NUMERICAL INTEGRATION OF SOME NEW UNIFIED PLASTICITY—CREEP FORMULATIONS

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SUMMARY

The usual constitutive description of metals at high temperature treats creep as a phenomenon which must be added to time independent phenomena. A new approach is now being advocated by some people, principally metallurgists. R. Lagneborg, J. H. Gittus, E. Hart, K. C. Valanis, F. A. Leckie, and A. R. S. Ponter; S. R. Bodner and Y. Partom; and A. Miller are just a few of the people involved. All of these theories are different. However, they have a common hypothesis. They all treat the inelastic strain as a unified quantity, incapable of being separated into time dependent and time independent parts. This paper examines the behavior of the differential formulations reported in the literature together with one proposed by the author. These formulations are capable of representing primary and secondary creep, cyclic hardening to a stable cyclic stress-strain loop, a conventional plasticity behavior, and a Bauchinger effect which may be creep induced and discernable either at fast or slow loading rates. These models are just beginning to attract attention and have not begun to become standardized in form; however, there are several aspects common to these formulations. One common difficulty is the design of simple material tests which allow determination of constants or functions in the theories. Although structural analysts could offer some help in this area, an aspect of more immediate concern to the computational structural analyst is numerical integration of the resulting equations.

The new unified formulations seem to lead to very non-linear systems of equations which are very well behaved in some regions and very stiff in other regions where the word "stiff" is used in the mathematical sense. Simple conventional methods of integrating incremental constitutive equations are observed to be totally inadequate. A method of numerically integrating the equations is presented.

Automatic step size determination based on accuracy and stability is a necessary expense. In the region where accuracy is the limiting condition the equations can be integrated directly. A forward Euler predictor with a trapezoidal corrector is used in the paper. In the region where stability is the limiting condition, direct integration methods become inefficient and an implicit integrator which is suited to stiff equations must be used. A backward Euler method is used in the paper. It is implemented with a Picard iteration method in which a Newton method is used to predict inelastic strainrate and speed convergence in a Newton-Raphson manner. This allows an analytic expression for the Jacobian to be used, where a full Newton-Raphson would require a numerical approximation to the Jacobian. The starting procedure for the iteration is an adaptation of time independent plasticity ideas.

Because of the inherent capability of the unified plasticity-creep formulations, it is felt that these theories will become accepted in the metallurgical community. Structural analysts will then be required to incorporate these formulations and must be prepared to face the difficult implementation inherent in these models. This paper is an attempt to shed some light on the difficulties and expenses involved.

1. Introduction

A conventional constitutive description of metals at high temperature uses an unnatural separation of inelastic strains into plastic and creep strains. This description works well when laboratory material tests mimic the final use load history. For complicated load histories this approach has definite weaknesses. Metallurgists have always recognized that dislocation generation and movement is a time dependent process and kinetics becomes even more important at high temperature since other physical processes become active and recovery from strain induced hardening becomes more pronounced with temperature

Bailey [1] and Orowan [2] years ago suggested that the competing processes of hardening and recovery be represented in constitutive equation for metals at high temperature. This idea was revived by Mitra and McLean [3] decades later and they experimentally verified the advantages of its basic form. Again little seemed to be done with the idea until rather recently. The approach is now gathering momentum and several full theories are now available with many people contributing bits and pieces.

2. Basic Skeletal Model

It is logical at this point to present the models proposed by various people. But the models are so diverse that it is convenient to place them into a common framework and discuss the basic behavior of this abstracted model. The behavior of the various models can then be explored by comparison with the skeletal model.

