SPECIAL FINITE ELEMENTS FOR THE ANALYSIS OF STRESS CONCENTRATIONS AND SINGULARITIES

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ABSTRACT

In matrix formulations, each region of stress concentration requires the use of a large number of small conventional finite elements, and often, the advantage of a fine mesh can be overcome by numerical errors. The situation can be particularly difficult in the presence of elastic stress singularities. A method has been developed to overcome this difficulty; a fairly large element replaces a large number of the conventional small elements in each region of concentration. The displacement or stress description of this large element includes terms that represent the nature of the local concentration. A method for developing such a large finite element and applying it is described and illustrated by typical examples. Some of the solutions are compared with conventional finite element solutions.

INTRODUCTION

Regions of stress concentrations present a special difficulty in finite element analysis. Each region of stress concentration has to be covered by a fine mesh of a large number of small elements. This involves large order matrix operations and hence large computing effort. Further, the advantages of refining the mesh may be overcome by numerical errors, Argyris [1]. The situation is particularly serious in the case of elastic stress singularities, as the finite element solution is incapable of identifying them.

To overcome this difficulty in finite element analysis, the authors proposed, Rao et al [2], a method in which, the fine mesh of standard elements in each region of concentration is replaced by one large special element, which has a vastly improved kinematic description compatible with the nature of the local concentration. It is possible to obtain a simple but effective description through a stress function in series which includes a term or terms representing the concentration. A particularly satisfactory position obtains when a biharmonic stress function in series can be written down satisfying all the boundary conditions at and in the vicinity of the source of concentration. Each such special element we call a primary element. The field, other than the primary elements, can be covered with a relatively coarse mesh of standard finite elements with simple descriptions, which we call secondary elements. The nodes on the boundary of the primary element are chosen to agree with the geometry of the neighbouring secondary elements. The stiffness matrices for the secondary elements are readily written down. The stiffness matrix of each primary element can be derived using the minimum potential energy principle. The stiffness matrices of all elements being known, the solution for displacements and stresses in the hybrid finite element representation of the structure can be completed by a standard matrix displacement procedure, Rao et al [2]. In brief, we divide the total structure into two types of domains, viz., (a) a set of large primary elements each containing a source of concentration and having a sophisticated kinematic description and (b) a coarse mesh of secondary elements each with a standard simple kinematic description.

To maintain simplicity in the analysis, the number of free constants in the description of the primary elements should be equal to the number of degrees of freedom of that element. The convergence of the solution has to be studied relative to the improvement of the kinematic description of the primary element by increasing its degrees of freedom and also by refining the secondary mesh. Experience shows that the refinement of the primary element is the more important factor, Rao et al [2].

An interesting extension of our procedure for concentrations is in the treatment of infinite and semi-infinite fields with concentration arising from awkward boundary conditions which preclude closed form solutions. In these cases a large primary element is made to extend from a finite intermediate boundary to infinity and its kinematic behaviour is described by a series which satisfies the conditions at infinity.

Two alternate formulations, to achieve the same goal as ours, have been proposed by Morley [3] and Yamamoto [4]. In these methods, effectively, two kinematic descriptions are superposed in the entire field. The first is a standard simple description in each element, as in conventional finite element analysis. The second is due to a set of continuous biharmonic stress function(s), such as suggested in our method, which include the stress singularities or finite concentrations in the problem and sweep across the complete domain. The present procedure appears to have an edge over the other two formulations in problems with more than one source of concentration and for large and infinite domains.

In this paper we briefly describe the proposed procedure, apply it to some typical problems of relevance to nuclear and reactor engineering, and compare these results with those by conventional finite elements solutions.

2. GENERAL PROCEDURE

Consider an arbitrarily shaped plate, with a source of concentration such as a crack, subjected to arbitrary loading (Fig.1). At the crack tips A and B the 'elastic stresses' are infinitely large and the local stress gradients are very large. Represent the plate by an array of finite elements, the crack being included in a large primary element.

In the matrix displacement procedures, the displacements in an element are expressed as

$$\{u\} = [u_n]\{\alpha_n\} \qquad \ldots \quad (1)$$

where u_n are simple functional forms and α_n are generalized displacements. Normally one uses simple forms like polynomials for u_n. We retain such descriptions for the secondary elements. But, for the primary element, we choose for u_n functions corresponding to the terms of an appropriate stress function. Then, one has, with a term wise correspondance with eq.(1) the element stresses

$$\{\sigma\} = [D_n]\{\alpha_n\} \qquad \ldots \quad (2)$$

and the element stress function

$$\phi = \{\phi_n\}^T\{\alpha_n\} \qquad \ldots \quad (3)$$

This stress function ϕ satisfies the differential equation and the local boundary conditions.

