LIMIT-ANALYSIS OF THICK AXI-SYMMETRIC VESSELS

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SUMMARY

In nuclear plant technology multiple load conditions must be taken into account including catastrophic events. When such extreme load conditions are considered a design based on purely elastic analysis turns out to be too much conservative. When ductility of the material is assessed, recourse to limit analysis is recommended, provided that changes in geometry at collapse do not significantly affect the stress distribution. When material characteristics do not correspond to theoretical assumptions of perfectly plastic behaviour, limit analysis can still contribute, when compared with experimental results, to a better understanding of structure reliability.

A method for limit analysis at plane and axi-symmetric problems has been recently proposed by the same authors. It is based on a finite element modeling of the stress yield through adoption of one (plane problems) or two (axi-symmetric problems) stress functions. A piecewise continuous equilibrated stress distribution is thus obtained and recourse to the static theorem of limit analysis is allowed, provided that plasticity condition is enforced all over the structure. Quadrilateral finite elements are obtained through assembly of triangular subfields, following a procedure which has been proposed by Fraejis de Veubeke (1965) for the definition of the strain field in the discretization of bending plates. Shape functions of stresses turn out to be linear in each triangular subfield, depending on the choice of a second order polynomial for the stress function.

Boundary conditions (loads and constraints) can be implicitly satisfied by suitably integrating them on the basis of the assumed stress polynomials. No explicit equilibrium equations have to be satisfied and the entire stress field is obtained as a linear function of the independent nodal variables. Once the yield condition is imposed, a redundant method is thus obtained. The maximization tool can be either linear or nonlinear programming, depending on the term of the yield condition. In the present paper V. Mises' domain in plane strain has been linearized so as to adopt standard linear programming codes for the solution of test problems.

In a previous paper (ASME Winter Meeting 1976) simple plane stress problems were solved in order to check basic assumption of the formulation and to estimate the efficiency of the method. A general purpose code has now been implemented for the solution of real life structural problems. The present contribution is intended to show some significant results obtained in the solution of axi-symmetric problem and to discuss some aspects which are susceptible to improve the numerical efficiency of the solution phase.

In the future other problems will be investigated. The more important of these is the applicability of limit analysis and therefore of the present approach to concrete pressure vessels. Since concrete has little ductility in compression and practically no ductility in tension and its flow rule is anyhow far from normality, a discrepancy between computed and experimental results is expected. However this investigation may suggest useful indications for design criteria.

1. Definition of the stress field

Let define the axi-symmetric structure of fig.1 in a cylindrical space frame x, y, θ , subject to axi-symmetric dead surface loads and to axi-symmetric active surface loads. The target is to obtain a safe estimate for the collapse multiplier of the active loads, working out the best statically admissible solution of the problem.

The first step is to define the most general equilibrated stress field for the structure, within the range of a previously chosen polinomial representation. The second step is to impose the respect of the appropriate yield condition at a suitable mesh of points. If the equilibrated stress field is piecewise modeled with convex functions, the respect of the yield condition at the corners of each subfield is such as to ensure static admissibility all over the structure. This last remark is in fact the guideline for present proposal.

First consider a slice of the structure (figs.1,3) defined by a unit central angle so that thickness $t=x$ and define a plane stress distribution in the x y frame, with nonzero components σ'_{xx} , σ'_{xy} , σ'_{yy} . If ϕ (x,y) is a given stress function in the Airy's sense, an equilibrated stress field is defined by

$$\sigma'_{xx} = \frac{1}{t}\,\frac{\partial^2 \phi}{\partial y^2} \qquad \sigma'_{xy} = -\frac{1}{t}\,\frac{\partial^2 \phi}{\partial x\,\partial y} \qquad \sigma'_{yy} = \frac{1}{t}\,\frac{\partial^2 \phi}{\partial x^2} \qquad (1)$$

Moreover define a second stress function ψ such as to give rise to an equilibrated plane stress distribution in the x, θ plane, with nonzero components

$$\sigma''_{xx} = \frac{1}{x}\,\frac{\partial\psi}{\partial x} + \frac{1}{x^2}\,\frac{\partial^2 \psi}{\partial \theta^2}\,, \quad \sigma''_{x\theta} = \frac{1}{x^2}\,\frac{\partial\psi}{\partial\theta} - \frac{1}{x}\,\frac{\partial^2 \psi}{\partial x\,\partial\theta}\,, \quad \sigma''_{\theta\theta} = \frac{\partial^2\psi}{\partial x^2} \qquad (2)$$

Axial symmetry of the stress distribution implies that $\sigma_{x\theta}$ must vanish, and therefore $\psi = \psi\,(x,y)$.

