THERMAL STRESSES DUE TO UNIFORM HEAT FLOW IN THICK PLATES WITH A CIRCULAR HOLE

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SUMMARY

It is generally known that uniform heat flow (with constant gradient) through a plate will not produce any thermal stress, if there is no hole in the plate and no mechanical constraint at the outer edge. On the other hand, the presence of a hole has the effect of disturbing the heat flow and thus will induce thermal stresses in the plate. Our discussion will be restricted to the case of uniform heat flow in an infinite plate with a circular hole. A two-dimensional solution of the problem was obtained by Florence and Goodier in 1959, in which it was found that the maximum stress did not depend on the plate thickness and the size of the hole.

In the present paper a three-dimensional solution for the same problem is obtained based on a thick plate theory (see: C. W. Lee, "Thermal Stresses in a Thick Plate," *International Journal of Solids and Structures*, pp. 605-615, 1970). Edge conditions at the hole are satisfied in the sense of Saint Venant, that is stress resultants and moments are satisfied instead of stresses. The maximum stress in the present solution is found to depend on the thickness-to-radius ratio.

As an alternative approach the same problem is also solved by using the method of thermal displacement potential function. In this approach a partial solution is obtained through a potential function to handle the temperature load. Another thick plate theory —L. H. Donnell, "A Theory for Thick Plates," *Proc. 2nd U.S. Nat. Congress of Applied Mechanics*, pp. 369-371, 1955,— is employed to give additional solution such that when superimposed to the former one all required edge conditions are satisfied. Identical results are reached from the two methods of solution.

Numerical results obtained from the present solution are compared with corresponding values from the two-dimensional solution. Such comparison leads to the conclusion that the two-dimensional solution should give fairly accurate results as long as the thickness-to-radius ratio remains small.

1. INTRODUCTION

The purpose of the study is to determine thermal stresses induced by uniform heat flow through an infinite plate with a circular hole. A two-dimensional solution of the problem was obtained by Florence and Goodier [1] using complex variable technique. In this solution the maximum stress was found to be independent of the plate thickness and the size of the hole.

In the present paper a three-dimensional solution for the same problem is obtained based on a recently developed thick plate theory [2]. Edge conditions at the hole are satisfied in the sense of Saint Venant; that is, stress resultants and moments are satisfied instead of stresses. At infinity, where the influence of the hole disappears, it is possible to satisfy the condition of vanishing stresses. The maximum stress in the present solution is found to depend on the thickness-to-radius ratio. As an alternative approach the problem is also solved by using the method of thermal displacement potential function. In this approach the thick plate theory developed by Donnell [3] is employed to solve the residue problem. Identical results are reached from the two methods of solution.

Cylindrical coordinates (r, θ, z) will be used throughout the paper. Some numerical results from the present solution will be given and compared with corresponding values from the two-dimensional solution.

2. BASIC EQUATIONS FOR THICK PLATES

The results of two earlier papers [2, 3] for thick plates will be utilized for the solution of the present problem. It is noticed that in the systematic study of equilibrium of a thick plate, Lure [4] separates the problem into two: extension (or compression) and bending. A division of general loading into symmetric and antisymmetric parts as discussed in the thick plate theory [3, 5] has the same effect. In the present problem only extension of the plate is involved and bending will be ignored. A bending problem of thick plates with a circular hole was studied previously [6], where the function associated with extension was assumed zero. It may be mentioned that the classical and Reissner plate theories treat only bending problems.

Basic equations necessary for the present study will be listed here from references [2, 3]. These equations are transformed into cylindrical coordinates (r, θ, z) and the bending part in each case is omitted.

