A COMPUTATIONAL METHOD FOR DIRECT INTEGRATION OF MOTION EQUATIONS OF STRUCTURAL SYSTEMS

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SUMMARY

The dynamic analysis of structural systems requires the solution of the matrix equations:

$$M\ddot{\delta}(t)+C\dot{\delta}(t)+K\delta(t) = F(t). \tag{1}$$

Many numerical methods are available for direct integration of equation (1) and their efficiency is due to the fulfillment of the following requirements:

— a reasonable order of accuracy must be obtained for the approximation of the response relevant to the first modes;

— the modal contributions relevant to the eigenvalues with large real part must be essentially neglected.

This paper presents a step-by-step numerical scheme for the integration of equation (1) which satisfies the requirements previously mentioned. The method is based on the reduction of problem (1) to the first order matrix equation:

$$\dot{y}(t) = Ay(t)+b \tag{2}$$

where the components of vector $y\ (t) = [\delta(t), \dot{\delta}(t)]^T$ are the nodal displacements and velocities, and matrix A and vector b are properly defined.

The basic assumption is that vector $\dot{y}(t)$ can be approximated by a linear interpolation between $t = t_k$ and $t = t_k+\theta h$, where h is time step and θ is a dimensionless parameter higher than one.

The vector $y(t)$ is approximated by a quadratic interpolation obtained by integration of $\dot{y}(t)$.

These hypotheses are similar to those relevant to the θ-Wilson scheme, but they are applied to equation (2) instead of equation (1) so that two different approximations are used for the nodal velocities, because $\dot{\delta}(t)$ appears both in vectors $y(t)$ and $\dot{y}(t)$.

The basic operations which define the integration formulas for each time step are the following:

— compute $\dot{y}(t_k+\theta h)$ by equation (2) where $y(t_k+\theta h)$ is obtained by its quadratic approximation;

— compute $y(t_k+h)$ and $\dot{y}(t_k+h)$ by their quadratic approximations.

In this way $y(t_k+h)$ and $\dot{y}(t_k+h)$ are defined in terms of $y(t_k)$ and $\dot{y}(t_k)$.

The initial conditions are given by $y(0)$ and $\dot{y}(0) = Ay(0)+b$.

This method is unconditionally stable provided that $\theta \geqslant 1$, and for $\theta = 1$ results in exactly the Crank-Nicolson formulation. The truncation error is $0(h^3)$ for each value of θ, like in the Crank-Nicolson scheme.

The procedure proposed allows to obtain an analytical expression for the approximation of the exponential matrix e^{At} which depends on the free parameter θ. This parameter can be used to make the scheme accurate with respect to rapidly varying solutions of "stiff systems". The optimal value of parameter θ has been computed according to "exponential fitting" procedure proposed by W. Liniger.

The efficiency of the numerical scheme described in the paper has been evaluated by means of numerical comparison with some widely used direct integration methods.

1. Introduction

The numerical methods for direct integration of linear differential equations provide approximate solutions which can be considered as sum of approximations of each modal contribution of the exact solution. Therefore the primary consideration in the choice of the numerical solution procedure is its ability to properly simulate all the modal contributions.

This requirement can be obviously verified using any conventional stable method with an integration time step h such that: $|h\lambda_n| \ll 1$ where λ_n is the time constant of maximum modulus relevant to the problem.

This condition results into a computing time waste if the problem considered, as frequently occurs in structural dynamic analysis, is characterized by widely spread time constants, and if the rapidly varying modal contributions are of no importance to the solution.

For this type of problems an efficient numerical scheme should have the following properties:

- the dominant modal contributions are approximated with a reasonable order of accuracy;
- the modal contributions relevant to large time constants are essentially neglected.

These requirements can be met using "exponential fitting procedures"[1, 2] .

The present paper describes a numerical scheme(*) for direct integration of linear systems of the form:

$$\dot{y}(t) = Ay(t) + b(t) \qquad (1)$$

which is based on the exponential fitting concept, and has a truncation error $O(h^3)$.