The first few generalized displacements are chosen to provide for the possible rigid body modes of the element and the corresponding elements of $[D_n]$ are zero.

Having established the kinematic descriptions for the primary element, one chooses a certain number of nodes ξ on the primary element (generally on the outer periphery) each with μ generalized displacements so that the kinematics of the element is also described in terms of $(\mu\xi - \lambda)$ independent nodal displacements, where λ are the number of displacements prescribed due to conditions like symmetries, antisymmetries etc.

Introducing the co-ordinates these ξ nodes in eq.(1) one has the nodal displacements

$$\{U\} = [U_n]\{\alpha_n\}$$

$$\text{or} \qquad \{\alpha_n\} = [U_n]^{-1}\{U\} \qquad \ldots \quad (4)$$

The series in ϕ of eq.(3) is truncated so that the matrix $[U_n]$ is rendered square, or in other words, one chooses $(\mu\xi - \lambda)$ number of α_n's in ϕ.

Using the general procedure of Rao et al [2], we have the primary element force-displacement relations as

$$\{F\} = [K_p]\{U\} \qquad \ldots \quad (5)$$

where, the primary element stiffness matrix is

$$[K_p] = [U_n]^{-1^T}[H][U_n]^{-1} \qquad \ldots \quad (6)$$

with

$$[H] = \int_{\vartheta} \left([D_n]^T [E]^{-1} [D_n]\right) dv$$

Having obtained the primary element stiffness matrix and knowing the secondary element stiffness matrices one can complete the solution through a matrix displacement procedure, Rao et al [2]. An extended description of this procedure in its more general form and a discussion of convergence criteria will be found in reference [2].

3. EXAMPLES

Some results from application of this method to perforated plates, cracked plates and bonded joints are presented.

3.1 PERFORATED PLATES

Fig.2 shows some details of the analysis of a square plate perforated

with a central circular hole and subjected to uniaxial loading. The quadrant A'ABEE' is covered by one primary element A'ACEE' and 30 secondary elements in the rest of the region ABECA, so that there are a total of 31 elements and 31 nodes. The solution involving a matrix order of 60 yields results in very close agreement with an analytical solution known to be very accurate, Schlack and Little [5]. For instance difference in peak stress values is only 0.12%. On the other hand a conventional finite element analysis by O'Connell [6] using 564 elements and 321 nodes in the quadrant gives an error of -10% in the peak stress (i.e. he obtains 5.7 σ_o instead of 6.3 σ_o)

3.2 PLATE WITH A CRACK

The second example (Figs. 3 and 4) is that of a plate with a central crack subjected to uniform edge tractions $\sigma_y = \sigma_o$. Using the symmetries of the problem only one quarter of the plate needs to be considered. A semicircular primary element is chosen with its centre at the crack tip. It is well known that

$$\phi = \sum_m A_m \, r^{\frac{m+1}{2}} \left[B \cdot \mathrm{Cos}\left(\frac{m+1}{2}\right)\theta - \mathrm{Cos}\left(\frac{m-1}{2}\right)\theta \right]$$

$$m = 2, 4, 6 \ldots\ldots ; \; B = \frac{m-3}{m+1} \qquad \ldots \quad (7)$$

$$m = 3, 5, 7 \ldots\ldots ; \; B = 1$$

is a biharmonic stress function satisfying the local conditions at the crack tip. Presuming plane strain, the corresponding displacement distributions are

$$Eu = K \, \mathrm{Cos}\theta + \sum_m A_m \, r^{\frac{m-1}{2}} \left[B \cdot \left\{ -(1+2\nu)(1-\nu) \, \frac{m+1}{2} \, \mathrm{Cos}\left(\frac{m+1}{2}\right)\theta \right\} + \left\{ (1+2\nu)(1-\nu) \, \frac{m+1}{2} - 4(1-\nu^2) \right\} \mathrm{Cos}\left(\frac{m-3}{2}\right)\theta \right]$$

$$Ev = -K \, \mathrm{Sin}\theta + \sum_m A_m \, r^{\frac{m-1}{2}} \left[B(1+2\nu)(1-\nu) \, \frac{m+1}{2} \, \mathrm{Sin}\left(\frac{m+1}{2}\right)\theta - \left\{ (1+2\nu)(1-\nu) \, \frac{m-3}{2} + 4(1-\nu^2) \right\} \mathrm{Sin}\left(\frac{m-3}{2}\right)\theta \right]$$

$$\ldots \quad (8)$$

We note that out of three possible rigid body modes, only translation in x-direction needs to be explicitly provided, because translation in y-direction and rotation about O are precluded by symmetry about the x-axis ($\theta = 0$) and this is implicitly accounted for by the stress function.