2.1 Thermal Stresses in Thick Plates

Refer to reference [2] in cylindrical coordinates (r, θ, z), if the temperature distribution is independent of z, namely

$$T = T(r, \theta) \tag{1}$$

then the function $\phi(r, \theta)$ associated with extension of the plate is determined from

$$\nabla^4\phi = \nabla^2 T = 0 \tag{2}$$

where $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

is used throughout the paper. In view of the fact that $\nabla^2 T = 0$ under the condition of steady state heat conduction, the series solution [2] for stresses and displacements becomes closed form. The stress components are given as

$$\frac{1-\nu}{E\alpha}\tau_{rr} = -\left(1-\nu\right)\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\phi - \frac{1}{2}\frac{\nu}{1+\nu}z^2\,\nabla^2\phi\right)$$

$$- \left(\frac{z^2}{2} - \frac{c^2}{6}\right)\frac{\partial^2 T}{\partial r^2} - \nu\left(\frac{z^2}{1+\nu} - \frac{c^2}{6}\right)\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right)$$

$$\frac{1-\nu}{E\alpha}\tau_{\theta\theta} = -\left(1-\nu\right)\frac{\partial^2}{\partial r^2}\left(\phi - \frac{1}{2}\frac{\nu}{1+\nu}z^2\,\nabla^2\phi\right) - \left(\frac{z^2}{2} - \frac{c^2}{6}\right)\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r}\frac{\partial^2 T}{\partial \theta^2}\right)$$

$$- \nu\left(\frac{z^2}{1+\nu} - \frac{c^2}{6}\right)\frac{\partial^2 T}{\partial r^2}$$

$$\frac{1-\nu}{E\alpha}\tau_{r\theta} = \left(1-\nu\right)\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\right)\left(\phi - \frac{1}{2}\frac{\nu}{1+\nu}z^2\,\nabla^2\phi\right)$$

$$- \left(\frac{1}{2}\frac{1-\nu}{1+\nu}z^2 - \frac{1-\nu}{6}c^2\right)\left(\frac{1}{r}\frac{\partial^2 T}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial T}{\partial\theta}\right)$$

$$\tau_{zz} = \tau_{rz} = \tau_{\theta z} = 0 \qquad (3)$$

The stress resultants and moments N_r, M_r etc., may be obtained as usual by integrating stresses across the plate thickness 2c.

2.2 Thick Plates under Normal Surface Loads

With minor change of notation from reference [3], the function $\psi(r, \theta)$ associated with extension of the plate is determined from

$$\nabla^4\psi = S(r, \theta) \qquad (4)$$

where $S(r, \theta)$ is related to the surface loads such that

$$z = \pm\, c : \quad \tau_{zz} = -\, S/2 \qquad (5)$$

If the function $S(r, \theta)$ is harmonic, that is if

$$\nabla^2 S = 0 \qquad (6)$$

then the series solution [3] for stresses and displacements again becomes closed form. The stress components are given as

$$\tau_{rr} = -\frac{\nu}{2}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\nabla^2\psi + \left(\frac{z^2}{4} - \frac{c^2}{12}\right)\left[\frac{\partial^2 S}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2}\right)\right]$$

$$\tau_{\theta\theta} = -\frac{\nu}{2}\frac{\partial^2}{\partial r^2}\nabla^2\psi + \left(\frac{z^2}{4} - \frac{c^2}{12}\right)\left(\frac{1}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \nu\frac{\partial^2 S}{\partial r^2}\right)$$

$$\tau_{zz} = -\frac{1}{2}S$$

$$\tau_{r\theta} = \frac{\nu}{2}\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\right)\nabla^2\psi + \left(1 - \nu\right)\left(\frac{z^2}{4} - \frac{c^2}{12}\right)\left(\frac{1}{r}\frac{\partial^2 S}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial S}{\partial \theta}\right)$$

$$\tau_{rz} = \tau_{\theta z} = 0 \tag{7}$$

Again the stress resultants and moments may be obtained by integration of stresses.

3. TEMPERATURE AND BOUNDARY CONDITIONS

Let a be the radius of the hole and $h = 2c$ the thickness of the plate. For a perfectly insulated hole, the uniform heat flow in the negative y-direction (Fig. 1) results in a temperature distribution

$$T = \tau\left(r + \frac{a^2}{r}\right)\sin\theta \tag{8}$$

where τ is the constant temperature gradient. Since the first term $T = \tau r \sin\theta = \tau y$ leads to zero stresses, only the second term

$$T = \frac{\tau a^2}{r}\sin\theta \tag{9}$$

is relevant in our further discussion.