The numerical method proposed was used to solve the second order equations

$$M\ddot{\delta} + C\dot{\delta} + K\delta = F(t) \qquad (2)$$

which can be reduced to a system of first order equations of type (1).

The efficiency of the numerical scheme was evaluated by comparison with some widely used direct integration methods.

2. Numerical method

2.1. Integration scheme

The numerical procedure proposed for the solution of the matrix equation

$$\begin{aligned} \dot{y}(t) &= Ay(t) + b(t) \\ y(0) &= y_o \end{aligned} \qquad (3)$$

is based on the assumption that vector $\dot{y}(t)$ can be approximated by a linear interpolation between $t = t_k$ and $t = t_k + \theta h$ where h is the integration time step and θ is adimensionless parameter:

$$\begin{aligned} \dot{y}(\tau) &= \frac{\tau}{\theta h}\dot{y}_\theta + \left(1-\frac{\tau}{\theta h}\right)\dot{y}_k \\ \tau = t - t_k\ ;\ 0 \le \tau \le \theta h;\ \dot{y}_\theta &= \dot{y}(t_k + \theta h);\ \dot{y}_k = \dot{y}(t_k) \end{aligned} \qquad (4)$$

Vector y(t) is obtained by integration of equation (4):

$$y(\tau) = \frac{\tau^2}{2\theta h}\dot{y}_\theta + \tau\left(1-\frac{\tau}{2\theta h}\right)\dot{y}_k + y_k \qquad (5)$$

The basic operations defining the integration formulas for each time-step are the following:

- compute $\dot{y}_\theta$ and y_θ by equations (3) and (5) assuming that y_k and $\dot{y}_k$ are known:

$$\begin{aligned} \dot{y}_\theta &= Ay_\theta + b_\theta \\ y_\theta &= \frac{\theta h}{2}\dot{y}_\theta + \frac{\theta h}{2}\dot{y}_k + y_k \end{aligned} \qquad (6)$$

(*) This work was performed in the frame of ELFI programme, supported by ENEL, to set up a finite element computer code for structural dynamic linear problems.

- compute $\dot{y}_{k+1}$ and y_{k+1} by equations (4) and (5):

$$\dot{y}_{k+1} = \frac{1}{\theta}\,\dot{y}_\theta + \left(1-\frac{1}{\theta}\right)\dot{y}_k$$
$$y_{k+1} = \frac{h}{2\theta}\,\dot{y}_\theta + h\left(1-\frac{1}{2\theta}\right)\dot{y}_k + y_k \tag{7}$$

Solving for $\dot{y}_\theta$ equations (6) and substituting into equations (7), the following relationship is established:

$$w_{k+1} = E\, w_k + S_k \tag{8}$$

where:

$$w_k = \begin{bmatrix} y_k \\ \dot{y}_k \end{bmatrix}$$

$$S_k = \begin{bmatrix} \left\{\frac{h^2}{4} A \left[I - \frac{\theta h}{2} A\right]^{-1} + \frac{h}{2\theta} I\right\} b_\theta \\ \left\{\frac{h^2}{2} A \left[I - \frac{\theta h}{2} A\right]^{-1} + \frac{1}{\theta} I\right\} b_\theta \end{bmatrix}; \quad E_k = \begin{bmatrix} \left\{\frac{h}{2\theta} A \left[I - \frac{\theta h}{2} A\right]^{-1} + I\right\} & \left\{\frac{h^2}{4} A \left[I - \frac{\theta h}{2} A\right]^{-1} + h\left(1-\frac{1}{2\theta}\right)I\right\} \\ \frac{1}{\theta} A \left[I - \frac{\theta h}{2} A\right]^{-1} & \left\{\frac{h}{2} A \left[I - \frac{\theta h}{2} A\right]^{-1} + \left(1-\frac{1}{\theta}\right)I\right\} \end{bmatrix}$$

In this way y_{k+1} and $\dot{y}_{k+1}$ are defined in terms of y_k and $\dot{y}_k$, and the computation starts with

$$y_o = y(0)$$
$$\dot{y}_o = Ay_o + b(0)$$

This recurrent relation can be used to study the stability and accuracy of the integration scheme. It can be shown that the truncation error is $O(h^3)$, and the method is A-stable [3] provided that $\theta \geq 1$. In particular for $\theta = 1$ the numerical scheme results in exactly the Crank-Nicolson formulation (Newmark method with $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$) [4,5].

