DESIGN STRESSES IN PROBABILISTIC FORM FOR ELLIPSOIDAL AND TOROIDAL PRESSURE VESSELS

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Design has customarily been based on applied loading, geometry, and handbook values for strength to give a deterministic solution. The engineering profession, however, has become increasingly concerned with the adequacy of design calculations. This concern indicates a need for critical evaluation of designs based on arbitrary multipliers, such as factors of safety or worst-case treatment.

Ellipsoids are frequently used for end closure of cylindrical pressure shells. Toroids of elliptic or circular cross-section, are widely used, e.g., for connecting two parallel legs in a U-shape. This paper gives equations for means and standard deviations of stresses developed in ellipsoids and toroids with internal pressure. Inherent are: (1) design variables are generally characterized by spectra of values (assumed to be normally distributed), rather than by unique values, and (2) a small, but finite, probability of failure must be recognized in any design. By coupling stresses due to applied loading as calculated by the given equations with strength available in a material, reliability (or the alternative probability of failure) can be calculated. Conversely, for a given reliability the appropriate size can be determined.

Appropriate illustrations of application of these equations are provided. The difficulty of relying on a factor of safety is demonstrated. As an example, consider a torus of circular cross-section with internal pressure. In one configuration with a given set of parameters using mean values and deterministic equations, the factor of safety is 1.70. The probabilistic equations indicate 2 probable failures in 100,000 vessels. If the standard deviation of the pressure is doubled, there are 12 probable failures in 100,000 vessels, although the factor of safety is still 1.70. In other words, if the range, but not the mean, of any parameter is changed, the failure rate changes, a change not predicted by the factor of safety.

1. Introduction

Design has customarily been based on applied loading, geometry, and handbook property values. None of these account for the spectra of values associated with design variables. The statistical nature of design parameters is usually ignored in conventional practice as demonstrated by efforts to find unique values representative of design parameters. Minimum guaranteed values, limit loads, and ultimate loads are examples of unique value representation.

Many believe that use of a safety factor greater than some preconceived magnitude (e.g., 2.5) will result in no failure. Actually, failure probability may vary from a satisfactory low to an intolerable high, since distributions exist in both capacity and demand requirements. The designer must be concerned with these as the variability may give different probabilities of failure for any one given safety factor.

2. Probabilistic Relationship

Assuming variables in a given problem are normally distributed, the algebra of normal functions [1]* applies. Reliability is defined as the probability that the available capacity C exceeds the demand D, i.e., $(C-D) > 0$. The difference between C and D can be related to the standard normalized variable z by

$$z = \frac{C - D}{\left(\sigma_C^2 + \sigma_D^2\right)^{\frac{1}{2}}} \tag{1}$$

where C is the mean capacity, D is the mean demand, σ_C is the standard deviation for capacity, and σ_D is the standard deviation for demand.

When demand exceeds capacity, i.e., when $D/C > 1$, distress (failure as defined for the particular situation) occurs. Eq. (1) is called a 'coupling equation' since it probabilistically couples capacity and demand functions through z, the coupling coefficient. Reliability, i.e., the probability of survival, is expressed as

$$R = \int_{-z}^{\infty} e^{-\left(\frac{Z}{2}\right)^2} dz = \int_{-\infty}^{z} e^{-\left(\frac{Z}{2}\right)^2} dz \tag{2}$$

