulting long arc to be $\gamma_1$, the partially doubled arc $\alpha_1$ corresponding to $\gamma_1$ has $a \in R(\alpha_1)$. Choosing $c \in (0, a)$ and cutting the left piece of $\gamma_2$ makes two pieces that correspond to the criteria (1c) and (2c). In this case, a cut can only be made at $c$ if $c \in L(\gamma_2)$ and thus $c \in L(\alpha_2)$, and if we cut at $c$ and choose $\gamma_1 = \alpha_1$ to be the resulting ordinary arc, then $a \in R(\alpha_1)$.

Supposing next that $\gamma_2$ has $a$ as its left endpoint and $b$ as its right endpoint, we have the same range of choices, but the requirements are different for cuts at $c \in (0, a)$. In this case, criteria (1b) and (2b) arise from cutting the left piece of $\gamma_2$ at $c$. Such a cut can only be made if $c \in L(\gamma_2)$ and thus in $L(\alpha_2)$, and the resulting partially doubled arc $\alpha_1$ has $c \in L(\alpha_1)$. Similarly, criteria (1c) and (2c) arise from cutting the right piece of $\gamma_2$. This can only be done if $c \in R(\gamma_2)$, and the resulting ordinary arc $\alpha_1$ has $a \in L(\alpha_1)$.

Finally, the arrows to a branched arc $\alpha_2$ are described in Proposition 2.5.18 and illustrated in Fig. 2.20. Again, besides the arrows pictured, there are also arrows where the diagrams shown in the figure are reflected through a vertical line. Also, in these pictures, there is no distinction between the case $b_L < b_R$ and $b_L > b_R$, although a choice must be made in the drawing and we have chosen $b_L > b_R$. The analogous arrows exist when $b_L < b_R$. A branched arc has only one representation as a type-B arc, and the proposition is a straightforward listing of the possibilities in Proposition 2.5.14 in this case.

**Proposition 2.5.18.** Suppose that $\alpha_2$ is a branched arc with branch point $a$, left endpoint $b_L > a$ and right endpoint $b_R > a$ such that $b_L \neq b_R$. Then $\alpha_1 \rightarrow \alpha_2$ if and only if $\alpha_1$ shares exactly one endpoint of $\alpha_2$, has one new endpoint $c$, and fulfills one of the following criteria:

1. $\alpha_1$ is a branched arc with branch point $a$ such that
   
   (a) $b_L$ is the left endpoint and $b_R > c > a$, so $c \neq b_L$ is the right endpoint of $\alpha_1$, and $L(\alpha_1) = L(\alpha_2)$ and $R(\alpha_1) = R(\alpha_2) \cap (0, c)$, or
   
   (b) $b_R$ is the right endpoint and $b_L > c > a$, so $c \neq b_R$ is the left endpoint of $\alpha_1$, and $L(\alpha_1) = L(\alpha_2) \cap (0, c)$ and $R(\alpha_1) = R(\alpha_2)$.

2. $\alpha_1$ is a partially doubled arc with internal endpoint $c \leq a$ and
   
   (a) external endpoint $b_L$ with $c \in R(\alpha_2)$ and $L(\alpha_1) = L(\alpha_2) \cap (0, c)$ and $R(\alpha_1) = (0, b_L) \setminus (L(\alpha_2) \cup \{c\})$, or

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(b) external endpoint $b_R$ with $c \in L(\alpha_2)$ and $L(\alpha_1) = L(\alpha_2) \cap (0, c)$ and $R(\alpha_1) = R(\alpha_2)$.

3. $\alpha_1$ is an ordinary arc with lower endpoint $c$ and

(a) upper endpoint $b_L$ with $c \neq b_R$, $c \not\in R(\alpha_2)$ and $L(\alpha_1) = L(\alpha_2) \cap (c, b_L)$, or

(b) upper endpoint $b_R$ with $c \neq b_L$, $c \not\in L(\alpha_2)$ and $R(\alpha_1) = R(\alpha_2) \cap (c, b_R)$.

### 2.5.4 Subarcs and forcing

We now define the notion of a subarc of a type-D arc. The main result is Theorem 2.5.24, below, which asserts that the subarc relation on arcs corresponds to the forcing relation on join-irreducible elements.

Because branched arcs and partially doubled arcs are long arcs in the type-B orbifold model, Definition 2.4.20 applies to both. A branched arc can have subarcs that are branched, partially doubled, or ordinary. A partially doubled arc can have subarcs that are partially doubled or ordinary. An ordinary arc can only have ordinary subarcs, by Definition 2.4.19.

**Definition 2.5.19** (Ordinary subarc of an ordinary arc). An ordinary type-D arc $\alpha_1$ is a subarc of another ordinary type-D arc $\alpha_2$ if and only if $\alpha_1$ is a subarc of $\alpha_2$ in the type-A sense defined in Section 2.3.

In the definition of subarcs $\alpha_1$ of a partially doubled arc $\alpha_2$ (and thus indirectly for subarcs of a branched arc), we choose a sequence of points. Details vary in different cases,
but the general idea is that these points alternate sides (left/right) of \( \alpha_2 \) and are on the opposite side of \( \alpha_1 \) as they are of \( \alpha_2 \). In different cases, it is convenient to describe this alternation/switching slightly differently (for example by requiring that they alternate sides of \( \alpha_1 \)).

**Definition 2.5.20** (Partially doubled subarc of a partially doubled arc). Let \( \alpha_1 \) be a partially doubled arc with internal endpoint \( c \) and external endpoint \( d \). Let \( \alpha_2 \) be a partially doubled arc with internal endpoint \( a \) and external endpoint \( b \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) if and only if one of the following conditions holds.

(i) \( b \geq d > a \geq c \) and there exist \( c_0, \ldots, c_k \) with \( a = c_0 > c_1 > \cdots > c_k = c \) (allowing \( k = 0 \) so that \( a = c_0 = c \)) such that

- \( R(\alpha_1) \setminus \{c_0, \ldots, c_k\} = (R(\alpha_2) \setminus \{c_0, \ldots, c_k\}) \cap (0, d) \),
- \( c_i+1 \in R(\alpha_2) \) if and only if \( c_i \notin R(\alpha_2) \) for \( 1 \leq i \leq k-1 \), and
- \( c_i-1 \in R(\alpha_1) \) if and only if \( c_i \in R(\alpha_2) \) for \( 1 \leq i \leq k \).

(ii) \( b > a \geq d > c \) and there exist \( c_1, \ldots, c_k \) with \( d > c_1 > \cdots > c_k = c \) (necessarily with \( k > 0 \)) such that

- \( R(\alpha_1) \setminus \{c_1, \ldots, c_k\} = (R(\alpha_2) \setminus \{c_1, \ldots, c_k\}) \cap (0, d) \),
- \( c_i+1 \in R(\alpha_2) \) if and only if \( c_i \notin R(\alpha_2) \) for \( 1 \leq i \leq k-1 \), and
- \( c_i-1 \in R(\alpha_1) \) if and only if \( c_i \in R(\alpha_2) \) for \( 2 \leq i \leq k \).

These two conditions are illustrated in Fig. 2.21. As with the earlier pictures of arrows, the figures illustrate which sides the points are relative to each other, and similar subarc relations exist with all pictures reflected through a vertical line. Also, each picture reflects a specific choice of the parity of \( k \), and changing that parity would change the picture by changing whether \( c_1 \) and \( c_k \) are on the same or opposite sides of \( \alpha_2 \).

**Definition 2.5.21** (Ordinary subarc of a partially doubled arc). Let \( \alpha_1 \) be an ordinary arc with upper endpoint \( d \) and lower endpoint \( c \). Let \( \alpha_2 \) be a partially doubled arc with internal endpoint \( a \) and external endpoint \( b \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) if and only if one of the following conditions holds.

(i) \( b \geq d > c > a \) and \( R(\alpha_1) = R(\alpha_2) \cap (c, d) \).

(ii) \( b \geq d > c = a > 1 \) and \( R(\alpha_1) = R(\alpha_2) \cap (c, d) \).
\[\begin{align*}
\text{(iii)} & \quad b \geq d > a > c \text{ and there exist } c_0, \ldots, c_k \text{ with } a = c_0 > c_1 > \cdots > c_k > c \text{ (allowing } k = 0 \text{ so that } a = c_0 > c) \text{ such that} \\
& \quad \bullet \ R(\alpha_1) \setminus \{c_0, \ldots, c_k\} = (R(\alpha_2) \setminus \{c_0, \ldots, c_k\}) \cap (c, d), \\
& \quad \bullet \ c_{i-1} \in R(\alpha_1) \text{ if and only if } c_i \not\in R(\alpha_1) \text{ for } 1 \leq i \leq k, \\
& \quad \bullet \ c_i \in R(\alpha_1) \text{ if and only if } c_i \not\in R(\alpha_2) \text{ for } 1 \leq i \leq k, \text{ and} \\
& \quad \bullet \text{ if } c = 1, \text{ then } c_k \in R(\alpha_1) \text{ if and only if } c \not\in R(\alpha_2). \\
\text{(iv)} & \quad b > a \geq d > c \text{ and there exist } c_1, \ldots, c_k \text{ with } d > c_1 > \cdots > c_k > c \text{ (allowing } k = 0 \text{ so that there are no } c_i \text{'s) such that} \\
& \quad \bullet \ R(\alpha_1) \setminus \{c_1, \ldots, c_k\} = (R(\alpha_2) \setminus \{c_1, \ldots, c_k\}) \cap (c, d), \\
& \quad \bullet \ c_{i-1} \in R(\alpha_1) \text{ if and only if } c_i \not\in R(\alpha_1) \text{ for } 2 \leq i \leq k, \\
& \quad \bullet \ c_i \in R(\alpha_1) \text{ if and only if } c_i \not\in R(\alpha_2) \text{ for } 1 \leq i \leq k, \text{ and} \\
& \quad \text{If } c = 1 \text{ and } k > 0, \text{ then } c_k \in R(\alpha_1) \text{ if and only if } c \not\in R(\alpha_2). \\
\end{align*}\]

These four conditions are illustrated in Fig. 2.22. The same considerations regarding reflections in a vertical line and the parity of \(k\) hold in this figure as in Fig. 2.21. In addition, in Fig. 2.22, we illustrate the additional requirements when \(c = 1\) by showing subarc relationships that occur for any \(c\) and subarc relationships that only occur for \(c > 1\) (or for \(c > 1\) or \(k = 0\)).

**Definition 2.5.22** (Branched subarc of a branched arc). Let \(\alpha_2\) be a branched arc with branch point \(a\), left endpoint \(b_L > a\) and right endpoint \(b_R > a\). Let \(\alpha_1\) be a branched arc with branch point \(c\), left endpoint \(d_L > c\) and right endpoint \(d_R > c\). Then \(\alpha_1\) is a subarc of \(\alpha_2\) if and only if \(a = c\), \(b_L \geq d_L\), \(b_R \geq d_R\), \(L(\alpha_1) = L(\alpha_2) \cap (0, d_L)\), and \(R(\alpha_1) = R(\alpha_2) \cap (0, d_R)\).

This definition is illustrated in Fig. 2.23, where again we have made an arbitrary decision about the relative heights of \(b_L, b_R, d_L,\) and \(d_R\). The only constraints are \(b_L \geq d_L > a\) and \(b_R \geq d_R > a\).
Figure 2.22: Ordinary subarcs $\alpha_1$ of a partially doubled arc $\alpha_2$ (Definition 2.5.21).

Figure 2.23: A branched subarc $\alpha_1$ of a branched arc $\alpha_2$ (Definition 2.5.22).
Figure 2.24: The partially doubled subarcs $\alpha_L$ and $\alpha_R$ of a branched arc $\alpha_2$ (Definition 2.5.23).

Figure 2.25: Ordinary subarcs $\alpha_1$ of a branched arc $\alpha_2$ with branch point 1 (Definition 2.5.23).

Definition 2.5.23 (Ordinary or partially doubled subarc of a branched arc). Let $\alpha_2$ be a branched arc with branch point $a$, left endpoint $b_L > a$ and right endpoint $b_R > a$. Let $\alpha_L$ be the partially doubled arc with $R(\alpha_L) = [R(\alpha_2) \cap (0, a)] \cup [(a, b_L) \setminus L(\alpha_2)]$ and $L(\alpha_L) = L(\alpha_2) \cap (0, a)$. Let $\alpha_R$ be the partially doubled arc with $R(\alpha_R) = R(\alpha_2)$ and $L(\alpha_R) = L(\alpha_2) \cap (0, a)$. The arcs $\alpha_L$ and $\alpha_R$ are illustrated in Fig. 2.24. If $a > 1$, then an ordinary or partially doubled arc $\alpha_1$ is a subarc of $\alpha_2$ if and only if $\alpha_1$ is a subarc of $\alpha_L$ or of $\alpha_R$ (or of both). If $a = 1$, then the ordinary or partially doubled subarcs of $\alpha_2$ consist of all subarcs of $\alpha_L$, all subarcs of $\alpha_R$, and all ordinary arcs $\alpha_1$ with upper endpoint $d$ and lower endpoint 1 having either $b_L \geq d$ and $L(\alpha_1) = L(\alpha_2) \cap (1, d)$ or $b_R \geq d$ and $R(\alpha_1) = R(\alpha_2) \cap (1, d)$. Ordinary subarcs in the case $a = 1$ are shown in Fig. 2.25.

The following theorem is the type-D analogue of Theorem 2.4.22.

Theorem 2.5.24. Let $j_1$ and $j_2$ be join-irreducible even-signed permutations. Then $j_1$ forces $j_2$ if and only if the type-D arc corresponding to $j_1$ is a subarc of the type-D arc corresponding to $j_2$.

As corollaries, we obtain type-D analogues of Corollaries 2.3.6 and 2.3.7. The following result is a rephrasing of Theorem 2.5.24.

Corollary 2.5.25. A set $U$ of type-D arcs corresponds to the set of uncontracted join-irreducible even-signed permutations of some congruence $\Theta$ on $D_n$ if and only if $U$ is closed under passing to subarcs.

Combining Corollary 2.5.25 with Theorems 2.5.7 and 2.5.8 and Corollary 2.2.3, we obtain the following result.
Corollary 2.5.26. If \( \Theta \) is a congruence on \( D_n \) and \( U \) is the set of type-D arcs corresponding to join-irreducible even-signed permutations not contracted by \( \Theta \), then \( \delta^D \) restricts to a bijection from the quotient \( D_n/\Theta \) (the set of even-signed permutations not contracted by \( \Theta \)) to the set of type-D noncrossing arc diagrams consisting only of arcs in \( U \).

The case of Theorem 2.5.24 where \( \alpha_2 \) is an ordinary arc is proved using Proposition 2.5.16 just as Theorem 2.3.5 is proved using Proposition 2.3.8. For the remaining cases, we prove each direction separately. One direction of the theorem is to show that if \( \alpha_1 \) is a subarc of \( \alpha_2 \), then \( j_1 \) forces \( j_2 \). We now prove this in two propositions, one for \( \alpha_2 \) partially doubled and one for \( \alpha_2 \) branched. In each proof, we use without comment the fact that the forcing relation on join-irreducible elements corresponds to the transitive closure of the \( \rightarrow \) relation on arcs. For convenience, we also refer to forcing relations between arcs, meaning forcing relations between the corresponding join-irreducible elements.

**Proposition 2.5.27.** If \( \alpha_2 \) is a partially doubled arc and \( \alpha_1 \) is a subarc of \( \alpha_2 \), then \( \alpha_1 \) forces \( \alpha_2 \).

**Proof.** We need to check every case of Definitions 2.5.20 and 2.5.21, so we organize the proof according to those cases. In each case, we use the notation from the definition; in particular, \( \alpha_2 \) has endpoints \( b > a \) and \( \alpha_1 \) has endpoints \( d > c \).

**Definition 2.5.20(i).** In this case, \( \alpha_1 \) is partially doubled and \( b \geq d > a \geq c \). We show by induction on \( k \) that \( \alpha_1 \) forces \( \alpha_2 \). The sequence of arrows constructed by induction is illustrated in Fig. 2.26. If \( k = 0 \) then \( b \geq d > a = c \), so either \( \alpha_1 \rightarrow \alpha_2 \) if \( b > d \) (Criterion (1a) of Proposition 2.5.17) or \( \alpha_1 = \alpha_2 \) if \( b = d \), and we’re done. If \( k > 0 \), then \( b \geq d > a > c \). Let \( \alpha' \) be the partially doubled arc obtained from \( \alpha_1 \) by making \( c_{k-1} \) an internal endpoint and putting \( c \) on the side of \( \alpha' \) where \( c \) is in \( \alpha_2 \). Then \( \alpha' \) is a subarc of \( \alpha_2 \), so by induction, \( \alpha' \) forces \( \alpha_2 \). Also, \( \alpha_1 \rightarrow \alpha' \) by Criterion (1b) of Proposition 2.5.17, and thus \( \alpha_1 \) forces \( \alpha_2 \).
Definition 2.5.20(ii). In this case, $\alpha_1$ is partially doubled and $b > a \geq d > c$. Let $\alpha''$ be a partially doubled subarc of $\alpha_2$ with endpoints $b$ and $c$ which has $\alpha_1$ as a subarc. We have just shown that $\alpha''$ forces $\alpha_2$. Also, $\alpha_1 \rightarrow \alpha''$ by Criterion (1a) of Proposition 2.5.17, and thus $\alpha_1$ forces $\alpha_2$. Fig. 2.27 illustrates the sequence of arrows from $\alpha_1$ to $\alpha_2$ in this case.

Definition 2.5.21(i). In this case, $\alpha_1$ is ordinary and $b \geq d > c > a$. Either $\alpha_1 \rightarrow \alpha_2$ (if $b = d$) by Proposition 2.5.17(2) or $\alpha_1 \rightarrow \alpha' \rightarrow \alpha_2$, where $\alpha'$ is the ordinary subarc of $\alpha_2$ with endpoints $c$ and $b$ ($\alpha_1 \rightarrow \alpha'$ by Proposition 2.5.16(2) and $\alpha' \rightarrow \alpha_2$ by Proposition 2.5.17(2)). The latter case is shown on the left side of Fig. 2.28.

Definition 2.5.21(ii). In this case, $\alpha_1$ is ordinary and $b \geq d > c = a > 1$. Let $\alpha''$ be a partially doubled subarc of $\alpha_2$ with external endpoint $b$ and internal endpoint $a'$ such that $a' < a = c$ as in Proposition 2.5.17(2). Then the previous paragraph shows that $\alpha_1$ forces $\alpha''$. Also, $\alpha'' \rightarrow \alpha_2$ as in the first case of this proof, so $\alpha_1$ forces $\alpha_2$. The sequence of arrows from $\alpha_1$ to $\alpha_2$ is shown on the right side of Fig. 2.28.

Definition 2.5.21(iii). In this case, $\alpha_1$ is ordinary and $b \geq d > a > c$. There is a partially doubled arc $\alpha'$ with endpoints $d$ and $c_k$ and $R(\alpha') \cap (c, d) = R(\alpha_1) \setminus \{c_k\}$. Then $\alpha'$ is a subarc of $\alpha_2$ by Definition 2.5.20(i), so $\alpha'$ forces $\alpha_2$. We will show that $\alpha_1$ forces $\alpha'$. If $c$ and $c_k$ are on the same side of $\alpha_2$, then there is an arrow $\alpha_1 \rightarrow \alpha'$. This case is illustrated in the top line of Fig. 2.29. If $c$ and $c_k$ are on the opposite side of $\alpha_2$, then Definition 2.5.21 requires that $c > 1$, so there exists at least one point $c'$ with $1 \leq c' < c$. If there exists $c' < c$ such that $c'$
Figure 2.29: Subarc implies forcing, Definition 2.5.21(iii).
and $c_k$ are on the same side of $\alpha_2$, then there is an ordinary arc $\alpha''$ with endpoints $c'$ and $d$ such that $\alpha_1 \to \alpha'' \to \alpha'$. This case is illustrated in the middle line of Fig. 2.29. Otherwise, there exists $c' < c$ such that $c'$ and $c_k$ are on opposite sides in $\alpha_2$, and there exists a partially doubled arc $\alpha''$ with endpoints $c'$ and $d$ with $R(\alpha'') \cap (c, d) = R(\alpha_1)$. Again, $\alpha_1 \to \alpha'' \to \alpha'$. This case is illustrated in the bottom line of Fig. 2.29.

**Definition 2.5.21(iv).** In this case, $\alpha_1$ is ordinary and $b > a \geq d > c$. Set $c_0 = a$ and let $\alpha'$ be an ordinary subarc of $\alpha_2$ described in Definition 2.5.21(iii) with endpoints $b$ and $c$ defined by the sequence $c_0, c_1, \ldots, c_k$. (If $k = 0$ and $c > 1$, then there are two such subarcs, one with $a$ to the left and one with $a$ to the right. Otherwise, there is only one such subarc, with $a$ to a side determined by whether $c_1 \in R(\alpha_2)$ when $k > 0$ or by whether $c \in R(\alpha_2)$ when $k = 0$ and $c = 1$. See Fig. 2.19.) Then $\alpha'$ forces $\alpha_2$, and also $\alpha_1 \to \alpha'$ as in Proposition 2.5.16.

**Proposition 2.5.28.** If $\alpha_2$ is a branched arc and $\alpha_1$ is a subarc of $\alpha_2$, then $\alpha_1$ forces $\alpha_2$.

**Proof.** Suppose $\alpha_2$ is a branched arc with branch point $a$, left endpoint $b_L > a$ and right endpoint $b_R > a$.

If $\alpha_1$ is a branched subarc of $\alpha_2$ with branch point $c$ and left endpoint $d_L > c$ and right endpoint $d_R > c$, then $a = c, b_L \geq d_L$, and $b_R \geq d_R$. There is a sequence of 0, 1 or 2 arrows from $\alpha_1$ to $\alpha_2$: If $b_L > d_L$, then there is an arrow from $\alpha_1$ to a branched arc $\alpha'$ with branch point $a$, left endpoint $b_L$, and right endpoint $d_R$ (Criterion (1a) of Proposition 2.5.18). If $b_R > d_R$, there is an arrow from $\alpha'$ to $\alpha_2$ (Criterion (1b) of Proposition 2.5.18). If $d_L = b_L$, then the first arrow in the sequence $\alpha_1 \to \alpha' \to \alpha_2$ is replaced by equality, and if $d_R = b_R$, then the second arrow is replaced by equality. We see that every branched subarc of $\alpha_2$ forces $\alpha_2$.

Suppose $\alpha_1$ is partially doubled or ordinary. Then in almost all cases, Definition 2.5.23 says that $\alpha_1$ is a subarc of $\alpha_L$ or $\alpha_R$ or both. We have already shown that every subarc of $\alpha_L$ (or $\alpha_R$) forces $\alpha_L$ (or $\alpha_R$). By Criterion (2) of Proposition 2.5.18, $\alpha_L \to \alpha_2$ and $\alpha_R \to \alpha$, so $\alpha_1$ forces $\alpha_2$. Finally, suppose $\alpha_1$ is not a subarc of $\alpha_L$ or $\alpha_R$. Then $a = 1$, and $\alpha_1$ is ordinary with endpoints 1 and $d$ and either $b_L \geq d$ and $L(\alpha_1) = L(\alpha_2) \cap (1, d)$ or $b_R \geq d$ and $R(\alpha_1) = R(\alpha_2) \cap (1, d)$. In either case, there is an ordinary arc $\alpha'$ with endpoints 1 and $b_L$ or $b_R$ and an arrow from $\alpha'$ to $\alpha_2$ (Criterion (3b) of Proposition 2.5.18) such that either $\alpha_1 = \alpha'$ or $\alpha_1 \to \alpha'$.

The other direction of the proof of Theorem 2.5.24 is to show that if $\alpha_1$ forces $\alpha_2$ then $\alpha_1$ is a subarc of $\alpha_2$. If $\alpha_1$ forces $\alpha_2$, then there is a sequence of arrows from $\alpha_1$ to $\alpha_2$. Thus we can finish the proof by showing that if $\alpha'$ is a subarc of $\alpha_2$ and $\alpha_1 \to \alpha'$, then $\alpha_1$ is also a subarc of $\alpha_2$. We do this in two propositions, one for $\alpha_2$ partially doubled and one for $\alpha_2$ branched. (Recall that the case where $\alpha_2$ is ordinary is already finished.)
Proposition 2.5.29. If α₂ is a partially doubled arc, α' is a subarc of α₂ and α₁ → α', then α₁ is also a subarc of α₂.

Proof. Suppose α₂ is a partially doubled arc with internal endpoint a and external endpoint b. The subarc α' may be either partially doubled or ordinary. Let c and d denote the endpoints of α', where c < d. There are six cases for α', described in Definitions 2.5.20 and 2.5.21. Within each case, the possibilities for α₁ are given by Proposition 2.5.16 or Proposition 2.5.17.

Case 1. α' is a partially doubled arc with \( b \geq d > a \geq c \). Let \( c₀, c₁, \ldots, cₖ \) be the sequence with \( a = c₀ > c₁ > \cdots > cₖ = c \) satisfying the conditions of Definition 2.5.20(i). We list the possibilities for α₁ in the order given in Proposition 2.5.17, shifting the letters in Proposition 2.5.17 so that the endpoints c and d of α' become the a and b in the proposition and α₁ has new endpoint e, corresponding to c in the proposition.

Subcase 1a. α₁ is partially doubled with internal endpoint c and external endpoint e such that \( d > e > c \), \( L(α₁) = L(α') \cap (0, c) \), and \( R(α₁) = R(α') \cap (0, c) \) as in Proposition 2.5.17(1a). Let \( ℓ = \{0, \ldots, k\} \) be the index such that \( c_{ℓ-1} ≥ e > cₖ \). If \( e > a \), so that \( ℓ = 0 \), then α₁ is a subarc of α₂ with \( b > e > a ≥ c \) and the sequence \( c₀, c₁, \ldots, cₖ \) satisfying the conditions of Definition 2.5.20(i). If \( e ≤ a \), so that \( ℓ > 0 \), then α₁ is a subarc of α₂ with \( b > a ≥ e ≥ c \) and the sequence \( cₖ, \ldots, c₁ \) satisfying the conditions of Definition 2.5.20(ii).

Subcase 1b. α₁ is partially doubled with internal endpoint e and external endpoint d such that \( d > c > e \), \( L(α₁) = L(α') \cap (0, e) \), and either \( R(α₁) = R(α') \cup \{e\} \setminus \{e\} \) if \( e \in R(α') \) or \( R(α₁) = R(α') \) if \( e \in L(α') \) as in Proposition 2.5.17(1b). If \( e \) and \( c \) are on the same side of α₂, then α₁ is a subarc of α₂ with \( b ≥ d > a > e \) and the sequence \( c₀, c₁, \ldots, c_{k-1}, e \) satisfying the conditions of Definition 2.5.20(i). (In this case, since \( a ≥ c > e \), the option of \( k = 0 \) is not possible.) If \( e \) and \( c \) are on opposite sides of α₂, then α₁ is a subarc of α₂ with \( b ≥ d > a > e \) and the sequence \( c₀, c₁, \ldots, c_k, c_{k+1} = e \) satisfying the conditions of Definition 2.5.20(i).

Subcase 1c. α₁ is partially doubled with internal endpoint e and external endpoint c such that \( d > c > e \), \( L(α₁) = L(α') \cap (0, e) \), and \( R(α₁) = R(α') \cap (0, c) \setminus \{e\} \) as in Proposition 2.5.17(1c). Then α₁ is a subarc of α₂ as in Definition 2.5.20(ii) with \( k = 1 \).

Subcase 1d. α₁ is ordinary with upper endpoint d and lower endpoint e such that \( d > e > c \) and \( R(α₁) = R(α') \cap (e, d) \) as in Proposition 2.5.17(2a). Let \( ℓ \in \{0, \ldots, k\} \) be the index such that \( c_{ℓ-1} > e ≥ cₖ \). If \( ℓ > 0 \) (i.e. if \( e < a \)) then the sequence \( c₀, \ldots, c_{ℓ-1} \) satisfies the conditions of Definition 2.5.21(iii) and thus α₁ is a subarc of α₂. (Note that since \( e > c \), also \( e > 1 \).) If \( ℓ = 0 \), (i.e. if \( e ≥ a \)) then α₁ is a subarc of α₂ as in Definition 2.5.21(i) or (ii). (Definition 2.5.21(ii) applies when \( e = a \). In this case, it must be true that \( a > 1 \), because if \( a = 1 \), it would force \( e = c = 1 \) so that there would be no arrow \( α₁ → α' \) according to Proposition 2.5.17.)
Subcase 1e. \( \alpha_1 \) is ordinary with upper endpoint \( c \) and lower endpoint \( e \) such that \( d > c > e \) and \( R(\alpha_1) = R(\alpha') \cap (e, c) \) as in Proposition 2.5.17(2b). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(iv) with \( k = 0 \) since \( e < c \leq a \).