2.2. Approximation to the exponential matrix

It can be shown that, by solving equation (8) for y_{k+1} the numerical method previously described is reduced to the following two-step integration scheme:

$$\left[I - \frac{\theta h}{2} A\right] y_1 = \left[I + (2-\theta)\frac{h}{2} A - \frac{h^2}{2}(\theta-1) A^2\right] y_o + $$
$$+ \frac{h}{2}\left\{\left[\left(2-\frac{1}{\theta}\right) I - (\theta-1)\, hA\right] b(o) + \frac{1}{\theta}\, b(\theta h)\right\} \tag{9}$$

$$\left[I - \frac{\theta h}{2} A\right] y_{k+1} = \left[\left(\frac{2\theta-1}{\theta}\right) I - \frac{h}{2\theta}\left(2\theta^2 - 2\theta - 1\right) A\right] y_k - \left(\frac{\theta-1}{\theta}\right)\left[I - (\theta-1)\frac{h}{2} A\right] y_{k-1}$$
$$+ \frac{h^2}{4} A \left[b(t_{k-1} + \theta h) + b(t_k + \theta h)\right] \tag{10}$$

$$k \geq 1$$

If eqs. (9) e (10) are written on the basis of eigenvectors $\varnothing_i$ of matrix A:

$$A\,\varnothing_i = \lambda_i\,\varnothing_i \qquad (i = 1,n)$$

using $y_k = \sum_{i=1}^{n} \varnothing_i\, x_k^{(i)}$; $b_k = \sum_{i=1}^{n} \varnothing_i\, s_k^{(i)}$, the following relations can be obtained for amplitudes $x_k^{(i)}$:

$$x_1 = \gamma x_o + s_o$$
$$x_{k+1} = 2\alpha x_k - (\alpha^2 - \beta^2)\, x_{k-1} + s_k \tag{11}$$

where:

$$\alpha = \frac{2\theta - 1 + q(2\theta^2 - 2\theta - 1)}{2\theta(1+\theta q)}$$

$$\beta = \frac{\sqrt{-q^2(4\theta^2 - 4\theta - 1) - 2q(2\theta - 1) + 1}}{2\theta(1+\theta q)}$$

$$\gamma = \frac{1-(2-\theta)q-2q^2(\theta-1)}{1+\theta q}$$

$$q = -\frac{\lambda h}{2} \qquad (\mathrm{Re}\ q > 0)$$

and s_o, s_k can be easily computed. The solution of equation (11) is

$$x_k = H_k(q,\theta)\, x_o + G_k(q,\theta)$$

where:

$$H_k(q,\theta) = \frac{1}{2\beta}\left\{ \gamma\left[(\alpha+\beta)^k - (\alpha-\beta)^k\right] - (\alpha^2-\beta^2)\left[(\alpha+\beta)^{k-1}-(\alpha-\beta)^{k-1}\right]\right\}$$

and $G_k(q,\theta)$ can be computed in terms of s_j ($j = 0,k$).

The modal amplitudes of the exact solution are:

$$\bar{x}_k = x_o\, e^{-2qk} + \int_o^{hk} e^{-\frac{2q}{h}(hk-\tau)} s(\tau)d\tau$$

so that $H_k(q,\theta)$ is an approximation of e^{-2qk}.