Both stress functions do thus respect axial symmetry of the problem and their superposition gives rise to the most general stress distribution within the range of the assumed polynomial degree.

2. The first stress function

Let the considered slice be subdivided into quadrilateral finite elements, and let moreover each element be subdivided into four triangular subelements. In each subelement define a function ϕ_i (i=1,4) corresponding to a complete third order polynomial, thus obtaining a linear stress distribution in the element as a function of 10 X 4 = 40 independent parameters. In each subfield these can be the coefficients of the polynomial, or, better, the set of local values of ϕ at the corner points and of its first derivatives at the corner points plus the external midside, as represented in fig.4.

It can be noted that each stress subfield is defined by ten parameters, three of which are arbitrary due to the differential relationship between potential stress function and stress component distribution. In geometrical terms this means that each potential surface has three rigid body degrees of freedom in the space ϕ, x, y. This freedom can be reduced by arbitrarily imposing that at the external corner points of the element the local values

$[\phi_i(x,y)]_{(p)}$ $\left[\frac{\partial\,\phi_i\,(x,\,y)}{\partial\,x}\right]_{(p)}$ and $\left[\frac{\partial\,\phi_i\,(x,\,y)}{\partial\,y}\right]_{(p)}$ are the same for both subelements facing at the corner.

Now looking at the internal sides with local reference frame s, n, it can be noted that boundary stresses

$$\sigma'_n = \frac{1}{t}\frac{\partial}{\partial s}\left(\left[\frac{\partial \phi_i(s, n)}{\partial s}\right]_{n=0}\right) \qquad \sigma'_t = \frac{1}{t}\frac{\partial}{\partial s}\left(\left[\frac{\partial \phi_i(s, n)}{\partial n}\right]_{n=0}\right) \qquad (3)$$

respectively normal and tangential to the side, must be equal on the two subelements facing that side. Due to preceding assumptions, on the two subelements functions $\left[\frac{\partial\phi(s, n)}{\partial s}\right]_{n=0}$ and $\left[\frac{\partial \phi(s, n)}{\partial n}\right]_{n=0}$ have the same value at the external edge of the side, and due to required continuity on stresses they must display the same derivative along s . It follows that also their values at the central point are the same, along with the value of the stress function $\phi(s, n) \equiv \phi(x, y)$.

If this remark is applied with reference to the four internal boundaries, the conclusion follows that the four boundary corners of the quadrilateral element have unique nodal variables for pairs of facing subelements, and that the internal node variables (central point) linearly depend on the external node variables. In fact it happens that 24 pairs of element nodal variables over forty are coupled by linear relations thus reducing the set of independent variables to sixteen. This means that the entire stress state of the element, as far as first stress function ϕ is concerned, can be expressed by the 3 X 4 = 12 boundary corner variables and by the 4 boundary side variables, altogether 16 variables, among which lie 3 arbitrary degrees of freedom of the element piecewise cubic potential surface ϕ .

Now look at the boundary between pairs of quadrilateral elements and notice that with reference to side s (with local reference frame n, s) boundary stresses σ_n , σ_t are defined as in eq. (3), and that their distribution is uniquely defined by their values at both edges of the side.

It can be noted also that function $\left[\frac{\partial\phi(n, s)}{\partial s}\right]_{n=0}$ on s is defined by the values $[\phi(n, s)]_{(p)}$, $\left[\frac{\partial\phi(n, s)}{\partial s}\right]_{(p)}$ at both end points, while function $\left[\frac{\partial\phi(n, s)}{\partial n}\right]_{n=0}$ on s is defined by the values $\left[\frac{\partial \phi(n, s)}{\partial n}\right]_{(p)}$ at end and middle points. It follows that σ_n and σ_t uniquely depend on the seven values of the stress function and of its first derivatives already defined as nodal variable of that side. Stress continuity, i.e. interelement equilibrium, can thus be enforced with no restriction on the effective degrees of freedom of the model, by simply imposing that nodal variables relevant to common boundary sides of each pair of elements coincide. This means that three arbitrary degrees of freedom are suppressed any time two elements are linked in a simply connected assembly of the structure (multiple connections are statically redundant because of the continuity of the equilibrated stress field). The assembled structure has thus three arbitrary degrees of freedom altogether.