The equations for the skeletal model are given as

$$\dot{\tilde{e}}_p = F\tilde{\xi} \tag{1}$$

$$\dot{\tilde{\sigma}} = \underset{\approx}{C}(\dot{\tilde{e}} - \dot{\tilde{e}}_p) \tag{2}$$

$$\dot{\tilde{\alpha}} = A_\alpha \dot{\tilde{e}}_p - \tilde{\alpha} f \tag{3}$$

$$\dot{R} = A_R|\dot{\tilde{e}}_p| - (R - R_o)g \tag{4}$$

where $\tilde{\sigma}$ is the external or applied stress, $\tilde{\alpha}$ is the rest or back stress corresponding to the center of the yield surface for kinematic hardening conventional plasticity, R is the drag stress corresponding to the radius of the yield surface for isotropic hardening conventional plasticity, $\tilde{\xi}$ is the internal stress and is the deviatoric part of the stress difference $(\tilde{\sigma} - \tilde{\alpha})$, $\dot{\tilde{e}}$ is the external strainrate, and $\dot{e}_p$ is the inelastic strainrate. (The subscript p has been used but both plasticity and creep are included.) The matrix $\underset{\approx}{C}$ is the matrix of elastic constants. The scalar functions F, A_α, A_R, f, and g are functions of applied stress $\tilde{\sigma}$ and internal variables $\tilde{\alpha}$ and R in general. The direction of the inelastic strainrate is consistent with the usual assumption that plastic strains and creep strains are in the direction of $\tilde{\xi}$.

The model is easily generalized to include other internal variables and the scalar functions generalized to matrices and so forth. The opposite approach is taken here. This is the simplest model which will demonstrate most of the observed behavior in metals at high temperature, and yet it is terribly complicated already. "Sufficient unto the day is the evil thereof." There are already more constants and functions than one would want to experimentally find without introducing general anisotropy into the list of material tests required.

3. Behavior of the Basic Model

The various models to be discussed below are characterized by various functional forms for F, A_α, A_R, f, and g. Some of them also omit one or both of the hardening variables. Those used in the various models are chosen either to model a particular metallurgical mechanism or to fit observed data.

The functions A_α, A_R, f, and g will be chosen to be constants for the explanation here. The function F must be chosen such that creep of a form observed in metals is represented. It contains a power function of stress in most of the models presented. It also must be a function of R if isotropic hardening is to be represented. The specific form beyond these general properties is not necessary here.

Temperature variations have been purposely excluded from the skeletal model as well as from the models taken from the literature. This is for simplicity.

This formulation then shows that for any given stress level with no hardening, i.e., $A_\alpha = A_R = 0$, creep strainrate is given by $\dot{\underset{\sim}{e}}_p = F\,\underset{\sim}{S}$, where $\underset{\sim}{S}$ is the deviatoric part of the applied stress, $\underset{\sim}{\sigma}$. The function F would be generated by creep tests. Note that no primary phase is modeled in this case. Steady state creep is established immediately. This is true for any form of $F(\underset{\sim}{S})$ if no hardening is present.

For the case where some hardening is present ($A_\alpha \neq 0$ and/or $A_R \neq 0$) then a transient phase is modeled where $\underset{\sim}{\alpha}$ and R build up to stationary values for a given steady applied stress state $\underset{\sim}{\sigma}$. We observe that if $\underset{\sim}{\alpha}$ and $R-R_o$ are initially zero then initially the recovery terms in Eqs. (3) and (4) are zero and the hardening terms, the first terms, are dominant and $\dot{\underset{\sim}{\alpha}}$ and $\dot{R}$ are large and positive. As $\underset{\sim}{\alpha}$ and R grow, the recovery (the second terms in Eqs. (3) and (4) become more dominant until $\dot{\underset{\sim}{\alpha}}$ and $\dot{R}$ approach zero. Steady state values of $\underset{\sim}{\alpha}$ and R and then given by

$$\underset{\sim}{\alpha}_{ss} = (A_\alpha/f)\,\dot{\underset{\sim}{e}}_{pss}$$

$$R_{ss} = R_o + (A_R/g)\left|\dot{e}_{pss}\right| \tag{5}$$

and the function F for this applied stress level is given by Eqs. (1) and (5) as

$$F(\underset{\sim}{S}, \alpha_{ss}, R_{ss}) = \dot{\underset{\sim}{e}}_{pss} \cdot \underset{\sim}{\xi}_{ss}/|\underset{\sim}{\xi}_{ss}|^2$$

so that measured values of $\underset{\sim}{\alpha}_{ss}$, R_{ss} and $\underset{\sim}{e}_p$ under creep at various $\underset{\sim}{S}$ could be used to determine the functional forms of A_α/f, A_R/g, and F. For a suddenly applied load then a transient period of high strainrate (primary creep) is followed by an exponential-like reduction in the rate as $\underset{\sim}{\xi}$ decreases to a steady state value. This secondary creep state is characterized by hardening and recovery processes which are in dynamic equilibrium.

