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An asymptotic approach to soil-structure dynamic interaction

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ABSTRACT

Asymptotic approach to a motion of a structure interacting with an elastic soil through a long strip foundation and distant point-supports is proposed. To derive the asymptotic integral equations of dynamic contact the explicit expression of the fundamental singular solution (the solution to the Lamb's problem) is used. Vertical motion is under consideration. Forced vibrations of a slender die and the transient behaviour of a system of circular dies are studied numerically.

1. INTRODUCTION

Dynamic interaction between a structure and a soil through a continuous foundation and a few point-supports is under consideration. Soil is simulated by a homogeneous isotropic and linearly-elastic half-space with free boundary. It is assumed that the characteristic period of the process is comparable with the period of the elastic wave propagation between the supports (or along the continuous foundation). The supports are distant, the continuous foundation is slender. The supports and the strip foundation are simulated by rigid frictionless dies (circular or slender).

Direct numerical analysis (FEM, BEM, FDM, etc.) of dynamic interaction between distant supports (or a strip foundation) and an elastic medium is connected with the principal difficulty caused by the degeneration of the contact area. To overcome this obstacle the proper asymptotic approach is proposed.

First the boundary integral equation of 3D elastodynamics is derived on the basis of the explicit expression [1] for fundamental singular solution (solution to the Lamb's problem). Then they are asymptotically simplified. Obtained equations of reduced dimension are solved numerically.

2. FUNDAMENTAL SINGULAR SOLUTION

The closed form of solution to Lamb's problem is represented in [1]. The vertical displacement at the points of a flat boundary ($z = 0$) caused by the surface point-force depending on time as Heaviside step function takes the form (for Poisson's ratio

$\nu < 0,263)$

$$G(t,r) = \frac{1-\nu}{2\pi\mu r}\left[\frac{1}{2}H\left(t-\frac{r}{c_1}\right)+\frac{1}{2}H\left(t-\frac{r}{c_2}\right)+q\left(\frac{c_2t}{r}\right)H\left(t-\frac{r}{c_1}\right)\times\right.$$

$$\left.\times\; H\left(\frac{r}{c_2}-t\right)-2q_3\left(\gamma^2-\frac{{c_2}^2t^2}{r^2}\right)^{-1/2}H\left(t-\frac{r}{c_2}\right)H\left(\frac{r}{c_R}-t\right)\right], \tag{1}$$

$$q(s) = -q_1\left(s^2-p_1\right)^{-1/2}+q_2\left(s^2-p_2\right)^{-1/2}-q_3\left(\gamma^2-s^2\right)^{-1/2},$$

where μ is a shear modulus, r is a distance, c_1 and c_2 are speeds of longitudinal and shear waves, c_R is a speed of Rayleigh's wave, and $\gamma = c_2/c_R$. For $\nu = 1/4$ we have

$$q_1 = \sqrt{3}/12, q_2 = \sqrt{3\sqrt{3}-5/12}, q_3 = \sqrt{3\sqrt{3}+5/12},$$

$$p_1 = 1/4, p_2 = (3-\sqrt{3})/4, \gamma = \sqrt{3+\sqrt{3}/2}, \alpha = c_2/c_1 = 1/\sqrt{3}\,.$$

The displacement due to harmonic point-force is $(sin\theta = 1/\gamma)$

$$G^{\omega}(r) = \frac{1-\nu}{2\pi\mu}\Big[\frac{1}{2r}\left(e^{i\omega r/c_1}+e^{i\omega r/c_2}\right)- \tag{2}$$

$$-\;\frac{i\omega}{c_2}\int_{\alpha}^{1} q(\tau)e^{i\omega\tau r/c_2}\,d\tau + 2q_3\frac{i\omega}{c_2}\int_{\theta}^{\pi/2} e^{i\omega\sin(\tau)r/c_R}\,d\tau\Big]\,.$$

3. MOTION OF A FEW CIRCULAR DIES.

Let the system of N circular frictionless dies be in contact with boundary $z = 0$ of the elastic half-space $z \leq 0$ (contact area $\Omega = \cup\Omega_i$, $i = 1,..N$, Ω_i – contact zone). Vertical motion of the dies is caused by the seismic impulse of the displacement of the boundary $z = 0$ $w_0 = w_0(t/T)$. The maximal die radius h is much smaller than the minimal distance l between the centres of the dies ($\varepsilon = h/l = o(1)$). The characteristic period T of the external action w_0 (and the process!) is assumed to be comparable with l/c_2.