Consider the error

$$\epsilon_k(q,\theta) = H_k(q,\theta) - e^{-2qk}$$

The order of accuracy of the numerical scheme guarantees that $|\epsilon_k(q,\theta)| \ll 1$ close to $q=0$ and A-stability guarantees that $|\epsilon_k(q,\theta)|$ is bounded if $\mathrm{Re} q > 0$ and $\theta \geq 1$. Generally this does not imply that $|\epsilon_k(q,\theta)| \ll 1$ for large values of the real part of q which are always present in equations characterized by widely spread time constants (stiff equations) [6] .

For this reason it may be convenient to use the parameter θ to perform a global exponential fitting for any value of the real part of q.

An optimal value of the parameter θ can be obtained by minimizing with respect to θ ($\theta \geq 1$) the maximum absolute error at the first step:

$$E(\theta) = \max_{o \leq q \leq +\infty} |\epsilon_1(q,\theta)|$$

Being $\lim_{q\to\infty} \gamma = -\infty$, the maximum value of $E(\theta)$ is obtained for $q = \infty$, and the optimal value of θ is $\theta_{opt} = 1$, so that the optimal scheme should be the Crank-Nicolson method. This choice is unsatisfactory because in this case $E(1) = 1$.

It seems therefore reasonable to start recurrent procedure using a method for which γ is bounded.

The natural choice is to perform the initial step with the Crank-Nicolson method (C.N. method) which has the same truncation error as the two-step procedure. In this case it is:

$$\gamma = \frac{1-q}{1+q}$$

and eq. (9) is replaced by:

$$\left[I - \frac{h}{2}A\right] y_1 = \left[I + \frac{h}{2}A\right] y_o + \frac{h}{2}\left[b(o) + b(h)\right] \qquad (13)$$

As a criterion for goodness of the approximation, one may choose to minimize the absolute error at the second step:

$$E(\theta) = \max_{o \leq q \leq +\infty} |\epsilon_2(q,\theta)| = \max_{o \leq q \leq +\infty} \left| e^{-4q} + \frac{q^2(3\theta-4)-q(\theta-3)-1}{(1+\theta q)(1+q)} \right|$$

In this case one obtains $\theta_{opt} = 1.2654$.

To evaluate the effectiveness of the exponential fitting numerical scheme here described (E.F. method) the error of approximate exponential has been compared with the error obtained by the C.N. method (figs. 1, 2, 3).

In particular figs. 1 and 2 show the behaviour of the error vs $|h\lambda|$ for real values of λ and fig. 3 shows the modulus of the approximate exponential for pure imaginary values of λ.

Figs. 1 and 2 indicate that C.N. scheme is more efficient than E.F. method for $|h\lambda| \lesssim 1$, even if for $|h\lambda| \ll 1$ the two procedures appear to have comparable accuracy. For $|h\lambda| \gg 1$ the E.F. scheme works better than C.N. method, and gives a satisfactory simulation of the exponential decay. In the case of pure imaginary values of λ, the E.F. method filters the amplitudes relevant to the high frequencies, as shown in fig. 3. This damping effect, which is not found in the C.N. scheme may be in some cases advantageous. Indeed this filtering property allows to use relatively large integration time steps to compute solutions in which the contribution of the high-frequency components are of no importance.

3. Numerical results

To demonstrate the efficiency of the E.F. method here proposed, two simple one-degree-of-freedom problems were solved, and the results compared with those obtained by θ-Wilson ($\theta = 1.4$) [7] and C.N. methods. The examples considered refer to the solution of the following equations:

$$2,5\cdot 10^4 . y + 1.025\cdot 10^3 \dot{y} + \ddot{y} = 2.5\cdot 10^4$$

$$y(o) = 10^{-3} \;;\; \dot{y}(o) = 24 \qquad (13)$$

$$\text{(exact solution: } y(t) = 1 - e^{-25t} + 10^{-3} e^{-1000t})$$

and:

$$1.8\cdot 10^4 y + 0.04\,\dot{y} + 0.2\ddot{y} = 1.8\cdot 10^4$$

$$y(o) = 1 \;;\; \dot{y}(o) = 0.3 \qquad (14)$$

$$\text{(exact solution: } y(t) = 1 + 10^{-3}\cdot e^{-0.1t} \sin 300t)$$

These problems were chosen to evaluate the performance of the methods considered both for stiff equations and for equations where high-frequency contributions which are of no importance to the solution are present.