Now consider the case of a system in secondary creep where the applied stress is then suddenly reduced to zero. If F were a homogeneous function of $\underset{\sim}{\sigma}$, then $\dot{\underset{\sim}{e}}_p$ would suddenly become zero. If F were a function of $\underset{\sim}{\xi}$ only then $\dot{\underset{\sim}{e}}_p$ would be nonzero and directed as $\underset{\sim}{\alpha}$, that is, opposite to the direction that the stress was originally applied. This would resemble an elastic strain recovery (and point out the inappropriateness of the name for $\dot{\underset{\sim}{e}}_p$ for this case). Even in this case $\dot{\underset{\sim}{e}}_p$ would be small since $\underset{\sim}{\alpha}$ is usually smaller than $\underset{\sim}{S}$ and F is roughly a power function of the stress. Then under the zero load condition, Eq. (3) gives an approximate value for $\underset{\sim}{\alpha}$ of $-\underset{\sim}{\alpha}f$ so that $\underset{\sim}{\alpha}$ would decay roughly exponentially for constant f as $\underset{\sim}{\alpha} \simeq \underset{\sim}{\alpha}$, exp(-t/f) and similarly for R. Annealing is then demonstrated by the model. Again more complicated functions for f and g are needed but the basic annealing behavior is apparent.

Other behavior demonstrated by the model is hardening under a cyclic load to a stationary loop which is of a different size for different strain rates and different strain limits and hold times. Creep induced Banschinger effect is apparent also. The list could be extended even further for this basic model, but at this point it becomes more fruitful to look at some specific forms for the functions as given in the various theories.

4. Survey of Various Models

Unfortunately, there isn't enough space here to present more than a list of authors of the various models and a short statement about their work. They are presented in alphabetical order to avoid the difficulties of attaching dates and unintended bias to the models. Some of these constitutive descriptions are called theories and others are called models by their authors. The word "model" will be used throughout for consistency.

The descriptive equations for the models are listed in Table I. The notation used is that presented here. The notations for constants and functions are only meant to convey that meaning so C_1, for example, is completely different for each model.

S. R. Bodner [4] discussed a simple viscoplastic model in 1968 using only Eqs. (1) and (2) with no hardening. This was expanded by Bodner and Partom [5] to include isotropic hardening. This model should perhaps be omitted here because of its lack of recovery behavior (g = 0) but is included because of the unique way of incorporating the isotropic hardening parameter R into the function F. They write the Z equation in terms of plastic work but if this is identified as $\underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{e}}_p$ and the transformation R = -C times

$\ell n(Z - Z_1)/Z_o$ is applied, then the forms shown in the table are the result. The model is particularly aimed at lower temperature behavior.

E. W. Hart [6] outlined his basic model in 1970. A more recent version [7] is stated for a general stress state. The equations used by Hart are computationally awkward in order to illuminate various metallurgically meaningful quantities. The variables $\underset{\sim}{\sigma}_f$, $\underset{\sim}{\sigma}_a$, $\underset{\sim}{\varepsilon}^*$, and $\underset{\sim}{\alpha}$ (in his notation) are eliminated by algebraic manipulations and our $\underset{\sim}{\alpha}$ is substituted for his $M\,\underset{\sim}{a}$. Note that an additional evolutionary scalar variable σ^* is used in this theory. But it does not play the role of the isotropic hardening parameter R since F is not a function of σ^*. In fact σ^* only influences the recovery rate on $\underset{\sim}{\alpha}$. The case where $\underset{\sim}{\alpha}$ is zero introduces problems in this model as noted in the form for f.