On the basis of the fundamental solution (1) we write out the boundary integral equation of elastodynamics:

$$w - w_0 = \sum_{i=1}^{N}\int_0^t\iint_{\Omega_i}\frac{\partial}{\partial(t-\tau)}G(t-\tau,|\underline{\mathbf{x}}-\underline{\mathbf{x}}'|)\sigma(\frac{\tau}{T},\underline{\mathbf{x}}')\,d\tau d\Omega(\underline{\mathbf{x}}'). \tag{3}$$

where w is the vertical displacement, σ is the contact pressure.

In order to estimate the displacement w at the point $\underline{\mathbf{x}} \in \Omega_k$ we rewrite eq.(3):

$$w - w_0 = \int_0^t\iint_{\Omega_k}\frac{\partial}{\partial(t-\tau)}G(t-\tau,|\underline{\mathbf{x}}-\underline{\mathbf{x}}'|)\sigma(\frac{\tau}{T},\underline{\mathbf{x}}')\,d\tau d\Omega(\underline{\mathbf{x}}') \tag{4}$$

$$+\;\sum_{i\neq k}\int_0^t\iint_{\Omega_i}\frac{\partial}{\partial(t-\tau)}G(t-\tau,|\underline{\mathbf{x}}-\underline{\mathbf{x}}'|)\sigma(\frac{\tau}{T},\underline{\mathbf{x}}')\,d\tau d\Omega(\underline{\mathbf{x}}').$$

The purpose of the asymptotic analysis is to simplify this expression with the use of the assumptions mentioned above. First we estimate the second term in the right side of eq.(4):

$$w - w_0 \sim \iint\limits_{\Omega_k} (\frac{\partial}{\partial t} G(t, |\underline{x} - \underline{x}'|)) * \sigma(\frac{t}{T}, \underline{x}')\, d\Omega(\underline{x}') + $$
$$+ \sum_{i \neq k} G(t, |\underline{x}_k - \underline{x}_i|) * [\frac{\partial}{\partial t} F_i(\frac{t}{T})], \quad \varepsilon \to 0, \tag{5}$$

where $F_i(\frac{t}{T}) = \iint\limits_{\Omega_i} \sigma(\frac{t}{T}, \underline{x}')\, d\Omega(\underline{x}')$, asterisk means the convolution with respect to time, $\underline{x}_i$ – radius-vector of the centre of i-die.

The first term in the right side of (5) can be simplified on the basis of the assumption on the slow variation of contact pressure in time. The period of elastic wave propagation along the die is much smaller than the period T, and we construct the equation of contact – an asymptotic expansion:

$$w - w_0 \sim \iint\limits_{\Omega_k} G^3(|\underline{x} - \underline{x}'|) * \sigma(\frac{t}{T}, \underline{x}')\, d\Omega(\underline{x}') + $$
$$+ \sum_{i \neq k} G(t, |\underline{x}_k - \underline{x}_i|) * [\frac{\partial}{\partial t} F_i(\frac{t}{T})] + \frac{(1-\nu)A}{2\pi\mu c_2} [\frac{\partial}{\partial t} F_k(\frac{t}{T})], \quad \varepsilon \to 0, \tag{6}$$

$$A = A(\nu) = -\frac{1}{2} - \frac{1}{2}\frac{c_2}{c_1} + \int_{\alpha}^{1} q(s) ds - 2q_3 arccos \frac{1}{\gamma} < 0 ,$$

$G^3 = (1-\nu)/(2\pi\mu r)$ is Boussinesq's solution [2].

Then we consider eq.(6) as an equation for unknown σ and use the well known solution of the corresponding static contact problem of the single die. Thus we obtain the relation between w and σ. The second Newton's law completes the statement of the problem

$$M_k \frac{d^2}{d^2 t} w_k(t) = -F_k(\frac{t}{T}) \tag{7}$$

M_k means the mass of k-die, w_k is the k-die displacement.