The boundary conditions to be satisfied in the present solution are

$$z = \pm c: \quad \tau_{zz} = \tau_{rz} = \tau_{\theta z} = 0 \tag{10}$$

$$r \to \infty: \quad \text{all stresses vanish} \tag{11}$$

$$r = a: \quad M_r = M_{r\theta} = V_r = N_r = N_{r\theta} = 0 \tag{12}$$

Notice that the number of conditions prescribed at the hole is five instead of the usual number of two at each edge for the classical plate theory, and three for Reissner plate theory.

4. SOLUTION BY DIRECT METHOD

To choose a biharmonic function $\phi(r, \theta)$ as indicated in eq. (2), we may assume

$$\phi = \left(\frac{A_1}{r} + A_2\, r \log \frac{r}{a}\right) \sin\theta \tag{13}$$

It is observed that conditions (10) are satisfied in view of equations (3). A direct substitution of equations (9) and (13) into (3), we obtain

$$\tau_{rr} = E\alpha \left\{\left(\frac{2A_1}{r^3} - \frac{A_2}{r}\right) - \frac{c^2}{1+\nu}\frac{1}{r^3}\left[\left(2\nu A_2 + \tau a^2\right)\frac{z^2}{c^2} - \frac{1+\nu}{3}\tau a^2\right]\right\} \sin\theta$$

$$\tau_{\theta\theta} = E\alpha \left\{-\left(\frac{2A_1}{r^3} + \frac{A_2}{r}\right) + \frac{c^2}{1+\nu}\frac{1}{r^3}\left[\left(2\nu A_2 + \tau a^2\right)\frac{z^2}{c^2} - \frac{1+\nu}{3}\tau a^2\right]\right\} \sin\theta$$

$$\tau_{r\theta} = E\alpha \left\{-\left(\frac{2A_1}{r^3} - \frac{A_2}{r}\right) + \frac{c^2}{1+\nu}\frac{1}{r^3}\left[\left(2\nu A_2 + \tau a^2\right)\frac{z^2}{c^2} - \frac{1+\nu}{3}\tau a^2\right]\right\} \cos\theta$$

$$\tau_{zz} = \tau_{rz} = \tau_{\theta z} = 0 \tag{14}$$

It is obvious that conditions (11) are satisfied. The satisfaction of edge conditions (12) results in

$$A_1 = \frac{\tau a^4}{4}, \qquad A_2 = \frac{\tau a^2}{2}$$

The final solution for thermal stresses then becomes

$$\tau_{rr} = -\frac{E\alpha\tau a}{2}\left(\frac{a}{r} - \frac{a^3}{r^3} + \frac{2}{3}\frac{a^3}{r^3}\frac{3z^2 - c^2}{a^2}\right)\sin\theta$$

$$\tau_{\theta\theta} = -\frac{E\alpha\tau a}{2}\left(\frac{a}{r} + \frac{a}{r^3} - \frac{2}{3}\frac{a}{r^3}\frac{3z^2 - c^2}{a^2}\right)\sin\theta$$

$$\tau_{r\theta} = \frac{E\alpha\tau a}{2}\left(\frac{a}{r} - \frac{a^3}{r^3} + \frac{2}{3}\frac{a^3}{r^3}\frac{3z^2 - c^2}{a^2}\right)\cos\theta$$

$$\tau_{zz} = \tau_{rz} = \tau_{\theta z} = 0 \tag{15}$$

The first two terms, which are independent of the plate thickness, agree with the two-dimensional solution [1]. The additional terms with the factor $(3z^2 - c^2)/a^2$ represent the three-dimensional effect.

It has been checked that the corresponding displacement components are single-valued, as required for the doubly connected region.