Once z has been determined, reliability is found directly from standard tables of the normal function. Failure probability is

$$Q = 1 - R \tag{3}$$

3. Ellipsoidal Head

Ellipsoidal heads are frequently used for end closure of cylindrical shells used as pressure and storage vessels. Such heads are usually half-ellipsoids formed by revolution about the minor axis (Fig. 1) so that the major axis is equal to the diameter of the cylinder. Deterministic equations for the maximum stresses developed in such heads are given in various places such as Freudenthal [2], Harvey [3] and Roark [4]. Recognizing variations in parameters and their natural tolerances (taken as ± 3 standard deviations), assuming normal distribution of parameters, and using the nomenclature given in Table 1,

maximum stresses in an ellipsoidal head can be written in probabilistic form:
Meridional Stress

$$\mu_{\sigma_1} = \frac{pa}{2h} \tag{4}$$

Standard Deviation: Meridional Stress

$$\sigma_{\sigma_1} = \frac{1}{2h}\left[\frac{p^2a^2\sigma_h{}^2 + h^2\left((p^2 + \sigma_p{}^2)\sigma_a{}^2 + a^2\sigma_p{}^2\right)}{h^2 + \sigma_h{}^2}\right]^{\frac{1}{2}} \tag{5}$$

Hoop Stress

$$\mu_{\sigma_2} = 2\mu_{\sigma_1}\left[1 - \frac{a^2 + \mu_a{}^2}{b^2 + \sigma_b{}^2}\right] \tag{6}$$

Standard Deviation: Hoop Stress

$$\sigma_{\sigma_2} = 2\left[\frac{\mu_{\sigma_1}{}^2 + \sigma_{\sigma_1}{}^2}{4}\left(\frac{1}{b^2+\sigma_b{}^2}\right)^2\left(\frac{(a^2+\sigma_a{}^2)^2\,(4b^2\sigma_b{}^2+2\sigma_b{}^4) + (b^2+\sigma_b{}^2)^2\,(4_a{}^2\sigma_a{}^2+2\sigma_a{}^4)}{(b^2 + \sigma_b{}^2)^2 + (4b^2\sigma_b{}^2 + 2\sigma_b{}^4)}\right)\right.$$
$$\left. + \left(1 - \frac{a^2 + \sigma_a{}^2}{b^2 + \sigma_b{}^2}\right)^2 \sigma_{\sigma_1}{}^2\right]^{\frac{1}{2}} \tag{7}$$

As an example, consider a series of ellipsoidal heads with dimensions given in Table 2. Application of eqs. (4) through (7) gives mean stresses and standard deviations for various ratios of the axes as shown in Table 2. Distribution of stresses and range within which 80% of stresses fall are shown in Fig. 2.

It is noted that the meridional stress (σ_1) is always tensile but the hoop stress (σ_2) becomes compressive when a/b is greater than 1.42 (deterministic value). For the set of parameters in Table 2, however, the value of a/b ranges between 1.30 and 1.65 for 80% of the cases. Compressive hoop stresses can lead to buckling in the ellipsoidal head. For an axis ratio (a/b) of 1.5, for example, the mean value of the hoop stress is a compression of 1.88 times the internal pressure. The range within which 80% of the hoop stresses will fall, however, varies from 1.18p (tension) to 4.93p (compression). One could have a mean compressive hoop stress which would not be large enough to make the ellipsoidal head buckle but the range of compressive hoop stress could be large enough to cause buckling. The likelihood of this can be predicted from probabilistic but not deterministic calculations.

4. Elliptic Toroid

Pressure vessels in the form of toroidal shells can be generated by rotating an elliptic cross section about an axis in the plane of the ellipse and parallel to one of the axes. Fig. 3 shows a section of an elliptic toroid formed by generation about an axis parallel to the major axis. Deterministic equations for maximum stresses developed in elliptic toroids are given in various places such as Freudenthal [2]. Assuming normal distribution of parameters and using the nomenclature given in Table 1, maximum stresses in an elliptic toroid can be written in probabilistic form:

Meridional Stress

$$\mu_{\sigma_1} = Y\left[\tfrac{1}{2}(4\mu_\chi^{\,2} - 2\sigma_\chi^{\,2})^{1/2}\right]^{1/2} \tag{8}$$