3. The second stress function

Let the second stress function of subfield i assume the form $\psi_i(x, y) = a_i xy + \frac{b_i}{2}x^2 + c_i x + d_i$ so that $\sigma_{xx} = \frac{1}{x}\frac{\partial \psi_i}{\partial x} = a_i\frac{y}{x} + b_i + \frac{c_i}{x}$, $\sigma_{\theta\theta} = \frac{\partial}{\partial x}\left(\frac{\partial\psi_i}{\partial x}\right) = b_i$. If the values of $\frac{\partial\psi_i(x, y)}{\partial x} = a_i xy + b_i x + c_i$ at boundary nodes are assumed as independent variables over the quadrilateral and if the value at the central node is obtained as an average of the boundary values, then functions $\sigma_{\theta\theta}(x, y)$, $\sigma_{xx}(x, y)$ can be represented (fig. 6) as a linear combination of these last.

It can be noted that only $\sigma_{\theta\theta}$ significantly depends on ψ, as its second derivative, σ_{xx} being contributed also by ϕ. This means that a suitable subset of nodal variables $\left[\frac{\partial\psi}{\partial x}\right]_{(i)}$ can be set to zero with no prejudice of the actual generality of the stress field formulation.

4. The superposed stress fields

Superposition of the two stress fields leads to

$$\sigma_{xx} = \sigma'_{xx} + \sigma''_{xx}, \quad \sigma_{xy} = \sigma'_{xy}, \quad \sigma_{yy} = \sigma'_{yy}, \quad \sigma_{\theta\theta} = \sigma''_{\theta\theta}$$

As shown above, the whole stress field can be expressed as a function of the nodal variables corresponding to corner points and to midside points of the mesh. With reference to element e and point p, corresponding components of the stress state can be expressed by

$$\{\sigma\}_{ep} = [W]_{ep} \{X\}_e \qquad (4)$$

where $\{\sigma\}_{ep} = \{\sigma_{xx}, \sigma_{xy}, \sigma_{yy}, \sigma_{\theta\theta}\}_{ep}$, $\{X\}_e = \{X_1, X_2, X_3, X_4\}_e$ and

$\{X\}_{ie} = \{[\phi]_{(i)}, \left[\frac{\partial\phi}{\partial x}\right]_{(i)}, \left[\frac{\partial\phi}{\partial y}\right]_{(i)}, \left[\frac{\partial\phi}{\partial n}\right]_{(i)}, \left[\frac{\partial\psi}{\partial x}\right]_{(i)}\}$. Vector $\{X\}_{ie}$ represents the five nodal variables of node-side i of element e, where the first three component coincide with function ϕ and its first x, y derivatives at node i, fourth component is the normal derivative at midside i, and fifth component is the first x derivative of ψ at node i. Matrix $[W]_{ep}$, which is 4 X 20, is not explicitly shown here for short, but it can be worked out on the basis of preceding statements.

If the whole set of nodal variables of the structure is ordered in vectors $\{X\}^I$, $\{X\}^B$ relevant respectively to inside nodes and boundary nodes, then eq. (4) can be written

$$\{\sigma\}_{ep} = [W]^I_{ep}\{X\}^I + [W]^B_{ep}\{X\}^B \qquad (5)$$

5. The boundary conditions

It has been shown that boundary stresses of each element side can be expressed as a function of nodal variables relevant to that side. If i is the first node of the side in a clockwise order, and k the second node, the following can be written

$$\{F\}_i = [G']_i \{X\}_i + [G'']_i \{X\}_k \qquad (6)$$

where $\{F\}_i = \{\sigma_{n(ii)}, \sigma_{t(ii)}, \sigma_{n(ik)}, \sigma_{t(ik)}, \left[\frac{\partial\psi}{\partial x}\right]_{(i)}\}$ and subscript (ik) relates the stress boundary component to point k of side i.

If all B sides which lie on the boundary of the structure are numbered sequentially (fig.3), then follows

$$\{\overline{F}\} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{B-1} \\ F_B \end{Bmatrix} = \begin{bmatrix} G_1^{'} & G_1^{''} & & & \\ & G_2^{'} & G_2^{''} & & \\ & & \ddots & & \\ & & & G_{B-1}^{'} & G_{B-1}^{''} \\ G_{B-1}^{''} & & & & G_B^{'} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{B-1} \\ X_B \end{Bmatrix} = [\overline{G}]^{-1} \{\overline{X}\}^B \tag{7}$$

where $\{X\}^B$ is the supervector which collects vectors $\{X\}_i$ pertaining to boundary sides, and $[\overline{G}]^{-1}$ is the equilibrium [5B X 5B] matrix.

It must be recalled from par.2 that piecewise function $\phi(x,y)$ is uniquely defined over the structure field through the values at nodal variables $\{X\}$. Due to rigid body degrees of freedom of the potential surface, three parameters of ϕ, i.e. three components of $\{X\}^B$ are redundant and can be arbitrarily assigned with no prejudice of the degrees of freedom of the stress field.