Subcase 1f. \( \alpha_1 \) is ordinary with upper endpoint \( d \) and lower endpoint \( e \) such that \( d > c > e \) and \( R(\alpha_1) = R(\alpha') \cap (e, d) \cup \{c \} \) if \( e \in L(\alpha') \) or \( R(\alpha_1) = R(\alpha') \cap (e, d) \) if \( e \in R(\alpha') \) as in Proposition 2.5.17(2c). We consider separately whether \( e \) and \( c \) are on the same or opposite side of \( \alpha_2 \). If \( e \) and \( c \) are on the same side of \( \alpha_2 \), then the sequence \( a = c_0, c_1, \ldots, c_k = c \) satisfies the conditions of Definition 2.5.21(iii) for \( \alpha_1 \) to be a subarc of \( \alpha_2 \). (The extra condition in the case \( e = 1 \) is precisely the fact that \( c \) and \( e \) are assumed to be on the same side of \( \alpha_2 \).) If \( e \) and \( c \) are on opposite sides of \( \alpha_2 \), then the sequence \( a = c_0, c_1, \ldots, c_{k-1} \) satisfies the conditions of Definition 2.5.21(iii) for \( \alpha_1 \) to be a subarc of \( \alpha_2 \). (In this case, since \( a \geq c > e \), the option of \( k = 0 \) is not possible.) We again need to check that if \( e = 1 \), then \( e \in R(\alpha') \) and \( e, c \) are on opposite sides of \( \alpha_2 \). (The extra condition in the case \( e = 1 \) is precisely the fact that \( c \) and \( e \) are assumed to be on the same side of \( \alpha_2 \).)

Case 2. \( \alpha' \) is a partially doubled arc with \( b > a \geq d > c \). Let \( c_1, \ldots, c_k \) be the sequence with \( d > c_1 > \cdots > c_k = c \) satisfying the conditions of Definition 2.5.20(ii). We emphasize that \( k \geq 0 \) in this case. We again list the possibilities for \( \alpha_1 \) in the order given in Proposition 2.5.17.

Subcase 2a. \( \alpha_1 \) is partially doubled with internal endpoint \( c \) and external endpoint \( e \) such that \( d > e > c \), \( L(\alpha_1) = L(\alpha') \), and \( R(\alpha_1) = R(\alpha') \cap (0, c) \) as in Proposition 2.5.17(1a). Let \( \ell \in \{1, \ldots, k\} \) be the index such that \( c_{\ell-1} \geq e > c_\ell \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with \( b > a > e > c \) and the sequence \( c_\ell, \ldots, c_k \) (which is nonempty since \( c_k = c \)) satisfying the conditions of Definition 2.5.20(ii).

Subcase 2b. \( \alpha_1 \) is partially doubled with internal endpoint \( e \) and external endpoint \( d \) such that \( d > c > e \), \( L(\alpha_1) = L(\alpha') \cap (0, e) \), and either \( R(\alpha_1) = R(\alpha') \cup \{e \} \setminus \{e \} \) if \( e \in R(\alpha') \) or \( R(\alpha_1) = R(\alpha') \) if \( e \in L(\alpha') \) as in Proposition 2.5.17(1b). If \( e \) and \( c \) are on the same side of \( \alpha_2 \), then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with \( b \geq d > a > e \) and the sequence \( c_1, \ldots, c_{k-1}, e \) (which may be just the one-term sequence \( e \)) satisfying the conditions of Definition 2.5.20(ii). If \( e \) and \( c \) are on opposite sides of \( \alpha_2 \), then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with \( b \geq d > a > e \) and the sequence \( c_1, \ldots, c_k = c, c_{k+1} = e \) satisfying the conditions of Definition 2.5.20(ii).

Subcase 2c. \( \alpha_1 \) is partially doubled with internal endpoint \( e \) and external endpoint \( c \) such that \( d > c > e \), \( L(\alpha_1) = L(\alpha') \cap (0, e) \), and \( R(\alpha_1) = R(\alpha') \cap (0, c) \setminus \{e \} \) as in Proposition 2.5.17(1c). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.20(ii) with \( k = 0 \).

Subcase 2d. \( \alpha_1 \) is ordinary with upper endpoint \( d \) and lower endpoint \( e \) such that \( d > e > c \) and \( R(\alpha_1) = R(\alpha') \cap (e, d) \) as in Proposition 2.5.17(2a). Let \( \ell \in \{1, \ldots, k\} \) be the index such
that \( c_{\ell - 1} > e \geq c_{\ell} \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with \( b > a \geq d > e \) and the sequence \( c_1, \ldots, c_{\ell - 1} \) (which may be empty) satisfying the conditions of Definition 2.5.21(iv). (Since \( e > c \geq 1 \), the last bullet point of Definition 2.5.21(iv) does not apply.)

**Subcase 2e.** \( \alpha_1 \) is ordinary with upper endpoint \( c \) and lower endpoint \( e \) such that \( d > c > e \) and \( R(\alpha_1) = R(\alpha') \cap (e, c) \) as in Proposition 2.5.17(2b). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(iv) with \( k = 0 \) since \( a > c > e \).

**Subcase 2f.** \( \alpha_1 \) is ordinary with upper endpoint \( d \) and lower endpoint \( e \) such that \( d > c > e \) and \( R(\alpha_1) = R(\alpha') \cap (e, d) \cup \{c\} \) if \( e \in L(\alpha') \) or \( R(\alpha_1) = R(\alpha') \cap (e, d) \) if \( e \in R(\alpha') \) as in Proposition 2.5.17(2c). We again consider separately whether \( e \) and \( c \) are on the same or opposite side of \( \alpha_2 \). If \( e \) and \( c \) are on the same side of \( \alpha_2 \), then the sequence \( c_1, \ldots, c_k = c \) satisfies the conditions of Definition 2.5.21(iv) for \( \alpha_1 \) to be a subarc of \( \alpha_2 \). (The extra condition in the case \( e = 1 \) and \( k > 0 \) is precisely the fact that \( c \) and \( e \) are assumed to be on the same side of \( \alpha_2 \).) If \( e \) and \( c \) are on opposite sides of \( \alpha_2 \), then the sequence \( c_1, \ldots, c_{k-1} \) satisfies the conditions of Definition 2.5.21(iv) for \( \alpha_1 \) to be a subarc of \( \alpha_2 \). (If \( k = 1 \), then this sequence is empty.) The extra condition when \( e = 1 \) and \( k - 1 > 0 \) is that \( e \in R(\alpha_2) \) if and only if \( c_{k-1} \in R(\alpha_2) \). This is true regardless of whether \( e = 1 \) because \( c \) and \( c_{k-1} \) are on opposite sides of \( \alpha_2 \) and \( c \) and \( e \) are on opposite sides of \( \alpha_2 \).

**Case 3.** \( \alpha' \) is an ordinary arc with upper endpoint \( d \) and lower endpoint \( c \) such that \( b \geq d > c > a \) and \( R(\alpha') = R(\alpha_2) \cap (c, d) \) as in Definition 2.5.21(i). There are two cases for \( \alpha_1 \) as in Proposition 2.5.16. In either case, \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(i).

**Case 4.** \( \alpha' \) is an ordinary arc with upper endpoint \( d \) and lower endpoint \( c \) such that \( b \geq d > c > a \) and \( R(\alpha') = R(\alpha_2) \cap (c, d) \) as in Definition 2.5.21(ii). There are again two cases for \( \alpha_1 \) as in Proposition 2.5.16. In the first case, where \( \alpha_1 \) shares its upper endpoint with \( \alpha' \), \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(i). In the second case, where \( \alpha_1 \) shares its lower endpoint with \( \alpha' \), \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(ii).

**Case 5.** \( \alpha' \) is an ordinary arc with upper endpoint \( d \) and lower endpoint \( c \) such that \( b \geq d > a > c \). Let \( c_0, \ldots, c_k \) be the sequence with \( a = c_0 > c_1 > \cdots > c_k > c \) satisfying the conditions of Definition 2.5.21(iii).

**Subcase 5a.** \( \alpha_1 \) has upper endpoint \( d \) and lower endpoint \( e \) such that \( d > e > c \) and \( R(\alpha_1) = R(\alpha') \cap (e, d) \) as in Proposition 2.5.16(1). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(i) if \( e > a \) or as in Definition 2.5.21(ii) if \( e = a \). (The condition \( a > 1 \) in Definition 2.5.21(ii) is satisfied because \( a > c \).) If \( a > e \), let \( \ell \in \{0, \ldots, k\} \) be the index such that \( c_{\ell} > e \geq c_{\ell+1} \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with the sequence \( a = c_0, \ldots, c_{\ell} \) satisfying the conditions of Definition 2.5.21(iii), so \( \alpha_1 \) is a subarc of \( \alpha_2 \). (The case \( e = 1 \) is impossible because \( e > c \).)
Subcase 5b. \( \alpha_1 \) has upper endpoint \( e \) and lower endpoint \( c \) such that \( d > e > c \) and \( R(\alpha_1) = R(\alpha') \cap (c, e) \) as in Proposition 2.5.16(2). If \( e > a \), then \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.21(iii), with the same sequence \( a = c_0, \ldots, c_k \). If \( a \geq e \), let \( \ell \in \{0, \ldots, k\} \) be the index such that \( c_\ell \geq e > c_{\ell+1} \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with the sequence \( c_\ell, \ldots, c_k \) (empty if \( \ell = k \)) satisfying the conditions of Definition 2.5.21(iv).

Case 6. \( \alpha' \) is an ordinary arc with upper endpoint \( d \) and lower endpoint \( c \) such that \( b > a \geq d > c \). Let \( c_1, \ldots, c_k \) be the sequence with \( d > c_1 > \cdots > c_k > c \) satisfying the conditions of Definition 2.5.21(iv) and for convenience, write \( c_0 = d \) and \( c_{k+1} = c \).

Subcase 6a. \( \alpha_1 \) has upper endpoint \( d \) and lower endpoint \( c \) such that \( d > e > c \) and \( R(\alpha_1) = R(\alpha') \cap (e, d) \) as in Proposition 2.5.16(1). Let \( \ell \in \{0, \ldots, k\} \) be the index such that \( c_\ell > e \geq c_{\ell+1} \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with the sequence \( c_1, \ldots, c_\ell \) (empty if \( \ell = 0 \)) satisfying the conditions of Definition 2.5.21(iv). (The case \( e = 1 \) is impossible because \( e > c \).)

Subcase 6b. \( \alpha_1 \) has upper endpoint \( e \) and lower endpoint \( c \) such that \( d > e > c \) and \( R(\alpha_1) = R(\alpha') \cap (c, e) \) as in Proposition 2.5.16(2). Let \( \ell \in \{1, \ldots, k+1\} \) be index the such that \( c_{\ell-1} \geq e > c_\ell \). Then \( \alpha_1 \) is a subarc of \( \alpha_2 \) with the sequence \( c_\ell, \ldots, c_k \) (empty if \( \ell = k+1 \)) satisfying the conditions of Definition 2.5.21(iv).

Proposition 2.5.30. If \( \alpha_2 \) is a branched arc, \( \alpha' \) is a subarc of \( \alpha_2 \) and \( \alpha_1 \to \alpha' \), then \( \alpha_1 \) is also a subarc of \( \alpha_2 \).

Proof. Let \( a \) be the branch point of \( \alpha_2 \), let \( b_L > a \) be the left endpoint and let \( b_R \) be the right endpoint. Let \( \alpha_L \) and \( \alpha_R \) be as in Definition 2.5.23. Suppose \( \alpha' \) is a subarc of \( \alpha_2 \). There are two possibilities for \( \alpha' \), given by Definitions 2.5.22 and 2.5.23.

Case 1. \( \alpha' \) is a branched arc with branch point \( a \). In this case, Definition 2.5.22 says that the left endpoint of \( \alpha' \) is \( d_L \) with \( b_L \geq d_L > a \), the right endpoint of \( \alpha' \) is \( d_R \) with \( b_R \geq d_R > a \), and \( L(\alpha') = L(\alpha_2) \cap (0, d_L) \) and \( R(\alpha') = R(\alpha_2) \cap (0, d_R) \). Given \( \alpha_1 \to \alpha' \), there are three subcases, described in Proposition 2.5.18.

Subcase 1a. \( \alpha_1 \) is a branched arc. In this case, it is immediate from Proposition 2.5.18 that \( \alpha_1 \) is a subarc of \( \alpha_2 \) as in Definition 2.5.22.

Subcase 1b. \( \alpha_1 \) is a partially doubled arc with internal endpoint \( c \leq a \) and external endpoint \( d_L \) or \( d_R \). If \( c = a \), then \( \alpha_1 \) arrows \( \alpha_L \) or \( \alpha_R \) (depending on whether its external endpoint is \( d_L \) or \( d_R \)), so \( \alpha_1 \) is a subarc of \( \alpha_2 \). Now assume \( c < a \).

Suppose first that \( \alpha_1 \) has endpoints \( c \) and \( d_L \). By Proposition 2.5.18(2a), \( c \in R(\alpha') \) and \( L(\alpha_1) = L(\alpha') \cap (0, c) \) and \( R(\alpha_1) = (0, d_L) \setminus (L(\alpha') \cup \{c\}) \). To satisfy Definition 2.5.23,
we need to show that \( \alpha_1 \) is a subarc of \( \alpha_L \). Since \( c \in R(\alpha') \), also \( c \in R(\alpha_2) \), and since also \( c < a \), we see that \( c \in R(\alpha_L) \). Also, since \( a \notin L(\alpha') \) and \( R(\alpha_1) = (0, d_L) \setminus (L(\alpha') \cup \{c\}) \), we have \( a \in R(\alpha_1) \). Thus \( \alpha_1 \) is a subarc of \( \alpha_L \) satisfying the conditions of Definition 2.5.20(i) with \( k = 1 \).

Suppose next that \( \alpha_1 \) has endpoints \( c \) and \( d_R \). By Proposition 2.5.18(2b), \( c \in L(\alpha') \) and \( L(\alpha_1) = L(\alpha') \cap (0, c) \) and \( R(\alpha_1) = R(\alpha') \). This time, we need to check that \( \alpha_1 \) is a subarc of \( \alpha_R \). We know \( c \in L(\alpha_2) \) so \( c \in L(\alpha_R) \). Also, \( a \notin R(\alpha') = R(\alpha_1) \). Thus \( \alpha_1 \) is a subarc of \( \alpha_R \) satisfying the conditions of Definition 2.5.20(i) with \( k = 1 \).

Subcase 1c. \( \alpha_1 \) is an ordinary arc with lower endpoint \( c \) and upper endpoint \( d_L \) or \( d_R \). If \( c = a = 1 \), then \( \alpha_1 \) is a subarc of \( \alpha_2 \) by Definition 2.5.23. Now assume that \( a \) and \( c \) are not both 1. We must show that \( \alpha_1 \) is a subarc of \( \alpha_R \) or \( \alpha_L \). If \( c = a > 1 \), then \( \alpha_1 \) is a subarc of \( \alpha_R \) or \( \alpha_L \) by Definition 2.5.21(i). If \( c > a \), then \( \alpha_1 \) is a subarc of \( \alpha_R \) or \( \alpha_L \) by Definition 2.5.21(i). If \( c < a \), then there are two cases, either \( c \in L(\alpha_2) \) or \( c \in R(\alpha_2) \). If \( c \in L(\alpha_2) \), then \( c \in L(\alpha_L) \) and \( a \in R(\alpha_1) \). Then \( R(\alpha_1) \setminus \{a\} = (R(\alpha_L) \setminus \{a\}) \cap (c, d_L) \) and so \( \alpha_1 \) is a subarc of \( \alpha_L \) by Definition 2.5.21(iii) with \( k = 0 \). (The extra condition if \( c = 1 \) is satisfied because \( a \in R(\alpha_1) \) and \( c \notin R(\alpha_L) \).) If \( c \in R(\alpha_2) \), then \( c \in R(\alpha_R) \) and \( a \in L(\alpha_1) \). Then \( R(\alpha_1) \setminus \{a\} = (R(\alpha_R) \setminus \{a\}) \cap (c, d_R) \) and so \( \alpha_1 \) is a subarc of \( \alpha_R \) by Definition 2.5.21(iii) with \( k = 0 \). (The condition for \( c = 1 \) is satisfied because \( a \notin R(\alpha_1) \) and \( c \in R(\alpha_R) \).)

Case 2. \( \alpha' \) is a partially doubled arc. Then \( \alpha' \) is a subarc of \( \alpha_L \) or \( \alpha_R \) (or of both). If \( \alpha_1 \to \alpha' \), then Proposition 2.5.29 says that \( \alpha_1 \) is a subarc of \( \alpha_L \) or \( \alpha_R \) (or of both). Thus \( \alpha_1 \) is a subarc of \( \alpha_2 \) by Definition 2.5.23.

Case 3. \( \alpha' \) is an ordinary arc. If \( \alpha' \) is a subarc of \( \alpha_L \) or \( \alpha_R \) (or of both), then \( \alpha_1 \) is a subarc of \( \alpha_2 \) by Proposition 2.5.29 as in Case 2. Otherwise \( a = 1 \), and \( \alpha' \) is ordinary with upper endpoint \( d \) and lower endpoint 1 and either \( b_L \geq d \) and \( L(\alpha_1) = L(\alpha_2) \cap (1, d) \) or \( b_R \geq d \) and \( R(\alpha_1) = R(\alpha_2) \cap (1, d) \). If \( \alpha_1 \to \alpha' \) then \( \alpha_1 \) is ordinary and there are two cases. In the first case, \( \alpha_1 \) has upper endpoint \( c \) and lower endpoint 1 with \( d > c > 1 \), so \( \alpha_1 \) is a subarc of \( \alpha_2 \) by Definition 2.5.23. In the second case, \( \alpha_1 \) has upper endpoint \( d \) and lower endpoint \( c \) with \( d > c > 1 = a \), so \( \alpha_1 \) is a subarc of \( \alpha_L \) or \( \alpha_R \) by Definition 2.5.21(i) and thus a subarc of \( \alpha_2 \) by Definition 2.5.23.

We have completed the proof of Theorem 2.5.24. We emphasize a consequence of the theorem, which is immediate in light of the congruence uniformity of the weak order. (See Section 2.2.2.)

**Corollary 2.5.31.** The subarc relation is a partial order on type-D arcs.
2.5.5 Superarcs

In using Theorem 2.5.24 and Corollaries 2.5.25 and 2.5.26, it is often necessary to find the superarcs of a given arc \( \alpha \) (meaning that arcs that have \( \alpha \) as a subarc). In this section, we describe superarcs by reversing Definitions 2.5.19 to 2.5.23.

**Superarcs of an ordinary arc.** Suppose \( \alpha \) is an ordinary arc with upper endpoint \( d \) and lower endpoint \( c \). A superarc of \( \alpha \) may be ordinary, partially doubled, or branched.

To construct an ordinary superarc of \( \alpha \), we push the top and/or bottom of the arc right or left of \( d \) and/or \( c \), independently, and then extend upward and/or downward to make a longer ordinary arc.

We construct a partially doubled superarc of \( \alpha \) as follows: Choose \( a \) and \( b \) with \( b > a \) and \( b \geq d \), and \( a \geq c \) to be the endpoints of the superarc. If \( b > d \), then push the arc to the left or right of the point \( d \) and extend it up to \( b \). If \( c \geq a \), then push \( \alpha \) to the left or right of \( c \), extend the arc down to \( \times \), push it onto \( a \), the new internal endpoint, and we’re done. If \( a > c \), choose a sequence of points \( a = c_0, c_1, \ldots, c_k \) with \( k \geq 0 \) and \( \min(a, d) > c_1 > \cdots > c_k > c \). The sequence \( c_0, \ldots, c_k \) must alternate sides of \( \alpha \), except that if \( a \geq d \), there is no requirement on \( c_0 = a \), which is on neither side of \( \alpha \). We then push the arc left or right of \( c \), as follows: If \( c > 1 \), we push the arc to either side of \( c \). If \( c = 1 \), we must push the arc to the left or right of the point \( c \), so that the arc is on the opposite side of \( c \) as of \( c_k \), except that if \( k = 0 \) and \( d \leq a = c_0 \), then we can push \( c \) to either side. Once we have pushed the arc to one side of \( c \), we extend it down to \( \times \). For each \( i \in \{1 \ldots, k\} \), we push the arc through \( c_i \) (so that the superarc and \( \alpha \) are on opposite sides of each \( c_i \)). Finally, push the arc onto \( a \), the new internal endpoint.

In every case, we can construct a branched superarc of \( \alpha \) by first constructing a partially doubled superarc \( \beta_X \) with internal endpoint \( a \) and external endpoint \( b \) as above. Then push \( \beta_X \) to the left or right of \( a \) and extend \( \beta_X \) to a branched arc \( \beta \) with branch point \( a \) such that \( \beta_X \) is either \( \beta_L \) or \( \beta_R \) in the sense of Definition 2.5.23. If \( c = 1 \) there is also a direct way to construct a branched superarc of \( \alpha \) without going through a partially doubled superarc: Push the bottom of \( \alpha \) left (or right) of \( 1 \) and extend it down to \( \times \), then add a right (or left) branch to make a branched superarc with branch point 1.

**Superarcs of a partially doubled arc.** Suppose \( \alpha \) is a partially doubled arc with internal endpoint \( c \) and external endpoint \( d \). A superarc of \( \alpha \) is either partially doubled or branched.

We construct a partially doubled superarc of \( \alpha \) as follows: Choose \( a \) and \( b \) with \( b > a \), \( b \geq d \), and \( a \geq c \) to be the endpoints of the superarc. If \( b > d \), then push the arc to the left or right of the point \( d \) and extend it up to \( b \). If \( a = c \), then we’re done. If \( a > c \), choose a sequence of points \( a = c_0, c_1, \ldots, c_{k-1} \) with \( k \geq 1 \) and \( \min(a, d) > c_1 > \cdots > c_{k-1} > c \). The
sequence $c_0, \ldots, c_{k-1}$ must alternate sides of $\alpha$, except that if $a \geq d$, there is no requirement on $c_0 = a$, which is on neither side of $\alpha$. Push the arc to the left or right of the point $c$, so that the arc is on the same side of $c$ as of $c_{k-1}$ (unless $k = 1$ and $a \geq d$, in which case the arc can be pushed to either side of $c$) and extend the arc down to $\times$. For each $i \in \{1 \ldots, k-1\}$, push the arc through $c_i$ (so that the superarc and $\alpha$ are on opposite sides of each $c_i$). Finally, push the arc onto the point $a$, the new internal endpoint.

To construct a branched superarc of $\alpha$, first construct a partially doubled superarc $\beta_X$ with internal endpoint $a$ and external endpoint $b$ as above. Then push $\beta_X$ to the left or right of $a$ and extend $\beta_X$ to a branched arc $\beta$ with branch point $a$ such that $\beta_X$ is either $\beta_L$ or $\beta_R$ in the sense of Definition 2.5.23.

**Superarcs of a branched arc.** We construct a superarc of a branched arc $\alpha$ by independently pushing one or both branches left or right of their endpoints and extending the branches upwards to form a branched arc.
3.1 Introduction

The primary goal of this section is to determine whether the shard intersection order on type $A_n$ is a sublattice of the shard intersection order on type $B_n$. This is one instance of Question 1.1.2.

We first establish a foundation in Coxeter groups of type $A$. Existing results relating permutations of $[n]$ and noncrossing arc diagrams on $n$ points are restated, and a bijection between noncrossing arc diagrams and permutation pre-orders is established. Permutation pre-orders were introduced and used to create an EL-labeling of $\Psi(A_n)$ in [4]. In Section 3.2.2, we present a characterization of the type-A weak order in terms of noncrossing arc diagrams that expands upon [1, Corollary 38]. We define two operations which have pairs of noncrossing arc diagrams as inputs in Definitions 3.3.8 and 3.3.10, and additional results show that the objects returned by both operations are themselves noncrossing arc diagrams. In Section 3.3.4, we restate existing results on $\Psi(A_n)$ in terms of noncrossing arc diagrams, including how the lattice $\Psi(A_n)$ is graded and how to go up by a cover in it. (Previous characterizations are stated in [4, 28].) These results build to the two main theorems in type A: that the operations we define on noncrossing arc diagrams characterize the meet and join in $\Psi(A_n)$ (Theorems 3.3.15 and 3.3.19).
In Section 3.4, we present type-B analogues of many objects and results from Section 3.3. We briefly consider the centrally symmetric model of noncrossing arc diagrams (which appears in [6, 1]), mainly to point out that the type-A operations defined in Section 3.3.3 preserve symmetry. A more thorough exploration of the orbifold model includes Definitions 3.4.3 and 3.4.5, type-B analogues of the type-A operations defined in Section 3.3.3. In Section 3.4.2, the results from the two models combine so that we are able to cut quickly to the chase: The analogous type-B operations also characterize the meet and join in $\Psi(B_n)$ (Theorems 3.3.15 and 3.3.19).

In Section 3.5, we are finally able to address Question 1.1.2 in the case where $B_n$ dominates $A_n$. Background for this section includes the characterizations of congruences on the weak order in terms of a partial pre-order on arcs (for type A in [35] and type B in Section 2.4), as well as results on surjective lattice homomorphisms from $(B_n, \leq)$ to $(A_n, \leq)$ presented in [38, Section 6]. Each of the four subsections contains the same progression of results for a single surjective lattice homomorphism $\eta$ from $B_n$ to $A_n$. First, we characterize the congruence defined by $\eta$ in terms of contracted and uncontracted type-B noncrossing arc diagrams. Next, we describe the natural inclusion $\zeta$ from $A_n$ to $B_n$ that is the inverse of the restriction of $\eta$ to the bottom elements of $(B_n, \leq)$ in the congruence it defines. We also provide the outputs of $\zeta$ for the same four permutations in $A_6 = S_7$ (in Examples 3.5.3 to 3.5.6, 3.5.10 to 3.5.13, 3.5.16 to 3.5.19 and 3.5.22 to 3.5.25). In Propositions 3.5.7, 3.5.14, 3.5.20 and 3.5.26, $\zeta$ is described in terms of type-A noncrossing arc diagram inputs and resulting type-B noncrossing arc diagram outputs. Finally, we state and prove that $\zeta$ either does (as in Theorems 3.5.8 and 3.5.21 and Proposition 3.5.26) or does not (as in Example 3.5.15) embed $\Psi(A_n)$ as a sublattice of $\Psi(A_n)$.

### 3.2 Preliminaries

In this section, we establish relevant background on finite Coxeter groups and lattice theory. We assume minimal foundational knowledge of algebra and combinatorics concepts. Standard references which provide an introduction to Coxeter groups, finite and otherwise, include Björner and Brenti [10] and Humphreys [21].

We follow the structure of Section 2.2, adding complementary background on lattice theory, finite Coxeter groups, posets of regions, and shards that will be relevant to the main results of this chapter.
3.2.1 Lattices and congruences

Throughout this section, we consider only finite lattices. A subset $U$ of a lattice $L$ is a sublattice if for all $x, y \in U$, $x \land_L y$ and $x \lor_L y$ are also in $U$. In other words, if $U$ is a sublattice of $L$, then it is a lattice using the restriction of $\land_L$ and $\lor_L$ to $U$.

Recall that lattice congruences on $L$ (equivalence relations that respect $\land_L$ and $\lor_L$) contract its edges, and that we say that a congruence $\Theta$ contracts a join-irreducible element $j \in L$ if it contracts the edge between $j$ and the only element it covers. The lattice quotient $L/\Theta$ is isomorphic to the subposet of $L$ consisting of the bottom (uncontracted) elements of the congruence classes of $\Theta$ (which are intervals in $L$), and $L/\Theta$ is also a lattice.

Consider two lattices $L$ and $M$ and a surjective lattice homomorphism $\eta : L \to M$. The set of fibers of $\eta$ is a congruence $\Theta$ on $L$, and $M$ is isomorphic to the lattice quotient $L/\Theta$. As an induced subposet of $L$, the set of bottom elements of fibers of $\eta$ is isomorphic to $M$. The set of bottom elements of fibers is also a join-subsemilattice of $L$ (a subposet in which the join operation holds), but it is not necessarily a sublattice [37, Theorem 9-5.8].

The map $\zeta : M \to L$ is said to embed $M$ as a sublattice of $L$ if $\zeta$ is one-to-one and $\zeta(x \land_M y) = \zeta(x) \land_L \zeta(y)$ and $\zeta(x \lor_M y) = \zeta(x) \lor_L \zeta(y)$ for all $x, y \in M$. In this case, $\zeta(M)$ is a sublattice of $L$ and the restriction of $\zeta$ to its image is an isomorphism.

3.2.2 Coxeter groups and the weak order

In this section, we begin by providing an introduction to Coxeter groups. A Coxeter group $W$ with generating set $S = \{s_1, \ldots, s_n\}$, also called the Coxeter system $(W, S)$, has the group presentation

$$\langle s_1, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1 \text{ for all } i, j \in [n]\rangle$$

such that each $m_{ii} = 1$ for each $i$ and $m_{ij} = m_{ji}$ for each pair $\{i, j\}$. In particular, since $m_{ii} = 1$, each generator $s_i$ is an involution. To be a finite Coxeter group, $(W, S)$ has additional requirements on the $m_{ij}$ for distinct pairs of generators.

The information of an irreducible finite Coxeter group (informally, a group that is not just a product of other smaller groups) is conveyed by its Coxeter diagram, a tree with vertices representing generators and labeled or unlabeled edges between generators $s_i$ and $s_j$ that have order $m_{ij}$ at least three. There are finitely many families of irreducible finite Coxeter groups, and three of these families are infinite: type A, type B, and type D. The family of type-A Coxeter groups is, in fact, the family of symmetric groups: a Coxeter group of type $A_n$ is the group $S_{n+1}$ of permutations of $[n+1] = \{1, \ldots, n+1\}$ where the generators are transpositions of the form $s_i = (i \ i+1)$. Similarly, the Coxeter groups of type $B_n$ and $D_n$ are the groups of signed permutations of $\pm[n]$ and even-signed permutations of $\pm[n]$, respectively.
respectively, with similar sets of generators. The Coxeter diagrams for $A_n$, $B_n$, and $D_n$ are pictured in Fig. 3.1.

Next, we state existing lattice theoretic results that are specific to Coxeter groups. In a Coxeter system $(W, S)$, the parabolic subgroup $W_{S'}$ is the set of elements in $W$ generated by a subset $S'$ of $S$. If $S' = \{s_i, s_j\}$, then the weak order restricted to $W_{S'}$ is the interval from the identity to the alternating product of $s_i$ and $s_j$ with length $m_{ij}$.