Using these displacement distributions, the primary element stiffness

matrix can be computed for a set of nodal displacements on its boundary. The solution for $a/b = 0.25$ computed with 5, 7 and 9 nodes on the primary element and 52, 54 and 57 secondary elements respectively, shows good convergence for the stress intensity factor, primary element strain energy and the total strain energy of the system (Table I). Solutions are also obtained for $a/b = 0.3$ and 0.4 with one 9-node primary element and 52 and 50 secondary elements respectively. The stress intensity factors so obtained are compared with values by other methods, Borris [7], Tables II and III. In one of these methods (Watwood [8]), a conventional finite element solution is obtained and the stress intensity factor is determined from the rate of increase of strain energy with crack length. It appears that the present hybrid procedure with matrices of orders about 80 yields results at least as good as those obtained by a conventional matrix displacement procedure with matrix orders of 956, Watwood [8].

3.3 COMPOUND PLATES

We apply the above procedure to a compound plate of two materials with elastic properties, E_1, ν_1 and E_2, ν_2, subjected to uniform edge tractions $\sigma_y = \sigma_0$ as shown in Fig.5a. The occurance and nature of concentrations at the bond and edge intersections can be determined by a corner function analysis, Rao [9]. For the combination $E_2/E_1 = 0.5$, $\nu_1 = \nu_2 = 0.3$, there are mild singularities of order $r^{-0.027}$ at A and B (Fig.5a). A biharmonic stress function satisfying the free edge conditions and continuity across the bond is given in Appendix I. A hybrid finite element system as in Fig.5b is convenient. Some significant results are given in Fig.6 and Table IV. It is seen that the solutions with 9 and 11 nodes on the primary element are in close agreement.

3.4 COMPOUND PLATE WITH A PARTIAL SEPARATION OF BOND

Another interesting problem analysed by the present method is a compound plate in which the bond has suffered partial separation in the centre. The normal stress across the bond is shown in Fig.7. There are very mild singularities at the ends of the bond. The significant singularity that at the crack tip, is of order $(r^{-0.5})$, the same as for a crack in a homogeneous plate. But in this case of a crack along a bond, the stress function indicates that as one approaches the crack tip, the stress oscillates between tension and compression with successively increasing amplitudes. This oscillation appears to have no physical significance as the oscillations are restricted to an extremely small region of radius about $2a \times 10^{-19}$.

4. CONCLUDING REMARKS

We have demonstrated the use of special elements around regions of stress concentrations and singularities to drastically reduce computing effort and yet obtain accurate solutions identifying the nature of the concentrations.

The method can be applied to problems with multiple concentrations. Extension of the method to more complex situations like elasto-plastic fields is under investigation.

5. ACKNOWLEDGMENTS

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REFERENCES

[1]. ARGYRIS,J.H., "Continua and discontinua", Proc. Conf. on Matrix Methods of Structural Analysis, WPAB,AFFDL-TR-66-80, 1966.

[2]. RAO,A.K., RAJU,I.S., KRISHNA MURTY,A.V., "A powerful hybrid method in finite element analysis" to appear in Int. Jnl. of Num. Meth. Engg. Also as IISc, Department of Aeronautical Engineering, Report AE 254 S.

[3]. MORLEY,L.S.D., "A finite element application of the modified Rayleigh-Ritz method", Int. Jnl. Num. Meth. Engg. 2, 85-98 (1970).

[4]. YAMAMOTO,Y., "Finite element approaches with the aid of analytical solutions", presented at Japan-U.S. Seminar on Matrix Methods of Structural Analysis and Design, August 25-30, 1969, Tokyo, Japan.

[5]. SCHLACK,A.L., LITTLE,R.W., "Elastostatic problem of a perforated square plate", Jnl. Engg. Mech. Div. Am. Soc. Civ. Engrs. ASCE, 90, 5, 171-187 (1964).

[6]. O'CONNELL,J.M., "Stress concentration factors around a penetration in a strip or slab of finite dimensions", Nucl. Eng. Design, 10, 3, 356-360 (1969).