5. SOLUTION BY POTENTIAL FUNCTION

A particular solution for thermal stresses is obtainable from the thermal displacement potential function Φ[7], which is determined from the differential equation

$$\nabla_1^2 \Phi = \frac{1 + \nu}{1 - \nu} \alpha T$$

where ∇_1^2 is the three-dimensional Laplace operator, or

$$\nabla_1^2 = \nabla^2 + \frac{\partial^2}{\partial z^2}$$

By substitution of eq. (9) for T, one has

$$\nabla_1^2 \Phi = \frac{1 + \nu}{1 - \nu} \frac{\alpha \tau a^2}{r} \sin \theta \tag{16}$$

Since we are concerned with a particular solution, any function Φ which satisfies equation (16) is acceptable. Let

$$\Phi = A\, r \log \frac{r}{a} \sin \theta \tag{17}$$

and a direct substitution into eq. (16) gives

$$A = \frac{1}{2} \frac{1 + \nu}{1 - \nu} \alpha \tau a^2 \tag{18}$$

The corresponding particular solution is then obtained [7] as follows.

$$\tau_{rr} = -\frac{1}{2} \frac{E\alpha}{1 - \nu} \frac{\tau a^2}{r} \sin \theta$$

$$\tau_{\theta\theta} = -\frac{1}{2} \frac{E\alpha}{1 - \nu} \frac{\tau a^2}{r} \sin \theta$$

$$\tau_{zz} = -\frac{E\alpha}{1 - \nu} \frac{\tau a^2}{r} \sin \theta$$

$$\tau_{r\theta} = \frac{1}{2} \frac{E\alpha}{1 - \nu} \frac{\tau a^2}{r} \cos \theta$$

$$\tau_{rz} = \tau_{\theta z} = 0 \tag{19}$$

Integration of the stresses in equations (19) across the plate thickness gives

$$M_r = M_{r\theta} = V_r = 0$$

$$N_r = -\frac{E\alpha}{1 - \nu} \frac{\tau a^2 c}{r} \sin \theta$$

$$N_{r\theta} = \frac{E\alpha}{1 - \nu} \frac{\tau a^2 c}{r} \cos \theta \tag{20}$$

Since the particular solution as obtained above satisfies only part of the boundary conditions (10) through (12), a complementary solution is to be sought next. This complementary solution, when superimposed with the above particular solution, will combinedly satisfy all boundary conditions (10) through (12). In view of the discussion, it is seen that the boundary conditions to be satisfied by the complementary solution are

$$z = \pm c : \quad \tau_{zz} = \frac{E\alpha}{1 - \nu} \frac{\tau a^2}{r} \sin\theta \ , \ \tau_{rz} = \tau_{\theta z} = 0 \tag{21}$$

$$r \to \infty : \quad \text{all stresses vanish} \tag{22}$$

$$r = a : \quad M_r = M_{r\theta} = V_r = 0$$

$$N_r = \frac{E\alpha}{1 - \nu} \tau ac \sin\theta$$

$$N_{r\theta} = - \frac{E\alpha}{1 - \nu} \tau ac \cos\theta \tag{23}$$

As noted earlier in the paper, Donnell's thick plate theory [3] is used to determine the required complementary solution. By observing equations (5) and (21), it is clear that

$$S = - \frac{2E\alpha}{1 - \nu} \frac{\tau a^2}{r} \sin\theta \tag{24}$$

which has been checked to satisfy eq. (6). By substituting eq. (24) into eq. (4), one has

$$\nabla^4 \psi = - \frac{2E\alpha}{1 - \nu} \frac{\tau a^2}{r} \sin\theta \tag{25}$$

A trial solution of equation (25) may be assumed as

$$\psi = \left(B_1 r + B_2 r^3\right) \log \frac{r}{a} \sin\theta \tag{26}$$

and it follows then

$$\nabla^4 \psi = \frac{16B_2}{r} \sin\theta \tag{27}$$

The constant B_2 is determined by equating equations (25) and (27), which results in

$$B_2 = - \frac{1}{8} \frac{E\alpha}{1 - \nu} \tau a^2 \tag{28}$$

The stress components are obtained next by substituting equations (24) and (26) into (7), thus

$$\tau_{rr} = \left[- \frac{\nu}{2} \left(- \frac{4B_1}{r^3} + \frac{8B_2}{r} \right) + \frac{3z^2 - c^2}{3} \frac{8(1 - \nu)B_2}{r^3} \right] \sin\theta$$

$$\tau_{\theta\theta} = \left[- \frac{\nu}{2} \left(\frac{4B_1}{r^3} + \frac{8B_2}{r} \right) - \frac{3z^2 - c^2}{3} \frac{8(1 - \nu)B_2}{r^3} \right] \sin\theta$$