A lattice congruence $\Theta$ of the weak order on $W$ is said to be generated by the minimal set, element-wise (with respect to the weak order) and set-wise (with respect to containment), of join-irreducible elements contracted by $\Theta$. The support of an element $w$ in a Coxeter group $W$, is the smallest set of generators $S' \subset S$ such that $w$ is in the parabolic subgroup $W_{S'}$. The degree of a join-irreducible element $j$ in $W$ is the size of its support. A congruence $\Theta$ generated by the set of join-irreducible elements $J = \{j_1, \ldots, j_k\}$ is said to be homogeneous of degree $d$ if each $j_i \in J$ has degree $d$.

### 3.2.3 Shards, congruences, and forcing

In this section, we add a few definitions and state results that will be of use in Sections 3.3 and 3.4.

For a Coxeter group $W$, the poset of regions of the Coxeter arrangement $\mathcal{A}(W)$ is isomorphic, as a lattice, to the right weak order on $W$. (See [11, 16, 17, 37].) A region $R$ covers another region $Q$ if there is exactly one hyperplane $H$ that separates $R$ from the base region $B$ but doesn’t separate $Q$ from $B$. If $R$ corresponds to $w \in W$ and $Q$ corresponds to $v \in W$, then $H$ is the hyperplane corresponding to the cover reflection of $w$ associated to the cover $v \preceq w$.) In this instance, $H$ is a facet-defining hyperplane of $R$ and of $Q$, since both $R \cap H$ and $Q \cap H$ have codimension 1. $H$ defines the common facet of the two regions. If a hyperplane $H$ is a facet-defining hyperplane of $R$ and a member of $R$’s separating set, it is a lower hyperplane of $R$. The set of lower hyperplanes for $R$ is denoted $\mathcal{L}(R)$.  

![Coxeter diagrams for $A_n$, $B_n$, and $D_n$.](image)
Recall that the shard intersection order of \( W \) was constructed by imposing the cutting relation between hyperplanes, taking closures of disconnected components to create shards, identifying elements of \( W \) with their lower shards, then ordering them by reverse containment of the intersections of lower shards. In [42, Theorem 2.10.5], Stump, Thomas, and Williams state a pair of conditions which, together, give an alternative characterization of the shard intersection order. The result is restated in [37, Theorem 9-7.24] in a manner that is particularly useful for the proofs of the main theorems, though it is stated for simplicial hyperplane arrangements and has an unfortunate typo where the order of a containment is reversed. Recall that Coxeter arrangements are simplicial, and for a Coxeter group \( W \), the ordering on the poset of regions of its arrangement \( \mathcal{A} \) is the right weak order. The following theorem is [37, Theorem 9-7.24] in the language of Coxeter groups, with the typo corrected. In it, \( \preceq \) denotes the shard intersection order on \( W \) and \( \leq \) denotes the weak order on \( W \).

**Theorem 3.2.1.** Suppose \( \mathcal{A} \) is the arrangement of a Coxeter group \( W \) and consider two regions regions \( Q \) and \( R \). Then \( Q \preceq R \) if and only if \( Q \leq R \) and \( \bigcap_{H \in \mathcal{L}(Q)} H \supseteq \bigcap_{H \in \mathcal{L}(R)} H \).

### 3.3 Type A

The Coxeter group of type \( A_{n-1} \) can be realized as the group \( S_n \) of permutations of \( [n] = \{1, \ldots, n\} \). Each simple reflection \( s_i \) is a transposition of the form \((i \ i+1)\). The right weak order on \( A_{n-1} \) is the partial order whose cover relations are given by swapping adjacent entries to put them out of order; it is also characterized in terms of inversion sets. An inversion of a permutation \( \pi = \pi_1 \cdots \pi_n \) is an ordered pair \((\pi_i, \pi_j)\) such that \( i < j \) and \( \pi_i > \pi_j \). The set of all inversions of \( \pi \), called its inversion set is denoted \( \text{inv}(\pi) \). A permutation \( \sigma \) is below another permutation \( \tau \) in the weak order if and only if \( \text{inv}(\sigma) \) is a subset of \( \text{inv}(\tau) \). To go up by a cover in the weak order from a permutation \( \sigma \), swap a pair of adjacent entries which are in increasing order in \( \sigma \) to create exactly one new inversion.

A descent of \( \pi \) is a pair \((\pi_i, \pi_{i+1})\) such that \( \pi_i > \pi_{i+1} \). (In other settings, the term "descent" denotes the position \( i \) where the descent occurs.) A descending run of \( \pi \) is a maximal sequence \( \pi_j \pi_{j+1} \cdots \pi_k \) such that \((\pi_i, \pi_{i+1})\) is a descent for all \( i = j, \ldots, k - 1 \).

#### 3.3.1 Permutations and noncrossing arc diagrams

We define noncrossing arc diagrams on \( n \) points, for a positive integer \( n \), following [35]. Begin by placing \( n \) distinct points on a vertical line and identifying the points with \( [n] = \{1, \ldots, n\} \) in order, with 1 at the bottom and \( n \) at the top. A noncrossing arc diagram consists of some (or no) curves called arcs, each of which satisfies the requirement that it connects
a point \( r \in [n] \) to a lower point \( p \), moving monotone downward from \( r \) to \( p \) and passing either left or right of each point between \( r \) and \( p \). Moreover, a collection of arcs constitutes a noncrossing arc diagram precisely when it satisfies two pairwise compatibility conditions. First, no pair of arcs may intersect except at their endpoints. Second, any endpoint shared by two arcs must be the upper endpoint of one arc and the lower endpoint of the other. Two arcs are combinatorially equivalent if they have the same upper and lower endpoints and pass left and right of the same points. Noncrossing arc diagrams are considered up to combinatorial equivalence. Fig. 3.2 shows all noncrossing arc diagrams on 4 points.

![Figure 3.2: 24 noncrossing arc diagrams on 4 points.](image)

Noncrossing arc diagrams on \( n \) points are shown in [35] to be in bijection with permutations of \([n]\). The bijection \( \delta \) from a permutation \( \pi \in S_n \) to its corresponding noncrossing arc diagram can be described as follows: Plot each entry \( \pi_i \) at the point \((i, \pi_i)\) in the plane, then draw a line segment from \((i, \pi_i)\) to \((i + 1, \pi_{i+1})\) for each descent \((\pi_i, \pi_{i+1})\). After drawing all necessary line segments, slide the points left or right into a vertical line, so that the line segments bend to avoid hitting vertices, as shown in Fig. 3.3.

In [4, Section 2.3], Bancroft introduces permutation pre-orders, which are in bijection with permutations, and uses them to find an EL-labeling on the shard intersection order of type A. Inspired by Bancroft’s work, we define a block in a noncrossing arc diagram as a connected component of the diagram considered as a graph.

A block \( B \) may consist of a single point, having no arcs. If \( B \) has at least one arc, then \( B \) has endpoints \( p = q_0 < q_1 < \cdots < q_s = r \) such that \( q_{i-1} \) and \( q_i \) are connected by an arc for all \( i \in \{1, \ldots, s\} \). Thus exactly one arc is attached to each of \( r \) and \( p \) and exactly two arcs have each \( q_i \) with \( i \in \{1, \ldots, s-1\} \) as a shared endpoint. Such a block corresponds to the descending run \( q_s \cdots q_0 \) in a permutation. Each point in the set \( \{q_1, \ldots, q_s = r\} \) is the upper endpoint of some arc in \( B \); we may refer to any of these points as an upper endpoint.
of $B$. Similarly, each point in the set $\{p = q_0, \ldots, q_{s-1}\}$ is the lower endpoint of some arc in $B$, and we may refer to any of these points as a lower endpoint of $B$. Notably, $B$ may have more than one upper endpoint; we distinguish the unique highest point $r$ of $B$ by calling it the top endpoint of $B$. Likewise, we call $p$ the bottom endpoint of $B$. If a block $B$ consists of a single point $q$, then $q$ is both the top and bottom endpoint of $B$.

Let $B$ and $B'$ be two blocks in the same noncrossing arc diagram $N$ with top and bottom endpoints $r$, $p$ and $r'$, $p'$ respectively. Suppose that $B$ and $B'$ overlap; that is, $[p, r] \cap [p', r'] \neq \emptyset$. Since the two blocks are disjoint topologically and overlapping, without loss of generality, $B$ must be left of $B'$. We say $B$ is immediately left of $B'$ if there are no points or arcs in $N$ which are right of $B$ and left of $B'$. In this case $B'$ is immediately right of $B$. A block $B_k$ is transitively left of a block $B$ if there exists a sequence of blocks $B = B_0, B_1, \ldots, B_{k-1}, B_k$ such that $B_i$ is immediately left of $B_{i-1}$ for each $i \in \{1, \ldots, k\}$. In this case $B$ is transitively right of $B_k$. We allow the case where $k = 0$, so that any block $B$ is both transitively left and transitively right of itself.

The transitively left/right relations can be carried from blocks in a noncrossing arc diagram to the underlying numbered points. We say that a point $q$ is transitively left of a block $B$ if and only if the block containing $q$ is transitively left of $B$. Moreover, a point $q$ is transitively left of a point $r$ if and only if the block containing $q$ is transitively left of the block containing $r$.

The language of blocks in the context of noncrossing arc diagrams was inspired by Bancroft’s permutation pre-orders in [4], and with the language of transitively left and right, we can make an explicit link between noncrossing arc diagrams and permutation pre-orders. In [4, Section 2.3], a block is defined as an equivalence class of a pre-order $P$ on $[n]$, in which $i \sim j$ if and only if $i \leq j \leq i$. $P$ can be thought of both as a pre-order on $[n]$ and as a partial
Figure 3.4: The permutation pre-order and noncrossing arc diagram for $\pi = 541237869$.

Bancroft defines a map $\mu$ that takes a permutation $\pi \in S_n$ to a pre-order $P$. Each descending run in $\pi$ is a block in $P$. For two blocks $B$ and $B'$ that overlap (if the highest and lowest number in one block create an interval that contains the highest or lowest number in the other block), $B \preceq B'$ if and only if the descending run corresponding to $B$ occurs before the descending run corresponding to $B'$ in the one-line notation of $\pi$. The partial order on blocks in $P$ is the transitive closure of such relations.

A pre-order $P$ on $[n]$ is a permutation pre-order if it satisfies two conditions: any pair of overlapping blocks must be comparable in $P$, and all covering relations in $P$ must be between overlapping blocks.

**Proposition 3.3.1.** [4, Proposition 2.7] $\mu$ is a bijection from permutations in $S_n$ to the set of permutation pre-orders on $[n]$.

The inverse of the map $\delta$ from permutations to noncrossing arc diagrams, denoted $\rho$, is described in [35, Section 3]. Given a noncrossing arc diagram $N$, $\rho$ recovers the permutation $\pi$ for which $\delta(\pi) = N$ as follows: Among blocks of $N$ which have nothing transitively left of them, remove the block which has the lowest bottom endpoint and write the endpoints of the block in descending order. Continue recursively on the remaining blocks, writing the permutation from left to right.

The following proposition is immediate, as the two composed maps are bijections. The permutation pre-order and noncrossing arc diagram corresponding to an example permutation in $S_9$ is pictured in Fig. 3.4.

**Proposition 3.3.2.** The composition $\mu \circ \rho$ is a bijection from noncrossing arc diagrams on $n$ points to permutation pre-orders on $[n]$.

Given a noncrossing arc diagram, the blocks in $N$ are precisely the blocks in $\mu(\rho(N))$. Moreover, if two blocks $B$ and $B'$ overlap, then $B$ is transitively left of $B'$ in $N$ precisely when $B \preceq B'$ in $\mu(\rho(N))$. Thus, the following proposition is immediate.
Proposition 3.3.3. The pre-order on \( n \) points in a noncrossing arc diagram \( N \) given by being transitively left is the permutation pre-order \( \mu(\rho(N)) \).

Before stating results relating noncrossing arc diagrams and the weak order in the next section, we discuss a map on permutations which can be expressed as an operation on noncrossing arc diagrams. The longest element, denoted \( w_0 \), in a Coxeter group of type \( A_{n-1} = S_n \) is the permutation \( n \, (n-1) \cdots 2 \, 1 \). It is an involution. Given a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \), the \( i \)th entry of the permutation \( w_0 \pi \) is \( (n + 1) - \pi_i \). Thus \( w_0 \pi \) is the result of inverting the entries of \( \pi \). The \( i \)th entry of \( \pi w_0 \) has \( i \)th entry is \( \pi_{n+1-i} \). Thus \( \pi w_0 \) is the entry-wise reverse of \( \pi \). Conjugating \( \pi \) by \( w_0 \) inverts both the values and the order on \( \pi \), replacing the \( i \)th term \( \pi_i \) with \( (n + 1) - \pi_{n+1-i} \). Moreover, conjugating by \( w_0 \) is an automorphism of the weak order. (See [10, Proposition 3.1.5].)

The following proposition is an immediate consequence of this description of conjugation by \( w_0 \) and the definition of the map \( \delta \) from \( S_n \) to noncrossing arc diagrams on \( n \) points.

Proposition 3.3.4. If \( N = \delta(\pi) \), the noncrossing arc diagram \( \delta(w_0 \pi w_0) \) is the image of \( N \) under the 180-degree rotation of the plane that sends each point \( i \) to \( n + 1 - i \).

3.3.2 Noncrossing arc diagrams and the weak order

The following lemma characterizes the inversions of a permutation \( \pi \) in terms of the transitively right relation in \( \delta(\pi) \). It is a generalization of [1, Lemma 37].

Lemma 3.3.5. Given \( q, r \in [n] \) with \( q < r \), the pair \( (r, q) \) is an inversion of \( \pi \) if and only if \( r \) is transitively left of \( q \) in \( \delta(\pi) \).

Proof. By definition, \( r \) is transitively left of \( q \) in \( \delta(\pi) \) in one of two cases: either the two points are in the same block, or the points are in disjoint blocks and the block containing \( r \) is transitively left of the block containing \( q \). In each case, \( r \) precedes \( q \) in \( \pi \), so \( (r, q) \) is an inversion of \( \pi \).

On the other hand, \( r \) is not transitively left of \( q \) in \( \delta(\pi) \) in one of two cases, both with \( r \) and \( q \) in disjoint blocks: either the block containing \( q \) is transitively left of the block containing \( r \), or the two blocks are not comparable under the transitively left relation. In each case, \( q \) comes before before \( r \) in \( \pi \), so \( (r, q) \) is not an inversion of \( \pi \).

The following remark is a restatement of the previous proposition via the half-turn rotation described in Proposition 3.3.4, since “transitively left” in \( \delta(\pi) \) becomes “transitively right” in \( \delta(w_0 \pi w_0) \).
Remark 3.3.6. Given \( q, r \in [n] \) with \( n + 1 - q > n + 1 - r \), the pair \((n + 1 - q, n + 1 - r)\) is an inversion of \( \pi \) if and only if \( n + 1 - r \) is transitively right of \( n + 1 - q \).

Recall that in the right weak order on \( S_n \), \( \sigma \leq \tau \) if and only if the inversion set of \( \sigma \) is a subset of the inversion set of \( \tau \). Thus, Lemma 3.3.5 can be used to characterize the weak order in terms of noncrossing arc diagrams.

Proposition 3.3.7. Let \( \pi \) and \( \pi' \) be permutations in \( S_n \). Then \( \pi \leq \pi' \) in the weak order if and only if \( \delta(\pi') \) is obtained from \( \delta(\pi) \) by a sequence of moves of the following forms, with each move resulting in a valid noncrossing arc diagram:

1. adding a new arc,
2. extending an arc upward so that it passes right of the previous upper endpoint,
3. extending an arc downward so that it passes left of the previous lower endpoint,
4. breaking a single arc into a pair of arcs with a shared endpoint, or
5. carrying out a pair of moves of types (2) and (3), (2) and (4), (3) and (4), or (4) twice, on two distinct arcs, such that two of the resulting arcs are combinatorially equivalent, then keeping one of the two equivalent arcs.

The moves described in Proposition 3.3.7 are not necessarily cover relations. Describing cover relations appears to be more tedious. The set of moves is invariant under the rotation described in Proposition 3.3.4: a move of type (1) or (4) is a rotation of a move of the same type; a move of type (2) becomes a move of type (3) and vice versa. Fig. 3.5 shows moves (1)–(4), and Fig. 3.6 shows each type of move in (5).

Proof. We first show that every cover in the weak order is one of the moves listed above.

Consider a permutation \( \pi = \cdots ij \cdots \), where \( ij \) is an ascent; that is, \( i < j \). Thus, the permutation \( \pi' = \cdots ji \cdots \) covers \( \pi \) in the weak order. Let \( N = \delta(\pi) \) and \( N' = \delta(\pi') \). We
consider the noncrossing arc diagrams resulting from several cases. Some cases are symmetric to one another via conjugation by $w_0$, the half-turn rotation as described in Proposition 3.3.4.

**Case 1.** If $\pi = ijk \cdots$, we consider three subcases.

(a) $k < i < j$: In this case, $jk$ is a descent in $\pi$. In $\pi'$, $ji$ and $ik$ are descents. $N'$ is obtained by breaking the arc from $j$ to $k$ in $N$ at the point $i$.

(b) $i < k < j$: In this case, $jk$ is a descent in $\pi$. In $\pi'$, $ji$ is a descent and $ik$ is an ascent. $N'$ is obtained by extending the arc from $j$ to $k$ in $N$ down to $i$ so that the arc passes to the left of $k$ and left of all points between $k$ and $i$.

(c) $i < j < k$: In this case, $jk$ is an ascent in $\pi$. In $\pi'$, $ji$ is a descent and $ik$ is an ascent. $N'$ is obtained by adding an arc from $j$ to $i$ that passes to the left of all points between $j$ and $i$.

**Case 2.** If $\pi = \cdots hijk \cdots$, we consider nine cases. The cases are ordered first by the comparison of $h$ with $i$ and $j$, then by the comparison of $k$ with $i$ and $j$.

(a) $h < i < j$, $k < i < j$: In this case, $hi$ is an ascent and $jk$ is a descent in $\pi$. In $\pi'$, $ji$ and $ik$ are descents and $hj$ is an ascent. $N'$ is obtained by breaking the arc from $j$ to $k$ in $N$ at the point $i$.

(b) $h < i < k < j$: In this case, $hi$ is an ascent and $jk$ is a descent in $\pi$. In $\pi'$, $hj$ and $ik$ are ascents and $ji$ is a descent. $N'$ is obtained by extending the arc from $j$ to $k$ in $N$ down to $i$, passing left of $k$ and passing right of exactly those points between $j$ and $i$ which precede $h$ in $\pi$.

(c) $h < i < j < k$: In this case, $hi$ and $jk$ are ascents in $\pi$. In $\pi'$, $hj$ and $ik$ are ascents and $ji$ is a descent. $N'$ is obtained by adding an arc from $j$ to $i$ that passes right of exactly those points between $j$ and $i$ which precede $h$ in $\pi$. 

Figure 3.6: Compound ways to go up in the weak order.
(d) $k < i < h < j$: In this case, $hi$ and $jk$ are descents in $\pi$. In $\pi'$, $hj$ is an ascent; $ji$ and $ik$ are descents. $N'$ is obtained by breaking the arc from $j$ to $k$ in $N$ at $i$, extending the arc from $h$ to $i$ up to $j$ so that it passes right of $h$ and right of exactly those points between $h$ and $j$ that precede $h$ in $\pi$, and keeping one of the resulting two copies of the arc from $j$ to $i$.

(e) $i < h < j$, $i < k < j$: In this case, $hi$ and $jk$ are descents in $\pi$. In $\pi'$, $hj$ and $ik$ are ascents and $ji$ is a descent. $N'$ is obtained by extending the arc from $h$ to $i$ up to $j$ so that it passes right of $h$ and right of exactly those points between $h$ and $j$ that precede $h$ in $\pi$, extending the arc from $j$ to $k$ down to $i$ so that it passes left of $k$, and keeping one of the resulting two copies of the arc from $j$ to $i$.

(f) $i < h < j < k$: This case is symmetric to Subcase (b).

(g) $k < i < j < h$: In this case, $hi$ and $jk$ are descents in $\pi$. In $\pi'$, $hj$, $ji$, and $ik$ are all descents. $N'$ is obtained by breaking the arcs from $h$ to $i$ and from $j$ to $k$ in $N$ at $j$ and $i$ respectively, then keeping one of the two resulting copies of the arc from $j$ to $i$.

(h) $i < k < j < h$: This case is symmetric to Subcase (d).

(i) $i < j < h$, $i < j < k$: This case is symmetric to Subcase (a).

Case 3. The case where $\pi = \cdots hi j$ is symmetric to Case 1.

Conversely, suppose that $N$ and $N'$ are noncrossing arc diagrams related by one of the moves in the statement of the proposition, and $\pi = \rho(N)$ and $\pi' = \rho(N')$. We wish to show that $\pi \leq \pi'$ in the weak order.

In [35], noncrossing arc diagrams are shown to encode the canonical join representations of permutations. Namely, a permutation $\pi$ with noncrossing arc diagram $N$ has canonical join representation $\bigvee_{\alpha \in N} \rho(\alpha)$.

Let $M = N \setminus (N \cap N')$ and $M' = N' \setminus (N \cap N')$. Thus $\pi$ and $\pi'$ have canonical join representations

$$\pi = \left( \bigvee_{\alpha \in N \cap N'} \rho(\alpha) \right) \lor \left( \bigvee_{\alpha \in M} \rho(\alpha) \right) \quad \text{and} \quad \pi' = \left( \bigvee_{\alpha \in N \cap N'} \rho(\alpha) \right) \lor \left( \bigvee_{\alpha \in M'} \rho(\alpha) \right).$$

The first joins in the two equations above are equal, so $\pi \leq \pi'$ if and only if $\rho(M) \leq \rho(M')$. Thus, moving forward, we may as well assume that $N$ and $N'$ have no arcs in common. We consider each operation in the statement of the proposition.
Case 1: adding a new arc. In this case, \( N \) has no arcs and \( N' \) has a single arc. Since \( \pi \) is the identity, \( \pi' \) is above \( \pi \) in the weak order.

Case 2: extending an arc up, right of the previous upper endpoint. See the left side of Fig. 3.7. In this case, \( N \) has one arc, \( \alpha \), which has upper endpoint \( j \), lower endpoint \( i \), and sets \( L \) and \( R \) of points left and right of \( \alpha \); \( N' \) has one arc, \( \alpha' \), which is the result of extending \( \alpha \) to a new upper endpoint \( j' \) so that it passes right of \( j \). The one-line notation of \( \pi \) is

\[
\pi = 1 \cdots i-1 \ L \ j \ i \ R \ j+1 \cdots n .
\]

Here and in all cases that follow, when we write a set in the one-line notation of a permutation, the elements of the set are understood to be in ascending order. So, all inversions of \( \pi \) arise from the relative positions of \( L, j, i, \) and \( R \).

Let \( L' \) and \( R' \) denote the set of points left and right of \( \alpha' \), respectively. Since \( \alpha' \) is the result of extending \( \alpha \) from \( j \) up to \( j' \), \( R' \) can be written as the disjoint union \( R \cup [R' \cap (j, j')] \). Similarly, \( L' \) can be written as the disjoint union \( L \cup \{j\} \cup [L' \cap (j, j')] \). (The previous upper endpoint \( j \) is in \( L' \) because \( \alpha' \) passes right of \( j \).) The one-line notation of \( \pi' \) is

\[
\pi' = 1 \cdots i-1 \ L \ j \ [L' \cap (j, j')] \ j' \ i \ R \ [R' \cap (j, j')] \ j'+1 \cdots n .
\]

The permutation \( \pi' \) has all the inversions of \( \pi \), since \( L, j, i, \) and \( R \) are all in the same relative positions in \( \pi' \) as in \( \pi \). Moreover, since \( j' \) and \( L \cap (j, j') \) have moved to the left of things smaller than them, \( \pi' \) is above \( \pi \) in the weak order.

Case 3: extending an arc down, left of the previous lower endpoint. See the right side of Fig. 3.7. In this case, \( N \) has one arc, \( \alpha \), which has upper endpoint \( j \) and lower endpoint \( i \); \( N' \) has one arc, \( \alpha' \), which is the result of extending \( \alpha \) to a new lower endpoint \( i' \) so that it passes left of \( i \). This case is symmetric to Case 2 under conjugation by \( w_0 \), half-turn rotation.
of \( N \) and \( N' \), in the same way that Lemma 3.3.5 and Remark 3.3.6 are symmetric.

**Case 4: breaking an arc.** See Fig. 3.8. In this case, \( N \) has one arc, \( \alpha \), which has upper endpoint \( j \), lower endpoint \( i \); \( N' \) has two arcs, \( \alpha' \) and \( \alpha'' \), which are the result of breaking \( \alpha \) at a point \( k \) between \( i \) and \( j \) such that \( \alpha' \) is above \( \alpha'' \). That is, \( k \) is the lower endpoint of \( \alpha' \) and the upper endpoint of \( \alpha'' \).

Let \( L \) and \( R \) denote the set of all points left and right of \( \alpha \), respectively. Either of these sets may be empty, but one of them must be nonempty since there must be a point between \( i \) and \( j \) at which \( \alpha \) can be broken. There are two subcases: \( k \) may be either left or right of \( \alpha \). Let \( L' \) and \( R' \) denote the set of points left and right of \( \alpha' \), and let \( L'' \) and \( R'' \) denote the set of points left and right of \( \alpha'' \).

If \( k \) is left of \( \alpha \), then \( L \) can be written as the disjoint union \( L'' \cup \{k\} \cup L' \) and \( R \) can be written as the disjoint union \( R'' \cup R' \). In this case, the one-line notation of \( \pi \) is

\[
\pi = 1 \cdots i-1 L'' \ k \ L' \ j \ i \ R'' \ R' \ j+1 \cdots n .
\]

In the other case, when \( k \) is right of \( \alpha \), the one-line notation of \( \pi \) is

\[
\pi = 1 \cdots i-1 L'' \ L' \ j \ i \ R'' \ k \ R' \ j+1 \cdots n .
\]

In either case, breaking \( \alpha \) at \( k \) results in the same one-line notation for \( \pi' \):

\[
\pi' = 1 \cdots i-1 L'' \ L' \ j \ k \ i \ R'' \ R' \ j+1 \cdots n .
\]

The relative positions of \( L'' \), \( L' \), \( j \), \( i \), \( R'' \) and \( R' \) are the same in \( \pi' \) as in \( \pi \). The only change from \( \pi \) to \( \pi' \) is that \( k \) has moved to the right of things larger than it (if \( k \) is left of \( \alpha \)) or to the left of things smaller than it (if \( k \) is right of \( \alpha \)). In either case, \( \pi' \) is above \( \pi \) in the weak order.

Figure 3.8: Breaking \( \alpha \).
Case 5: carrying out a pair of moves. $N$ has two distinct arcs, $\alpha$ and $\beta$. Without loss of generality, let $\alpha$ denote the arc with the lower top endpoint. Let $L_\alpha$ and $L_\beta$ denote the set of points left of $\alpha$ and $\beta$, respectively, and similarly for $R_\alpha$ and $R_\beta$. We consider each combination of operations in previous cases.

Because $\alpha$ and $\beta$ together bound the resulting arc(s), we make the simplifying assumption moving forward that the lowest endpoint between the two arcs is $i = 1$ and the highest endpoint between the two arcs is $j = n$. This eliminates the terms $1 \cdots i - 1$ and $j + 1 \cdots n$ at the beginning and end of the one-line notation of both $\pi$ and $\pi'$.

Case 5a: extending $\alpha$ up, right and extending $\beta$ down, left. See Fig. 3.9. In this case, $N'$ consists of one arc, $\alpha'$, which is the result of extending $\alpha$ up, passing to the right of the upper endpoint of $\alpha$ and is also the result of extending $\beta$ down, passing to the left of the lower endpoint of $\beta$. We write $L'$ and $R'$ for the set of points left and right of $\alpha'$, respectively. Let $s$ and $i$ denote the upper and lower endpoints of $\alpha$, and let $j$ and $t$ denote the upper and lower endpoints of $\beta$.

There are only two configurations of $\alpha$ and $\beta$ which allow for $\alpha'$ to be as described: either the two arcs do not overlap or $\alpha$ is immediately left of $\beta$.

If $\alpha$ and $\beta$ do not overlap, then $i < s < t < j$. In this case, $L'$ can be written as the disjoint union $L_\alpha \cup \{s\} \cup [L' \cap (s,t)] \cup L_\beta$ and $R'$ can be written as the disjoint union $R_\alpha \cup [R' \cap (s,t)] \cup \{t\} \cup R_\beta$. In this case the one-line notation of $\pi$ is

$$
\pi = L_\alpha \ s \ i \ R_\alpha \ s + 1 \cdots t - 1 \ L_\beta \ j \ t \ R_\beta.
$$

The one-line notation of $\pi'$ is

$$
\pi' = L_\alpha \ s \ [L' \cap (s,t)] \ L_\beta \ j \ i \ R_\alpha \ [R' \cap (s,t)] \ t \ R_\beta.
$$

Since $L_\alpha$, $s$, $i$, and $R_\alpha$ have the same relative order in $\pi'$ as in $\pi$ and likewise for $L_\beta$, $j$, $t$,
and $R_\beta$, all inversions of $\pi$ are also inversions of $\pi'$. From $\pi$ to $\pi'$, the entries $s$ and $j$ and all entries in $L' \cap (s, t)$ and $L_\beta$ move to the left of things smaller than them, so $\pi'$ is above $\pi$ in the weak order.