[7]. BORRIS,W.G., Manual on Fatigue of Structures - Fundamentals and Physical Aspects, AGARD-MAN-8-70, June 1970.

[8]. WATWOOD, V.B., "The finite element method for prediction of crack behaviour". Nucl. Engg. and Design. 11, 323-332 (1969).

[9]. RAO,A.K., "Stress concentrations at interfaces corners" presented at the XII Int. Cong. on Appl. Mech. Stanford, August 1968. To appear in ZAMM.

APPENDIX I

Primary element stress functions for bonded interfaces of dissimilar materials:

Consider two materials with elastic constants E_1, ν_1 and E_2, ν_2 bonded along an interface OC and free along the edges OA, OB (Fig.8). The plate thickness t is identical and constant. Extracting a primary element which includes the bond (Fig.8) we can write down the stress function for the two materials satisfying the interface conditions of bonding as, (Rao [9])

$$\phi_i = \sum_{\lambda} r^{\lambda+1} \left[A_i \cos(\lambda+1)\theta + B_i \cos(\lambda-1)\theta + C_i \sin(\lambda+1)\theta + D_i \sin(\lambda-1)\theta \right] \quad .. \quad (9)$$

where

$$\begin{aligned} A_i &= \left[1 - \frac{\lambda_m + 1}{4} (1+\nu_i) \right] \cdot A_m - \frac{\lambda_m + 1}{4} E_i \cdot t \cdot B_m \\ B_i &= \frac{\lambda_m + 1}{4} (1+\nu_i) \cdot A_m + \frac{\lambda_m + 1}{4} \cdot E_i \cdot t \cdot B_m \\ C_i &= \left[1 + \frac{\lambda_m - 1}{4} (1+\nu_i) \right] \cdot C_m - \frac{\lambda_m - 1}{4} \cdot E_i \cdot t \cdot D_m \\ D_i &= -(1+\nu_i) \cdot \frac{\lambda_m - 1}{4} \cdot C_m + \frac{\lambda_m - 1}{4} \cdot E_i \cdot t \cdot D_m \end{aligned} \quad .. \quad (10)$$

where subscript i refers to the two fields 1,2

A_m, B_m, C_m, D_m are a set constants relevant to the eigenvalue λ_m. The stress free conditions along OA and OB (Fig.8) can be written as

$$\phi_{rr_i} = 0 \quad ; \quad \left(\frac{1}{r}\phi_\theta\right)_{r_i} = 0 \quad \text{on} \quad \theta = \alpha_i \quad ; \quad i = 1,2 \quad .. \quad (11)$$

Substituting ϕ_i into eq.(11) would lead to a secular matrix equation

$$[D]\{A\} = 0$$

The eigenvalues of matrix $[D]$ yield λ_m and the eigenvectors for a λ_m determine the relationship between A_m, B_m, C_m and D_m. The eigenvalues for a bond normal to a straight free edge and for the crack tip at the separation along a bond are given in Table V for the case $E_2/E_1 = 0.5$ and $\nu_1 = \nu_2 = 0.3$. Thus, with the use of eigenvalues given in Table V and eq.(10), eq.(9) provides the required stress function.

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APPENDIX II

Notation:

1. Scalars

2a	side of square plate in example 1, crack length in example 2 and width of compound plates.
2b	width of the plate in example 2 and height of the compound plates in examples 3 and 4.
c	radius of the hole in example 1.
E	Young's modulus
K	stress intensity factor
U_p	primary element strain energy
U_T	total strain energy of all elements
(r,θ)	polar coordinates
(x, y)	Cartesian coordinates
α_n	free constants in the stress function for a primary element
ξ	number of nodal points on the external periphery of a primary element
μ	degrees of freedom per node on a primary element
λ	degrees of freedom prescribed on the primary element due to condition like symmetries, antisymmetries etc.
$\int_v$	volume integral over a primary element.

2. Vectors and Matrices

$[D_n]$	matrix containing the functional forms of stresses, eq.(2)
$[E]$	elasticity matrix defined as $\{\sigma\} = [E]\{\epsilon\}$
$[K_p]$	stiffness matrix of a primary element
$\{u\}$	vector of displacements for a primary element (in the functional form)
$\{U\}$	vector of displacements at all modes of a primary element.
$\{\sigma\}, \{\epsilon\}$	vectors of stresses and strains.

Table I: Central Crack in a Rectangular Plate (Plane Strain): Convergence of Solution by various Criteria.

$a/b = 0.25$ and $L/b = 2.125$

Refer Figs. 3 and 4.