$$\tau_{zz} = -\frac{8B_2}{r}\sin\theta$$

$$\tau_{r\theta} = \left[\frac{\nu}{2}\left(-\frac{4B_1}{r^3}+\frac{8B_2}{r}\right) - \frac{3z^2 - c^2}{3}\frac{8(1-\nu)B_2}{r^3}\right]\cos\theta$$

$$\tau_{rz} = \tau_{\theta z} = 0 \qquad (29)$$

With the value of B_2 as given in eq. (28), all boundary conditions (21) through (23) are satisfied if we choose

$$B_1 = \frac{E\alpha\tau a^4}{4\nu} \qquad (30)$$

The final solution for stress components is obtained by superposition of equations (19) and (29), with constants B_1 and B_2 as given in equations (28) and (30). These results are identical to equations (15) as obtained by the first method. It is clear that for the present problem the first method is simpler.

6. DISCUSSION

(1) We have listed two sets of basic equations for thick plates in Section 2 of the paper. The stress components of equations (3) are useful when temperature load is known. Equations (7) are suitable for mechanical surface loads. Proper conditions should be met, of course, when these equations are to be employed. For the convenience of further reference, a third set of basic equations for extension of thick plates will be listed here. These equations are obtained by Lure [4], and are applicable when neither temperature nor mechanical surface load is present. In the present notation, the stress components are given as

$$\tau_{rr} = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\phi - \frac{1}{2}\frac{\nu}{1+\nu}\frac{3z^2 - c^2}{3}\nabla^2\phi\right)$$

$$\tau_{\theta\theta} = \frac{\partial^2}{\partial r^2}\left(\phi - \frac{1}{2}\frac{\nu}{1+\nu}\frac{3z^2 - c^2}{3}\nabla^2\phi\right)$$

$$\tau_{r\theta} = -\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\right)\left(\phi - \frac{1}{2}\frac{\nu}{1+\nu}\frac{3z^2 - c^2}{3}\nabla^2\phi\right)$$

$$\tau_{zz} = \tau_{rz} = \tau_{\theta z} = 0 \qquad (31)$$

where the function $\phi(r, \theta)$ is determined from

$$\nabla^4\phi = 0 \qquad (32)$$

(2) The maximum thermal stress for the present problem is the value of $\tau_{\theta\theta}$, equations (15), at $r = a$, $\theta = \pm\pi/2$, and $z = 0$, its numerical value becomes

$$\sigma_{max} = E\alpha\tau a\left(1 + \frac{1}{3}\frac{c^2}{a^2}\right)$$

The first term agrees with the two-dimensional solution, and the correction term depends on the thickness-to-radius ratio of the plate. Figure 2 gives the comparison of σ_{max} as a function of the ratio c/a. The correction reaches 33% when the ratio c/a = 1, and probably the present solution should not be used for c/a approaching unity or larger.

7. REFERENCES

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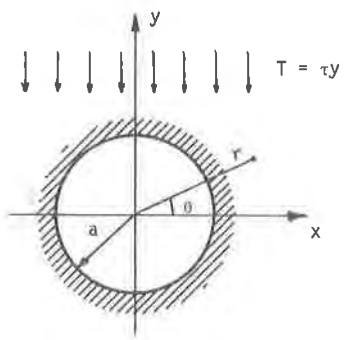

Figure 1. Temperature Distribution

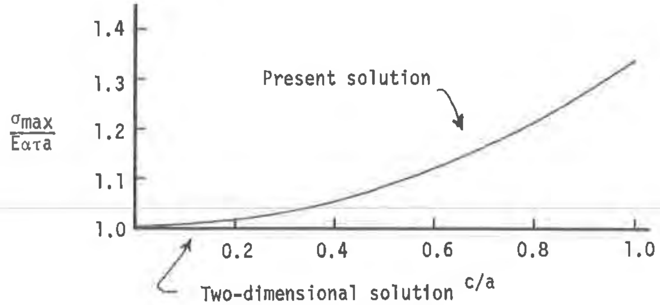

Figure 2. Maximum Thermal Stress