If $\alpha$ is immediately left of $\beta$, then $i < t < s < j$ and $R_\alpha \cap L_\beta \cap (t, s) = \emptyset$. In particular, this implies that $L_\alpha \cap (t, s) = L_\beta \cap (t, s)$ and likewise $R_\alpha \cap (t, s) = R_\beta \cap (t, s)$. As a result, $L'$ can be written as the disjoint union $L_\alpha \cup \{s\} \cup [L_\beta \cap (s, j)]$ and $R'$ can be written as the disjoint union $[R_\alpha \cap (i, t)] \cup \{t\} \cup R_\beta$. In this case, the one-line notation of $\pi$ is

$$\pi = L_\alpha \ s \ i \ [R_\alpha \cap (i, t)] \ [L_\beta \cap (s, j)] \ j \ t \ R_\beta.$$ 

The one-line notation of $\pi'$ is

$$\pi' = L_\alpha \ s \ [L_\beta \cap (s, j)] \ j \ i \ [R_\alpha \cap (i, t)] \ t \ R_\beta.$$ 

$L_\alpha$, $s$, $i$, and $[R_\alpha \cap (i, t)]$ have the same relative order in $\pi'$ as in $\pi$ and likewise for $[L_\beta \cap (s, j)]$, $j$, $t$, and $R_\beta$. From $\pi$ to $\pi'$, the entry $j$ and all entries in $L_\beta \cap (s, j)$ move to the left of entries smaller than them, so $\pi'$ is above $\pi$ in the weak order.

In either case, $\pi'$ is above $\pi$ in the weak order.

**Case 5b: extending $\alpha$ up, right and breaking $\beta$.** See the left side of Fig. 3.10. In this case, $N'$ consists of two arcs, $\alpha'$ and $\alpha''$. The result of extending $\alpha$ to a new upper endpoint $j$ so that it passes right of the upper endpoint of $\alpha$ is called $\alpha'$, and it is also the higher arc which occurs as a result of breaking $\beta$ at the lower endpoint of $\alpha$. The lower arc which occurs as a result of breaking $\beta$ at the lower endpoint of $\alpha$ is called $\alpha''$.

Let $t$ and $s$ denote the upper and lower endpoints of $\alpha$ and let $j$ and $i$ denote the upper and lower endpoints of $\beta$. In order for the two moves to result in arcs combinatorially

![Figure 3.10: Extending $\alpha$ and breaking $\beta$.](image)
Figure 3.11: Breaking $\alpha$ and breaking $\beta$, $\alpha$ left of $\beta$.

equivalent to $\alpha'$, the endpoints must fulfill the string of inequalities $i < s < t < j$ and $\alpha$ must be immediately left of $\beta$. In particular, this implies that $L_\alpha = L_\beta \cap (s, t)$ and $R_\alpha = R_\beta \cap (s, t)$.

As a result, $L'$ can be written as the disjoint union $L_\alpha \cup \{t\} \cup [L_\beta \cap (t, j)]$ and $L''$ can be written as $L_\beta \cap (i, s)$. Similarly, $R'$ can be written as the disjoint union $R_\alpha \cup [R_\beta \cap (t, j)]$ and $R''$ can be written as $R_\beta \cap (i, s)$. The one-line notation of $\pi$ is

$$\pi = L'' L_\alpha t s [L_\beta \cap (t, j)] j i R'' R_\alpha [R_\beta \cap (t, j)] .$$

The one-line permutation of $\pi'$ is

$$\pi = L'' L_\alpha t [L_\beta \cap (t, j)] j s i R'' R_\alpha [R_\beta \cap (t, j)] .$$

$L_\alpha$, $t$, $s$, and $R_\alpha$ have the same relative order in $\pi'$ as in $\pi$, so all inversions coming from $\alpha$ are also inversions of $\pi'$. In fact, the only change in the one-line notation is the movement of $s$ to the right, past entries greater than $s$. Thus $\pi'$ is above $\pi$ in the weak order.

Case 5c: extending $\alpha$ down, left and breaking $\beta$. See the right side of Fig. 3.10. In this case, $N'$ consists of two arcs, $\alpha'$ and $\alpha''$. The higher arc which occurs as a result of breaking $\beta$ at the upper endpoint of $\alpha$ is called $\alpha'$. The result of extending $\alpha$ to a new lower endpoint $i$ so that it passes left of the lower endpoint of $\alpha$ is called $\alpha''$, and it is also the lower arc which occurs as a result of breaking $\beta$ at the upper endpoint of $\alpha$. This case is dual to Case 5b under conjugation by $w_0$, half-turn rotation of $N$ and $N'$, as Case 3 is symmetric to Case 2.

Case 5d: breaking $\alpha$ and $\beta$. See Fig. 3.11. In this case, $N'$ consists of three arcs $\alpha'$, $\alpha''$, and $\alpha'''$ such that $\alpha'$ and $\alpha''$ are the result of breaking $\alpha$, and $\alpha''$ and $\alpha'''$ are the result of breaking $\beta$. Let $t$ and $i$ denote the upper and lower endpoints of $\alpha$ and let $j$ and $s$ denote the upper and lower endpoints of $\beta$. In order for the two moves to result in an equivalent arc $\alpha''$, $\alpha$ must be broken at $s$ and $\beta$ must be broken at $t$.

Moreover, the endpoints must fulfill the string of inequalities $i < s < t < j$ (so $\alpha'$
is below $\alpha''$, which is below $\alpha'''$) and $\alpha$ must be immediately left or immediately right of $\beta$. Within the interval $(s, t)$, $L_\alpha$ and $L_\beta$ must agree and $R_\alpha$ and $R_\beta$ must agree; that is, $R_\alpha \cap L_\beta \cap (s, t) = \emptyset$ and $L_\alpha \cap R_\beta \cap (s, t) = \emptyset$. As a result, the set $L' \cup L'' \cup L'''$ can be written as the disjoint union $[L_\alpha \cap (i, s)] \cup [L_\alpha \cap (s, t)] \cup [L_\beta \cap (t, j)]$. Likewise, $R' \cup R'' \cup R'''$ can be written as the disjoint union $[R_\alpha \cap (i, s)] \cup [R_\beta \cap (s, t)] \cup [R_\beta \cap (t, j)]$.

In the first case, suppose that $\alpha$ is immediately left of $\beta$. The one-line notation of $\pi$ is

$$\pi = L_\alpha \ t \ i \ [R_\alpha \cap (i, s)] \ [L_\beta \cap (t, j)] \ j \ s \ R_\beta .$$

The one-line notation of $\pi'$ is

$$\pi' = L_\alpha \ [L_\beta \cap (t, j)] \ j \ t \ i \ [R_\alpha \cap (i, s)] \ R_\beta .$$

Since $L_\alpha$, $t$, $i$, and $R_\beta \cap (i, s)$ have the same relative order in $\pi'$ as in $\pi$ and likewise for $L_\beta \cap (t, j)$, $j$, $s$, and $R_\beta$, the inversions of $\pi$ arising from $\alpha$ and $\beta$ are also inversions of $\pi'$. From $\pi$ to $\pi'$, the entries in $L_\beta \cap (t, j)$, $j$ and $s$ move to the left past entries smaller than them.

In the other case, suppose that $\alpha$ is immediately right of $\beta$. The one-line notation of $\pi$ is

$$\pi = [L_\alpha \cap (i, s)] \ L_\beta \ j \ s \ t \ i \ R_\alpha \ [R_\beta \cap (t, j)] .$$

The one-line notation of $\pi'$ is

$$\pi' = [L_\alpha \cap (i, s)] \ L_\beta \ j \ t \ s \ i \ R_\alpha \ [R_\beta \cap (t, j)] .$$

From $\pi$ and $\pi'$, $t$ moves to the left of $s$, which is smaller than it.

In either case, $\pi'$ is above $\pi$ in the weak order.

\[\square\]

### 3.3.3 Cooperating and matting

We now present two new constructions, each of which takes in two noncrossing arc diagrams and outputs a single noncrossing arc diagram. In Section 3.3.4, we prove that these constructions correspond to the meet and join in the shard intersection order on type-A Coxeter groups.

Recall that, for a block $B$ with at least one arc, the upper endpoint of any arc in $B$ is an upper endpoint of $B$ and the lower endpoint of any arc in $B$ is a lower endpoint of $B$.

**Definition 3.3.8.** Given two noncrossing arc diagrams $N_1$ and $N_2$ on $n$ points, the **cooperative noncrossing arc diagram** of $N_1$ and $N_2$, denoted $cn(N_1, N_2)$, consists of all arcs $\alpha$
constructed as follows. An arc $\alpha$ in $\text{cn}(N_1, N_2)$ exists only when:

1. there is a pair of points $\{p, r\}$ in $[n]$ such that $p < r$, and there exists a block $B$ of $N_1$ and a block $C$ of $N_2$ such that $r$ is an upper endpoint of both $B$ and $C$ and $p$ is a lower endpoint of both $B$ and $C$, and there is no point $t$ between $p$ and $r$ that is an endpoint of both blocks, and

2. for every $q \in (p, r)$, $B$ and $C$ pass weakly to the same side of $q$. That is, either there exist arcs $\beta$ in $B$ and $\gamma$ in $C$ both passing left [resp. right] of $q$, or there exists an arc $\beta$ in $B$ passing left [resp. right] of $q$ and arcs $\gamma_1$ and $\gamma_2$ in $C$ sharing $q$ as an endpoint, or there exists an arc $\gamma$ in $C$ passing left [resp. right] of $q$ and arcs $\beta_1$ and $\beta_2$ in $B$ sharing $q$ as an endpoint.

If both requirements above are satisfied, $\alpha$ is the arc from $r$ to $p$ which passes to the same side of each $q \in (p, r)$ as $B$ and $C$.

Less formally, we describe the process for drawing each arc $\alpha$ in $\text{cn}(N_1, N_2)$, as pictured in Figs. 3.12 and 3.13. First, identify two points $r$ and $p$ that are upper and lower endpoints, respectively, of the same pair of blocks $B$ and $C$ and that have no endpoint of both $B$ and $C$ between them. Begin to draw an arc from $r$ down to $p$ that agrees with all arcs along the way. That is, at each point $q$ between $r$ and $p$ about which $B$ and $C$ weakly agree, draw $\alpha$ so that it agrees with both blocks as it passes $q$, as in Fig. 3.12. If there exists a point between $r$ and $p$ that $B$ and $C$ pass to opposite sides of, as in Fig. 3.13, give up on drawing the arc from $r$ to $p$.

![Figure 3.12: Creating cn(N_1, N_2) for N_1 = \delta(3412) and N_2 = \delta(3421).](image)

**Lemma 3.3.9.** For any noncrossing arc diagrams $N_1$ and $N_2$ on $n$ points, $\text{cn}(N_1, N_2)$ is a noncrossing arc diagram.
weak agreement
left of 3
disagreement
at 2
no arc from
4 to 1

Figure 3.13: Creating $\text{cn}(N_1, N_2)$ for $N_1 = \delta(4123)$ and $N_2 = \delta(2431)$.

Proof. It is clear from Definition 3.3.8 that each curve in $\text{cn}(N_1, N_2)$ is an arc. We wish to show that any pair of arcs in $\text{cn}(N_1, N_2)$ is compatible.

Suppose $B$ is a block of $N_1$ and $C$ is a block of $N_2$ and suppose that $B$ and $C$ share an endpoint $r$ which is the upper endpoint of an arc $\beta \in B$ and $\gamma \in C$. Since $N_1$ is a noncrossing arc diagram, $\beta$ is the only arc in $N_1$ that has $r$ as its upper endpoint. Likewise, $\gamma$ is the only arc in $N_2$ that has $r$ as its upper endpoint. We conclude that for any $r \in [n]$, there is at most one arc in $\text{cn}(N_1, N_2)$ that has $r$ as its upper endpoint. The proof that for any $p \in [n]$, there is at most one arc in $\text{cn}(N_1, N_2)$ which has $p$ as its lower endpoint runs symmetrically. Hence, no two arcs in $\text{cn}(N_1, N_2)$ share an upper endpoint or a lower endpoint.

Suppose now, for proof by contradiction, that two arcs $\alpha_1$ and $\alpha_2$ in $\text{cn}(N_1, N_2)$ intersect along their interiors. Then there must be some $i$ and $j$ with $1 \leq i < j \leq n$ such that $\alpha_1$ is weakly left of $i$ and weakly right of $j$ and $\alpha_2$ is weakly right of $i$ and weakly left of $j$. That is, $\alpha_1$ must be left of $\alpha_2$ at $i$ and $\alpha_1$ must be right of $\alpha_2$ at $j$. If there are many pairs of points which satisfy the conditions above, consider any pair $i$ and $j$ among them.

Let $B_1$ and $C_1$ denote the blocks of $N_1$ and $N_2$, respectively, that result in $\alpha_1$. Similarly, let $B_2$ and $C_2$ denote the blocks of $N_1$ and $N_2$, respectively, that result in $\alpha_2$. Each pair of blocks $B$ in $N_1$ and $C$ in $N_2$ gives rise to at most one arc in $\text{cn}(N_1, N_2)$ along the interval $(i, j)$. Since $\alpha_1$ and $\alpha_2$ are distinct arcs along $(i, j)$, the pair $(B_1, C_1)$ must be distinct from $(B_2, C_2)$. We consider two cases.

In the first case, $B_1 \neq B_2$ and $C_1 \neq C_2$. Because $B_1$ and $B_2$ are distinct blocks, one block must be strictly left of the other at the height of $i$ and strictly right at the height of $j$, regardless of whether either block has $i$ or $j$ as an endpoint; likewise for $C_1$ and $C_2$. Since $\alpha_1$ is left of $\alpha_2$ at $i$, $B_1$ must be left of $B_2$ at $i$ and $C_1$ must be left of $C_2$ at $i$. Since $\alpha_1$ is right of $\alpha_2$ at $j$, $B_1$ must be right of $B_2$ at $j$ and $C_1$ must be right of $C_2$ at $j$. Thus $B_1$ must cross $B_2$ and $C_1$ must cross $C_2$. This is a contradiction to the fact that $N_1$ and $N_2$ are noncrossing arc diagrams: for instance, in order for $B_1$ to cross $B_2$, the two blocks must either share an endpoint (as both an upper and lower endpoint of both blocks) or have arcs which cross in
their interiors.

In the second case, without loss of generality, $B_1 = B_2$ and $C_1 \neq C_2$. Since $\alpha_1$ is left of $\alpha_2$ at $i$, $C_1$ must be left of $C_2$ at $i$. Since $\alpha_1$ is right of $\alpha_2$ at $j$, $C_1$ must be right of $C_2$ at $j$. Again, this is a contradiction to the fact that $N_2$ is a noncrossing arc diagram, since $C_1$ and $C_2$ cannot cross. \hfill \Box

The requirements for an arc to exist in a cooperative noncrossing arc diagram are stringent. By contrast, in the following operation on noncrossing arc diagrams, the requirements for an arc to exist are less restrictive.

**Definition 3.3.10.** Given two noncrossing arc diagrams $N_1$ and $N_2$ on $n$ points, the *matted noncrossing arc diagram* of $N_1$ and $N_2$, denoted $\text{mn}(N_1, N_2)$, is constructed as follows:

1. Consider the union of $N_1$ and $N_2$ on the same set of $n$ points. Call each connected component of the resulting graph a **woven block**. If two blocks $B \in N_1$ and $C \in N_2$ intersect at an endpoint or the interior of arcs in each block, they are in the same woven block.

2. For each woven block $W$ with highest and lowest endpoints $r$ and $p$, there is a corresponding block $\text{mat}(W)$, the result of “matting” $W$, described as follows:
   
   - Each endpoint of $W$ is an endpoint of $\text{mat}(W)$.
   - If all arcs in $W$ pass to one side of a point $q \in (p, r)$, an arc in $\text{mat}(W)$ passes to that side of $q$.
   - If two arcs in $W$ pass to opposite sides of a point $q \in (p, r)$, then $q$ is an endpoint of $\text{mat}(W)$.

3. If steps (1) and (2) yield a valid noncrossing arc diagram, stop. If not, then form woven blocks from the arcs created and repeat step (2).

Less formally, we describe the process for “matting” $W$, as pictured in Figs. 3.14 and 3.15. Draw a block of arcs from each endpoint $r'$ in $W$ to the next-highest endpoint $p'$ by agreeing with all arcs along the way. That is, for each point between $p'$ and $r'$ such that all arcs in $W$ pass to the same side, the block agrees with all arcs; for each point about which two arcs in $W$ “disagree” (passing to opposite sides) the block has that point as an added endpoint and continues down from the point. In both figures, there is a single woven block in the union of $N_1$ and $N_2$, so the result of matting the block necessarily is a valid noncrossing arc diagram.
Figure 3.14: Creating \( mn(N_1, N_2) \) for \( N_1 = \delta(1342) \) and \( N_2 = \delta(2341) \).

Figure 3.15: Creating \( mn(N_1, N_2) \) for \( N_1 = \delta(1423) \) and \( N_2 = \delta(2341) \).

However, if \( N_1 \) and \( N_2 \), drawn together, have more than one woven block, it is possible for the resulting set of “matted blocks” not to constitute a valid noncrossing arc diagram. In Figs. 3.16 and 3.17, the set of arcs resulting from matting the two original woven blocks are not a valid diagram; rather, the arcs constitute a single new woven block which is matted again to give a valid noncrossing arc diagram. In both figures, the resulting noncrossing arc diagram corresponds to the permutation 54321, which is the longest element \( w_0 \) of the Coxeter group \( A_4 \). This occurs because these are, in some sense, the smallest examples whose matted blocks together do not give a valid noncrossing arc diagram. In general, having to mat twice does not necessarily force the resulting matted noncrossing arc diagram to correspond to \( w_0 \).

In both figures, one of the original woven blocks consists of two arcs, one in each of the original noncrossing arc diagrams, which share an upper endpoint and disagree about some points between them. When this original woven block is matted, it gets “tangled” on a point the two arcs disagree about, which is an upper endpoint of the other matted block. This shared upper endpoint is what precludes the set of matted blocks from constituting a valid noncrossing arc diagram.

**Lemma 3.3.11.** For any noncrossing arc diagrams \( N_1 \) and \( N_2 \) on \( n \) points, \( mn(N_1, N_2) \) is a noncrossing arc diagram.

**Proof.** By definition, the process described in Definition 3.3.10 must eventually yield a matted
noncrossing arc diagram. Since, each time a matting step fails to produce a valid noncrossing arc diagram, the process of taking a union and matting decreases the number of blocks, the matting process terminates in a finite number of repetitions.

3.3.4 Shard intersection order

Recall that a Coxeter group of type $A_n$ can be realized as the symmetric group $S_{n+1}$. We first discuss the geometric realization of the type $A_n$ Coxeter group.

The hyperplane corresponding to the generator $s_i = (i \ i+1)$ is $H_i = \{ \vec{x} \in \mathbb{R}^{n+1} : x_i = x_{i+1} \}$. As a generalization, the transposition $(i \ j)$ with $i < j$ corresponds to the hyperplane $H_{ij} = \{ \vec{x} \in \mathbb{R}^{n+1} : x_i = x_j \}$. The natural choice for the base region $B$ in the Coxeter arrangement $A(A_n)$ is the region containing $[1, 2, \ldots, n+1]$.

The action of reflection across $H_{ij}$ on any vector $\vec{x} \in \mathbb{R}^{n+1}$ outside of the hyperplane swaps the entries $x_i$ and $x_j$. Given a permutation $\pi \in S_{n+1}$, acting on the left by $(i \ j)$ swaps the values $i$ and $j$, and acting on the right by $(i \ j)$ swaps the entries $\pi_i$ and $\pi_j$. Thus, to have the poset of regions for $A(A_n, B)$ agree with the right weak order on $A_n$, the region $R$ in the hyperplane arrangement which corresponds to a permutation $\pi = \pi_1 \cdots \pi_{n+1}$ is the region containing the vector $[(\pi^{-1})_1, \ldots, (\pi^{-1})_{n+1}]$. This is the region $\{ \vec{x} \in \mathbb{R}^{n+1} : x_{\pi_1} \leq x_{\pi_2} \leq \cdots \leq x_{\pi_{n+1}} \}$.

Consider a permutation $\pi = \pi_1 \cdots \pi_{n+1}$. The region corresponding to $\pi$ has the
hyperplane $H_{ij}$ in its separating set if and only if $(j, i)$ is an inversion of $\pi$. Furthermore, $H_{ij}$ is a lower hyperplane of the region if and only if $(j, i)$ is a descent of $\pi$. The intersection of the lower hyperplanes of the region is the intersection of all subspaces

$$\{ \vec{x} \in \mathbb{R}^{n+1} : x_{\pi_r} = x_{\pi_{r+1}} = \cdots = x_{\pi_{s-1}} = x_{\pi_s} \}$$

such that $\pi_r \pi_{r+1} \cdots \pi_{s-1} \pi_s$ a descending run of $\pi$.

A shard $\Sigma$ in $\mathcal{A}(A_n)$ is defined by a set of linear inequalities:

$$\Sigma = \{ \vec{x} \in \mathbb{R}^{n+1} : x_p = x_q, x_p \leq x_q \forall q \in R, x_p \geq x_q \forall q \in L, p < r \}$$

where the sets $R$ and $L$ partition the interval $(p, r)$. Such a shard exists for every choice of $p < r$ and every choice of $L$ and $R$ partitioning $(p, r)$. Each shard corresponds to a join-irreducible permutation in the weak order, and each join-irreducible permutation corresponds to a noncrossing arc diagram on $n + 1$ points with a single arc. The noncrossing arc diagram corresponding to the shard $\Sigma$ above has a single arc from $p$ to $r$ which passes left of all $q \in R$ and right of all $q \in L$. The sets $L$ and $R$ give the position of points $q$ relative to the arc from $p$ to $r$, in contrast with our earlier language which describes arcs passing right or left of points. Thus, for example, $q \in L$ if $q$ is left of the arc, meaning the arc passes right of $q$.

Each arc $\alpha$ encodes the defining inequalities of its corresponding shard, denoted $\Sigma_{\alpha}$, and this idea can be extended easily to blocks with many arcs. A block $B$ with endpoints $p < q_1 < \cdots < q_k < r$ corresponds to the shard intersection:

$$\Gamma_B = \bigcap_{\alpha \in B} \Sigma_{\alpha} = \{ \vec{x} \in \mathbb{R}^{n+1} : x_p = x_{q_1} = \cdots = x_{q_k} = x_r, x_p \leq x_i \forall i \in R, x_p \geq x_i \forall i \in L \}$$

where $R$ and $L$ partition the set $(p, r) \setminus \{q_1, \ldots, q_k\}$, with $i \in R$ if an arc in $B$ passes left of $i$ and $i \in L$ if an arc in $B$ passes right of $i$. The shard intersection order on $A_2$, realized both as the set of permutations on $[3]$ and as the set of noncrossing arc diagrams on 3 points, is pictured in Fig. 3.18. The shard intersection order on $A_3$, realized as the set of noncrossing arc diagrams on 4 points, is pictured in Fig. 3.19.

As noted in [4, Proposition 2.11] and [28, Observation 2], the shard intersection order lattice on $A_n$ is graded, and the rank of a permutation $\pi$ is equal to the number of descents in $\pi$. This is the type-A case of [33, Proposition 1.1]. By construction, each arc in the noncrossing arc diagram corresponding to $\pi$ represents exactly one descent in $\pi$. The following proposition restates this observation in terms of noncrossing arc diagrams.
Proposition 3.3.12. For any permutation $\pi \in S_n$, the rank of $\pi$ in $\Psi(S_n) = \Psi(A_{n-1})$ is equal to the number of arcs in $\delta(\pi)$.

In the paragraph following [4, Proposition 2.11], Bancroft explains how to go up by a cover from a shard intersection $\Gamma$ in the shard intersection order, phrased in terms of permutation pre-orders. That explanation is restated as the following proposition.

Proposition 3.3.13. Suppose $\sigma$ and $\tau$ are permutations in $S_n$. Then $\sigma \prec \tau$ in $\Psi(S_n)$ if and only if $\mu(\tau)$ can be obtained from $\mu(\sigma)$ by combining two blocks which are unrelated or related by a cover in $\mu(\sigma)$.

The previous proposition holds in part because blocks of $\mu(\sigma)$ that are unrelated or related by a cover can be combined without changing any order relations between the combined blocks and other blocks of $\mu(\sigma)$.

Recall that we consider an isolated point to be a block with no arcs. We define two moves on a pair of blocks in a noncrossing arc diagram, depending on whether the pair overlaps.

Let $N$ be a noncrossing arc diagram on $n$ points with each block $B$ having bottom endpoint $p$ and top endpoint $r$. Consider two blocks $B_1$ and $B_2$ which do not overlap, so $(p_1, r_1) \cap (p_2, r_2) = \emptyset$, and are unrelated by the transitively left relation; let $B_1$ denote the lower block. A link move on $B_1$ and $B_2$ adds a single arc from $p_2$ to $r_1$, as long as it gives a valid noncrossing arc diagram (that is, it goes monotone down from $p_2$ to $r_1$ without crossing any arc along the way). The results of two valid link moves on $\delta(24178536)$ are pictured in the middle of Fig. 3.20.

For the other move, consider two blocks $B_1$ and $B_2$ which overlap, so $(p_1, r_1) \cap (p_2, r_2) \neq \emptyset$, and $B_1$ is immediately left of $B_2$. A merge move on $B_1$ and $B_2$ breaks as few arcs as
Figure 3.19: The shard intersection order on $A_3$.

possible to create pairs of combinatorially equivalent arcs from the lower of $r_1$ and $r_2$ to the higher of $p_1$ and $p_2$, then identifies equivalent arcs to give a valid noncrossing arc diagram. The block resulting from the merge move has all endpoints from both $B_1$ and $B_2$ as endpoints, and it passes left of a non-endpoint if and only if at least one of $B_1$ and $B_2$ passes left of that point. (Because $B_1$ is immediately left of $B_2$, there can be no point that the two blocks pass to opposite sides of.) The simplest version of a merge move is done by breaking any arc in $N$ at an isolated point immediately left or right of the arc. The results of two valid link moves on $\delta(24178536)$ are pictured on the right of Fig. 3.20.

**Proposition 3.3.14.** Let $N$ and $N'$ be two noncrossing arc diagrams on $n$ points. $N'$ covers $N$ in $\Psi(A_{n-1})$ precisely when $N'$ is the result of doing a valid link move or a valid merge move on two blocks of $N$.

**Proof.** A pair of non-overlapping blocks $B_1$ and $B_2$ in $N$ such that neither block is transitively left of the other corresponds to a pair of unrelated blocks in the permutation pre-order $\mu(\rho(N))$. Adding an arc connecting the two blocks to create a valid noncrossing arc diagram
combines the two unrelated blocks in $\mu(\rho(N))$ to give a new permutation pre-order $\mu(\rho(N'))$. A link move corresponds to combining two unrelated blocks in the permutation pre-order associated with $N$.

If $B_1$ and $B_2$ overlap and $B_1$ is immediately left of $B_2$ in $N$, then $B_2$ covers $B_1$ in the permutation pre-order $\mu(\rho(N))$. The block in $\mu(\rho(N'))$ resulting from the merge move on $B_1$ and $B_2$ is the combination of the two corresponding blocks in $\mu(\rho(N))$, and because the resulting block respects the left/right information of the two original blocks, all blocks covering or covered by $\mu(\rho(B_1))$ or $\mu(\rho(B_2))$ in $\mu(\rho(N))$ will still cover or be covered by the merged block in $\mu(\rho(N'))$. A merge move corresponds to combining two blocks which are related by a single cover in the permutation pre-order associated with $N$.

Thus, this proposition is a direct translation of moves, in terms of permutation pre-orders, which go up by a cover in $\Psi(A_{n-1})$ as stated in Proposition 3.3.13.

**Theorem 3.3.15.** Given two permutations $\sigma$ and $\tau$ in $S_n$ with noncrossing arc diagrams $N_1 = \delta(\sigma)$ and $N_2 = \delta(\tau)$, their meet in $\Psi(A_{n-1})$ is the permutation corresponding to $cn(N_1, N_2)$.

**Proof.** Let $\pi = \rho(cn(N_1, N_2))$. The proof consists of two parts. In the first, we prove that the permutation $\pi$ is below both $\sigma$ and $\tau$ in the shard intersection order. In the second, we prove that an element strictly above $\pi$ cannot be below both $\sigma$ and $\tau$; since $\Psi(A_{n-1})$ is a lattice, this suffices to show that $\pi$ is the (unique) maximal element below both $\sigma$ and $\tau$ and is thus their meet.

We first prove that $\pi$ is weakly below both $\sigma$ and $\tau$ in the shard intersection order. Consider an arc $\alpha$ in $cn(N_1, N_2)$ with upper endpoint $r$ and lower endpoint $p$. By definition of

\[ \text{Figure 3.20: Link moves and merge moves on } \delta(24178536). \]
We will show in either case that \( \mu \Gamma B \) and the shard intersection associated with diagrams, say \( \sigma N \) two sequences of arcs disagree. In particular, an arc in one of the original noncrossing arc diagrams, say \( N \), must not pass left of each point in \( R \) and weakly agree with \( \sigma \) at each point between \( r \) and \( p \). Since \( B \) weakly agrees with \( \alpha \) at each point between \( r \) and \( p \), each inequality in \( \Sigma \alpha \) also holds in the intersection. Moreover, since \( B \) may have additional endpoints between \( r \) and \( p \), above \( r \), or below \( p \), it is possible that \( \Gamma B \) has strict equalities in addition to those defining \( \Sigma \alpha \). So, the shard \( \Sigma \alpha \) contains the shard intersection \( \Gamma B \). Because such containment holds for each arc in \( \text{cn}(N_1, N_2) \) and its corresponding block in \( N_1 \), the shard intersection for \( \text{cn}(N_1, N_2) \) contains the shard intersection for \( N_1 \). Thus, \( \pi \) is below \( \sigma \) in \( \Psi(A_{n-1}) \), and an identical argument can be made to prove that \( \pi \) is also below \( \tau \) in \( \Psi(A_{n-1}) \).