Nodes on primary element	No. of secondary elements	Matrix order	Stress intensity C_m	Primary element strain energy U_P/E	Total strain energy of all elements U_T/E	Total computing time on CDC 3600 in seconds
5	52	76	0.3825	0.1339	2.0417	27.4
7	54	80	0.3850	0.1403	2.0442	42.3
9	57	86	0.3865	0.1426	2.0447	60.3

Stress intensity factor $K = \sigma_o \frac{b}{b-a}\sqrt{\frac{\pi b}{2}} \cdot C_m$

Table II: Central Crack in a Rectangular Plate (Plane Strain): Computing Effort.

Refer Figs. 3 and 4

Crack length a/b	Present method with one 9-node primary element				Finite element energy method. Watwood [8]		
	No. of elements	Matrix order	Computing time on CDC 3600 in seconds	Stress intensity C_m	No. of elements	Matrix order	Stress intensity C_m
0.25	58	86	64.0	0.388	470	956	0.398
0.30	53	80	81.0	0.404	470	956	0.405
0.40	51	76	96.8	0.431	-	-	-

Stress intensity factor $K = \sigma_o \frac{b}{b-a}\sqrt{\frac{\pi b}{2}} \cdot C_m$

Table III Central Crack in a Rectangular Plate (Plane Strain) Stress Intensity Factor.

Refer Figs. 3 and 4

Crack length a/b	Stress intensity C_m						
	Finite element solutions		Analytical solutions [7]				
	Present method	Watwood [8]	Westergaard	Dixon	Greenspan	Isida	Fredderson
0.25	0.388 (58)	0.398 (470)	0.385	0.387	0.388	0.392	0.390
0.30	0.404 (52)	0.405 (470)	0.398	0.402	0.404	0.410	0.400
0.40	0.431 (50)	-	0.408	0.414	0.420	0.430	0.432

Figures in brackets indicate number of elements used in a quadrant. In the present method one of the elements is a 9 node primary element. Stress intensity factor $K = \sigma_o \frac{b}{b-a}\sqrt{\frac{\pi b}{2}}\, C_m$.

Table IV: Compound Plate: Convergence of Solution.

$b/a = 2 \quad E_2/E_1 = 0.5, \quad \nu_1 = \nu_2 = 0.3$

Refer Fig.5

Nodes on primary element	Number of secondary elements	Matrix order	Primary element strain energy U_p/E	Coefficient A_1
5	30	54	1.8764	-0.840
7	34	62	2.1544	-0.891
9	36	66	2.2220	-0.933
11	38	70	2.2698	-0.944

Singular part of σ_θ is $A_1 r^{-0.027}$

Total computing time for all 4 cases is 8 mts and 56 secs. on CDC 3600 computer.

Table V: Eigenvalues for Interface Corners in Compound Plates

$E_2/E_1 = 0.5$; $\nu_1 = \nu_2 = 0.3$

No. of eigenvalue	Normal edge (See Fig.5)		Separated bond (See Fig.7)	
	Real Part	Imaginary Part	Real Part	Imaginary Part
1	0.973254	0.0	0.5	0.037310
2	1.895827	0.255310	1.5	0.037310
3	2.847175	0.652315	2.0	0.0
4	3.866469	0.773202	2.5	0.037310
5	4.866593	1.005703	3.0	0.0
6	5.884532	1.048850	3.5	0.037310
7	6.886007	1.227581	4.0	0.0
8	7.899461	1.237843	4.5	0.037310
9	8.900434	1.390455	5.0	0.0
10	9.910861	1.382422	5.5	0.037310