My own model is presented here as a composite of the other models so that the numerical difficulties of integration of a unified theory could be illustrated. It has both isotropic and kinematic hardening. It has nonlinear recovery behavior for $\underset{\sim}{\alpha}$ and R, and has a power function of stress in the form for F. A unique feature is that it is based on a (yet unpublished) time independent plasticity theory with a nonlinear hardening rate, similar in behavior to that in [8] but the one here is much simpler to implement.

R. Lagneborg [9, 10] developed a model based on dislocation density, Since his is a uniaxial model it cannot be compared directly. The table entries are a multiaxial extension where we have made ρ^2 in his Eq. (5b)[10], to be $|\underset{\sim}{\rho}|\underset{\sim}{\rho}$ and identified our $\underset{\sim}{\alpha}$ as $C|\underset{\sim}{\rho}|^{-0.5}\underset{\sim}{\rho}$. This extension has the curious behavior that $\underset{\sim}{\alpha}$ does not harden at the same rate in all directions, but moves at twice the rate when $\dot{\underset{\sim}{e}}_p$ is perpendicular to $\underset{\sim}{\alpha}$ than when it is parallel. This should not be blamed on Lagneborg since it is only the result of the extension. The model is particularly notable in its strong physical motivation. A notable point here is that the form for A does not admit the condition $|\underset{\sim}{\alpha}| = 0$. This is only a fault of the identification of $\underset{\sim}{\alpha}$ with $\underset{\sim}{\rho}$ where the equation for $\underset{\sim}{\rho}$ does not have this defect. This leads one to question the appropriateness of the variable $\underset{\sim}{\alpha}$.

A. Miller [11-12] has a model based on a combination of metallurgical arguments and emperical observations. This model has been applied to Zircalloy particularly. It has both isotropic and kinematic hardening and is computationally unique in the transcendental form for F since $|\dot{\underset{\sim}{e}}_p|$ occurs on the right side of the equation through its relation to V.

P. R. Paslay and C. H. Wells [13] applied the concepts of Lagneborg together with other metallurgical arguments to arrive at a uniaxial model. This has been extended here to the multiaxial case in the manner explained above for Lagneborg's model and has the same difficulty. Again the difficulties are not present when the equation is stated in terms of their variable $\underset{\sim}{\rho}$ rather than $\underset{\sim}{\alpha}$.

A. R. S. Ponter and F. A. Leckie [14] have proposed two models, one with isotropic and the other with kinematic hardening. They use the creep potential ideas of Rice [15] to good advantage. Their models are deduced from

5 Eqs. (29) [14].

The ideas of Lagneborg and those of Rice are also reflected in the Robinson-Pugh-Corum Model [16]. This model is unique in its use of the Mises function $\xi \cdot \xi - R_o^2$. This model is notable in the small number of constants required.

There are others who have contributed to the literature in this area, several with their own models which for one reason or another cannot be directly compared. G. M. Brown [17], J. H. Gitters [18-19], and K. C. Valanis [20] are particularly notable.

5. Implementation of the Models

Of the models presented, only those of Bodner and Partom and Hart [21] have been implemented in structural computations. Miller's model has been programmed into a subroutine and is presently being incorporated into a structural program. A difficulty common to all the models is that the equations have regions, particularly the high stress regions, where the equations are "stiff" in the mathematical sense. Miller has handled this stiffness by using the methods of C. W. Gear [21] in an adaptive general purpose computer integration program. This has been very successful and is recommended as a technique for examining the behavior of the various models. This model is now being incorporated into a finite element structural computer program at General Electric in Schenectady, New York.

Kumar and Mukherjee [22] who incorporated Hart's model into a structural model did not use a stiff integrator. They used a fourth order Runge-Kutta method which is only conditionally stable. They mention that "the choice of a time step size is critical and variable time steps are necessary." The problems worked were only one dimensional in space and the stress was monotone in time at each point. Only 100 nodal points were used for these stress relaxation types of problems but the execution time was reported at under 40 seconds on the IBM 360/168. It remains to be seen whether this approach is viable when the problem involves rapid loading to high stress levels with stress reversals over some regions.