On the other hand the set of boundary stresses represented by vectors $\{\overline{F}\}_i$ must ensure overall equilibrium of the body. As the structure has three degrees of freedom in the x, y space, then three components of $\{\overline{F}\}$ cannot be arbitrarily assigned but depend on the others, through equilibrium. It follows that three rows of $[\overline{G}]^{-1}$ are linearly dependent on the other 5B-3 rows and can thus be suppressed. If three components of $\{\overline{X}\}_B$ are set to zero and relevant columns are also suppressed, then the following relation becomes true

$$\{X\}^B = [G]\ \{F\} \tag{8}$$

where the [(5B-3) X (5B-3)] matrix $[G]$ is the inverse of the reduced $[\overline{G}]^{-1}$.

The boundary of the structure can be partially constrained. This means that boundary stress components, grouped in subvector $\{F\}^F$, corresponding to suppressed kinematic degrees of freedom of the structure, can assume any value, i.e. are "free".

Remaining boundary stress components can be referred to distributed surface loads, the distribution of which is uniquely defined by edge values normal and tangential to the side and follows the same law as the corresponding internal stresses. Part of them will be "dead" loads $\{F\}^D$, and part will be "live" loads $\lambda\{F\}^L$, where (scalar) λ is the load multiplier to be maximized and $\{F\}^L$ represents a given distribution.

Taking into account eq. (8) and these last remarks, eq. (5) can now be written

$$\{\sigma\}_{ep} = [W]_{ep}^{I}\ \{X\}^I + [W]_{ep}^{B}\ [G]\{F\} = [W]_{ep}^{I}\ \{X\}^I + [W]_{ep}^{B}\ [G]^F\ \{F\}^F + \lambda\ [W]_{ep}^{B}\ [G]^L\ \{F\}^L + [W]_{ep}^{B}\ [G]^D\ \{F\}^D \tag{9}$$

6. The yield condition

Present applications were developed under the hypotesis of utilizing linear programming as optimization tool for solving the problem, even if this is not the only available choice. This implies that all governing relations are linear with respect to unknows. If the yield condition is nonlinear, it must be previously linearized, that means that the yield domain can be represented by an hyper-Polyhedron in the 4-space of the stress components.

Imposing the respect of the yield condition means checking if the stress vector $\{\sigma\}_{ep}$ at any point of any element is "inside" the domain. With reference to the linearized domain which contains the origin and f^{th} face of which is defined by the normal unit vector $\{n\}_f$ and by the distance k_f of the face from the origin, to be "inside" means that $\{n\}_f^T \{\sigma\}_{ep} \leq k_f$ for all of the F faces of the hyper polyhedron, i.e. $[n]\{\sigma\}_{ep} \leq \{k\}$ where $[n]^T = [n_1 \dots n_F]$, $\{k\}^T = \{k_1 \dots k_F\}$

If P points have to be checked for each of the E elements of the mesh, in order to ensure the respect of the yield condition all over the structure, then

$$[N]\{\sigma\} \leq \{k\} \tag{10}$$

where $[N] = \text{diag}\ ([n])$, $\{k\}^T = \{k^T \dots k^T\}$, $\{\sigma\}^T = \{\sigma_1^T \dots \sigma_E^T\}$ and $\{\sigma_e^T\} = \{\sigma_{e1}^T \dots \sigma_{eP}^T\}$

With reference to eqs. (4) and (9), if $[W]^T = [W_1^T \dots W_E^T]$, and $[W]_e^T = [W_{e1}^T \dots W_{eP}^T]$ then eq. (10) can be written

$$[N][W]^I\{X\}^I + [N][W]^B[G]^F\{F\}^F + \lambda[N][W]^L[G]^L\{F\}^L \leq \{k\} - [W]^B[G]^D\{F\}^D \tag{11}$$

or, adopting simple formal substitutions,

$$[T]\{X\}^I + [U]\{F\}^F + \lambda\{L\} \leq \{k\} - \{D\} \tag{12}$$

7. The linear program

Static theorem of limit analysis states that collapse multiplier cannot exceed any statically admissible multiplier, i.e. a load multiplier associated with an equilibrated stress field such as not to violate the yield condition at any point of the structure. A safe estimate of the collapse factor is then obtained when maximizing the static multiplier within the class of the assumed solutions.