Tables I and II quote the relative errors (%) relevant to time step sizes h = 0.25; 0.5 and 1.

The errors quoted in table I show that the E.F. scheme works better than the other methods, and gives acceptable results even with the largest integration time step (h = 1.).

The C.N. method is more accurate than the θ- Wilson method but gives unsatisfactory results for large integration time steps.

The θ - Wilson method produces errors larger than those obtained with C.N. and E.F. methods in spite of its smaller local truncation error. These results confirm that the local error may not be a meaningful parameter to evaluate global accuracy of numerical methods for stiff equations.

Table II indicates that the accuracy of C.N. and E.F. methods is comparable. This result suggests that the different approximations of the exponential relevant to the two methods do not affect the results because in this case the exponential part of the exact solution is multiplied by a small coefficient and therefore is of no importance.

It can be also noticed that the errors relevant to the C.N. and E.F. methods weakly depend on the time step size. On the contrary, the θ - Wilson method produces errors which considerably increase with time step size.

4. Conclusions

The paper presented the main properties of a numerical procedure for integration of stiff equations which is based on the exponential fitting concept. This method was used to solve two simple one-degree-of-freedom problems, and the results were compared with those obtained by the C.N. and θ- Wilson methods.

Even if a more extensive numerical experimentation on a meaningful class of differential equations is necessary to fully evaluate the performance of the method, nevertheless from the results available at this time the following conclusions can be drawn:

i) the particular strategy of exponential fitting used is quite adequate to solve stiff equations, and gives better results than the C.N. and θ- Wilson methods

ii) the method, although being designed for stiff equations, works well enough to solve problems characterized by high-frequency modal contributions which are of no importance to the overall solution.

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TABLE I

RELATIVE ERROR (%) IN THE SOLUTION OF EQ. (13) FOR DIFFERENT INTEGRATION TIME STEPS

	Crank-Nicolson Method			θ-Wilson Method			Exponential Fitting Method		
t	h=0.25	h=0.5	h=1	h=0.25	h=0.5	h=1	h=0.25	h=0.5	h=1
1	6.9	52.3	-85.	-73.6	906.	-6893.	1.1	-2.9	-85
2	0.4	27.4	72.4	5.2	-1.92	3978.	0.	-0.03	4.9
3	-0.047	14.3	-61.7	-0.3	-68.0	-2279.	0.	0.015	3.9
4	-0.074	7.4	52.5	0.013	-11.5	1109.	0.	0.	-0.74
5	-0.072	3.8	-44.7	0.	2.9	-464.	0.	0.	-0.12
6	-0.07	1.9	38.1	0.	1.3	150.	0.	0.	0.055
7	-0.06	1.0	-32.4	0.	0.02	-20.9	0.	0.	0.
8	-0.06	0.48	27.6	0.	-0.09	-20.5	0.	0.	0.
9	-0.056	0.21	-23.5	0.	-0.02	25.6	0.	0.	0.
10	-0.052	0.07	20.0	0.	0.	-19.3	0.	0.	0.