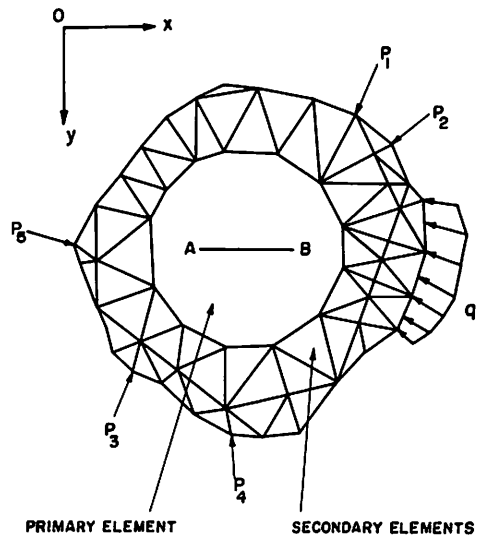


FIG.1 Arbitrary plate with a crack

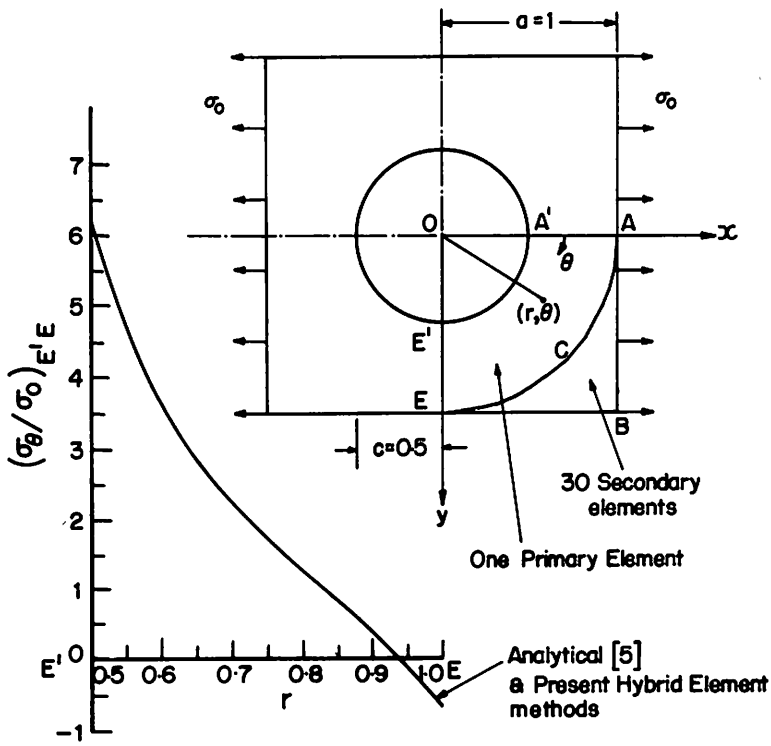


FIG.2 Hollow square under uniform extension

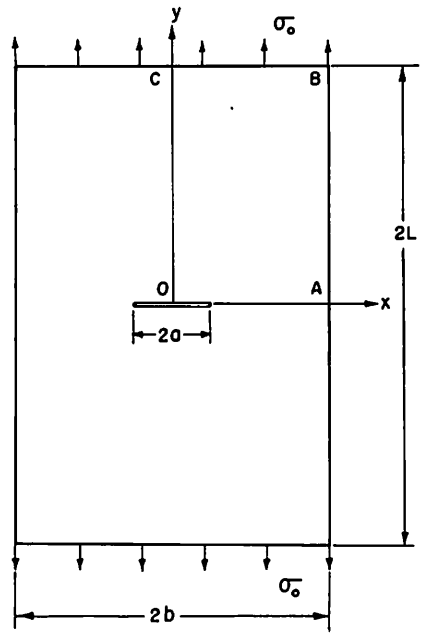
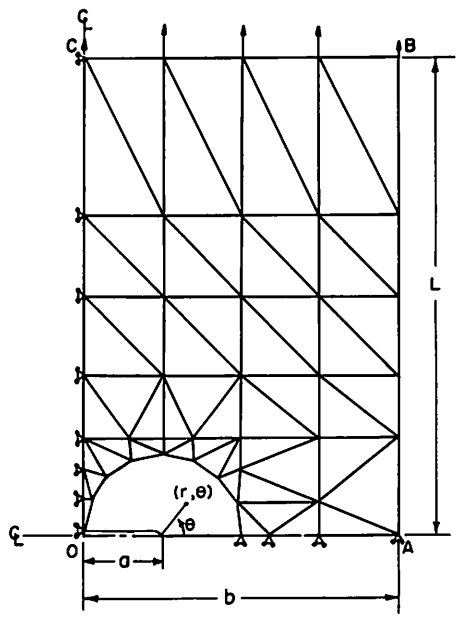