The static multiplier being λ , then the goal is to maximize λ with respect to the free variables $\{X\}^I$, $\{F\}^F$, λ , under contraint that the yield condition is respected anywhere, i.e.

$$\max_{\{X\}^I, \{F\}^F, \lambda} \left(\lambda \,\middle|\, [T]\{X\}^I + [U]\{F\}^F + \lambda\{L\} \leq \{k\} - \{D\}\ ,\ \lambda \geq 0\right)$$

The above turns out to be a problem of linear programming which can be represented by the following tableau, where all of the free variables are splitted into two, $\{X\}^I = \{\overset{+}{X}\}^I - \{\bar{X}\}^I$, $\{F\}^F = \{\overset{+}{F}\}^F - \{\bar{F}\}^F$ in order to impose them to be non negative ($\{\overset{+}{X}\}^I \geq 0$, $\{\bar{X}\}^I \geq 0$, $\{\overset{+}{F}\}^F \geq 0$, $\{\bar{F}\}^F \geq 0$)

min / max		$\{\overset{+}{X}\}^I$	$\{\bar{X}\}^I$	$\{\overset{+}{F}\}^F$	$\{\bar{F}\}^F$	λ	$\leq$
$\geq$	$\{y\}$	$[T]$	$-[T]$	$[U]$	$-[U]$	$\{L\}$	$\{k\}-\{D\}$
		0	0	0	0	1	

or

min / max		$\{x\}$	$\leq$
$\geq$	$\{y\}$	$[M]$	$\{b\}$
		$\{a\}$	

(13) (14)

Right tableau is read

$$\max_{\{x\}} \left(\Omega_p = \{x\}^T\{a\} \;\middle|\; [M]\{x\} \leq \{b\} \;,\; \{x\} \geq \{0\} \right) \qquad (15)$$

$$\min_{\{y\}} \left(\Omega_D = \{y\}^T\{b\} \;\middle|\; [M]^T\{y\} \geq \{a\} \;,\; \{y\} \geq \{0\} \right) \qquad (16)$$

or even

maximize $\Omega_p = \{x\}^T\{a\}$ with respect to $\{x\}$ variables, provided that $\{x\}$ components are nonnegative and $[M]\{x\} \leq \{b\}$, and, respectively, minimize $\Omega_D \equiv \{y\}^T\{b\}$ with respect to $\{y\}$, provided that $\{y\} \geq \{0\}$ and $[M]^T\{y\} \geq \{a\}$.

Linear programming properties ensure that the two problems are dual to each other and that at respective solutions, $\Omega_p \equiv \Omega_D$. Moreover solution of the primal, eq. (15), provides both solution vectors $\{x\}$ and $\{y\}$, and vice-versa. These properties allow to choose, between the two formulations, the one which implies better conditions in terms of cost. Theoretical considerations and experience show that the cheaper formulation is that one where there are fewer restrictions than variables.

In present case the more favourable formulation is the dual eq. (16). Correspondingly tableau (13) leads to

$$\min_{\{y\}} \left(\Omega_p = (\{k\} - \{D\})\{y\}^T \;\middle|\; [T]^T\{y\} = 0 \;,\; [U]^T\{y\} = 0 \;,\; \{L\}^T\{y\} \geq 1,\; \{y\} \geq \{0\} \right) \qquad (17)$$

Example of fig.3 shows that adopting a check mesh of twelwe points per element, corresponding to the vertices of the four triangular subfields, and a linearized yield domain with 14 planes, eq. (17) leads to 6048 variables and 126 restrictions.

It can be computed that there are as many $\{y\}$ variables as yielding planes in the limit domain for each of the stress points: they are understood as "plastic multipliers" relevant to each of the plastic "modes" of the structure and are such as to define a plastic mechanism which turns out to be the collapse mechanism (within the class of possible mechanisms) at solution. The corresponding strain velocities do not define a strictly compatible field, even if some overall compatibility rules are respected, exactly in the same sense as in displacement models the stress field is not strictly equilibrated even if some overall equilibrium relations are satisfied.

Condition $\{L\}^T\{y\} \geq 1$ imposes a lower bound to the total dissipated power $\{L\}^T\{y\}$, otherwise no plastic mechanism is possible and $\lambda = 0$.

Conditions $[T]^T\{y\} = 0$ and $[U]^T\{y\} = 0$ can be interpreted as compatibility rules in the average sense as told above.

8. Applications

Present formulation is applicable to both plane stress and axi-symmetric problems. First of the two applications does not imply the presence of $\sigma_{\theta\theta}$ components and thus no $\psi(x, y)$ function is needed. Moreover thickness t of the reference slice of par.1 does not coincide with x but is given by $t = t\ (x, y)$.

This led to implement a general purpose user oriented computer program intended to solve both problems on option. This program generates first the linear programming numerical model of eq. (17) on the basis of input data and of multiple standard choices. The model is stored on mass storage area (or tape) in one of the optional standard input formats. In a second phase the model is read from the mass storage area, and the solution of the mathematical problem is obtained by making recourse to one of the available standard linear programming packeges. In present applications three different standard codes were tested, namely ILONA, Marie-Claire and FMPS. Both solution vectors (primal and dual) are then recorded on a physical support.