The viability of various integrators is illustrated with Eqn (4) which has been decoupled by assuming $|\dot{\underset{\sim}{e}}_p|$ to be constant and then non-dimensionalizing to the form:

$$y' = 1 - y^3 \quad , \; y_o \geq 0$$

One method of integrating this is to use a simple backward Euler method as

$$y_{n+1} = y_n + (1 - y_n^3)\,\Delta t$$

This equation is absolutely stable as defined by Dalquist [23] when the equation of first variation gives $|y_{n+1}|$ less than $|y_n|$. This gives the stability condition:

$$\Delta t < 2/y_n^2$$

6 When $y_n = 100$ for example, the time step may become so expensive that this method must be abandoned.

The forward Euler method simply uses y^3_{n+1} on the right side of the equation. The resulting cubic scalar equation could be solved directly but an iterative method must be used on the real equations so we apply the simplest iterative method, the Picard method as

$$y^{i+1}_{n+1} = y_n + \left[1 - \left(y^i_{n+1}\right)^3\right] \Delta t$$

where the superscript i refers to the iterate number. In spite of the fact that the forward Euler method is an A-stable method and y_{n+1} is positive for any positive y_n and Δt, this implementation of the method will not always converge. As an example, with $\Delta t = 2$ and $y_n = 100$, an overflow will be encountered in four iterations or less, even if the initial guess for y^i_{n+1} is within 0.001 of the desired value 3.66349. A similar thing happens for $y_n = 0.01$.

The full Newton-Raphson iteration method applied to the Euler forward difference integrator does converge if the analytic form for the Jacobian is used. In fact, it will even converge if the cubic term is replaced by a general positive power p.

A promising technique then seems to be to use a simple integrator such as a Runge-Kutta or even a simple Euler backward difference method as long as stability (and accuracy) does not require a time step which is too small. In that case a backward Euler method with a Newton-Raphson iteration would be used.

6. Stability of the Explicit Method

A vector unknown can be defined as $\underset{\sim}{y}_n = \{\underset{\sim}{S}_n \vdots \underset{\sim}{\alpha}_n \vdots R\}^T$. The backward Euler method can then be written as

$$\underset{\sim}{y}_{n+1} = \underset{\approx}{A}\,\underset{\sim}{y}_n + \underset{\sim}{b}$$

where the matrix $\underset{\approx}{A}$ is a function of $\underset{\sim}{y}_n$. Absolute stability requires that the usual norm of $\bar{\underset{\sim}{y}}_{n+1} = \underset{\approx}{A}\,\underset{\sim}{y}_n$ be less than that of $\underset{\sim}{y}_n$. This in turn requires that all the eigenvalues of $\underset{\approx}{A}$ have magnitudes less than one. But if $\underset{\sim}{S}$ and $\underset{\sim}{\alpha}$ have six components, then $\underset{\approx}{A}$ is a 13 x 13 matrix and only finding the largest eigenvalue will be expensive. A cheaper method is simply to compute $\bar{\underset{\sim}{y}}_{n+1}$ with a single step, compute its norm, and compare this with the norm of $\underset{\sim}{y}_n$. If $|\bar{\underset{\sim}{y}}_{n+1}| < f|\underset{\sim}{y}_n|$ where $f < 1$ is an experimentally determined parameter which produces acceptable accuracy, then the desired integration is $\underset{\sim}{y}_{n+1} = \bar{\underset{\sim}{y}}_{n+1} + \underset{\sim}{b}$. If not, then the process could be repeated again with the smallest economically acceptable step size. This step size could be used to subincrement inside the constitutive subroutine. This process must be repeated at every step.

The advantage of this method is that it is virtually free if the step size is acceptable and cheaper than finding the eigenvalue of $\underset{\approx}{A}$ with the largest magnitude in any case.