We now prove that \( \pi \) is the maximal element of \( \Psi(A_{n-1}) \) which is below both \( \sigma \) and \( \tau \). Suppose \( \mu \) covers \( \pi \) in the shard intersection order, and let \( N_3 = \delta(\mu) \). By Proposition 3.3.12, \( N_3 \) has one more arc than \( \text{cn}(N_1, N_2) \), and by Proposition 3.3.14, the additional arc is either the result of doing a valid link move or a valid merge move on a pair of blocks in \( \text{cn}(N_1, N_2) \). We will show in either case that \( \mu \) must not be below at least one of \( \sigma \) and \( \tau \).

**Case 1.** \( N_3 \) is the result of doing a valid link move on a pair of blocks in \( \text{cn}(N_1, N_2) \). Let \( \alpha \) be the arc, from \( r \) to \( p \), which is added to \( \text{cn}(N_1, N_2) \) to yield \( N_3 \), and let \( L \) and \( R \) denote the points left and right of \( \alpha \), respectively. Since \( \alpha \) is not in \( \text{cn}(N_1, N_2) \), there must not be sequences of arcs, one in \( N_1 \) and one in \( N_2 \), both connecting \( r \) to \( p \) and passing weakly left of each point in \( R \) and weakly right of each point in \( L \).

If there is no sequence of arcs in, without loss of generality, \( N_1 \), which connects the two points, then the descent \((r, p)\) in \( \mu \) is not a pair in a descending run of \( \sigma \). In this case, \( H_{pr} \) is a lower hyperplane of the region \( Q \) associated to \( \mu \) but not a lower hyperplane of the region \( P \) associated to \( \sigma \), so \( \bigcap_{H \in \mathcal{L}(Q)} H \) does not contain \( \bigcap_{H \in \mathcal{L}(P)} H \). Because the intersection of lower hyperplanes for \( Q \) does not contain the intersection of lower hyperplanes for \( P \) and containment of lower hyperplanes is one of the conditions for \( \mu \leq \sigma \) as stated in Theorem 3.2.1, \( \mu \) is not below \( \sigma \) in the shard intersection order.

Suppose there are sequences of arcs, one in \( N_1 \) and one in \( N_2 \), which connect \( r \) and \( p \). Since \( \alpha \) is not in \( \text{cn}(N_1, N_2) \), there must be a point \( q \) between \( p \) and \( r \) about which the two sequences of arcs disagree. In particular, an arc in one of the original noncrossing arc diagrams, say \( N_2 \), must pass to the opposite side of \( q \) as \( \alpha \). If \( \alpha \) passes right of \( q \), the pair \((q, p)\) is an inversion of \( \mu \) but not of \( \tau \); if \( \alpha \) passes left of \( q \), the pair \((r, q)\) is an inversion of \( \mu \).
but not of \( \tau \). In either case, \( \mu \) is not below \( \tau \) in the weak order. According to Theorem 3.2.1, in order to have \( \mu \preceq \tau \), \( \mu \) must be below \( \tau \) in the weak order. Thus \( \mu \) is not below \( \tau \) in the shard intersection order.

**Case 2.** \( N_3 \) is the result of doing a valid merge move on a pair of blocks in \( \text{cn}(N_1, N_2) \). For any merge move, there must be at least one point at which a block is broken; let \( q \) denote one such point in the merge move from \( \text{cn}(N_1, N_2) \) to \( N_3 \), and let \( \alpha \) denote the arc from \( r \) to \( p \) in \( \text{cn}(N_1, N_2) \) which is broken at \( q \).

The arc \( \alpha \) in \( \text{cn}(N_1, N_2) \) which passes \( q \) must pass to the left or right side of \( q \). In order for \( \alpha \) to do so, the blocks in both \( N_1 \) and \( N_2 \) which give rise to \( \alpha \) must pass weakly to the same side \( q \) as \( \alpha \); in particular, at least one arc from these blocks the same side of \( q \) as \( \alpha \).

Suppose without loss of generality that the block in \( N_1 \) has such an arc. By Lemma 3.3.5, since \( q \) in the same block as \( r \) and \( p \) in \( N_1 \), it is not both transitively left and transitively right of \( r \) and \( q \). So, one of the pairs \((r, q)\) and \((q, p)\) is not an inversion of \( \sigma \). Since \( q \) is both transitively left and transitively right of \( r \) and \( p \) in \( N_3 \), both pairs are inversions of \( \mu \). In either case, the inversion set of \( \sigma \) does not contain the inversion set of \( \mu \), so \( \mu \) is not below \( \sigma \) in the weak order. Thus \( \mu \) is not below \( \sigma \) in the shard intersection order.

Next, we build toward a theorem similar to Theorem 3.3.15, that the join in \( \Psi(A_n) \) of two permutations is given by their matted noncrossing arc diagram. To do this, we present statements of increasing strength leading to the desired theorem.

Expanding on the notation of the shard intersection \( \Gamma_B \) corresponding to a block \( B \), we let \( \Gamma_C = \bigcap_{\alpha \in C} \Sigma_\alpha \) denote the shard intersection corresponding to any collection \( C \) of arcs. With this convention in place, the first lemma is immediate.

**Lemma 3.3.16.** If \( N \) is a noncrossing arc diagram consisting of a set of blocks \( \{B_1, \ldots, B_k\} \), then the shard intersection corresponding to \( N \) is

\[
\Gamma_N = \bigcap_{i \in [k]} \Gamma_{B_i}.
\]

**Proposition 3.3.17.** Let \( N_1 \) and \( N_2 \) be two noncrossing arc diagrams on \( n \) points. If each diagram consists of a single nontrivial block, then the join of \( \rho(N_1) \) and \( \rho(N_2) \) in \( \Psi(A_{n-1}) \) is \( \text{mn}(N_1, N_2) \).

**Proof.** Since the join in the shard intersection order corresponds to the intersection of shard intersections, proving the proposition amounts to proving that \( \Gamma_{N_1} \cap \Gamma_{N_2} = \Gamma_{\text{mn}(N_1, N_2)} \).

Let \( B_1 \) and \( B_2 \) denote the nontrivial blocks in \( N_1 \) and \( N_2 \) respectively. The shard
intersection corresponding to $N_1$ is

$$
\Gamma_1 = \{ \vec{x} \in \mathbb{R}^n : x_{b_0} = x_{b_1} = \cdots = x_{b_k}, x_{b_0} \leq x_i \forall i \in R_1, x_{b_0} \geq x_i \forall i \in L_1 \}
$$

where $b_0 < b_1 < \cdots < b_k$ are the endpoints of $B_1$, and $R_1$ and $L_1$ are the points right and left of $B_1$ respectively. Likewise, the shard intersection corresponding to $N_2$ is

$$
\Gamma_2 = \{ \vec{x} \in \mathbb{R}^n : x_{c_0} = x_{c_1} = \cdots = x_{c_l}, x_{c_0} \leq x_i \forall i \in R_2, x_{c_0} \geq x_i \forall i \in L_2 \}
$$

where $c_0 < c_1 < \cdots < c_l$ are the endpoints of $B_2$, and $R_2$ and $L_2$ are the points right and left of $B_2$ respectively. We consider two cases.

**Case 1:** $B_1$ and $B_2$ are disjoint. In this case, each block constitutes its own woven block. The matting process returns the two original blocks, so $\text{mn}(N_1, N_2)$ is simply the union of $N_1$ and $N_2$. The shard intersection corresponding to $\text{mn}(N_1, N_2)$ in this case is

$$
\Gamma_{\text{mn}(N_1, N_2)} = \{ \vec{x} \in \mathbb{R}^n : x_{b_0} = \cdots = x_{b_k}, x_{b_0} \leq x_i \forall i \in R_1, x_{b_0} \geq x_i \forall i \in L_1 \}
\cap \{ \vec{x} \in \mathbb{R}^n : x_{c_0} = \cdots = x_{c_l}, x_{c_0} \leq x_i \forall i \in R_2, x_{c_0} \geq x_i \forall i \in L_2 \}
= \Gamma_1 \cap \Gamma_2
$$

**Case 2:** $B_1$ and $B_2$ are not disjoint. In this case, $B_1$ and $B_2$ form a single woven block $\mathcal{W}$, and $\text{mn}(N_1, N_2) = \text{mat}(\mathcal{W})$. Let $p_0, \ldots, p_m$ denote the endpoints of $\text{mat}(\mathcal{W})$. By definition, each endpoint $p_m$ of $\text{mat}(\mathcal{W})$ must be at least one of the following:

- an endpoint of $B_1$ ($p_m = b_i$ for some $i \in \{1, \ldots, k\}$),
- an endpoint of $B_2$ ($p_m = c_j$ for some $j \in \{1, \ldots, l\}$), or
- a point of disagreement between $B_1$ and $B_2$ (either $p_m \in L_1$ and $R_2$ or $p_m \in R_1$ and $L_2$).

The shard intersection corresponding to $\text{mn}(N_1, N_2) = \text{mat}(\mathcal{W})$ is

$$
\Gamma_{\text{mn}(B_1, B_2)} = \{ \vec{x} \in \mathbb{R}^n : x_{p_0} = x_{p_1} = \cdots = x_{p_m},
\quad x_{p_0} \leq x_i \forall i \in (R_1 \setminus L_2) \cup (R_2 \setminus L_1),
\quad x_{p_0} \geq x_i \forall i \in (L_1 \setminus R_2) \cup (L_2 \setminus R_1) \}
$$

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The shard intersection corresponding to $\rho(N_1) \lor \rho(N_2)$ is

$$
\Gamma_1 \cap \Gamma_2 = \{ \vec{x} \in \mathbb{R}^n : x_{b_0} = \cdots = x_{b_k}, x_{c_0} = \cdots = x_{c_l}, \forall i \in R_1, x_{b_0} \geq x_i \forall i \in L_1 \}
\cap \{ \vec{x} \in \mathbb{R}^n : x_{c_0} = \cdots = x_{c_l}, x_{c_0} \leq x_i \forall i \in R_2, x_{b_0} \geq x_i \forall i \in L_2 \}.
$$

We claim that the intersection $\Gamma_1 \cap \Gamma_2$ is contained in the subspace defined by the string of equations $x_{b_0} = \cdots = x_{b_k} = x_{c_0} = \cdots = x_{c_l}$. If $B_1$ and $B_2$ share an endpoint, then the claim is immediate. Otherwise, $B_1$ and $B_2$ must cross without sharing an endpoint, so there exists a pair of distinct points $i$ and $j$ such that $x_{b_0} \leq x_i$ in $\Gamma_1$ and $x_{c_0} \geq x_i$ in $\Gamma_2$, and $x_{b_0} \geq x_j$ in $\Gamma_1$ and $x_{c_0} \leq x_j$ in $\Gamma_2$. Thus, in $\Gamma_1 \cap \Gamma_2$ it must be that $x_{b_0} = x_i = x_j = x_{c_0}$, and the claim follows.

Now, the claim implies that

$$
\Gamma_1 \cap \Gamma_2 = \{ \vec{x} \in \mathbb{R}^n : x_{b_0} = \cdots = x_{b_k} = x_{c_0} = \cdots = x_{c_l}, x_{b_0} \leq x_i \forall i \in R_1, x_{b_0} \geq x_i \forall i \in L_1, x_{b_0} \leq x_i \forall i \in R_2, x_{b_0} \geq x_i \forall i \in L_2 \}.
$$

For any $i$ in $L_1$ and $R_2$ or in $R_1$ and $L_2$, the inequalities $x_{b_0} \leq x_i$ and $x_{b_0} \geq x_i$ combine to give $x_{b_0} = x_i$. Thus $\Gamma_1 \cap \Gamma_2 = \Gamma_{\text{mat}(B_1, B_2)}$. $\square$

**Lemma 3.3.18.** If the woven block $W$ is a union of $m$ blocks $B_1, \ldots, B_m$, then

$$
\Gamma_{\text{mat}(W)} = \bigcap_{i \in [m]} \Gamma_{B_i}
$$

**Proof.** The proof proceeds by induction on the number of blocks in $W$. The nontrivial base case, when $m = 2$, is addressed by Proposition 3.3.17.

Suppose for each $k < m$, for any woven block $W$ consisting of $k$ blocks, $\Gamma_{\text{mat}(W)} = \bigcap_{i \in [k]} \Gamma_{B_i}$.

We wish to prove that if $W$ is the union of $m$ blocks, then $\Gamma_{\text{mat}(W)} = \bigcap_{i \in [m]} \Gamma_{B_i}$. Without loss of generality, $B_{m-1}$ and $B_m$ intersect, and at least one of the two blocks intersects with another block in $W$. We begin by considering the intersection of shard intersections for all blocks in $W$ and explicitly consider the shard intersections for $B_{m-1}$ and $B_m$ separately from the remaining blocks:

$$
\bigcap_{i \in [m]} \Gamma_{B_i} = \left( \bigcap_{i \in [m-2]} \Gamma_{B_i} \right) \cap (\Gamma_{B_{m-1}} \cap \Gamma_{B_m})
$$

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By Proposition 3.3.17, the intersection of $\Gamma_{B_{m-1}}$ and $\Gamma_{B_m}$ can be rewritten to give

$$\bigcap_{i \in [m]} \Gamma_{B_i} = \left( \bigcap_{i \in [m-2]} \Gamma_{B_i} \right) \cap \left( \Gamma_{\text{mat}(B_{m-1}, B_m)} \right)$$

Since $\text{mat}(B_{m-1}, B_m)$ is a block, the right side of the equation above is the intersection of shard intersections corresponding to $m - 1$ blocks. Thus by the induction hypothesis,

$$\bigcap_{i \in [m]} \Gamma_{B_i} = \Gamma_{\text{mat}(B_1, \ldots, B_{m-2}, \text{mat}(B_{m-1}, B_m))}$$

It remains to show that $\text{mat}(B_1, \ldots, B_{m-2}, \text{mat}(B_{m-1}, B_m)) = \text{mat}(W)$.

We begin by considering $\text{mat}(B_{m-1}, B_m)$, with endpoints $b_0 < b_1 < \cdots < b_r$. Each $b_i$ is either an endpoint of at least one of the blocks or a point of disagreement between the blocks, meaning $B_{m-1}$ and $B_m$ pass to opposite sides of $b_i$. Let $R_M$ and $L_M$ denote the set of points right and left of $\text{mat}(B_{m-1}, B_m)$, respectively. A point $c \in (b_0, b_r) \setminus \{b_1, \ldots, b_{r-1}\}$ is in $R_M$ if one of $B_{m-1}$ and $B_m$ passes left of $c$ and the other block is not right of it; similarly, $c$ is in $L_M$ if one of the blocks passes right of $c$ and the other block is not left of it.

Next, we consider $\text{mat}(W)$. It has endpoints $p_0 < p_1 < \cdots < p_s$, each of which is either an endpoint of some original block $B_i$, for some $i \in \{1, \ldots, m\}$ or a point of disagreement between two blocks — there are two blocks $B_i$ and $B_j$ such that $B_i$ has an arc left of $p_k$ and and $B_j$ has an arc right of $p_k$. The set of points right of $\text{mat}(W)$, which we denote $R_w$, consists of all non-endpoints that are right of some block $B_i$ and are not left of another block $B_j$. Letting $R_i$ and $L_i$ denote the set of all points right and left of the block $B_i$, we can represent $R_w$ as follows:

$$R_w = \{ r \in (p_0, p_s) \setminus \{p_1, \ldots, p_{s-1}\} : \bigcup_{i \in [k]} r \in R_i \setminus (\bigcup_{j \neq i} L_j) \}.$$ 

The set of all points left of $\text{mat}(W)$, denoted $L_w$ can be represented similarly:

$$L_w = \{ l \in (p_0, p_s) \setminus \{p_1, \ldots, p_{s-1}\} : \bigcup_{i \in [k]} l \in L_i \setminus (\bigcup_{j \neq i} R_j) \}.$$ 

Claim: There is some $j \in \{1, \ldots, m - 2\}$ such that $\text{mat}(B_{m-1}, B_m)$ intersects $B_j$.

Proof. At least one of $B_{m-1}$ and $B_m$, say $B_m$, intersects $B_j$ for some $j \in \{1, \ldots, m - 2\}$. We will show that $\text{mat}(B_{m-1}, B_m)$ also intersects $B_j$.

If $B_m$ intersects with $B_j$ at an endpoint $q$, then $\text{mat}(B_{m-1}, B_m)$ will also have $q$ as an
endpoint and thus also intersect with $B_j$ at $q$. If $B_m$ and $B_j$ intersect in the interior of two arcs, there are distinct points $k$ and $l$ such that $B_m$ is left of $B_j$ at $k$ and right of $B_j$ at $l$. We consider three cases of $B_j$’s behavior: when $B_j$ passes right of $k$ and left of $l$, when it has $k$ as an endpoint and passes left of $l$ (or has $l$ as an endpoint and passes right of $k$), and when it has both $k$ and $l$ as endpoints.

**Case 1.** If $B_j$ passes right of $k$ and left of $l$, then $B_m$ must be weakly left of $k$ and weakly right of $l$. $\text{mat}(B_{m-1}, B_m)$ may agree with $B_m$ at either or both points; if it does not agree with $B_m$ at a point, it differs by having the point as an endpoint. Regardless, it will also be weakly left of $k$ and weakly right of $l$ and thus still intersects $B_j$ in the interior of two arcs.

**Case 2.** If $B_j$ has $k$ as an endpoint and passes left of $l$, then $B_m$ must pass left of $k$ and be weakly right of $l$. $\text{mat}(B_{m-1}, B_m)$ may agree with $B_m$ at either or both points; if it does not agree with $B_m$ at a point, it differs by having the point as an endpoint. If $k$ is an endpoint of $\text{mat}(B_{m-1}, B_m)$, then the block intersects $B_j$ at an endpoint. Otherwise, it passes left of $k$ and weakly right of $l$ and thus still intersects $B_j$ in the interior of two arcs.

**Case 3.** If $B_j$ has $k$ and $l$ as endpoints, then $B_m$ must pass left of $k$ and right of $l$. $\text{mat}(B_{m-1}, B_m)$ may agree with $B_m$ at either or both points; if it does not agree with $B_m$ at a point, it differs by having the point as an endpoint. If either $k$ or $l$ is an endpoint of $\text{mat}(B_{m-1}, B_m)$, then the block intersects $B_j$ at an endpoint. Otherwise, it passes left of $k$ and right of $l$ and thus still intersects $B_j$ in the interior of two arcs. \(\square\)(Claim)

The claim shows that the set of blocks \{\(B_1, \ldots, B_{m-2}, \text{mat}(B_{m-1}, B_m)\)\} is a single woven block. Let $B_T = \text{mat}(B_1, \ldots, B_{m-2}, \text{mat}(B_{m-1}, B_m))$, the result of matting this new woven block. $B_T$ has endpoints $c_0 < c_1 < \cdots < c_t$, where each $c_i$ is an endpoint of a block $B_i$ for $i \in \{1, \ldots, m-2\}$, an endpoint of $\text{mat}(B_{m-1}, B_m)$, or a point of disagreement between either $B_i$ and $B_j$ with $i, j \in \{1, \ldots, m-2\}$ or between some $B_i$ and $\text{mat}(B_{m-1}, B_m)$. Let $R_T$ and $L_T$ denote the sets of points right and left of $B_T$, respectively. A point $q \in (c_0, c_t) \setminus \{c_1, \ldots, c_{t-1}\}$ is in $R_T$ if it is either in $R_i$ for some block $B_i$ and in neither $L_M$ nor $L_j$ for any block $B_j$ or if $q$ is in $R_M$ and not in $L_i$ for any block $B_i$. Likewise, $q$ is in $L_T$ if it is either in $L_i$ for some block $B_i$ and in neither $R_M$ nor $R_j$ for any block $B_j$ or if $q$ is in $L_M$ and not in $R_i$ for any block $B_i$.

It remains to show that the sets of endpoints, points right of and points left of the twice-matted block $B_T$ exactly match the corresponding sets for $\text{mat}(\mathcal{W})$. The bottom endpoint of $B_T$, $c_0$, is the bottom endpoint of either some $B_i$ with $i \in \{1, \ldots, m-2\}$ or of $\text{mat}(B_{m-1}, B_m)$, in which case it is the bottom endpoint of either $B_{m-1}$ or $B_m$. By similar reasoning, $c_t$ is the top endpoint of some $B_i$ for $i \in \{1, \ldots, m\}$. If $c_k$ is an endpoint in $(c_0, c_t)$, there are several possibilities.
Case 1. $c_k$ is an endpoint of some $B_i$ with $i \in \{1, \ldots, m - 2\}$. It is obvious in this case that $c_k$ is also an endpoint of $\mat(W)$.

Case 2. $c_k$ is an endpoint of $\mat(B_{m-1}, B_m)$. In this case, $c_k$ may be an endpoint of $B_{m-1}$ or $B_m$ and it is clearly an endpoint of $\mat(W)$. Otherwise, $c_k$ may be a point of disagreement between $B_{m-1}$ and $B_m$. Since both blocks are in $W$, $c_k$ is also a point of disagreement between two blocks of $W$ and thus an endpoint of $\mat(W)$.

Case 3. $c_k$ is a point of disagreement between $B_i$ and $B_j$ with $i$ and $j$ in $\{1, \ldots, m - 2\}$. Since the blocks are in $W$, $c_k$ is also a point of disagreement in $W$ and thus an endpoint of $\mat(W)$.

Case 4. $c_k$ is a point of disagreement between some $B_i$ and $\mat(B_{m-1}, B_m)$ with $i$ in $\{1, \ldots, m - 2\}$. In order for $\mat(B_{m-1}, B_m)$ to pass to the opposite side of $c_k$ as $B_i$, at least one of $B_{m-1}$ and $B_m$ must pass to the opposite side of $c_k$ as $B_i$. Thus $c_k$ is a point of disagreement between $B_i$ and at least one of $B_{m-1}$ and $B_m$. Since all three of the blocks are in $W$, $c_k$ is also a point of disagreement in $W$ and thus an endpoint of $\mat(W)$.

The above cases cover all ways that endpoints can occur in $\mat(W)$: as an endpoint of an original block or as a point of disagreement between two blocks in $W$, where each original block may be in either $\{B_1, \ldots, B_{m-2}\}$ or $\{B_{m-1}, B_m\}$.

Let $R_T$ denote the the set of points right of the twice-matted block $B_T$. A point $q \in (c_0, c_k) \setminus \{c_1, \ldots, c_{k-1}\}$ may be in $R_T$ if it is in $R_i$ for some block $B_i$ with $i \in \{1, \ldots, m-2\}$ and in neither $L_M$ nor $L_j$ for any $j \neq i$ in $\{1, \ldots, m - 2\}$. If $q$ is not in $L_M$ and not an endpoint of $B_T$, $q$ must not be left of $B_{m-1}$ or $B_m$. Alternatively, $q$ may be in $R_T$ if it is in $R_M$ and not in $L_i$ for any $i \in \{1, \ldots, m - 2\}$. If $q$ is in $R_M$, it must be right of either $B_{m-1}$ or $B_m$ and not left of either block. In either case, $q$ is in $R_i$ for some $i \in \{1, \ldots, m\}$ and not in any $L_j$. This is precisely what is required for $q$ to be in $R_w$. The argument that $L_T$ (the set of points left of $B_T$)and $L_w$ are equal sets follows by the same logic.

**Theorem 3.3.19.** Given two permutations $\sigma$ and $\tau$ in $S_n$ with noncrossing arc diagrams $N_1 = \delta(\sigma)$ and $N_2 = \delta(\tau)$, their join in $\Psi(A_{n-1})$ is the permutation corresponding to $\mn(N_1, N_2)$.

**Proof.** By definition of $\mn(N_1, N_2)$, the matting process in step (2) may need to be repeated several times before the resulting set of arcs is a noncrossing arc diagram.

Suppose $\mn(N_1, N_2)$ requires $k$ repetitions of the matting process to result in a noncrossing arc diagram. Let $M^i$ denote the $i$th iteration of the matting process, so $M^0 = N_1 \cup N_2$ and $M^k = \mn(N_1, N_2)$. Thus, $\Gamma_{\mn(N_1, N_2)} = \Gamma_{M^k}$. Since the matting process must be repeated $k$ times, $M^k$ is a noncrossing arc diagram and $M^{k-1}$ is not a valid noncrossing arc diagram.
diagram. Since \( M^{k-1} \) is not a noncrossing arc diagram, it consists of some number of woven blocks. As \( M^k = \text{mat}(M^{k-1}) \) and by Lemma 3.3.18, we can write \( \Gamma_{M^k} \) as the intersection of the shard intersections corresponding to the blocks making up \( M^{k-1} \):

\[
\Gamma_{M^k} = \Gamma_{\text{mat}(M^{k-1})} = \bigcap_{B \in M^{k-1}} \Gamma_B .
\]

If \( k \geq 2 \), then the shard intersection corresponding to each block \( B \) in \( M^{k-1} \) can likewise be written as an intersection of the shard intersections corresponding to all blocks of \( M^{k-2} \) which are part of the woven block that is matted to form \( B \). That is,

\[
\Gamma_{M^k} = \bigcap_{B \in M^{k-1}} \left( \bigcap_{C \in M^{k-2}(B)} \left( \bigcap_{\text{mat}(W) \in M^1} \Gamma_{\text{mat}(W)} \right) \right) .
\]

Continuing in this manner, we can write \( \Gamma_{M^k} \) as an intersection of shard intersections coming from blocks in the first matting of \( N_1 \cup N_2 \). We write each such block as \( \text{mat}(W) \), where \( W \) is a woven block of \( N_1 \cup N_2 \).

\[
\Gamma_{M^k} = \bigcap_{B \in M^{k-1}} \left( \bigcap_{C \in M^{k-2}(B)} \left( \bigcap_{\text{mat}(W) \in M^1} \Gamma_{\text{mat}(W)} \right) \right) ,
\]

and by Lemma 3.3.18, the shard intersection \( \Gamma_{\text{mat}(W)} \) for each woven block can be written as an intersection of the shard intersections of the blocks in \( W \):

\[
\Gamma_{M^k} = \bigcap_{B \in M^{k-1}} \left( \bigcap_{C \in M^{k-2}(B)} \left( \bigcap_{\text{mat}(W) \in M^1} \Gamma_{\text{mat}(W)} \right) \right) ,
\]

Each block in the union of \( N_1 \) and \( N_2 \) is part of some woven block, and each woven block in \( M^i \) becomes part of some matted block \( M^{i+1} \) for each \( i \in \{0, \ldots, k-1\} \). Thus, the repeated intersection notation in the previous equation can be collapsed to give

\[
\Gamma_{M^k} = \bigcap_{B_i \in N_1 \cup N_2} \Gamma_{B_i} = \bigcap_{B_i \in N_1} \Gamma_{B_i} \cap \bigcap_{C_j \in N_2} \Gamma_{B_i} .
\]

Finally, by Lemma 3.3.16, the intersections on the right side of the equation above can be rewritten as \( \Gamma_{N_1} \) and \( \Gamma_{N_2} \). Since \( M^k \) was defined to be \( \text{mn}(N_1, N_2) \), we conclude that \( \Gamma_{\text{mn}(N_1, N_2)} = \Gamma_{N_1} \cap \Gamma_{N_2} \), so the proof is complete. \( \Box \)
3.4 Type B

3.4.1 Signed permutations and noncrossing arc diagrams

We describe two models of noncrossing arc diagrams of type B in Section 3.4.1. A signed permutation of $\pm [n]$ is an automorphism on the set $\{\pm 1, \pm 2, \ldots, \pm n\}$, with the requirement that for every $i$ in the set, $\pi(-i) = -\pi(i)$, or in the shorthand established in type A, $\pi_{-i} = -\pi_i$. The Coxeter group of type $B_n$ can be realized as the group of signed permutations of $\pm [n]$. As in type A, we will consider noncrossing arc diagrams corresponding to signed permutations and describe certain operations on noncrossing arc diagrams.

A symmetric model

As with a permutation, we can construct one-line notation for a signed permutation $\pi$ of $\pm [n]$ by writing what each of the elements in the set maps to under $\pi$. To mark the symmetry of the signed permutation, we include a vertical line $|$ between $\pi_{-1}$ and $\pi_1$. We refer to this as the long one-line notation for a signed permutation. We discuss the short one-line notation in the next section.

We will construct noncrossing arc diagrams on $2n$ points on a vertical line, numbered from $-n$ at the bottom, in order up to $-1$, then from 1 up to $n$ at the top so that there is a half-turn rotation of the plane that sends each point $i$ to $-i$. The map $\delta$ from type A can be modified to map signed permutations on $\pm [n]$ to noncrossing arc diagrams on these $2n$ points. Specifically, $\delta$ plots points $(i, \pi_i)$ on the Cartesian plane, then creates a line segment between a point and the point immediately to its right if the left point is above the right point. Once all necessary line segments have been drawn, the map brings all the points into a vertical line, letting all the line segments bend so that they stay to the correct side of the points they pass left or right of. This construction is presented in work by Barnard and Reading [6] as well as Albertin and Pilaud [1]. Additionally, recent work in Chapter 2 presents type-B lattice-theoretic results analogous to those proven for type A in [35].