A third step consists of the interpretation of the output solution vectors, leading to a complete description of the reference stress distribution and of the rigid-plastic collapse mechanism.

Simple but significant tests were run both in plane stresses and in axi-symmetric situations, with reference to test problems the solution of which is largely known as well in analytical and in numerical aspects.

The reference plane stress problem is the square slab with central circular hole of fig. 8 , subject to a tensile uniform load $\bar{\sigma}$ acting on a pair of opposite sides. For this test a previously proposed linearization of the Mises' domain was adopted, shown in fig. 10 . A mesh of 8 elements, corresponding to 32 linear subfields, covers a quarter of the structure. A safe estimate of .68 was obtained for the collapse multiplier, together with a strain velocity distribution such as to define the mechanism.

For axi-symmetric problems the Mises' yield domain $f(\sigma_{ik}) = 0$ was linearized as follows in the 4-space σ_{xx}, σ_{xy}, σ_{yy}, $\sigma_{\theta\theta}$. Corresponding to three values of σ_{xy} , three nonlinear subdomains were defined, namely $f_o(\sigma_{xx}, 0, \sigma_{yy}, \sigma_{\theta\theta})$, $f_+(\sigma_{xx}, \bar{\sigma}/\sqrt{3}, \sigma_{yy}, \sigma_{\theta\theta})$ and $f_-(\sigma_{xx}, \pm\bar{\sigma}/\sqrt{3}, \sigma_{yy}, \sigma_{\theta\theta})$. In the principal 3-space σ_{xx}, σ_{yy}, $\sigma_{\theta\theta}$ the three subdomains correspond to co-axial circular cylinders, the axis of which forms equal angles with the reference frame. Each subdomain was linearized in an exagonal cylinder as shown in fig. 9 Finally six hyper-planes of the required hyperpolyhedron were obtained by coupling planes of surfaces f_o and f_+ (each hyperplane contains a pair of corresponding parallel faces of the coupled surfaces) and six by coupling f_o and f_- . Two more hyperfaces were imposed parallel to principal axes, i.e. normal to σ_{xy} , at distance $\pm\bar{\sigma}/\sqrt{3}$ from the origin, thus obtaining a 14-face hyper polyhedron.

The procedure, although being cumbersome in representing, was simply performed, due to strict analogy with linearization of fig. 10 .

As application test for axi-symmetric situations, a circular thick plate, uniformly loaded and simply supported at the boundary, was considered. A double layer of elements was adopted, so as to allow a cross-rectangular distribution of stresses along the thickness, while different mesh densities were adopted along the radius. Solution obtained with 4 elements gave a collapse multiplier .016 , which is satisfactorily close to theoretical results quoted in the literature.

Numerical efficiency is not, so far, comparable to linear elastic solutions. Strong improvement is, however expected in the future, from oriented codes of linear programming, due to increasing interest in limit analysis.

References

|1| FRAEJIS de VEUBEKE, B., "Bending and Stretching of Plates-Special Models for Upper and Lower Bounds", Proceedings of the First Conference on Matrix Methods in Structural Mechanics, Wright Patterson Air Force Base, Ohio, Oct.1965, pp.863-886.

|2| ZAVELANI-ROSSI, A., PEANO, A., BINDA, L., "Lower Bounds to Collapse Pressure of Axi-Symmetric Vessels", The Winter Annual Meeting of the American Society of Mechanical Engineers, New York, N.Y., December 5-10, 1976.

|3| ANDERHEGGEN, E., KNOPFEL, H., "Finite Element Limit Analysis Using Linear Programming", Int. Journal of Solids and Structures, Vol.8, pp.1413-1431, 1972.

|4| ZAVELANI-ROSSI, A., "A New Linear Programming Approach to Limit Analysis", Proceedings of the Int. Conference on Variational Methods in Engineering, Southampton University Press, Vol.2, 1973.

|5| ZAVELANI-ROSSI, A., "A Compact Linear Programming Procedure For Optimal Design in Plane Stress", Journal of Structural Mechanics, Vol.2, No.4, pp.301-324, 1973.

|6| ZAVELANI-ROSSI, A., "Finite Element Techniques in Plane Limit Problems", Journal of the Italian Association of Theoretical and Applied Mechanics, Vol.9, No.4, pp.312-324, Dec.1974.

|7| LYSMER, J., "Limit Analysis of Plane Problems in Soil Mechanics", Journal of the Soil Mechanics and Foundations Divisions, ProceedingsASCE, Vol.96, No.SM4, pp.1311-1334, July 1970.