Standard Deviation: Meridional Stress

$$\sigma_{\sigma_1} = \left[(\mu_y^{\,2} + \sigma_y^{\,2})\left(\mu_\chi - \tfrac{1}{2}(4\mu_\chi^{\,2} - 2\sigma_x^{\,2})\right)^{1/2} + \left(\tfrac{1}{2}(4\mu_\chi^{\,2} - 2\sigma_\chi^{\,2})^{1/2}\right)\sigma_y^{\,2}\right]^{1/2} \tag{9}$$

where

$$Y = \frac{pb}{2h}$$

$$\sigma_y = \frac{1}{2h}\left[\frac{p^2b^2\sigma_h^{\,2} + h^2\left((p^2+\sigma_p^{\,2})\sigma_b^{\,2} + b^2\sigma_p^{\,2}\right)}{h^2 + \sigma_h^{\,2}}\right]^{1/2}$$

$$\mu_\chi = \frac{2}{1 + \dfrac{b^2 + \sigma_b^{\,2}}{a^2 + \sigma_a^{\,2}}}$$

$$\sigma_\chi = \left[\frac{2}{1 + \dfrac{b^2 + \sigma_b^{\,2}}{a^2 + \sigma_a^{\,2}}}\right]\left[\frac{\left(\dfrac{1}{a^2+\sigma_a^{\,2}}\right)^2\left(\dfrac{(b^2+\sigma_b^{\,2})^2(4_a^{\,2}\sigma_a^{\,2}+2\sigma_a^{\,4}) + (a^2+\sigma_a^{\,2})^2(4b^2\sigma_b^{\,2}+2\sigma_b^{\,4})}{(a^2 + \sigma_a^{\,2})^2 + (4a^2\sigma_a^{\,2} + 2\sigma_a^{\,4})}\right)}{\left(1 + \dfrac{b^2+\sigma_b^{\,2}}{a^2+\sigma_a^{\,2}}\right)^2 + \left(\dfrac{1}{a^2+\sigma_a^{\,2}}\right)\left(\dfrac{(b^2+\sigma_b^{\,2})^2(4a^2\sigma_a^{\,2}+2\sigma_a^{\,4})+(a^2+\sigma_a^{\,2})^2(4b^2\sigma_b^{\,2}+2\sigma_b^{\,4})}{(a^2 + \sigma_a^{\,2})^2 + (4a^2\sigma_a^{\,2}+2\sigma_a^{\,4})}\right)}\right]^{1/2}$$

Hoop Stress

$$\mu_{\sigma_2} = YZ \tag{10}$$

Standard Deviation: Hoop Stress

$$\sigma_{\sigma_2} = \left[(\mu_y^{\,2}+\sigma_y^{\,2})\sigma_z^{\,2}+z^2\sigma_y^{\,2}\right]^{1/2} \tag{11}$$

where

$$Z = \frac{2R_o - r}{R_o - r}$$

$$\sigma_z = \left(\frac{1}{R_o - r}\right)\left[\frac{(2R_o-r)^2(\sigma R_o^2+\sigma_r^2) + (R_o-r)^2(4\sigma_{R_o}^2 + \sigma_r^2)}{(R_o - r)^2 + (\sigma_{R_o}^2 + \sigma_r^2)}\right]^{1/2}$$

5. Circular Toroid

For the special case in which the axis ratio is 1, the ellipse becomes a circle. Such a toroid, or a part thereof, is a most useful shape and is widely used in vessel construction, e.g., connecting two parallel legs of a U-shaped vessel. Deterministic equations for the maximum developed in circular toroids are given in various places such as Freudenthal [2], Harvey [3], and Roark [4]. Assuming normal distribution of parameters and using the nomenclature given in Table 1, the maximum stresses in a circular toroid can be written in probabilistic form:

Meridional Stress

$$\mu_{\sigma_1} = \frac{pr}{2h} \tag{12}$$

Standard Deviation: Meridional Stress

$$\sigma_{\sigma_1} = \frac{1}{2h}\left[\frac{p^2r^2\sigma_h^2 + h^2\left((p^2+\sigma_p^2)\sigma_r^2 + r^2\sigma_p^2\right)}{h^2 + \sigma_h^2}\right]^{1/2} \tag{13}$$