The symmetry of signed permutations is preserved by $\delta$ to yield a symmetry of noncrossing arc diagrams on $2n$ points. Recall that for a permutation, descents are pairs of consecutive entries $(\pi_i, \pi_{i+1})$ such that $\pi_i \geq \pi_{i+1}$. For signed permutations, the idea of descents is altered to account for their symmetry.

If for some $i \in \{1, \ldots, n-1\}$ $\pi_i$ is greater than $\pi_{i+1}$, the symmetry of signed permutations implies that $\pi_{-i-1}$ is also greater than $\pi_{-i}$. Together, the two pairs $(\pi_i, \pi_{i+1})$ and $(\pi_{-i-1}, \pi_{-i})$ are considered a single type-B descent. Additionally, an entry $\pi_j$ satisfying $\pi_{i+1} < \pi_j < \pi_i$ occurs after $\pi_{i+1}$ if and only if the entry $\pi_{-j} = -\pi_j$ satisfies $\pi_1 < \pi_j < \pi_{-1}$. 

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and occurs before $\pi_{-i-1}$. This type of descent, therefore, corresponds to a pair of arcs $\{\alpha, -\alpha\}$ which are symmetric to one another under the half-turn rotational symmetry of the numbered points. The arc $\alpha$ from $\pi_i$ to $\pi_{i+1}$ passes right of a point $\pi_j$ if and only if the arc symmetric to it, $-\alpha$, from $\pi_{-i-1}$ to $\pi_{-i}$ passes left of $\pi_{-j}$.

The other kind of type-B descent is a pair $(\pi_1, \pi_{-1})$ such that $\pi_1$ is negative. An entry $\pi_j$ satisfying $\pi_1 < \pi_j < \pi_{-1}$ occurs after $\pi_1$ if and only if the entry $\pi_{-j}$ satisfies $\pi_1 < \pi_j < \pi_{-1}$ and occurs before $\pi_{-1}$. Therefore, this type of descent corresponds to a single arc from $\pi_{-1}$ to $\pi_1$ which is itself symmetric under the half-turn symmetry of the numbered points, passing right of $\pi_j$ if and only if it passes left of $\pi_{-j}$.

We see that $\delta$ maps the set of signed permutations bijectively to the set of centrally symmetric noncrossing arc diagrams on $2n$ points. For example, the symmetric noncrossing arc diagrams of type $B_2$ are shown in Fig. 3.21.

The idea of a block translates naturally from type A to the symmetric construction of type-B noncrossing arc diagrams. In this construction, there may be a centrally symmetric pair of type-A blocks, one having only positive endpoints and the other having only negative endpoints. Alternatively, a centrally symmetric pair of blocks may overlap one another: each block has at least one negative and at least one positive endpoint. Finally, a block may itself be centrally symmetric, consisting of a single symmetric arc connecting a centrally symmetric pair of collections of type-A arcs, one connected to the top of the symmetric arc and one connected to the bottom of it.

We now consider how the cooperation and matting processes (Definitions 3.3.8 and 3.3.10) apply to pairs of symmetric noncrossing arc diagrams on $2n$ points. When $N_1$ and $N_2$ are centrally symmetric, the blocks $B$ in $N_1$ and $C$ in $N_2$ pass weakly left of a point $i$ if and only if the blocks or their symmetric partners pass weakly right of $-i$. Similarly, a woven block $W$ in $N_1 \cup N_2$ passes right of a point $i$ if and only if either $W$ or its symmetric partner passes left of $-i$. Thus the following lemmas are immediate.

**Lemma 3.4.1.** If $N_1$ and $N_2$ are symmetric noncrossing arc diagrams on $2n$ points, then
\( cn(N_1, N_2) \) is a symmetric noncrossing arc diagram on \( 2n \) points.

**Lemma 3.4.2.** If \( N_1 \) and \( N_2 \) are symmetric noncrossing arc diagrams on \( 2n \) points, then \( mn(N_1, N_2) \) is a symmetric noncrossing arc diagram on \( 2n \) points.

**An orbifold model**

As mentioned in Section 3.4.1, we can construct a short one-line notation for each signed permutation \( \pi \) of \( \pm [n] \) along with its long one-line notation. The long one-line notation is somewhat redundant since \( \pi_{-i} = -\pi_i \) for each \( i \in [n] \), and the short one-line notation eliminates this redundancy. The **short one-line notation** or **window notation** of \( \pi \) consists only of the images of \( 1 \) through \( n \) in order, \( \pi_1 \cdots \pi_n \).

As we consider type-B analogues of noncrossing arc diagrams, a natural question arises: Is there a way to exploit the symmetry of signed permutations to make more “compact” noncrossing arc diagrams? Fortunately, the answer to this question is yes, and though the construction of these diagrams requires some care, it also sets us up for a satisfyingly short and self-evident answer to the central question at hand: Is \( \Psi(A_n) \) a sublattice of \( \Psi(B_n) \)?

With that motivation, let’s consider this new construction of noncrossing arc diagrams.

Following Chapter 2, we take advantage of the symmetry of the type-B noncrossing arc diagrams on \( 2n \) points by “modding out” by the half-turn rotation under which all such diagrams are symmetric. The underlying structure of the new construction consists of \( n + 1 \) distinct points on a vertical line, with the lowest point marked by the symbol \( \times \) (the **orbifold point**) and the \( n \) points above \( \times \) identified with \( 1 \) through \( n \) in order from bottom to top.

Recall that type-B descents occur in one of two forms: as a pair \((\pi_i, \pi_{i+1})\) and \((\pi_{i-1}, \pi_{i-1})\) or as a single descent \((\pi_{-1}, \pi_1)\). We define three kinds of type-B arcs based on three types of type-B descents. The first two types are ways in which a descent as a pair may occur.

In the first type of descent, \( \pi_i \) and \( \pi_{i+1} \) have the same sign. This results in a symmetric pair of arcs where one arc has only positive endpoints and the other arc has only negative endpoints. When we mod out by half-turn symmetry, this pair becomes a single arc from \( |\pi_i| \) to \( |\pi_{i+1}| \). We this a **type-A arc**, as it meets exactly the criteria that curve must meet to be an arc in the type-A sense.

In the second type of descent, \( \pi_i \) and \( \pi_{i+1} \) have opposite signs. This results in a symmetric pair of arcs where each arc in the pair will have one positive and one negative endpoint, so the two arcs overlap. When we mod out by half-turn symmetry, this pair becomes a single arc which has \( \pi_i \) as an endpoint, goes monotone down to the right of the point \( \times \), passes in a half-circle clockwise around \( \times \), and goes monotone up to the endpoint \( |\pi_{i+1}| \),
never crossing itself. By definition of signed permutations, \( \pi_i \neq |\pi_{i+1}| \), so the arc has two distinct endpoints. We call any arc meeting this description a **long arc**.

The third type of descent is of the form \((\pi_{-1}, \pi_1)\). The resulting arc from \(\pi_{-1}\) to \(\pi_1\) is itself centrally symmetric. When we mod out by half-turn symmetry, this arc becomes an arc from \(\pi_{-1}\) to the point \(\times\). We call such an arc an **orbifold arc**.

Any type-A, long, or orbifold arc is a **type-B arc**. A collection of type-B arcs constitutes a **type-B noncrossing arc diagram** if and only if each pair of type-B arcs satisfies the same compatibility conditions as those stated in type A: no crossing along interiors of arcs and no sharing of upper endpoints or lower endpoints. We include \(\times\) in consideration as a lower endpoint in considering compatibility: if there are two orbifold arcs in a collection of arcs, it is not a noncrossing arc diagram. We also use the convention that both endpoints of a long arc to be upper endpoints.

Orbifold representations of all type-B noncrossing arc diagrams for \(B_2\) and \(B_3\) are in Fig. 3.22 and Fig. 3.23. To see the relationship between the symmetric and orbifold constructions, compare Fig. 3.22 with Fig. 3.21.

As with arcs, a **type-B block** in a type-B noncrossing arc diagram may occur in one of three forms.

The first form is the simplest: The block may consist only of type-A arcs, in which case we call it a **type-A block**. Its numbered endpoints \(p, q_1, \ldots, q_s, r\), with \(p < q_1 < \cdots < q_s < r\), correspond to the descending run \(rq_s \cdots q_1 p\) or \((-p)(-q_1) \cdots (-q_s)(-r)\) in the window notation of a signed permutation \(\pi\). Which descending run appears in the window notation depends on the block’s position relative to other blocks in a given diagram, and we will introduce the language needed to make this statement less vague shortly.

In the second form, the block has one orbifold arc with upper endpoint \(q_1\), along with some (or no) type-A arcs; we call this an **orbifold block**. Its numbered endpoints \(q_1, \ldots, q_s\), with \(q_1 < \cdots < q_s\), correspond to the descending run \(q_s \cdots q_1(-q_1) \cdots (-q_s)\) in the long one-line notation with the line \(|\) between \(q_1\) and \((-q_1)\). In the window notation, the first \(s\) entries are the second half of the descending run, \((-q_1) \cdots (-q_s)\).

In the third form, the block has one long arc along with some (or no) type-A arcs; we
call this a long block. Its numbered endpoints $p, q_1, \ldots, q_l, q_{l+1}, \ldots, q_{s-1}, r$, all distinct and with $p = q_0 > q_1 > \cdots > q_l > 0 < q_{l+1} < \cdots < q_s = r$ such that $p, q_1, \ldots, q_l$ are on the left side of the block and $q_{l+1}, \ldots, q_{s-1}, r$ are on the right side of the block, correspond to the descending run $r q_{s-1} \cdots q_{l+1} (-q_l) \cdots (-q_1) (-p)$ in the window notation. It is possible that there are no type-A arcs on the left side of the block (i.e. $l = 0$) and/or no type-A arcs on the right side of the block (i.e. $l + 1 = s$).

Our next objective is to present type-B versions of cooperative and matted noncrossing arc diagrams in this orbifold construction. Recall that in type A, the cooperation process included going from an upper endpoint shared by a pair of blocks to a lower endpoint shared by the pair of blocks. However, because the orbifold model of noncrossing arc diagrams conflates the top and bottom of the symmetric model, defining and using upper and lower endpoints is perhaps more trouble than it is worth. Instead, we will refer to endpoints of blocks in generality — an endpoint of a nontrivial block is an endpoint of some arc in the block. For an orbifold block, we count $\times$ as an endpoint.

**Definition 3.4.3.** Let $N_1$ and $N_2$ be two type-B noncrossing arc diagrams on $n$ points. The type-B cooperative noncrossing arc diagram of $N_1$ and $N_2$, denoted $cn_B(N_1, N_2)$, consists of all arcs $\alpha$ constructed as follows. An arc $\alpha$ exists only when there exists a pair of
points \( r \in [n] \) and \( p \in [n] \cup \{\times\} \), each an endpoint of a block \( B \) in \( N_1 \) and a block \( C \) in \( N_2 \), satisfying one of the following conditions:

1. \( p \in [n] \) such that \( r \) and \( p \) are connected by a sequence of type-A arcs in \( B \) and also by a sequence of type-A arcs in \( C \) that pass weakly to the same side of each point between \( r \) and \( p \), but the two sequences do not share any endpoint between \( r \) and \( p \).

2. \( p = \times \) and \( B \) and \( C \) are orbifold blocks such that \( r \) is the lowest numbered endpoint shared by the two blocks, and both blocks pass weakly to the same side of every numbered point below \( r \).

3. \( p \in [n] \) and \( B \) and \( C \) are long blocks such that \( r \) and \( p \) are the lowest endpoints shared by the right and left sides respectively of the two blocks, the right sides of \( B \) and \( C \) pass weakly to the same side of every numbered point below \( r \), and the left sides of \( B \) and \( C \) pass weakly to the same side of every numbered point below \( p \).

4. \( p \in [n] \) and \( B \) and \( C \) are a long block and an orbifold block such that \( r \) and \( p \) are endpoints on the right side and left side respectively of the long block and endpoints of the orbifold block, and the orbifold block passes weakly to the same side of each numbered point below \( r \) as the right side of the long block and weakly to the same side of each numbered point below \( p \) as the left side of the long block, but the orbifold block does not share any endpoint below \( p \) on the left side of the long block or below \( r \) on the right side of the long block.

If the requirements above are satisfied, \( \alpha \) is the arc that follows \( B \) and \( C \) from \( r \) to \( p \) and passes to the same side of each point between \( r \) and \( p \) (possibly including \( \times \)) as the appropriate side of \( B \) and of \( C \).
Examples of arcs resulting from each condition of Definition 3.4.3 are included in Fig. 3.24.

**Lemma 3.4.4.** Let $N_1$ and $N_2$ be type-B noncrossing arc diagrams on $n$ points, and let $M_1$ and $M_2$ be the corresponding symmetric noncrossing arc diagrams on $2n$ points. Then $\text{cn}(M_1, M_2)$ is the symmetric noncrossing arc diagram on $2n$ points corresponding to $\text{cn}_B(N_1, N_2)$.

**Proof.** In requirement (1) of Definition 3.4.3, the arcs connecting $r$ and $p$ are all type-A. Each of these arcs corresponds to a pair of arcs in $M_1$ and $M_2$, and the weak agreement of the sequence of arcs connecting $r$ and $p$ (and those connecting $-r$ and $-p$ in $M_1$ and $M_2$) is precisely the weak agreement required for an arc to exist in the type-A $\text{cn}(M_1, M_2)$. The type-A cooperative process results in a non-overlapping pair of arcs in $\text{cn}(M_1, M_2)$ whose image under modding out by rotational symmetry is the arc from $r$ to $p$ which agrees with the type-A arcs connecting them.

In requirement (2), the symmetric versions of $B$ and $C$ are centrally symmetric blocks which weakly agree between $r$ and $-r$ but share no endpoint between $r$ and $-r$. The type-A cooperative process results in a single centrally symmetric arc from $r$ to $-r$ in $\text{cn}(M_1, M_2)$ whose image under modding out by rotational symmetry is the arc from $r$ to $\times$ which agrees with the arcs connecting them.

In requirement (3), the symmetric versions of $B$ and $C$ are centrally symmetric pairs of blocks whose right blocks have endpoints $r$ and $-p$ and whose left blocks have endpoints $p$ and $-r$. The right blocks weakly agree between $r$ and $-p$ but do not share an endpoint between $r$ and $-p$. The type-A cooperative process results in an overlapping centrally symmetric pair of arcs whose right arc goes from $r$ to $-p$ and whose left arc goes from $p$ to $-r$. The image of this pair of arcs under modding out by rotational symmetry is the long arc from $r$ to $p$ which has $r$ as its right endpoint and $p$ as its left endpoint and which agrees with the right sides of $B$ and $C$ below $r$ and the left sides of the two blocks below $p$.

In requirement (4), the symmetric version of either $B$ or $C$ is an overlapping centrally symmetric pair of blocks whose right block has endpoints $r$ and $-p$, and the symmetric version of either $C$ or $B$ is a centrally symmetric block which has $-r$, $-p$, and $p$, and $r$ among its endpoints. As in (3), the type-A cooperative process results in an overlapping centrally symmetric pair of arcs whose right arc goes from $r$ to $-p$ and agrees with the symmetric block and the right block of the overlapping pair. The the image of this pair of arcs under modding out by rotational symmetry is the long arc from $r$ to $p$ which has $r$ as its right endpoint and $p$ as its left endpoint and which agrees with $B$ and $C$ below $r$ on the right side and below $p$ on the left side.

Next, we define the matted noncrossing arc diagram in type B. As in type A, its
construction consists of matting woven blocks. Before stating the definition, we describe the
three forms that type-B woven blocks may take.

A collection $C$ of type-B arcs on $n$ points which is connected may take one of three
forms: It is type-A if it has no orbifold or long arcs. It is bilateral if it contains at least one
long block, does not contain an orbifold block, and it has well-defined right and left sides: the
right side of $C$ is the union of the right sides of all long blocks in $C$ and any type-A blocks
that are connected to the right side of a long block, the left side of $C$ is defined similarly,
and no part of any arc on the right side is left of any arc on the left side (and vice versa).
Otherwise, the collection is called unilateral.

Definition 3.4.5. Let $N_1$ and $N_2$ be two type-B noncrossing arc diagrams on $n$ points.
The type-B matted noncrossing arc diagram of $N_1$ and $N_2$, denoted $\text{mn}_B(N_1, N_2)$, is
constructed as follows:

1. Consider the union of $N_1$ and $N_2$ on the same set of $n$ numbered points and orbifold
   point $\times$. As in type A, the connected components of this union are called woven
   blocks.

2. For each woven block $W$, there is a corresponding block $\text{mat}_B(W)$, the result of “matting”
   $W$, described as follows:

   - Each endpoint of $W$ (including possibly $\times$) is an endpoint of $\text{mat}_B(W)$.
   - If $W$ is a type-A woven block, then $\text{mat}_B(W)$ agrees with $\text{mat}(W)$ as defined in
     Definition 3.3.10.
   - If $W$ is bilateral with top-left endpoint $l$ and top-right endpoint $r$, then $\text{mat}_B(W)$
     is a long block. If all arcs in the right side of $W$ pass to the same side of a point
     $q \in [1, r)$, the right side of $\text{mat}_B(W)$ passes to that side of $q$; likewise, if all arcs
     in the left side of $W$ pass to the same side of a point $q \in [1, l)$, the left side of
     $\text{mat}_B(W)$ passes to that side of $q$. If two arcs in the right side of $\text{mat}_B(W)$ pass
     to opposite sides of a point $q \in [1, r)$, then $q$ is an endpoint of the right side of
     $\text{mat}_B(W)$; similarly, if two arcs in the left side of $W$ pass to opposite sides of a
     point $q \in [1, l)$, then $q$ is an endpoint of the left side of $\text{mat}_B(W)$.
   - If $W$ is unilateral with highest endpoint $r$, then $\text{mat}_B(W)$ is an orbifold block. If
     all arcs in $W$ pass to the same side of a point $q \in [1, r)$, an arc in $\text{mat}_B(W)$
     passes to that side of $q$. If any arcs in $W$ pass to opposite sides of a point $q \in [1, r)$,
     $q$ is an endpoint of $\text{mat}_B(W)$.

3. If steps (1) and (2) yield a valid noncrossing arc diagram, stop. If not, then form woven
   blocks from the arcs created and repeat step (2).
Lemma 3.4.6. Let $N_1$ and $N_2$ be type-B noncrossing arc diagrams on $n$ points, and let $M_1$ and $M_2$ be the corresponding symmetric noncrossing arc diagrams on $2n$ points. Then $\text{mn}(M_1, M_2)$ is the symmetric noncrossing arc diagram on $2n$ points corresponding to $\text{mn}_B(N_1, N_2)$.

Proof. Any woven block $\mathcal{W}$ in the orbifold setting corresponds to either a single centrally symmetric woven block $\mathcal{V}$ or to a centrally symmetric pair of woven blocks $\{\mathcal{V}, -\mathcal{V}\}$.

We consider the three types of woven blocks in the union of $N_1$ and $N_2$ and confirm that for each type, the matting process described in Step (2) of Definition 3.4.5 agrees with the matting process in the symmetric setting, as described in Definition 3.3.10.

A type-A woven block $\mathcal{W}$ in the orbifold setting corresponds to a non-overlapping symmetric pair of woven blocks $\{\mathcal{V}, -\mathcal{V}\}$ in the symmetric setting. The matting process on $\mathcal{V}$ is exactly the matting process in type A, and the matting process on $-\mathcal{V}$ yields a block which is symmetric to $\text{mat}(\mathcal{V})$. The image of this pair of blocks under modding out by rotational symmetry is the type-A block on $n$ points exactly matching $\text{mat}(\mathcal{V})$.

A bilateral woven block $\mathcal{W}$ in the orbifold setting corresponds to an overlapping symmetric pair of woven blocks $\{\mathcal{V}, -\mathcal{V}\}$ in the symmetric setting. Because $\mathcal{W}$ has right
and left sides which remain strictly right and left of one another, $V$ and $-V$ are disjoint and one of the woven blocks must be right of the other. Let $V$ denote the right woven block, so it has highest endpoint $r$ and lowest endpoint $-l$; the left woven block $-V$ has highest endpoint $l$ and lowest endpoint $-r$. The two blocks $\text{mat}(V)$ and $\text{mat}(-V)$ will be an overlapping symmetric pair of blocks, where $\text{mat}(V)$ has top endpoint $r$ and bottom endpoint $-l$, and $\text{mat}(V)$ is right of $\text{mat}(-V)$. The image of this pair under modding out by rotational symmetry is the long block with top-left endpoint $l$ and top-right endpoint $r$ whose right side agrees with the portion of $\text{mat}(V)$ along the interval $[1, r]$ and whose left side agrees with the portion of $\text{mat}(-V)$ along the interval $[1, l]$.

A unilateral woven block $W$ in the orbifold setting corresponds to a centrally symmetric woven block $V$ in the symmetric setting. Because $V$ is centrally symmetric with top endpoint $r$ and bottom endpoint $-r$, $\text{mat}(V)$ is also centrally symmetric and has the same top and bottom endpoints. The image of $\text{mat}(V)$ under modding out by rotational symmetry is the orbifold arc whose highest endpoint is $r$ which agrees with the top half of $\text{mat}(V)$.

### 3.4.2 Shard intersection order of type B

Recall that a Coxeter group of type $B_n$ can be realized as the group of signed permutations of $\pm[n]$. As we did in type A, we begin by discussing the geometric realization of the type $B_n$ Coxeter group.

The hyperplane corresponding to the generator $s_0 = (-1, 1)$ is $H_1 = \{ \vec{x} \in \mathbb{R}^n : x_1 = -x_1 \}$. For $i \in \{1, \ldots, n-1\}$, the hyperplane corresponding to the generator $s_i = (i \ i+1)$ is $H_i = \{ \vec{x} \in \mathbb{R}^n : x_i = x_{i+1} \}$, as in type A. The natural choice for the base region $B$ in the Coxeter arrangement $A(B_n)$ is the region containing $[1, 2, \ldots, n+1]$. We often write $x_{-i}$ instead of $-x_i$, as this notation agrees nicely with the symmetry of signed permutations. As in type A, in order to have the action of reflection across hyperplanes agree with the right weak order on $B_n$, the region corresponding to the signed permutation $\pi$ with window notation $\pi_1 \cdots \pi_n$ is $\{ \vec{x} \in \mathbb{R}^n : x_{\pi_1} \leq x_{\pi_2} \leq \cdots \leq x_{\pi_n} \}$.

Just as we have characterized the meet and join in the shard intersection order of type A in terms of noncrossing arc diagrams, we want to understand the shard intersection order of type B in terms of noncrossing arc diagrams, first in the centrally symmetric model and then in the orbifold model. The set of all permutations of $\pm[n]$ is a Coxeter group of type $A_{2n-1}$. The signed permutations are precisely the permutations that are fixed under the map that sends $\pi_{-n} \cdots \pi_{-1} \pi_1 \cdots \pi_n$ to $(-\pi_n) \cdots (-\pi_1)(-\pi_{-1}) \cdots (-\pi_{-n})$. This map is conjugation by $w_0$ in the type-A Coxeter group, which is an automorphism of the shard intersection order. It is a known lattice theoretic result that the set of fixed points of a
lattice automorphism $\eta : L \to L$ forms a sublattice of $L$. In light of this result, the following proposition is immediate.

**Proposition 3.4.7.** The shard intersection order on the Coxeter group of type $B_n$ is a sublattice of the shard intersection order on the Coxeter group of type $A_{2n-1}$.

Proposition 3.4.7 says that the meet and join operations in $\Psi(B_n)$ are precisely the meet and join in $\Psi(A_{2n-1})$. The two theorems that follow are therefore immediate from Theorems 3.3.15 and 3.3.19 and Proposition 3.4.7.

**Theorem 3.4.8.** Given two signed permutations $\sigma$ and $\tau$ of $\pm [n]$ with centrally symmetric noncrossing arc diagrams $M_1 = \delta(\sigma)$ and $M_2 = \delta(\tau)$, their meet in $\Psi(B_n)$ is the signed permutation corresponding to $\pi_{\text{cn}}(M_1, M_2)$.

**Theorem 3.4.9.** Given two signed permutations $\sigma$ and $\tau$ of $\pm [n]$ with centrally symmetric noncrossing arc diagrams $M_1 = \delta(\sigma)$ and $M_2 = \delta(\tau)$, their join in $\Psi(B_n)$ is the signed permutation corresponding to $\pi_{\text{mn}}(M_1, M_2)$.

With this understanding of the shard intersection order of type B in terms of symmetric noncrossing arc diagrams, we now synthesize the results above using the orbifold construction of type-B noncrossing arc diagrams. The reason for this is that the orbifold construction has advantages that can be leveraged to better understand the shard intersection order of type B. The first and most obvious advantage is that the grading of $\Psi(B_n)$ can be framed more succinctly in the orbifold construction than the symmetric construction. As stated by Petersen in [28, Observation 4], the rank of a signed permutation in $\Psi(B_n)$ is given by its descent number, the number of type-B descents. As discussed in Section 3.4.1, each type-B descent in a signed permutation $\pi$ corresponds either to a single symmetric arc or to a symmetric pair in the symmetric noncrossing arc diagram. Thus, from the symmetric perspective, the shard intersection order is graded by the sum of the number of symmetric arcs and the number of pairs of symmetric arcs. In contrast, in the orbifold construction, each type-B descent corresponds to exactly one arc. The following proposition is a restatement of Petersen’s observation from the simpler orbifold perspective, and is immediate.

**Proposition 3.4.10.** For any $\pi \in B_n$, the rank of $\pi$ in $\Psi(B_n)$ is equal to the number of arcs in the type-B noncrossing arc diagram corresponding to $\pi$.

The shard intersection order of type $B_2$, realized both as symmetric noncrossing arc diagrams on 4 points and as type-B noncrossing arc diagrams on 2 points, is pictured in Fig. 3.27. Even in this small example, the grading of $\Psi(B_2)$ is more easily seen in the orbifold construction.
The second, less obvious advantage is that the orbifold construction is suggestive of (and indeed matches) a nice embedding of the lattice $\Psi(A_n)$ into the $\Psi(B_n)$, which we discuss in Section 3.5.1. As a preface to that discussion, we note that 24 of the 48 type-B noncrossing arc diagrams on 3 points in $\Psi(B_3)$, as seen in Fig. 3.28, look like type-A noncrossing arc diagrams on 4 points, except that the lowest point in the diagram is $\times$ instead of the numbered point 1.

Combining the results of Lemmas 3.4.4 and 3.4.6, which state that cooperation and matting commute with modding out by half-turn symmetry, with Proposition 3.4.7, we obtain the following theorems in type B which are analogous to Theorems 3.3.15 and 3.3.19.

**Theorem 3.4.11.** Given two signed permutations $\sigma$ and $\tau$ with type-B noncrossing arc diagrams $N_1 = \delta^\circ(\sigma)$ and $N_2 = \delta^\circ(\tau)$, their meet in $\Psi(B_n)$ is the signed permutation corresponding to $\text{cn}_B(N_1, N_2)$.

**Theorem 3.4.12.** Given two signed permutations $\sigma$ and $\tau$ with type-B noncrossing arc diagrams $N_1 = \delta^\circ(\sigma)$ and $N_2 = \delta^\circ(\tau)$, their join in $\Psi(B_n)$ is the signed permutation corresponding to $\text{mn}_B(N_1, N_2)$.
In this section, we discuss all four surjective lattice homomorphisms from the weak order on $B_n$ to the weak order on type $A_n$. The homomorphisms we consider are discussed in depth in [38, Section 6]. Three of these homomorphisms have nice properties which suggest them as good candidates to induce embeddings of the type-A shard intersection order lattice into the shard intersection order lattice on type $B$, as we now explain.

Recall that the fibers of a lattice homomorphism are intervals, so for any surjective lattice homomorphism $\eta : L \to M$, there is a natural inclusion $\zeta : M \to L$ in the opposite direction that sends an element $m \in M$ to the bottom element of its fiber in $\eta$. In an important example, the homomorphism from the weak order $(W, \leq)$ to a Cambrian lattice $\Theta_c(W)$, the inclusion is actually an embedding as a sublattice of the shard intersection order.
order [33, Proposition 8.7]. As stated in [33, Proposition 8.21], a necessary condition for $\zeta$ to be an embedding as a sublattice of the shard intersection order is that the congruence associated with $\eta$ be homogeneous of degree two. This condition is not met by one of the four homomorphisms we will discuss, so the remaining homomorphisms are the only possible candidates. Namely, the three homomorphisms discussed in Sections 3.5.1 and 3.5.3 are homogeneous of degree two.

**Remark 3.5.1.** While this dissertation’s main focus is the relationship between shard intersection orders of types A and B, there is a more general underlying question: If a finite Coxeter group $W$ dominates another Coxeter group $W'$, is the shard intersection order on $W'$ a sublattice of the shard intersection order on $W$? Dominance can be understood by considering the Coxeter diagrams of the two groups: $W$ dominates $W'$ if the two diagrams have the same number of vertices and the diagram for $W'$ can be obtained from that of $W$ by lowering or erasing the label between two adjacent vertices or by erasing edges.

It is possible that the Coxeter diagrams of $W$ and $W'$ differ only in the subgraph induced by two generators $s_i$ and $s_j$ which are adjacent in the diagram for $W$. In this case, we might expect that a homomorphism from $W$ to $W'$ that induces a congruence on $W$ with a join-irreducible generator outside of the standard parabolic subgroup generated by $s_i$ and $s_j$ does not fit nicely with the operations on diagrams which correspond to dominance.