7. Newton-Raphson forward Euler Method

The Newton-Raphson forward Euler method when applied directly to the differential equation $\dot{\underset{\sim}{y}} - \underset{\sim}{F}(\underset{\sim}{y},t)$ is an expensive method since $\underset{\sim}{y}$ has nine scalar components for a planar problem or 13 for a three-dimensional problem. This means the resulting non-symmetric Jacobian has 81 or 169 components which must be formed to solve:

$$[\underset{\approx}{I} - \partial \underset{\sim}{F}/\partial \underset{\sim}{y}]^{i}_{n+1}\ \delta \underset{\sim}{y} = \underset{\sim}{y}_{n} - \underset{\sim}{y}^{i}_{n+1} + \underset{\sim}{F}^{i}_{n+1}\ \Delta t$$

and obtain the next iterate as $\underset{\sim}{y}^{i+1}_{n+1} = \underset{\sim}{y}^{i}_{n+1} + \delta \underset{\sim}{y}$. A more economical approach is to form the scalar equations for the time rates of $|\underset{\sim}{\S}|$, $|\underset{\sim}{\alpha}|$ and R. These equations involve the relative directions of the vectors in terms such as $\dot{\underset{\sim}{e}} \cdot \underset{\sim}{\S} \,/\, |\dot{\underset{\sim}{e}}||\underset{\sim}{\S}|$ and $\underset{\sim}{\alpha} \cdot \underset{\sim}{\S} \,/\, |\underset{\sim}{\alpha}||\underset{\sim}{\S}|$ which are taken to be constant over the time of integration. The Jacobian which must be reformed at every iteration has only eight non-trivial components. This allows $|\dot{\underset{\sim}{e}}_{p}|$ in particular to be found reliably and rather economically. The solutions for these magnitudes are then used in the Euler equations for $\dot{\underset{\sim}{S}}$ and $\dot{\underset{\sim}{\alpha}}$ but now the matrices which multiply $\underset{\sim}{y}_{n+1}$ only contain the scalar magnitudes which have been found already so that no iteration is needed here.

At this point it may be profitable to repeat the entire process where values of $\dot{\underset{\sim}{e}} \cdot \underset{\sim}{\S} \,/\, |\dot{\underset{\sim}{e}}||\dot{\underset{\sim}{\S}}|$ and $\underset{\sim}{\alpha} \cdot \underset{\sim}{\S} \,/\, |\underset{\sim}{\alpha}||\underset{\sim}{\S}|$ have been averaged over the time step.

8. Summary and Unanswered Questions

Eight unified creep-plasticity models of various authors were stated in isothermal form in a common framework or skeletal model. The behavior of this model is explored and seen to have many attractive features. But an intriguing question arose concerning whether $\underset{\sim}{\alpha}$ or the dislocation density vector $\underset{\sim}{\rho}$ should be used. Another point concerns the wide variety of functional forms which can be applied to the skeletal model as noted in Table I. Random searching for functions which fit test data seems to be an expensive way to build a model. Applied mechanicians must rely on materials experts for physical motivation for these functional forms.

The rate equations which result from these models are noted to be mathematically stiff in many interesting regions. Suggested methods of numerically integrating these equations are motivated and presented. Since this is a report on work in progress, the suggestions are merely that, not proven fact. In particular, even if the constitutive equations can be successfully integrated, this does not imply that the finite element program containing a theory of this type can successfully converge.

This merely points out that researchers working with structural computer programs will not run out of meaningful work for some time.