Hoop Stress

$$\mu_{\sigma_2} = \mu_{\sigma_1}\left[\frac{2R_o - r}{R_o - r}\right] \tag{14}$$

Standard Deviation: Hoop Stress

$$\sigma_{\sigma_2} = \left[\left(\mu_{\sigma_1}^2 + \sigma_{\sigma_1}^2\right)\left(\frac{1}{R_o - r}\right)^2\left(\frac{(2R_o-r)^2(\sigma R_o^2+\sigma r^2)+(R_o-r)^2(4\sigma R_o^2+\sigma r^2)}{(R_o-r)^2 + (\sigma R_o^2+\sigma r^2)}\right) + \left(\frac{2R_o - r}{R_o - r}\right)^2 \sigma_{\sigma_1}^2\right]^{1/2} \tag{15}$$

As an example, consider a series of circular toroids with the dimensions given in Table 3. Application of eqs. (12) through (15) gives mean stresses and standard deviations for various ratios of the centroid radius to the circle radius (R_o/r) as shown in Table 3. Distribution of stresses and range within which 90% of stresses fall are shown in Fig. 4. For a radius ratio (R_o/r) of 3.0, the mean value of the ratio of hoop stress to internal pressure is 12.50, a value also given by deterministic calculation. The 80% range of hoop stress/pressure ratio runs from 10.39 to 14.61. The 90% range of hoop stress/pressure ratio runs from 9.80 to 15.20. These ranges can not be obtained from deterministic calculations.

6. Inadequacy of Safety Factor

Consider one configuration of the circular toroid given in Table 3, i.e., one with a radius ratio of 3.0. The safety factor and reliability (from eq. 2) are given in Table 4 for various pressures using two different materials. If the vessel is made from AISI 1035, using a pressure of 16 MPa with a standard deviation of 0.80 MPa, the safety factor is 1.70 and the reliability is 0.99998 (2 failures per 100,000). If the standard deviation of the pressure doubles, the reliability decreases to 0.99988 (12 failures per 100,000). This increase in anticipated failures by a factor of six is not predicted by the safety factor which remains at 1.70.

If the standard deviation of one (or more) of the parameters changes, with no change in the respective mean, there will be an accompanying change in the reliability (and probable failure rate). This change will not be predicted by the safety factor which does not change.

If a circular toroid of a specified radius ratio is to be made from a given steel to sustain a given pressure, there is a question of necessary wall thickness. Some answers to this question are given in Table 5. Three bases were taken for determining wall thickness: selection of a safety factor; worst case; and selection of a reliability. A safety factor of one, implying a reliability of 0.50 (1 failure for every 2 vessels) obviously gives the smallest value of wall thickness. Worst case treatment gives a single value of wall thickness with no indication of probable failure rate. The equivalent safety factor is 1.68. For selected levels of reliability (and probable failure rates), wall thickness, standard deviation and range (3 standard deviations), and equivalent safety factors are calculated. A reliability of 0.999 (1 failure per 1000 vessels) and worst case treatment give the same mean wall thickness, a chance coincidence. Going from a reliability of 0.9999 to a reliability of 0.99999 implies a decrease in probable failure rate by a factor of 10 while the equivalent safety factor changes from 1.90 to 2.11. Use of worst case treatment and "reasonable" safety factors gives wall thicknesses which do not differ greatly from values given by selected reliabilities. The degree of under- or over-design is not known, however, unless one approaches from a desired level of reliability. Over-design is not necessarily wrong but it is desirable to know the degree of over-design and thus be able to estimate the "price" to be paid. Under-design is obviously unacceptable.