We consider each homomorphism in [38, Section 6] in terms of the orbifold construction of type-B noncrossing arc diagrams. Each homomorphism corresponds to a congruence generated by a small set of join-irreducible signed permutations. In [35], noncrossing arc diagrams are shown to be very useful in understanding congruences of the weak order on Coxeter groups of type A. A direct consequence of [35, Theorem 4.4], is that any congruence on $(A_n, \leq)$ can be described completely by listing the “smallest” arcs it contracts (in the sense of subarcs). An analogous result for type-B noncrossing arc diagrams is stated in Theorem 2.4.22, and we will use it several times in the sections that follow.

### 3.5.1 Simion’s homomorphism

In [41], Simion describes an operation on a signed permutation $\pi = \pi_{-n} \cdots \pi_{-1} | \pi_1 \cdots \pi_n$ of $\pm [n]$. The map is denoted $\eta_\sigma$ in [38], but we denote it instead by $\eta_0$ for reasons that will be apparent shortly. Simion’s operation on $\pi$ is described as follows: replace the vertical line $|$ between $\pi_{-1}$ and $\pi_1$ with a zero, read off the sequence of nonnegative integers from left to right, then add one to each entry of the sequence. This gives a permutation of $[n+1]$, and in [38, Theorem 6.1], the operation is shown to be a surjective lattice homomorphism from the weak order on $B_n$ to the weak order on $A_n$. 

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Figure 3.29: Contracted arcs that generate the congruence defined by $\eta_0$.

**Example 3.5.2.** $\eta_0(2(-5)1(-4)(-3)) = 456132$

<table>
<thead>
<tr>
<th>3 4 -1 5 -2</th>
<th>2 -5 1 -4 -3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 4 -1 5 -2 0 2 -5 1 -4 -3</td>
<td></td>
</tr>
<tr>
<td>becomes 0</td>
<td></td>
</tr>
<tr>
<td>read sequence $\geq 0$</td>
<td></td>
</tr>
<tr>
<td>3 4 5 0 2 1</td>
<td></td>
</tr>
<tr>
<td>add one</td>
<td></td>
</tr>
<tr>
<td>4 5 6 1 3 2</td>
<td></td>
</tr>
</tbody>
</table>

[38, Theorem 6.1] also states that the congruence defined by $\eta_0$ is generated by the join-irreducible elements $s_0 s_1$ and $s_1 s_0 s_1$, which are the signed permutations $2(-1)3\cdots n$ and $1(-2)3\cdots n$. These arcs in $B_3$ are pictured in Fig. 3.29, and in larger type-B Coxeter groups, the noncrossing arc diagrams are only changed by adding numbered points above 3. It is easy to verify (using Section 2.4.2) that the superarcs of these two arcs are exactly the long arcs. (This recovers [38, Proposition 6.3] much more easily.) Thus by Theorem 2.4.22 and Corollary 2.4.24, the uncontracted elements in the congruence on the weak order on $B_n$ corresponding to $\eta_0$ are *exactly* those which have only type-A and orbifold arcs in their noncrossing arc diagrams. These type-B noncrossing arc diagrams on $n$ points look like type-A noncrossing arc diagrams on $n + 1$ points, except that $\times$ is occupying the space of the numbered point 1. This is not a coincidence.

The natural inclusion $\zeta_0$ from $A_n$ to $B_n$ which sends each permutation $\pi$ to the bottom element of the congruence defined by $\eta_0$ that maps to $\pi$ is described as follows: First, subtract one from each entry of $\pi$. The resulting permutation $\sigma$ of $\{0, \ldots, n\}$ will have three parts, $\sigma_1 \cdots \sigma_{j-1}$, then $\sigma_j = 0$, and then $\sigma_{j+1} \cdots \sigma_{n+1}$, so the first and third parts together constitute a permutation of $[n]$. Next, write the window notation of a signed permutation of $\pm[n]$, $(-\sigma_{j-1}) \cdots (-\sigma_1) \sigma_{j+1} \cdots \sigma_{n+1}$. (See Examples 3.5.3 to 3.5.6.) The long one-line notation of $\zeta_0(\pi)$ is

$(-\sigma_{n+1}) \cdots (-\sigma_{j+1}) \sigma_1 \cdots \sigma_{j-1} \ | \ (-\sigma_{j-1}) \cdots (-\sigma_1) \sigma_{j+1} \cdots \sigma_{n+1}$.

The signed permutation $\zeta_0(\pi)$ has the fewest possible inversions among signed permutations that map to $\pi$, because both the positive entries left of $|$ and the positive entries right of $|$ are as far right as possible.

**Example 3.5.3.** $\zeta_0(1452736) = 3 4 1 6 2 5$
Example 3.5.4. \( \zeta_0(4521736) = (-1)(-4)(-3)625 \)

Example 3.5.5. \( \zeta_0(4251736) = (-4)(-1)(-3)625 \)

Example 3.5.6. \( \zeta_0(4517326) = (-4)(-3)6215 \)

Proposition 3.5.7. The map \( \zeta_0 \) on permutations corresponds to the following operation on noncrossing arc diagrams: For any \( \pi \in A_n = S_{n+1} \), the type-B noncrossing arc diagram \( \delta^o(\zeta_0(\pi)) \) is identical to the type-A noncrossing arc diagram \( \delta(\pi) \), except that the numbered point 1 in \( \delta(\pi) \) is replaced by the orbifold point \( \times \) and the numbered points 2, \ldots, \( n+1 \) are renumbered as 1, \ldots, \( n \).

The operation on noncrossing arc diagrams corresponding to \( \zeta_0 \) for Examples 3.5.3 to 3.5.6 is pictured in Fig. 3.30.

Proof. Let \( \pi \) be a permutation in \( S_{n+1} \), and let \( j \) denote the position of 1 in \( \pi \). The long one-line notation of \( \zeta_0(\pi) \) is

\[
(-\sigma_{n+1}) \cdots (-\sigma_{j+1}) \ \sigma_1 \cdots \sigma_{j-1} \ | \ (-\sigma_{j-1}) \cdots (-\sigma_1) \ \sigma_{j+1} \cdots \sigma_{n+1} ,
\]

where \( \sigma_i = \pi_i - 1 \) for all \( i \in [n+1] \setminus \{j\} \).
If $j = 1$, then 1 is in a trivial descending run in $\pi$ and thus there is no arc with lower endpoint 1 in $\delta(\pi)$. In this case, the window notation for $\zeta_0(\pi)$ is $(\pi_2 - 1)(\pi_3 - 1) \cdots (\pi_{n+1} - 1)$. Because all terms in the window notation for $\zeta_0(\pi)$ are positive, all blocks in $\delta^\circ(\eta_0(\pi))$ are type-A. Consider any block $B$ above 1 in $\delta(\pi)$, which corresponds to the descending run $\pi_1 \cdots \pi_k$. In $\zeta_0(\pi)$, this descending run corresponds to the descending run $(\pi_k - 1) \cdots (\pi_1 - 1)$ beginning in the $k - 1$th position. This corresponds to a type-A block which is just $B$ shifted down by one, since $\pi_j - 1$ is before $\pi_k - 1$ in the window of $\zeta_0(\pi)$ and thus transitively left of $\delta^\circ(\zeta_0(\rho^\circ(B)))$ in $\delta^\circ(\zeta_0(\pi))$ if and only if $\pi_j$ is before $\pi_k$ in $\pi$ and thus transitively left of $B$ in $\delta(\pi)$. This occurs in Example 3.5.3 and is pictured in the leftmost part of Fig. 3.30.

If $j > 1$, then 1 is in a nontrivial descending run, since $\pi_{j-1}$ must be greater than 1. Let $\pi_k \cdots \pi_{j-1} \pi_j$ be the descending run ending in 1, so the block in $\delta(\pi)$ which contains 1 has top endpoint $\pi_k$. The first $j - k - 1$ terms in the window notation of $\zeta_0(\pi)$ are $(-\pi_{j-1} + 1) \cdots (-\pi_k + 1)$ which corresponds to an orbifold block in $\delta^\circ(\zeta_0(\pi))$ with top endpoint $\pi_k - 1$. In fact, the orbifold block is identical to the block containing 1 in $\delta(\pi)$ except that the numbered point 1 has been replaced by $\times$, since a positive term $\pi_l - 1$ is in the window of $\zeta_0(\pi)$ and thus transitively right of the orbifold block in $\delta^\circ(\zeta_0(\pi))$ if and only if $\pi_l$ is after 1 in $\pi$ and thus transitively right of the block containing 1 in $\delta(\pi)$. Any block $B$ that is transitively right of the block containing 1 in $\delta(\pi)$ corresponds to a descending run in the window notation of $\zeta_0(\pi)$, which corresponds to a type-A block in $\delta^\circ(\zeta_0(\pi))$ which, by the same reasoning as in the case where $j = 1$, is $B$ shifted down by one. Consider a block $C$ that is transitively left of the block containing 1 in $\delta(\pi)$ corresponding to the descending run $\pi_r \cdots \pi_q$, where $q < k$. In $\zeta_0(\pi)$, this corresponds to the descending run $(-\pi_r + 1) \cdots (-\pi_q + 1)$ in the window notation, after $-\pi_k + 1$. This descending run corresponds to a type-A block with endpoints $(\pi_r - 1), \ldots, (\pi_q - 1)$ to the left of the orbifold block in $\delta^\circ(\zeta_0(\pi))$. The block is $C$ shifted down by one, since a negative term $-\pi_l + 1$ in the window of $\zeta_0(\pi)$ is after $-\pi_q + 1$ and thus $\pi_l - 1$ is left of $\delta^\circ(\zeta_0(\rho^\circ(C)))$ in $\delta^\circ(\zeta_0(\pi))$ if and only if $\pi_l$ is before $\pi_q$ and thus left of $C$ in $\delta(\pi)$.

Regardless of the position of 1 in $\pi$, all blocks in $\delta(\pi)$ with bottom endpoint above 1 are shifted down by 1 in $\delta^\circ(\zeta_0(\pi))$, and the block containing 1 in $\delta^\circ(\zeta_0(\pi))$ has its bottom endpoint 1 replaced with $\times$. This occurs in Examples 3.5.4 to 3.5.6 and is pictured in every part of Fig. 3.30 except the leftmost part.

Because of the relationship between $\eta_0$ and $\zeta_0$, we can also describe what the operation on type-B noncrossing arc diagrams corresponding to $\eta_0$ does to diagrams with no long block. For a type-B noncrossing arc diagram $\delta^\circ(\pi)$ with no long arcs, the operation corresponding to $\eta_0$ can be described as follows: turn $\times$ into the numbered point 1, and add one to the label of each originally numbered point.
Because $\eta_0$ and $\zeta_0$ are so efficiently captured by noncrossing arc diagrams, they can be used to consider whether $\zeta_0$ embeds $\Psi(A_n)$ as a sublattice of $\Psi(B_n)$, as shown in Fig. 3.31.

**Theorem 3.5.8.** $\zeta_0$ embeds the shard intersection order on the Coxeter group of type $A_n$ as a sublattice of the shard intersection order on the Coxeter group of type $B_n$.

**Proof.** Consider two type-A noncrossing arc diagrams, $N_1$ and $N_2$, on $n + 1$ points. We will show that it does not matter in what order we do two operations: mapping from $A_n$ to $B_n$ via $\zeta_0$ and taking the meet using the appropriate type of cooperative noncrossing arc diagram.

We first consider $\zeta_0(\text{cn}(N_1, N_2))$. There will be an arc in $\text{cn}(N_1, N_2)$ precisely when the requirements stated in Definition 3.3.8 are met. In particular, there will be an arc in $\text{cn}(N_1, N_2)$ whose lower endpoint is 1 if and only if both of the original diagrams have arcs whose lower endpoint is 1 and the two blocks containing 1 weakly agree up to their next-
highest shared endpoint, which we will call \(i\). By Proposition 3.5.7, \(\zeta_0(\text{cn}(N_1, N_2))\) matches the type-A cooperative noncrossing arc diagram except that the point 1 is replaced by the point \(\times\) and one is subtracted from all other numbered points.

Next, we consider \(\text{cn}_B(\zeta_0(N_1), \zeta_0(N_2))\). Again by Proposition 3.5.7, \(\zeta_0(N_1)\) matches the original type-A noncrossing arc diagram except that the point 1 is replaced by the point \(\times\), and one is subtracted from all other numbered points, and similarly for \(\zeta_0(N_2)\). In particular, each image may contain at most one orbifold arc in addition to a collection of type-A arcs. So, constructing the type-B cooperative noncrossing arc diagram \(\text{cn}_B(\zeta_0(N_1), \zeta_0(N_2))\) consists mostly of considering type-A cooperation, with perhaps one exception. If both \(\zeta_0(N_1)\) and \(\zeta_0(N_2)\) have orbifold arcs and the two orbifold blocks weakly agree up to their next-highest shared point, then \(\text{cn}_B(\zeta_0(N_1), \zeta_0(N_2))\) contains an orbifold arc with that point as its upper endpoint. In fact, the upper endpoint of this orbifold arc is \(i - 1\), corresponding to the arc from \(i\) to 1 in the type-A cooperative noncrossing arc diagrams. Moreover, the type-A cooperation above \(\times\) in \(\text{cn}_B(\zeta_0(N_1), \zeta_0(N_2))\) agrees exactly with the cooperation in \(\text{cn}(N_1, N_2)\), except that the labels on all numbered points are one less than in \(\text{cn}(N_1, N_2)\).

We have shown that \(\text{cn}_B(\zeta_0(N_1), \zeta_0(N_2)) = \zeta_0(\text{cn}(N_1, N_2))\). So, \(\zeta_0\) embeds the shard intersection order on \(S_{n+1}\) as a meet-sublattice of the shard intersection order on \(B_n\). In particular, \(\zeta_0\) is an order isomorphism from \(S_{n+1}\) to its image. The map \(\zeta_0\) would fail to embed \(S_{n+1}\) as a sublattice if there were two elements \(\sigma, \tau \in S_{n+1}\) such that \(\zeta_0(\sigma \lor \tau)\) is strictly larger than \(\zeta_0(\sigma) \lor \zeta_0(\tau)\). However, [33, Proposition 7.8] says that the image of \(\zeta_0\) is also a join-sublattice of the shard intersection order on \(B_n\). Thus \(\zeta_0(\sigma) \lor \zeta_0(\tau)\) is in the image of \(\zeta_0\), and since \(\zeta_0\) is an order isomorphism onto its image, we conclude that \(\zeta_0(\sigma \lor \tau) = \zeta_0(\sigma) \lor \zeta_0(\tau)\). We conclude that \(\zeta_0\) embeds \(S_{n+1}\) as a sublattice.

### 3.5.2 A nonhomogeneous homomorphism

Another map from \(B_n\) to \(A_n\), described in [38, Section 6.2], consists of a straightforward operation on a signed permutation. The map is denoted \(\eta_v\) in [38], but we denote it instead by \(\eta_{-1}\) for reasons that will be apparent shortly. Given a signed permutation \(\pi = \pi_{-n} \cdots \pi_{-1} \mid \pi_1 \cdots \pi_n\), the operation is described as follows: read off the sequence of terms greater than or equal to \(-1\) from left to right, replace \(-1\) with zero, then add one to each entry of the sequence.

**Example 3.5.9.** \(\eta_{-1}(2(-5)1(-4)(-3)) = 451632\)

\[
\begin{array}{ccccccc}
3 & 4 & -1 & 5 & -2 & | & 2 & -5 & 1 & -4 & -3 \\
3 & 4 & -1 & 5 & & 2 & 1 \& \\
3 & 4 & 0 & 5 & & 2 & 1 \& \\
4 & 5 & 1 & 6 & & 3 & 2 & \\
\end{array}
\]

read sequence \(\geq -1\)

\(-1\) becomes 0

add one
As with $\eta_0$, the output is a permutation of $[n + 1]$. Although $\eta_{-1}$ is a surjective lattice homomorphism from the weak order on $B_n$ to the weak order on $A_n$ [38, Theorem 6.4] and its description is comparable in simplicity to that of $\eta_0$, its associated congruence is not homogeneous of degree 2. Later in this section, we will show that the corresponding map $\zeta_{-1}$ does not embed $\Psi(A_n)$ as a sublattice of $\Psi(B_n)$. First, we will explore details of $\eta_{-1}$ and $\zeta_{-1}$, as they are foundational to understanding the maps discussed in the next section.

[38, Theorem 6.4] states that the congruence defined by $\eta_{-1}$ is generated by the join-irreducible elements $s_0s_1s_0$, $s_1s_0$, $s_1s_0s_1s_2$, and $s_2s_1s_0s_1s_2$, which are the signed permutations $(-2) (-1) 3 \cdots n$, $(-2) 13 \cdots n$, $13 (-2) 4 \cdots n$, and $12 (-3) 4 \cdots n$. The arcs in $B_3$ are pictured in Fig. 3.32, and in larger type B Coxeter groups, the noncrossing arc diagrams are only changed by adding numbered points above 3. (Because the generators $s_1s_0s_1s_2$ and $s_2s_1s_0s_1s_2$ each have degree 3, the congruence on $(B_n, \leq)$ associated with $\eta_{-1}$ is not homogeneous of degree 2.) It is easy to verify (using Section 2.4.2) that the superarcs of these four arcs are exactly the orbifold arcs with upper endpoint above 1 and long arcs with both left and right endpoints above 1. (This recovers [38, Proposition 6.6] much more easily.) Thus by Theorem 2.4.22 and Corollary 2.4.24, the uncontracted elements in the congruence on the weak order on $B_n$ corresponding to $\eta_{-1}$ are exactly those which have only type-A arcs and long arcs with left or right endpoint 1 in their noncrossing arc diagrams.

The natural inclusion $\zeta_{-1}$ from $A_n$ to $B_n$ which sends each permutation $\pi$ to the bottom element of the congruence defined by $\eta_{-1}$ that maps to $\pi$ is described as follows: First, subtract one from each entry of $\pi$ greater than 1 and subtract two from the entry 1. The resulting permutation $\sigma$ of $\{-1, 1, \ldots, n\}$ will have three parts, $\sigma_1 \cdots \sigma_{j-1}$, then $\sigma_j = (-1)$, and then $\sigma_{j+1} \cdots \sigma_{n+1}$, so the first and third parts together constitute a permutation of $[n]$. Next, write the window notation of a signed permutation of $\pm [n]$, depending on the relative position of 1 and $(-1)$. Let $i$ denote the position of 1 in $\sigma$. If $j = 1$, then the one line-notation of the signed permutation is simply $\sigma_{j+1} \cdots \sigma_{n+1}$. (See Example 3.5.10.) If $j > 1$ and $i = j - 1$, then the one line-notation of the signed permutation is $(-\sigma_j) \cdots (-\sigma_1) \sigma_{j+1} \cdots \sigma_{n+1}$. (See Example 3.5.11.) If $j > 1$ and $i < j - 1$, then the one-line notation of the signed permutation is $\sigma_{i+1} \cdots \sigma_{j-1} (-\sigma_i) \cdots (-\sigma_1) \sigma_{j+1} \cdots \sigma_{n+1}$. (See Example 3.5.12.) If $j > 1$ and $i > j$, then the one-line notation of the signed permutation is $\sigma_{j+1} \cdots \sigma_{i-1} \sigma_i (-\sigma_{j-1}) \cdots (-\sigma_1) \sigma_{i+1} \cdots \sigma_{n+1}$.
Example 3.5.10. \(\zeta_{-1}(1452736) = 3\;4\;1\;6\;2\;5\)
\[
\begin{array}{cccccc}
1 & 4 & 5 & 2 & 7 & 3 & 6 \\
-1 & 3 & 4 & 1 & 6 & 2 & 5 \\
3 & 4 & 1 & 6 & 2 & 5
\end{array}
\]
\(j = 1, i = 4\)

Example 3.5.11. \(\zeta_{-1}(4521736) = (-1)(-4)(-3)6\;2\;5\)
\[
\begin{array}{cccccc}
4 & 5 & 2 & 1 & 7 & 3 & 6 \\
3 & 4 & 1 & -1 & 6 & 2 & 5 \\
-1 & -4 & -3 & 6 & 2 & 5
\end{array}
\]
\(j = 4, i = 3\)

Example 3.5.12. \(\zeta_{-1}(4251736) = 4(-1)(-3)6\;2\;5\)
\[
\begin{array}{cccccc}
4 & 2 & 5 & 1 & 7 & 3 & 6 \\
3 & 1 & 4 & -1 & 6 & 2 & 5 \\
4 & -1 & -3 & 6 & 2 & 5
\end{array}
\]
\(j = 4, i = 2\)

Example 3.5.13. \(\zeta_{-1}(4517326) = 6\;2\;1(-4)(-3)5\)
\[
\begin{array}{cccccc}
4 & 5 & 1 & 7 & 3 & 2 & 6 \\
3 & 4 & -1 & 6 & 2 & 1 & 5 \\
6 & 2 & 1 & -4 & -3 & 5
\end{array}
\]
\(j = 3, i = 6\)

Proposition 3.5.14. The map \(\zeta_{-1}\) on permutations corresponds to the following operation on noncrossing arc diagrams: For any \(\pi \in A_n = S_{n+1}\), the type-B noncrossing arc diagram \(\delta^0(\zeta_{-1}(\pi))\) is identical to the type-A noncrossing arc diagram \(\delta(\pi)\), except that the numbered point 1 is replaced by the orbifold point \(\times\), the numbered points 2, \ldots, \(n+1\) are renumbered as 1, \ldots, \(n\), and if \(\delta(\pi)\) contains an arc \(\alpha\) which passes left [resp. right] of 2 and has 1 as its lower endpoint, then this arc is replaced by a long arc whose left [resp. right] side agrees with \(\alpha\) shifted down by one and whose right [resp. left] endpoint is 1.

The operation on noncrossing arc diagrams corresponding to \(\zeta_{-1}\) for Examples 3.5.10 to 3.5.13 is pictured in Fig. 3.33.

Proof. Let \(\pi\) be a permutation in \(S_{n+1}\). Let \(j\) denote the position of 1 in \(\pi\) and let \(i\) denote the position of 2 in \(\pi\).

If \(j = 1\) or if \(j > 1\) and \(i = j - 1\), the argument follows exactly as in the proof of Proposition 3.5.7. This occurs in Examples 3.5.10 and 3.5.11 and is pictured in the two left
part of Fig. 3.33. In the case where \( j > 1 \) and \( i = j - 1 \), the orbifold block has 1 as its lowest numbered endpoint.

If \( j > 1 \) and \( i < j - 1 \), then 1 is in a nontrivial descending run and 2 precedes the descending run ending in 1. The one-line notation of \( \zeta_{-1}(\pi) \) is \( \sigma_i \cdots \sigma_{j-1}(-\sigma_i) \cdots (-\sigma_1)\sigma_{j+1} \cdots \sigma_{n+1} \), where \( \sigma_m = \pi_{m-1} - 1 \) for all \( m \in [n+1] \setminus j \). Let \( \pi_k \cdots \pi_{j-1} \pi_j \) be the descending run ending in 1, so the block in \( \delta(\pi) \) which contains 1 has top endpoint \( \pi_k \) and an arc which passes right of 2. Let \( \pi_l \cdots \pi_{i-1} \pi_i \) be the descending run ending in 2, so the block in \( \delta(\pi) \) which contains 2 has top endpoint \( \pi_l \). Within the window notation of \( \zeta_{-1}(\pi) \) is the descending run \( \pi_{j-1}(\pi_{j-1}-1)(-\pi_{i-1}+1)(-\pi_{i-1}+1)(-\pi_{i+1}+1) \cdots (\pi_{j+1}+1) \) which, since \( -\pi_{i+1} = -1 \), corresponds to a long block in \( \delta(\zeta_{-1}(\pi)) \) with top-left endpoint \( \pi_l - 1 \) and top-right endpoint \( \pi_k - 1 \). In fact, the left and right sides of the long block are identical to the blocks containing 2 and 1 in \( \delta(\pi) \) shifted down by one, except that the arc with lower endpoint 1 is replaced by a long arc with left endpoint 1.

Any block \( B \) that is transitively right of the block containing 1 in \( \delta(\pi) \) corresponds to a descending run \( (\pi_{s-1} - 1) \cdots (\pi_{t-1}) \) beginning in the \( (s-1) \)th position of the window notation of \( \zeta_{-1}(\pi) \), where \( s > j \). So, \( B \) corresponds to a type-A block that is transitively right of the right side of the long block in \( \delta(\zeta_{-1}(\pi)) \). This block is \( B \) shifted down by one, since \( \pi_{w-1} \) is after \( \pi_{t-1} \) in the window of \( \zeta_{-1}(\pi) \) and thus transitively right of \( \delta(\zeta_{-1}(\rho^{\sigma}(B))) \) in \( \delta(\zeta_{-1}(\pi)) \) if and only if \( \pi_{w} \) is after \( \pi_{t} \) in \( \pi \) and thus transitively right of \( B \) in \( \delta(\pi) \). Any block \( C \) that is transitively left of the block containing 2 in \( \delta(\pi) \) corresponds to the descending run \( \pi_{r} \cdots \pi_{q} \) where \( q < i \). In \( \zeta_{-1}(\pi) \), this corresponds to the descending run \( (-\pi_{r} - 1) \cdots (-\pi_{q} + 1) \) in the window notation, after \( (-\pi_{i} + 1) \). This corresponds to a type-A block with endpoints \( (\pi_{r} - 1), \ldots, (\pi_{q} - 1) \) transitively left of the left side of the long block in \( \delta(\zeta_{-1}(\pi)) \). The block is identical to \( C \) shifted down by one, since a negative term \(-\pi_{w} + 1\) in the window.
of $\zeta_{-1}(\pi)$ is after $-\pi_q + 1$ and thus $\pi_w - 1$ is left of $\delta^0(\zeta_{-1}(\rho^2(C)))$ in $\delta^0(\zeta_{-1}(\pi))$ if and only if $\pi_w$ is before $\pi_q$ and thus left of $C$ in $\delta(\pi)$. Any block $D$ that is transitively right of the block containing 2 and transitively left of the block containing 1 in $\delta(\pi)$ corresponds to the descending run $\pi_u \cdots \pi_v$ where $i < u$ and $v < k$. In $\zeta_{-1}(\pi)$, this corresponds to the descending run $(\pi_u - 1) \cdots (\pi_v - 1)$ in the window notation, before $(\pi_k - 1)$. This corresponds to a type-A block with endpoints $(\pi_u - 1), \ldots, (\pi_v - 1)$ transitively right of the left side and transitively left of the right side of the long block in $\delta^0(\zeta_{-1}(\pi))$. The block is $D$ shifted down by one, since a positive term $\pi_w - 1$ in the window of $\zeta_{-1}(\pi)$ is before $\pi_u - 1$ and thus transitively left of $\delta^0(\zeta_{-1}(\rho^2(D)))$ in $\delta^0(\zeta_{-1}(\pi))$ if and only if $\delta_w$ is before $\delta_u$ and thus transitively left of $D$ in $\delta(\pi)$. This occurs in Example 3.5.12 and is pictured in the center-right part of Fig. 3.33.

If $j > 1$ and $i > j$, then 1 is in a nontrivial descending run and 2 follows the descending run ending in 1. The one-line notation of $\zeta_{-1}(\pi)$ is $\sigma_{j+1} \cdots \sigma_i(-\sigma_{j-1}) \cdots (-\sigma_1)\sigma_{i+1} \cdots \sigma_{n+1}$, where $\sigma_m = \pi_m - 1$ for all $m \in [n + 1] \setminus \{j\}$. Let $\pi_k \cdots \pi_{j-1}\pi_j$ be the descending run ending in 1, so the block in $\delta(\pi)$ which contains 1 has top endpoint $\pi_k$ and an arc which passes left of 2. Let $\pi_1 \cdots \pi_i$ be the descending run ending in 2, so the block in $\delta(\pi)$ which contains 2 has top endpoint $\pi_i$. Within the window notation of $\zeta_{-1}(\pi)$ is the descending run $(\pi_1 - 1) \cdots (\pi_i - 1)(-\pi_{j-1} + 1) \cdots (-\pi_k + 1)$, which corresponds to a long block in $\delta^0(\zeta_{-1}(\pi))$ with top-left endpoint $\pi_k - 1$ and top-right endpoint $\pi_i - 1$. In fact, the left and right sides of the long block are identical to the blocks containing 1 and 2 in $\delta(\pi)$ shifted down by one, except that the arc with lower endpoint 1 is replaced by a long arc with right endpoint 1.