|8| PEANO, A., "Limit Analysis via Stress Functions", Limit Analysis Using Finite Elements, ASME Winter Annual Meeting, N.Y., 1976.

|9| FRAEIJS de VEUBEKE, B., "Stress Function Approach", Proceedings of the World Congress on Finite Element Methods in Structural Mechanics, Bournemouth 1975, pp.J1-J48.

|10| FRAEIJS de VEUBEKE, B., "Discretization of Rotational Equilibrium in the Finite Element Method", Meeting on Mathematical Aspects of Finite Element Methods, Roma, 1975.

|11| ANDERHEGGEN, E., "Finite Element Analysis Assuming Rigid-Ideal-Plastic Material Behaviour", ASME Winter Annual Meeting, N.Y., 1976.

|12| BIRON, A., "On Results and Limitations of Lower Bound Limit Analysis through Nonlinear Programming", ASME Winter Annual Meeting, N.Y., 1976.

|13| HUTULA, D.N., "Finite Element Limit Analysis of Two-Dimensional Plane Structures", ASME Winter Annual Meetingn, N.Y., 1976.

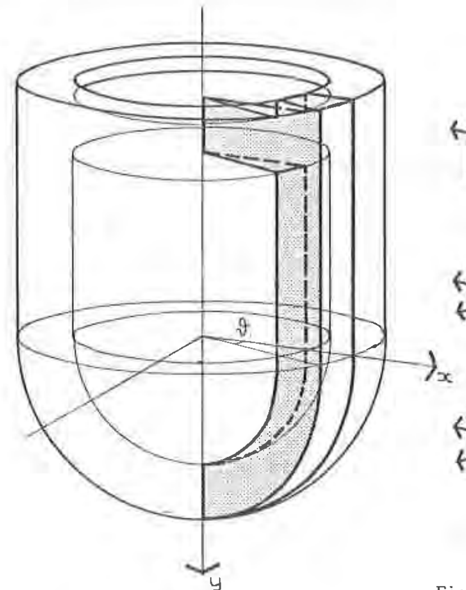

Fig.1

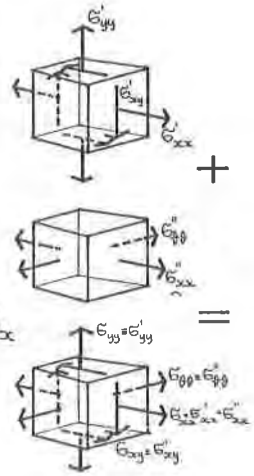

Fig.2

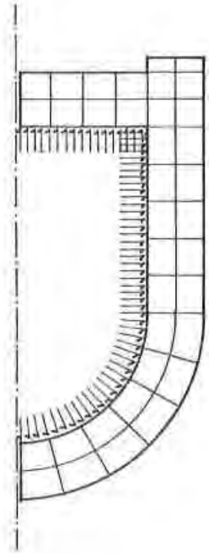

Fig.3

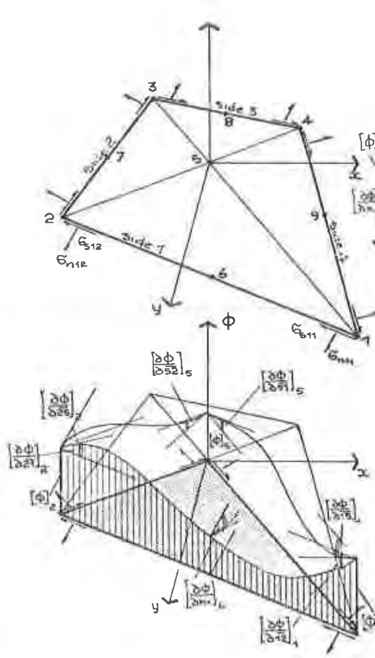

Fig.4

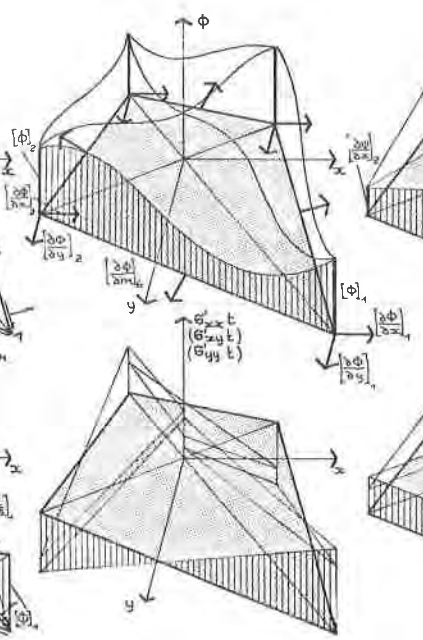

Fig.5

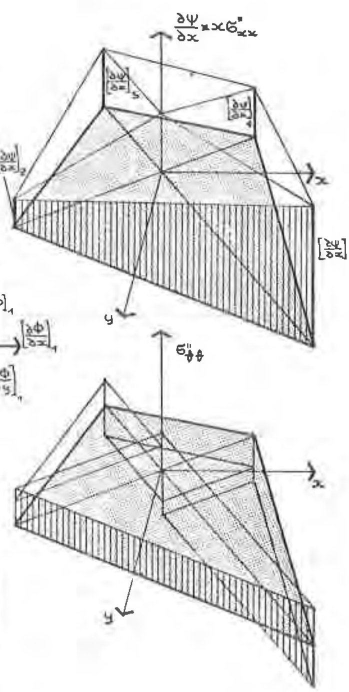

Fig.6

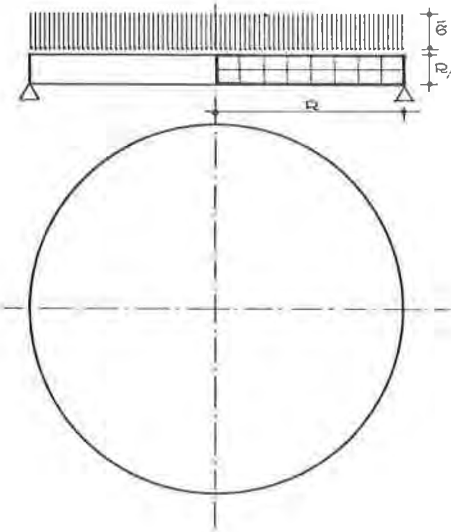

Fig.7

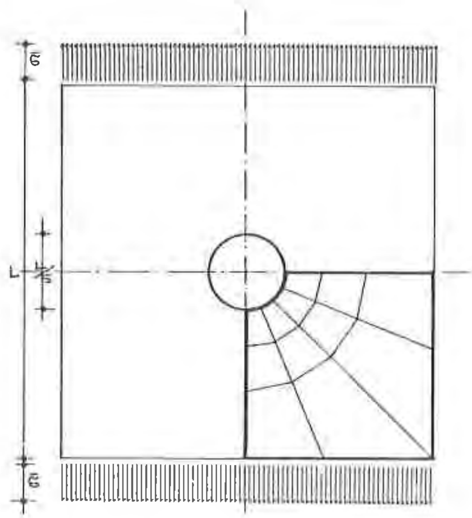

Fig.8

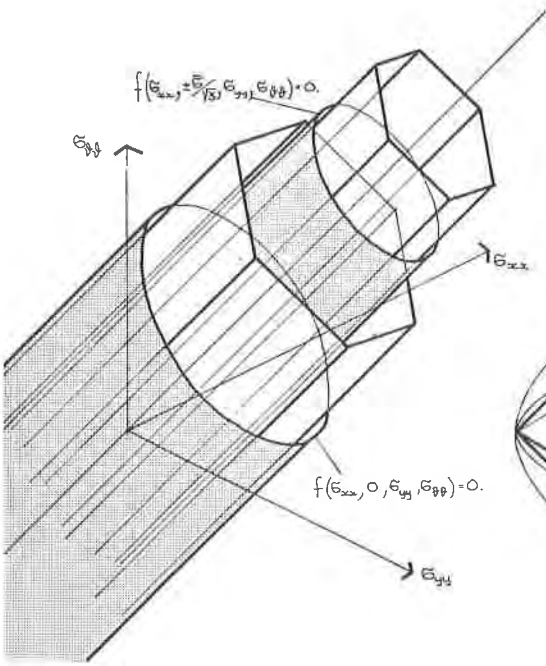

Fig.9

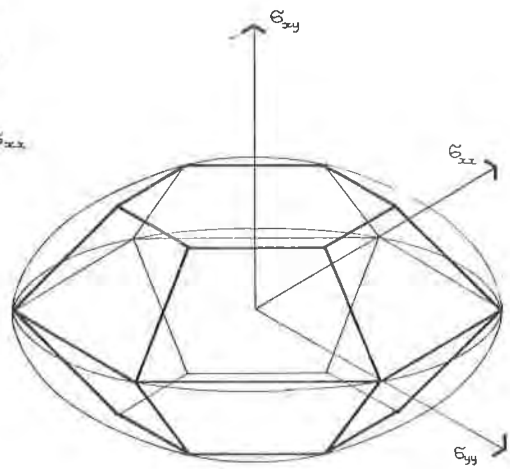

Fig.10