7. Additional Comments

Obviously a number of factors have been neglected in the above calculations, such as stress concentration factors, load factors (dynamic, impact, etc.), temperature factors, forming or manufacturing stress factors, corrosion factors, and notch sensitivity factors. These were omitted so that the central idea of the probabilistic approach might be clearer. The probabilistic approach does not eliminate consideration of such factors. They must be included. Appropriate stress concentration factors must be used to determine effective magnitude of applied loading. Temperature factors should be applied to the strength of the material used. In like manner, all necessary modifications should be integrated into the design solution.

It is often argued that failure probabilities of the order of 1/10,000 or 1/1,000,000 can have no real meaning since they are far out of a range which can be verified by experiment and experience. While lack of possible verification is real, the argument ignores the basic point that the probabilistic approach is not always used to provide absolute values but is often used to provide a scale against which safety of different designs can be compared on a rational basis.

References

[1] Haugen, E.B., "Probabilistic Approaches to Design", John Wiley and Sons, New York (1968).

[2] Freudenthal, A.M., "Introduction to the Mechanics of Solids", New York, John Wiley and Sons (1966).

[3] Harvey, J.F., "Theory and Design of Modern Pressure Vessels", Second Edition, New York, Van Nostrand Reinhold (1974).

[4] Roark, R.J. and Young W.C., "Formulas for Stress and Strain", Fifth Edition, New York, McGraw-Hill Book Co., (1975).

TABLE 1

NOMENCLATURE

	mean	standard deviation
Ellipse		
semi-major axis	a	σ_a
semi-minor axis	b	σ_b
Circle		
radius	r	σ_r
Toroid		
radius of centroid	R_o	σ_{R_o}
Common Symbols		
wall thickness	h	σ_h
internal pressure	p	σ_p
meridional stress	σ_1	σ_{σ_1}
hoop stress	σ_2	σ_{σ_2}
yield stress	σ_y	σ_{σ_y}

Table 2

Ellipsoidal Head

Parameters for Illustration

parameter	mean	stan. dev.	a/b	stress/pressure ratio: meridional stress, σ_1 mean	meridional stress, σ_1 stan. dev.	hoop stress, σ_2 mean	hoop stress, σ_2 stan. dev.
h, cm	3.00	0.15					
b, cm	30.00	1.50					
a, cm	30.00	1.50	1	5.00	0.43	5.00	0.83
	36.00	1.80	1.2	6.00	0.52	3.36	1.25
	42.00	2.10	1.4	7.00	0.61	0.28	1.93
	45.00	2.25	1.5	7.50	0.65	-1.88	2.38
	60.00	3.00	2.0	10.00	0.87	-20.00	5.90
	75.00	3.75	2.5	12.50	1.09	-53.12	11.94
	90.00	4.50	3.0	15.00	1.30	-105.00	21.10
pressure, MPa	p	0.05p					

TABLE 3

Circular Toroid

Parameters for Illustration

parameter	mean	stan. dev.	Ro/r	stress/pressure ratio: meridional stress, σ_1 mean	meridional stress, σ_1 stan. dev.	hoop stress, σ_2 mean	hoop stress, σ_2 stan. dev.
h, cm	3.00	0.15					
r, cm	30.00	1.50					
Ro, cm	60.00	3.00	2	5.00	0.43	15.00	2.35
	90.00	4.50	3	5.00	0.43	12.50	1.65
	120.00	6.00	4	5.00	0.43	11.67	1.46
	180.00	9.00	6	5.00	0.43	11.00	1.31
	240.00	12.00	8	5.00	0.43	10.71	1.25
	300.00	15.00	10	5.00	0.43	10.56	1.22
pressure, MPa	p	0.05p					

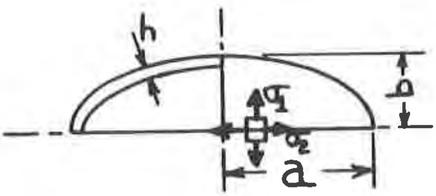

Fig. 1 Ellipsoidal Head for a Pressure Vessel Showing Location and Orientation of Maximum Stress.