Any block $B$ that is transitively right of the block containing 2 in $\delta(\pi)$ corresponds to a descending run $(\pi_s - 1) \cdots (\pi_t - 1)$ beginning in the $(s - 1)$th position of the window notation of $\zeta_{-1}(\pi)$, where $s > i$. So, $B$ corresponds to a type-A block that is transitively right of the right side of the long block in $\delta^0(\zeta_{-1}(\pi))$. This block is $B$ shifted down by one, since $\pi_w - 1$ is after $\pi_t - 1$ in the window of $\zeta_{-1}(\pi)$ and thus transitively right of $\delta^0(\zeta_{-1}(\rho^2(B)))$ in $\delta^0(\zeta_{-1}(\pi))$ if and only if $\pi_w$ is after $\pi_t$ in $\pi$ and thus transitively right of $B$ in $\delta(\pi)$. Any block $C$ that is transitively left of the block containing 1 in $\delta(\pi)$ corresponds to the descending run $\pi_r \cdots \pi_q$ where $q < k$. In $\zeta_{-1}(\pi)$, this corresponds to the descending run $(-\pi_r + 1) \cdots (-\pi_q + 1)$ in the window notation, after $(-\pi_k + 1)$. This corresponds to a type-A block with endpoints $(\pi_r - 1), \ldots, (\pi_q - 1)$ transitively left of the left side of the long block in $\delta^0(\zeta_{-1}(\pi))$. The block is identical to $C$ shifted down by one, since a negative term $-\pi_w + 1$ in the window of $\zeta_{-1}(\pi)$ is after $-\pi_q + 1$ and thus $\pi_w - 1$ is left of $\delta^0(\zeta_{-1}(\rho^2(C)))$ in $\delta^0(\zeta_{-1}(\pi))$ if and only if $\pi_w$ is before $\pi_q$ and thus left of $C$ in $\delta(\pi)$. Any block $D$ that is transitively right of the block containing 1 and transitively left of the block containing 2 in $\delta(\pi)$ corresponds to the descending run $\pi_u \cdots \pi_v$ where $j < u$ and $v < l$. In $\zeta_{-1}(\pi)$, this corresponds to the descending run $(\pi_u - 1) \cdots (\pi_v - 1)$ in the window notation, before $(\pi_l - 1)$. This corresponds to a type-A
block with endpoints \((\pi_u - 1), \ldots, (\pi_w - 1)\) transitively right of the left side and transitively left of the right side of the long block in \(\delta^\circ(\zeta_1(\pi))\). The block is \(D\) shifted down by one, since a positive term \(\pi_w - 1\) in the window of \(\zeta_1(\pi)\) is before \(\pi_u - 1\) and thus transitively left of \(\delta^\circ(\zeta_1(\rho^\circ(D)))\) in \(\delta^\circ(\zeta_1(\pi))\) if and only if \(\delta_w\) is before \(\delta_u\) and thus transitively left of \(D\) in \(\delta(\pi)\). This occurs in Example 3.5.13 and is pictured in the rightmost part of Fig. 3.33.

The subposet \(\zeta_1(S_4)\) of \(\Psi(B_3)\), as shown in Fig. 3.34, is not a sublattice of \(\Psi(B_3)\). To prove this, we present a pair of permutations \(\sigma\) and \(\tau\) which demonstrate that \(\zeta_1(\sigma \land \tau)\) is strictly below \(\zeta_1(\sigma) \land \zeta_1(\tau)\). In fact, there are two such pairs, and we leave the similar reasoning of the other pair as an exercise for the reader.

**Example 3.5.15.** Consider the permutations \(\sigma = 3142\) and \(\tau = 3241\). By Theorem 3.3.15, \(\sigma \land \tau\) in \(\Psi(A_3)\) is the permutation corresponding to the noncrossing arc diagram \(\text{cn}(\delta(\sigma), \delta(\tau))\). 

![Figure 3.34: The subposet \(\zeta_1(S_4)\) of \(\Psi(B_3)\).](image)
As shown in the top of Fig. 3.35, the meet of $\sigma$ and $\tau$ is the permutation $1\ 2\ 3\ 4$, the identity in $A_3$. The signed permutation $\zeta_-(\sigma \wedge \tau)$ is the identity in $B_3$, the signed permutation $1\ 2\ 3$.

The images of $\sigma$ and $\tau$ under $\zeta_-$ are the signed permutations $\zeta_-(\sigma) = 3\ 1\ (-2)$ and $\tau = 3\ (-1\ (-2))$. By Theorem 3.4.11, $\zeta_-(\sigma) \wedge \zeta_-(\tau)$ in $\Psi(B_3)$ is the signed permutation corresponding to $\text{cn}_B(\delta^o(\zeta_-(\sigma)), \delta^o(\zeta_-(\tau)))$. As shown in the bottom of Fig. 3.35, the meet of $\zeta_-(\sigma)$ and $\zeta_-(\tau)$ is the signed permutation $1\ 3\ (-2)$, which is strictly above the identity in $\Psi(B_3)$. The signed permutation $1\ 3\ (-2)$ is not in the image of $\zeta_-$, since both endpoints of the long arc in $\delta^o(1\ 3\ (-2))$ are above 1.

### 3.5.3 Two more homogeneous homomorphisms

Two additional maps from $B_n$ to $A_n$ are described in [38, Section 6.3], both of which are a mix of the two maps introduced in the two preceding sections. We use the same notation as [38].

One, denoted $\eta_\delta$, agrees with $\eta_0$ when when 1 is in the window notation of a signed permutation $\pi$, inserting 0 between $\pi_{-1}$ and $\pi_1$ then reading off the nonnegative sequence of terms and adding one to each term. When $-1$ is in the window notation of $\pi$, $\eta_\delta$ agrees with $\eta_{-1}$, reading off the sequence of terms $\geq -1$, changing $-1$ to zero, then adding one to each term.

The other, denoted $\eta_\epsilon$, agrees with $\eta_{-1}$ when 1 is in the window notation of $\pi$ and with $\eta_0$ when -1 is in the window notation of $\pi$.  

---

Figure 3.35: Example showing that $\zeta_-(A_3)$ is not a sublattice of $\Psi(B_3)$. 

As shown in the top of Fig. 3.35, the meet of $\sigma$ and $\tau$ is the permutation $1\ 2\ 3\ 4$, the identity in $A_3$. The signed permutation $\zeta_-(\sigma \wedge \tau)$ is the identity in $B_3$, the signed permutation $1\ 2\ 3$.

The images of $\sigma$ and $\tau$ under $\zeta_-$ are the signed permutations $\zeta_-(\sigma) = 3\ 1\ (-2)$ and $\tau = 3\ (-1\ (-2))$. By Theorem 3.4.11, $\zeta_-(\sigma) \wedge \zeta_-(\tau)$ in $\Psi(B_3)$ is the signed permutation corresponding to $\text{cn}_B(\delta^o(\zeta_-(\sigma)), \delta^o(\zeta_-(\tau)))$. As shown in the bottom of Fig. 3.35, the meet of $\zeta_-(\sigma)$ and $\zeta_-(\tau)$ is the signed permutation $1\ 3\ (-2)$, which is strictly above the identity in $\Psi(B_3)$. The signed permutation $1\ 3\ (-2)$ is not in the image of $\zeta_-$, since both endpoints of the long arc in $\delta^o(1\ 3\ (-2))$ are above 1.
As both homomorphisms can be thought of as a combination of a homogeneous lattice homomorphism and a nonhomogeneous lattice homomorphism, it may come as a surprise that both of these hybrid maps are homogeneous.

A hybrid map, $\eta_\delta$

[38, Theorem 6.7] states that the congruence defined by $\eta_\delta$ is generated by the join-irreducible elements $s_0s_1s_0$ and $s_1s_0s_1$, the signed permutations $(-2)(-1)3\cdots n$ and $1(-2)3\cdots n$. These arcs in $B_3$ are pictured in Fig. 3.36, and in larger type-B Coxeter groups, the noncrossing arc diagrams are only changed by adding numbered points above 3. It is easy to verify (using Section 2.4.2) that the superarcs of these two arcs are exactly the orbifold arcs that pass right of 1 and the long arcs whose right endpoint is 1. (This recovers [38, Proposition 6.9] much more easily.) Thus by Theorem 2.4.22 and Corollary 2.4.24, the uncontracted elements in the congruence on the weak order on $B_n$ corresponding to $\eta_\delta$ are exactly those which have only the following types of arcs in their noncrossing arc diagrams:

- type-A arcs,
- orbifold arcs with an endpoint at 1 or that pass left of 1, and/or
- long arcs whose left endpoint is 1.

The natural inclusion $\zeta_\delta$ from $A_n$ to $B_n$ which sends each permutation $\pi$ to the bottom element of the congruence defined by $\eta_\delta$ that maps to $\pi$ is described as follows: Let $j$ denote the position of 1 in $\pi$, and let $i$ denote the position of 2 in $\pi$. If $j < i$, then $\zeta_\delta$ behaves as $\zeta_0$, by first subtracting one from each entry of $\pi$ to give a permutation $\sigma_1\cdots\sigma_{n+1}$ of $\{0,\ldots,n\}$ whose $j$th entry is 0. In this case, the signed permutation $\zeta_\delta(\pi) = \zeta_0(\pi)$, with window notation $(-\sigma_j-1)\cdots(-\sigma_1)\sigma_{j+1}\cdots\sigma_{n+1}$. (See Examples 3.5.16 and 3.5.19.) If $i < j$, then $\zeta_\delta$ behaves as $\zeta_{-1}$, by first subtracting one from each entry of $\pi$ greater than 1 and subtracting two from 1 to give a permutation $\sigma_1\cdots\sigma_{n+1}$ of $\{-1,1,\ldots,n\}$ whose $j$th entry is $-1$. In this case, the signed permutation $\zeta_\delta(\pi) = \zeta_{-1}(\pi)$, with window notation $\sigma_{i+1}\cdots\sigma_{j-1}(-\sigma_1)\cdots(-\sigma_1)\sigma_{j+1}\cdots\sigma_{n+1}$, though the first set of positive entries may be empty if $i = j - 1$. (See Examples 3.5.17 and 3.5.18.)
Example 3.5.16. \( \zeta(1452736) = 341625 \)

\[
\begin{array}{cccccc}
1 & 4 & 5 & 2 & 7 & 3 & 6 \\
0 & 3 & 4 & 1 & 6 & 2 & 5 \\
3 & 4 & 1 & 6 & 2 & 5
\end{array}
\]
\( j = 1, i = 4 \)

Example 3.5.17. \( \zeta(4521736) = (-1)(-4)(-3)625 \)

\[
\begin{array}{cccccc}
4 & 5 & 2 & 1 & 7 & 3 & 6 \\
3 & 4 & 1 & -1 & 6 & 2 & 5 \\
-1 & -4 & -3 & 6 & 2 & 5
\end{array}
\]
\( i = 3, j = 4 \)

Example 3.5.18. \( \zeta(4251736) = 4(-1)(-3)625 \)

\[
\begin{array}{cccccc}
4 & 2 & 5 & 1 & 7 & 3 & 6 \\
3 & 1 & 4 & -1 & 6 & 2 & 5 \\
4 & -1 & -3 & 6 & 2 & 5
\end{array}
\]
\( i = 2, j = 4 \)

Example 3.5.19. \( \zeta(4517326) = (-4)(-3)6215 \)

\[
\begin{array}{cccccc}
4 & 5 & 1 & 7 & 3 & 2 & 6 \\
3 & 4 & 0 & 6 & 2 & 1 & 5 \\
-4 & -3 & 6 & 2 & 1 & 5
\end{array}
\]
\( j = 3, i = 6 \)

Proposition 3.5.20. The map \( \zeta \) on permutations corresponds to the following operation on noncrossing arc diagrams: For any \( \pi \in A_n = S_{n+1} \), the type-B noncrossing arc diagram \( \delta^\circ(\zeta_\delta(\pi)) \) is identical to the type-A noncrossing arc diagram \( \delta(\pi) \), except that the numbered point 1 is replaced by the orbifold point \( \times \), the numbered points \( 2, \ldots, n+1 \) are renumbered as \( 1, \ldots, n \), and if \( \delta(\pi) \) has an arc \( \alpha \) which passes right of 2 and has 1 as its lower endpoint, then this arc is replaced by a long arc whose right side agrees with \( \alpha \) shifted down by one and whose left endpoint is 1.

The operation on noncrossing arc diagrams corresponding to \( \zeta \) for Examples 3.5.16 to 3.5.19 is pictured in Fig. 3.37.

Proof. Let \( \pi \) be a permutation in \( S_{n+1} \). Let \( j \) denote the position of 1 in \( \pi \) and let \( i \) denote the position of 2 in \( \pi \).

If \( j < i \), then the descending run in \( \pi \) that ends in 1 (regardless of whether it is trivial) occurs before the descending run that ends in 2. Because of this, there is no arc in \( \delta(\pi) \) which has 1 as its lower endpoint and either passes right of or has 2 as its upper endpoint. In this case, \( \delta^\circ(\zeta_\delta(\pi)) \) is identical to the type-A noncrossing arc diagram \( \delta(\pi) \), except that the
numbered point 1 is replaced by the orbifold point $\times$ and the numbered points $2, \ldots, n + 1$ are renumbered as $1, \ldots, n$. This occurs in Examples 3.5.16 and 3.5.19 and is pictured in the leftmost and rightmost parts of Fig. 3.37. The proof of this is the same as the proof of Proposition 3.5.7.

If $i = j - 1$, then 1 and 2 are in the same nontrivial descending run. Because of this, there is an arc in $\delta(\pi)$ from 2 to 1. In this case, $\delta^\circ(\zeta_\delta(\pi))$ is as described in the previous case. This occurs in Example 3.5.17 and is pictured in the center-left part of Fig. 3.37. The proof of this is the same as the proof of Proposition 3.5.14 where $j > 1$ and $i = j - 1$.

If $i < j - 1$, then the nontrivial descending run in $\pi$ that ends in 1 occurs before the descending run that ends in 2. Because of this, there is an arc in $\delta(\pi)$ that passes right of 2 and has 1 as its lower endpoint. In this case, $\delta^\circ(\zeta_\delta(\pi))$ is identical to the type-A noncrossing arc diagram $\delta(\pi)$, except that the numbered point 1 is replaced by the orbifold point $\times$, the numbered points $2, \ldots, n + 1$ are renumbered as $1, \ldots, n$, and the arc $\alpha$ which passes right of 2 and has 1 as its lower endpoint is replaced by a long arc whose right side agrees with $\alpha$ shifted down by one and whose left endpoint is 1. This occurs in Example 3.5.18 and is pictured in the center-right part of Fig. 3.37. The proof of this the same as the proof of Proposition 3.5.14 where $j > 1$ and $i < j - 1$.

Theorem 3.5.21. $\zeta_\delta$ embeds the shard intersection order on the Coxeter group of type $A_n$ as a sublattice of the shard intersection order on the Coxeter group of type $B_n$.

Proof. As in the proof of Theorem 3.5.8, it is enough to show that $\zeta_\delta(\text{cn}(N_1, N_2)) = \text{cn}_B(\zeta_\delta(N_1), \zeta_\delta(N_2))$.

Consider two type-A noncrossing arc diagrams $N_1$ and $N_2$ on $n + 1$ points. We will show that it does not matter in what order we do two operations: mapping from $A_n$ to $B_n$ via $\zeta_{-1}$ and taking the meet using the appropriate type of cooperative noncrossing arc diagram.
Figure 3.38: The sublattice $\zeta_\delta(S_1)$ of $\Psi(B_3)$.

It is clear from the definitions that $\text{cn}_B(\zeta_\delta(N_1), \zeta_\delta(N_2)) = \zeta_\delta(\text{cn}(N_1, N_2))$ unless at least one of $N_1$ and $N_2$ has an arc that passes right of 2. Thus there are four cases left to consider, without loss of generality where $N_1$ has an arc passing right of 2 and where $N_2$ has: no arc with lower endpoint 1; an arc passing left of 2; an arc with upper endpoint 2; or an arc passing right of 2.

In the first case, $N_1$ has an arc $\alpha$ passing right of 2 and $N_2$ has no arc with lower endpoint 1. In this case, no two blocks in $N_1$ and $N_2$ share 1 as a lower endpoint. Thus there is no arc with lower endpoint 1 in $\text{cn}(N_1, N_2)$ and hence no orbifold or long arc in $\zeta_\delta(\text{cn}(N_1, N_2))$. Because $\alpha$ in $N_1$ passes right of 2, $\zeta_\delta(N_1)$ has a long arc whose right side agrees with $\alpha$ shifted down by one and whose left endpoint is 1. Because $N_2$ has no arc with lower endpoint 1, $\zeta_\delta(N_2)$ has no orbifold or long arc. Thus by Definition 3.4.3, the only
arcs in $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2}))$ must be type-A. Moreover, each arc in $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2}))$ is an arc in $\text{cn}(N_{1}, N_{2})$ shifted down by one, as in the operation $\zeta_{\delta}$ on noncrossing arc diagrams. Therefore, in this case $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2})) = \zeta_{\delta}(\text{cn}(N_{1}, N_{2}))$.

In the second case, $N_{1}$ has an arc $\alpha$ passing right of 2 and $N_{2}$ has an arc $\beta$ passing left of 2. In this case, the two blocks in $N_{1}$ and $N_{2}$ which share 1 as a lower endpoint pass to opposite sides of 2, and thus there is no arc in $\text{cn}(N_{1}, N_{2})$ whose lower endpoint is 1. Thus there is no orbifold or long arc in $\zeta_{\delta}(\text{cn}(N_{1}, N_{2}))$. As in the previous case, $\zeta_{\delta}(N_{1})$ has a long arc whose right side agrees with $\alpha$ shifted down by one and whose left endpoint is 1. Because $\beta$ in $N_{2}$ passes left of 2, $\zeta_{\delta}(N_{2})$ has an orbifold arc which passes left of 1. Because the orbifold block containing $\zeta_{\delta}(\beta)$ passes to the opposite side of 1 ad the long block containing $\zeta_{\delta}(\alpha)$, the requirement in (4) of Definition 3.4.3 is not met, so there is no long arc in $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2}))$. Moreover, each arc in $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2}))$ is an arc in $\text{cn}(N_{1}, N_{2})$ shifted down by one, as in the operation $\zeta_{\delta}$ on noncrossing arc diagrams. Therefore, also in this case $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2})) = \zeta_{\delta}(\text{cn}(N_{1}, N_{2}))$.

In the third case, $N_{1}$ has an arc $\alpha$ passing right of 2 and $N_{2}$ has an arc $\beta$ from 2 to 1. In this case, the two blocks in $N_{1}$ and $N_{2}$ which share 1 as a lower endpoint pass weakly to the same side of 2, so there is an arc in $\text{cn}(N_{1}, N_{2})$ exactly when these two blocks share another endpoint $i > 2$ and pass weakly to the same side of each point between 2 and $i$. If $\text{cn}(N_{1}, N_{2})$ has no such arc, then all of its arcs have lower endpoints above 1, so $\zeta_{\delta}(\text{cn}(N_{1}, N_{2}))$ has only type-A arcs. If $\text{cn}(N_{1}, N_{2})$ contains such an arc, which we denote $\gamma$, then by Definition 3.3.8, $\gamma$ has upper endpoint $r$, lower endpoint 1, and passes to the right of 2. In $\zeta_{\delta}(\text{cn}(N_{1}, N_{2}))$, $\gamma$ is replaced by a long arc whose right side agrees with $\gamma$ shifted down by one and whose left endpoint is 1. In $\zeta_{\delta}(N_{1})$, as in previous cases, $\alpha$ is replaced by a long arc whose right side agrees with $\alpha$ shifted down by one and whose left endpoint is 1. In $\zeta_{\delta}(N_{2})$, the lower endpoint 1 of $\beta$ is replaced by the point $\times$. By the requirement (4) of Definition 3.4.3, $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2}))$ has a long arc with left endpoint 1 if and only if the right side of the long block in $\zeta_{\delta}(N_{1})$ and the orbifold block in $\zeta_{\delta}(N_{2})$ share an endpoint and the orbifold block weakly agrees with the right side of the long block below the shared endpoint. This shared endpoint is $i - 1$, and the resulting arc, if it exists, is precisely $\zeta_{\delta}(\gamma)$. All other arcs in $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2}))$ are type-A, exactly the arcs in $\text{cn}(N_{1}, N_{2})$ with lower endpoints above 1 shifted down by one by $\zeta_{\delta}$. Therefore, also in this case $\text{cn}_{B}(\zeta_{\delta}(N_{1}), \zeta_{\delta}(N_{2})) = \zeta_{\delta}(\text{cn}(N_{1}, N_{2}))$.

In the final case, $N_{1}$ has an arc $\alpha$ passing right of 2 and $N_{2}$ has an arc $\beta$ which also passes right of 2. In this case also, the two blocks in $N_{1}$ and $N_{2}$ which share 1 as a lower endpoint pass weakly to the same side of 2, so there is an arc in $\text{cn}(N_{1}, N_{2})$ with lower endpoint 1 exactly when these two blocks share another endpoint $i > 2$ and pass weakly to the same side of each point between 2 and $i$. If $\text{cn}(N_{1}, N_{2})$ has no such arc, then all of
its arcs have lower endpoints above 1, so $\zeta_\delta(cn(N_1, N_2))$ has only type-A arcs. If $cn(N_1, N_2)$ contains such an arc, which we denote $\gamma$, then by Definition 3.3.8, $\gamma$ has upper endpoint $r$, lower endpoint 1, and passes to the right of 2. In $\zeta_\delta(cn(N_1, N_2))$, $\gamma$ is replaced by a long arc whose right side agrees with $\gamma$ shifted down by one and whose left endpoint is 1. In $\zeta_\delta(N_1)$, as in previous cases, $\alpha$ is replaced by a long arc whose right side agrees with $\alpha$ shifted down by one and whose left endpoint is 1. In $\zeta_\delta(N_2)$, as in previous cases, $\beta$ is also replaced by a long arc whose right side agrees with $\beta$ shifted down by one and whose left endpoint is 1. By requirement (3) of Definition 3.4.3, $cn_B(\zeta_\delta(N_1), \zeta_\delta(N_2))$ has a long arc with left endpoint 1 if and only if the right sides of the long blocks in $\zeta_\delta(N_1)$ and $\zeta_\delta(N_2)$ share an endpoint and weakly agree below their shared endpoint. This shared endpoint is $i - 1$, and the resulting arc, if it exists, is precisely $\zeta_\delta(\gamma)$. All other arcs in $cn_B(\zeta_\delta(N_1), \zeta_\delta(N_2))$ are type-A, exactly the arcs in $cn(N_1, N_2)$ with lower endpoints above 1 shifted down by one by $\zeta_\delta$. Therefore, also in this case $cn_B(\zeta_\delta(N_1), \zeta_\delta(N_2)) = \zeta_\delta(cn(N_1, N_2))$.

We have shown that $cn_B(\zeta_\delta(N_1), \zeta_\delta(N_2)) = \zeta_\delta(cn(N_1, N_2))$ for any pair $N_1$ and $N_2$ of noncrossing arc diagrams.

A second hybrid map, $\eta_\epsilon$

[38, Theorem 6.10] states that the congruence defined by $\eta_\epsilon$ is generated by the join-irreducible elements $s_0s_1$ and $s_1s_0$, which are the signed permutations $2 (-1) 3 \cdots n$ and $(-2) 1 3 \cdots n$. These arcs in $B_3$ are pictured in Fig. 3.39, and in larger type-B Coxeter groups, the noncrossing arc diagrams are only changed by adding numbered points above 3. It is easy to verify (using Section 2.4.2) that the superarcs of these two arcs are exactly the orbifold arcs that pass left of 1 and the long arcs with left endpoint above 1. Thus by Theorem 2.4.22 and Corollary 2.4.24, weak order on $B_n$ corresponding to $\eta_\epsilon$ are exactly those which have only the following types of arcs in their noncrossing arc diagrams:

- type-A arcs,
- orbifold arcs with an endpoint at 1 or that pass right of 1, and/or
- long arcs whose left endpoint is 1.

The natural inclusion $\zeta_\epsilon$ from $A_n$ to $B_n$ which sends each permutation $\pi$ to the bottom element of the congruence defined by $\eta_\epsilon$ that maps to $\pi$ is described as follows: Let $j$ denote the position of 1 in $\pi$, and let $i$ denote the position of 2 in $\pi$. If $j < i$, then $\zeta_\epsilon$ behaves as $\zeta_{-1}$, by subtracting one from each entry of $\pi$ greater than 1 and subtracting two from 1 to give a permutation $\sigma_1 \cdots \sigma_{n+1}$ of $\{-1, 1, \ldots, n\}$ whose $j$th entry is $-1$. In this case, the signed permutation $\zeta_\epsilon(\pi) = \zeta_{-1}(\pi)$, with window notation $\sigma_{j+1} \cdots \sigma_{i-1} \sigma_{i}(-\sigma_{j-1}) \cdots (-\sigma_{1}) \sigma_{i+1} \cdots \sigma_{n+1}$.
(See Examples 3.5.22 and 3.5.25.) If \( i < j \), then \( \zeta \) behaves as \( \zeta_0 \), by subtracting one from each entry of \( \pi \) to give a permutation \( \sigma_1 \cdots \sigma_{n+1} \) of \( \{0, \ldots, n\} \) whose \( j \)th entry is 0. In this case, the signed permutation \( \zeta(\pi) = \zeta_0(\pi) \), with window notation \( (-\sigma_{j-1}) \cdots (-\sigma_1)\sigma_{j+1} \cdots \sigma_{n+1} \).

(See Examples 3.5.23 and 3.5.24.)

**Example 3.5.22.** \( \zeta(1452736) = 3 \ 4 \ 1 \ 6 \ 2 \ 5 \)

\[
\begin{array}{ccccccc}
1 & 4 & 5 & 2 & 7 & 3 & 6 \\
-1 & 3 & 4 & 1 & 6 & 2 & 5 \\
3 & 4 & 1 & 6 & 2 & 5 \\
\end{array}
\]

\( j = 1, \ i = 4 \)

**Example 3.5.23.** \( \zeta(4521736) = (-1) (-4) (-3) 6 \ 2 \ 5 \)

\[
\begin{array}{ccccccc}
4 & 5 & 2 & 1 & 7 & 3 & 6 \\
3 & 4 & 1 & 0 & 6 & 2 & 5 \\
-1 & -4 & -3 & 6 & 2 & 5 \\
\end{array}
\]

\( i = 3, \ j = 4 \)

**Example 3.5.24.** \( \zeta(4251736) = (-4) (-1) (-3) 6 \ 2 \ 5 \)

\[
\begin{array}{ccccccc}
4 & 2 & 5 & 1 & 7 & 3 & 6 \\
3 & 1 & 4 & 0 & 6 & 2 & 5 \\
-4 & -1 & -3 & 6 & 2 & 5 \\
\end{array}
\]

\( i = 2, \ j = 4 \)

**Example 3.5.25.** \( \zeta(4517326) = 6 \ 2 \ 1 (-4) (-3) 5 \)

\[
\begin{array}{ccccccc}
4 & 5 & 1 & 7 & 3 & 2 & 6 \\
3 & 4 & -1 & 6 & 2 & 1 & 5 \\
6 & 2 & 1 & -4 & -3 & 5 \\
\end{array}
\]

\( j = 3, \ i = 6 \)

**Proposition 3.5.26.** The map \( \zeta \) on permutations corresponds to the following operation on noncrossing arc diagrams: For any \( \pi \in \mathbb{A}_n = \mathbb{S}_{n+1} \), the type-B noncrossing arc diagram \( \delta^\circ(\zeta(\pi)) \) is identical to the type-A noncrossing arc diagram \( \delta(\pi) \), except that the numbered point 1 is replaced by the orbifold point \( \times \), the numbered points \( 2, \ldots, n+1 \) are renumbered as \( 1, \ldots, n \), and if \( \delta(\pi) \) has an arc \( \alpha \) which passes left of 2 and has 1 as its lower endpoint, then this arc is replaced by a long arc whose left side agrees with \( \alpha \) shifted down by one and whose right endpoint is 1.
The operation on noncrossing arc diagrams corresponding to \( \zeta_\epsilon \) for Examples 3.5.22 to 3.5.25 is pictured in Fig. 3.40.

**Proof.** Let \( \pi \) be a permutation in \( S_{n+1} \). Let \( j \) denote the position of 1 in \( \pi \) and let \( i \) denote the position of 2 in \( \pi \).

If \( j = 1 \), then 1 is in a trivial descending run and thus not the lower endpoint of any arc in \( \delta(\pi) \). This occurs in Example 3.5.22 and is pictured in the leftmost part of Fig. 3.40. In this case, the argument follows exactly as in the proofs of Propositions 3.5.7 and 3.5.14.

If \( 1 < j < i \), then the nontrivial descending run in \( \pi \) that ends in 1 occurs before the descending run that ends in 2 and thus the arc \( \alpha \) in \( \delta(\pi) \) which has 1 as its lower endpoint passes left of 2. In this case, \( \delta^o(\zeta_\epsilon(\pi)) \) is identical to \( \delta(\pi) \), except that the numbered point 1 is replaced by the orbifold point \( \times \), the numbered points \( 2, \ldots, n+1 \) are renumbered as \( 1, \ldots, n \), and the arc \( \alpha \) is replaced by a long arc whose left side agrees with \( \alpha \) shifted down by one and whose right endpoint is 1. This occurs in Example 3.5.25 and is pictured in the rightmost part of Fig. 3.40. The proof of this is the same as the proof of Proposition 3.5.14 where \( j > 1 \) and \( i > j \).

If \( i < j \), then 2 is either part of or precedes the nontrivial descending run in \( \pi \) that ends in 1 and thus the arc in \( \delta(\pi) \) which has 1 as its lower endpoint either has 2 as its upper endpoint or passes right of 2. In this case, \( \delta^o(\zeta_\epsilon(\pi)) \) is identical to \( \delta(\pi) \), except that the numbered point 1 is replaced by the orbifold point \( \times \), the numbered points \( 2, \ldots, n+1 \) are renumbered as \( 1, \ldots, n \). This occurs in Examples 3.5.23 and 3.5.24 and is pictured in the two middle parts of Fig. 3.40. The proof of this is a the same as the proof of Proposition 3.5.7. \( \square \)

Comparing Proposition 3.5.26 with Proposition 3.5.20, we see that \( \zeta_\epsilon \) and \( \zeta_\delta \) are related by reflection in a vertical line. Specifically, the map \( \zeta_\epsilon \) corresponds to reflecting the type-A
Theorem 3.5.27. $\zeta_\epsilon$ embeds the shard intersection order on the Coxeter group of type $A_n$ as a sublattice of the shard intersection order on the Coxeter group of type $B_n$. 

noncrossing arc diagram, applying $\zeta_\delta$, and then reflecting the result. Thus the proof of the following theorem is symmetric to the proof of Theorem 3.5.21.

Figure 3.41: The sublattice $\zeta_\epsilon(S_4)$ of $\Psi(B_3)$. 

REFERENCES