TABLE II

RELATIVE ERROR (%) IN THE SOLUTION OF EQ. (14) FOR DIFFERENT INTEGRATION TIME STEPS

	Crank-Nicolson Method			θ-Wilson Method			Exponential Fitting Method		
t	h=0.25	h=0.5	h=1	h=0.25	h=0.5	h=1	h=0.25	h=0.5	h=1
1	-0.35	-0.15	-0.09	1.74	5.53	-13.9	-0.26	-0.14	-0.09
2	-0.22	-0.05	-0.01	0.67	3.65	10.9	-0.17	-0.04	0.
3	-0.12	-0.02	0.07	0.31	2.30	-8.89	-0.1	0.03	0.06
4	-0.16	-0.05	-0.02	0.08	1.34	7.09	-0.18	-0.05	-0.02
5	-0.18	-0.10	-0.07	-0.03	0.75	-5.63	-0.23	-0.10	-0.07
6	-0.08	-0.03	0.	0.02	0.50	4.35	-0.16	-0.03	0.
7	-0.01	0.02	0.04	0.05	0.35	-3.34	-0.12	0.	0.04
8	-0.04	-0.03	-0.01	0.	0.17	2.63	-0.18	-0.05	-0.02
9	-0.05	-0.06	-0.05	-0.04	0.07	-2.09	-0.21	-0.08	-0.05
10	0.01	0.	0.	0.	0.07	1.61	-0.16	-0.03	0.

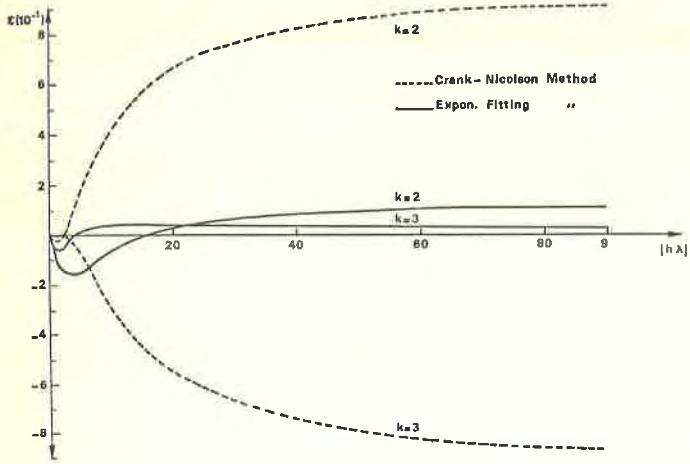

Fig. 1 - Error of the approximation of exp (λhk) for negative real values of λ .

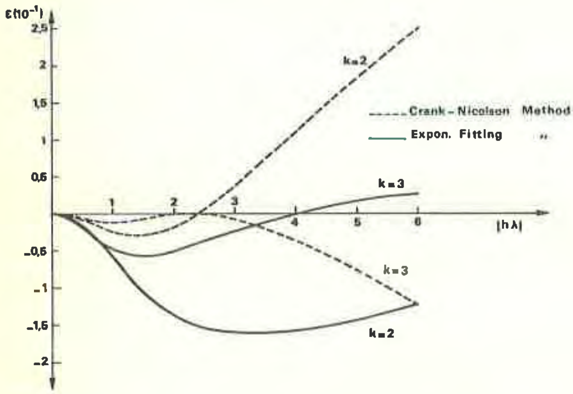

Fig. 2 - Error of the approximation of exp (λhk) for negative real values of λ .

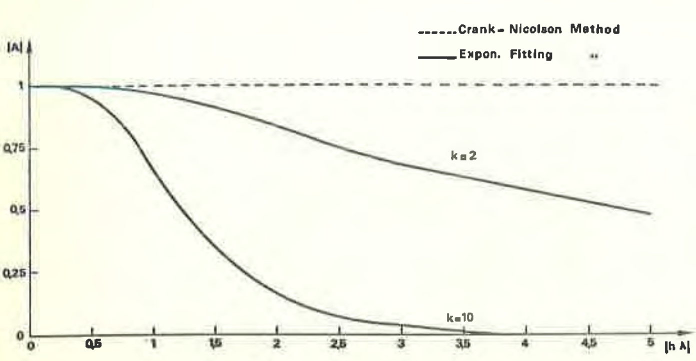

Fig. 3 - Amplitude of the approximation of exp (iλhk).