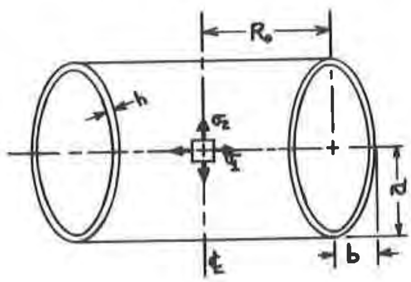

Fig. 3 Elliptic Toroidal Pressure Vessel Showing Location and Orientation of Maximum Stresses.

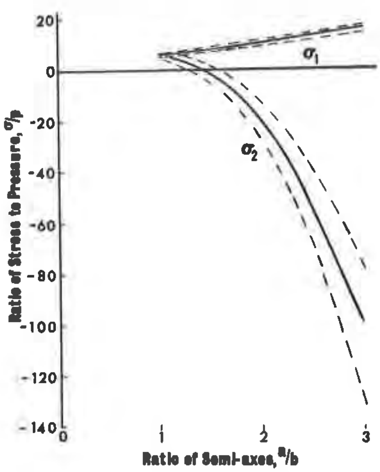

Fig. 2 Meridian (σ_1) and hoop (σ_2) stresses due to internal pressure in an ellipsoidal head as a function of the ratio of the ellipse axes showing mean and range within which 80% of stresses will occur.

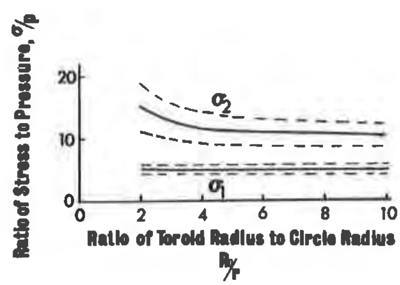

Fig. 4 Meridional (σ_1) and hoop (σ_2) stresses due to internal pressure as a function of the ratio of centroid radius to the circle radius showing mean and range within which 90% of stresses will occur.

TABLE 4

Circular Toroid

Safety Factor and Reliability for Various Conditions

pressure, p MPa		Material	yield strength MPa		safety factor	reliability	failures per 10^5
mean	stan. dev.		mean	stan. dev.			
16.0	0.80	AISI 1035	340.0	21.0	1.70	0.99998	2
16.0	0.80	Type 301	310.0	33.0	1.55	0.99536	464
16.0	1.60	AISI 1035	340.0	21.0	1.70	0.99988	12
16.0	1.60	Type 301	310.0	33.0	1.55	0.99230	770
16.0	0.80	AISI 1035	340.0	23.1	1.70	0.99997	3
16.0	0.80	Type 301	310.0	36.3	1.55	0.99287	713
16.0	0.80	AISI 1035	340.0	25.2	1.70	0.99994	6
16.0	0.80	Type 301	310.0	39.6	1.55	0.98960	1040
20.0	1.00	AISI 1035	340.0	21.0	1.36	0.9893	
20.0	1.00	Type 301	310.0	33.0	1.24	0.9006	
20.0	2.00	AISI 1035	340.0	21.0	1.36	0.9777	
20.0	2.00	Type 301	310.0	33.0	1.24	0.8778	

$R_o/r = 3$, Table 3

TABLE 5

Circular Toroid

Wall Thickness Required for Various Conditions

	mean	standard deviation	range
safety factor = 1.00 (reliability = 0.500)	1.935	---	---
safety factor = 1.75	3.387	---	---
safety factor = 2.00	3.871	---	---
worst case (safety factor = 1.68)	3.270	---	---
reliability = 0.999 (safety factor = 1.68)	3.270	0.164	0.490
reliability = 0.9999 (safety factor = 1.90)	3.673	0.184	0.551
reliability = 0.99999 (safety factor = 2.11)	4.082	0.204	0.612

R_o = 90.0 cm, r = 30.0 cm, p = 16.0 MPa, h, 5% standard deviations Type 301 steel