FIG.3 Plate with a central crack



NUMBER OF ELEMENTS 57, NODES 43

FIG.4 Plate with a central crack - Finite element subdivision

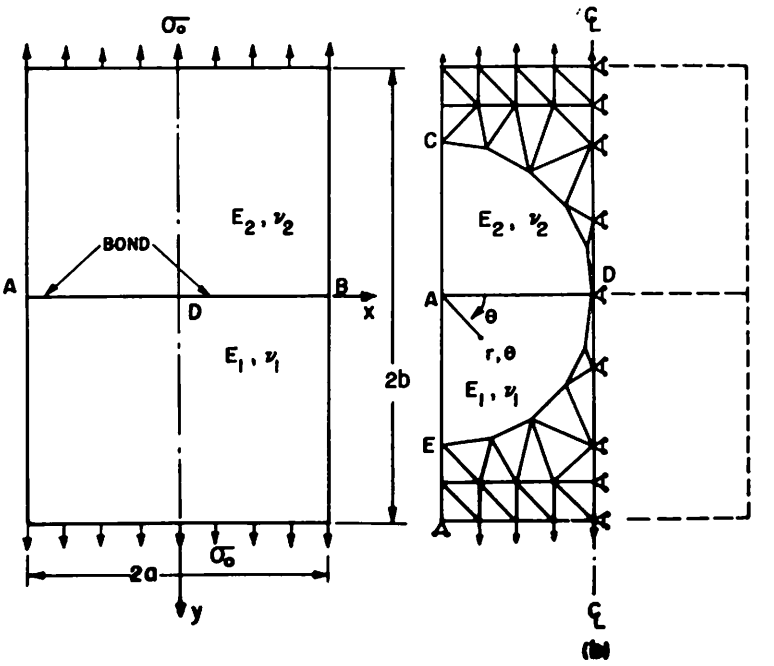


FIG.5 Compound plate - Finite element idealisations

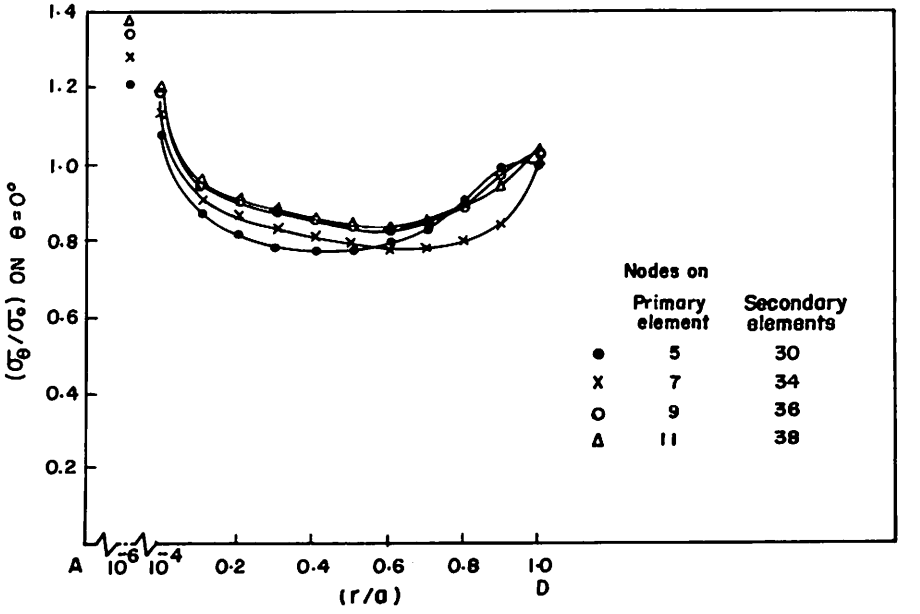


FIG.6 Compound plate - σ_θ variation along the bond

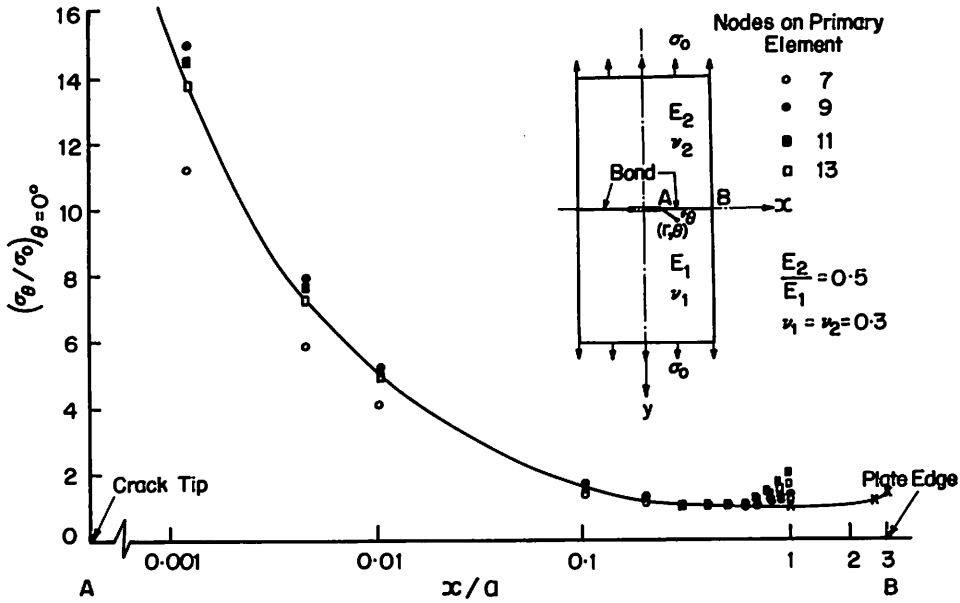


FIG.7 Compound plate with partial separation of bond - σ_θ variation along bond.

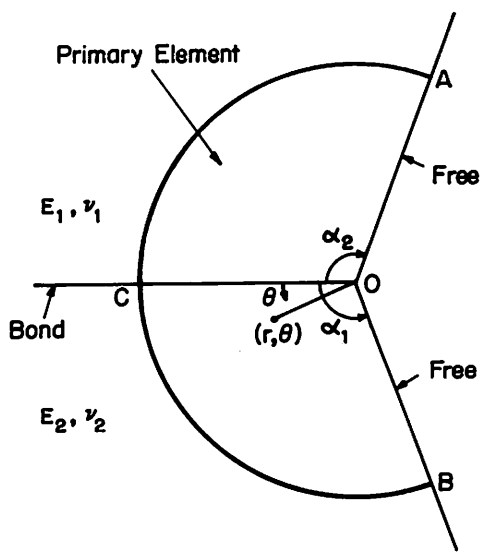


FIG.8 Primary element for a compound plate with inclined free edges.

DISCUSSION

C J. A. SWANSON, U. S. A.

It appears that the dividing line between the finite element method and the boundary collocation methods is becoming very hazy. A slight enlargement of the region around the crack tip reduces the entire problem to a boundary point collocation problem, a technique which has been used very successfully in the analysis of fracture mechanics problems. Perhaps it is time for a computer program to merge the two techniques such that the advantages of each may be exploited.

This paper also illustrates an obvious general principle which is: the closer a assumed stress distribution is to the true stress distribution, the better the results, and particularly if the exact stress function is used, exact results will be obtained if the boundary conditions can be properly satisfied. The use of exact stress functions is a characteristic of boundary collocation methods.

A A. K. RAO, India

Dr. Swanson suggests that the FEM and collocation methods are getting close to one another. We agree, but it is too narrow a view to take for collocation is merely one way of restricting errors in a solution. When the finite element methods first caught attention, many people tended to think of them as being different from and replacing classical continuum procedures. Having been in the Argyris school at the time the FEM methods were being developed, I know that this was not the view in the groups that pioneered the methods. Now there is even wider recognition that the underlying principles are identical. The matrix language and the computer show advantages that encourage the use of finite element formulation. But a good knowledge and intelligent application of continuum solutions along with FEM can be strikingly effective. Many classical procedures can be consciously translated into a matrix formulation.