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TABLE I

Basic Model	$\underline{F}$	$\underline{A_\alpha}$	$\underline{f}$	$\underline{A_R}$	$\underline{g}$
Bodner-Partom	$C_1\left\{J^{-1}\exp\left[-\left(\frac{n+1}{n}\right)\left(\frac{z^2}{3J}\right)^n\right]\right\}^{0.5}$ where: $z = C_3 + C_4\exp(-R/C_5)$	N.A.	N.A.	C_2J	0
Hart	$C_1\lvert\underset{\sim}{\S}\rvert^n$	C_2	$C_3\sigma^{*m}\left\{\ln\frac{\sigma^*}{\lvert\underset{\sim}{\alpha}\rvert}\right\}^{C_4}\Big/\lvert\underset{\sim}{\alpha}\rvert$ where $\overline{\ln\sigma^*} = C_3D(\sigma^*,\alpha) - E(\sigma^*,T)$	N.A.	N.A.
Krieg	$C_1\lvert\underset{\sim}{\S}/R\rvert^n$	$C_2+C_3\exp(-\underset{\sim}{\S}\cdot\underset{\sim}{\alpha}/C_3)$	$C_4\lvert\underset{\sim}{\alpha}\rvert^2$	C_5	$C_6(R-R_o)^2$
Extended Lagneborg	$C_1\lvert\underset{\sim}{\S}\rvert^n$	$C_2\left\{\underset{\approx}{I} - \frac{\underset{\sim}{\alpha}\otimes\underset{\sim}{\alpha}}{2\underset{\sim}{\alpha}\cdot\underset{\sim}{\alpha}}\right\}\Big/\lvert\underset{\sim}{\alpha}\rvert$	$C_3\lvert\underset{\sim}{\alpha}\rvert^2$	N.A.	N.A.
Miller	$C_1\left\{\sinh\left[\lvert\underset{\sim}{\S}\rvert^{1.5}(R+V)^{0.75}\right]\right\}^n\Big/\lvert\underset{\sim}{\S}\rvert$ where: $V = C_{10} + C_{11}\log\left(C_{12}\lvert\underset{\sim}{\dot{e}}_p\rvert\right) + \sum_{i=1}^{m} C_{13}^i\exp\left[-\left(\log C_{14}^i\lvert\underset{\sim}{\dot{e}}_p\rvert\right)^2\right]$	C_2	$C_3\left[\sinh\left(C_4\lvert\underset{\sim}{\alpha}\rvert\right)\right]^n\Big/\lvert\underset{\sim}{\alpha}\rvert$	$C_5\left(C_6 + \lvert\underset{\sim}{\alpha}\rvert - C_7R^3\right)$	$\frac{C_8\left[\sinh\left(C_9R^3\right)\right]^n}{R-R_o}$
Extended Pasley-Wells	$C_1e^{C_2\lvert\underset{\sim}{\S}\rvert}A(\underset{\sim}{\sigma})\Big/\lvert\underset{\sim}{\S}\rvert$	$\frac{C_1m^2}{\lvert\underset{\sim}{\alpha}\rvert}\left(\underset{\approx}{I} - \frac{\underset{\sim}{\alpha}\otimes\underset{\sim}{\alpha}}{2\underset{\sim}{\alpha}\cdot\underset{\sim}{\alpha}}\right)$	$\frac{m^2A(\underset{\sim}{\alpha})}{\underset{\sim}{\alpha}\cdot\underset{\sim}{\alpha}}$	N.A.	N.A.

TABLE I continued

Basic Model	$\underset{\sim}{F}$	A_α	$\underset{\sim}{f}$	A_R	g
Ponter-Leckie Isotropic*	$A\left(\lvert\underset{\sim}{S}\rvert - R\right)\Big/\lvert\underset{\sim}{S}\rvert,\ \lvert\underset{\sim}{S}\rvert = R$	N.A.	N.A.	$A_R(R)$	$g(R)$
	$0 \qquad ,\ \lvert\underset{\sim}{S}\rvert \leq R$				
Ponter-Leckie Kinematic*	$A\left(\lvert\underset{\sim}{s}\rvert - R_o\right)\Big/\lvert\underset{\sim}{s}\rvert,\ \lvert\underset{\sim}{s}\rvert = R_o$	$A_\alpha(\underset{\sim}{\alpha})$	$f(\underset{\sim}{\alpha})$	N.A.	N.A.
	$0 \qquad ,\ \lvert\underset{\sim}{s}\rvert \leq R_o$				
Robinson-Pugh-Corum	$C_1\left(\underset{\sim}{s} \cdot \underset{\sim}{s} - R_o^2\right)^n$	$C_2\Big/\lvert\underset{\sim}{\alpha}\rvert$	$C_3\lvert\underset{\sim}{\alpha}\rvert^3$	N.A.	N.A.

Notation:

$C_1, \ldots C_{14}$ -- denotes constants

$A(\cdot), B(\cdot), \ldots E(\cdot)$ -- denotes functions

*There is some question about whether these are the inequalities intended by the authors.

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