Q D. R. J. OWEN, U. K.

In view of the increasing interest in the application of the finite element method in the study of fundamental material behaviour, I would like to ask the authors if they believe their approach could be employed in the analysis of other metallurgical singularities such as dislocations.

A A. K. RAO, India

Yes, we believe this can be done and we are looking into applications to this field. We are also extending the method to elasto-plastic analysis and to situations for which exact local solutions are not available.

Q D. COSTES, France

I did not understand whether you use in the vicinity of the crack the linear elastic fracture mechanics.

A A. V. KRISHNA MURTY, India

Yes, we have used the usual linear elastic fracture mechanics approach.

Q K. S. PISTER, U. S. A.

How do you decide on the location of the boundary between primary and secondary elements ? How sensitive is the location of the boundary relative to the numerical values of the field ?

A A. K. RAO, India

We will confess that we experimented a bit in our eagerness to make the primary element as large as possible and have found some unsatisfactory shapes for it. In one case, when we included an axis of symmetry as an edge for the primary element, large shears were developed in the inter-model regions on that edge. The size and shape of a primary element would depend very much on the type of functions we use. With the polar trigonometric functions we have used in this paper a near circular element is ideal, but not essential. As the shape deviates substantially, we could have increasing errors on lines which are to be traced by a rapid change in the length of the radius vector. In principle, as in continuum solutions, rapidly changing functions and functional sequences that exhibit "linear interdependence" indicate relatively regular regions. There is no real limitation on size as such.