Let introduce the following dimensionless arguments:

$$\tau = \frac{t}{T}, \; R_{ik} = \frac{|\underline{x}_i - \underline{x}_k|}{l},$$

the dimensionless parameters:

$$b = \frac{Tc_2}{l}, \; \delta_j = \frac{M_j}{M}, \; \theta_j = \frac{h_j}{h}, \; \beta = \frac{Mc_2^2}{l^2 h\mu}$$

and the dimensionless functions:

$$g(R_{ik}, \tau) = G(|\underline{x}_i - \underline{x}_k|, t) \frac{2\pi\mu |\underline{x}_i - \underline{x}_k|}{(1-\nu)}, \; v_k(\tau) = \frac{w_k(t/T)}{h}, \; v_0(\tau) = \frac{w_0(t/T)}{h}$$

here M is minimal mass of the die. After some supplemental asymptotical transformations we obtain the final equations for the dimensionless displacements $v_k(\tau),\ k = 1,..N$

$$\begin{aligned}
&\frac{\delta_k}{\theta_k^2}\beta\ddot{v}_k(\tau) + \varepsilon\frac{8}{\pi(1-\nu)}|A|\frac{\theta_k^2}{b}\dot{v}_k(\tau) + \frac{4}{1-\nu}v_k(\tau) = \\
&\frac{4}{1-\nu}v_0(\tau) + \varepsilon\frac{8}{\pi(1-\nu)}|A|\frac{\theta_k^2}{b}\dot{v}_0(\tau) - \\
&-\frac{2}{\pi}\varepsilon\sum_{i\neq k} R_{ik}\frac{\delta_i}{\theta_i}\beta\ g(R_{ik},\tau)\star\ \dddot{v}_i\ (\tau) + O(\varepsilon^2),\ \varepsilon\rightarrow 0
\end{aligned} \tag{8}$$

Here upperpoint means the derivative with respect to dimensionless time τ, the asterisk '$\star$' means the convolution with respect to dimensionless time τ.

Calculations are carried out for the system of two equal dies (N=2). Equations (8) is solved numerically for a range of values of the parameter β for Poisson's ratio $\nu = 1/4$ and for $\varepsilon = 0.1$. The figure 1 illustrates the dependency $v(\tau)$.

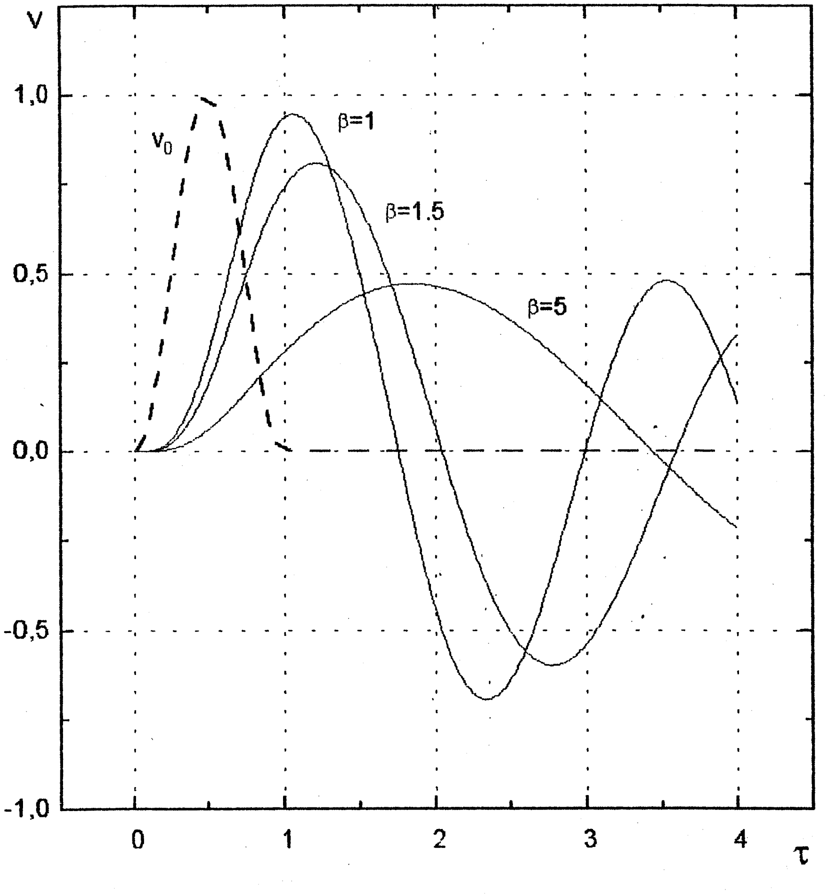

Fig.1 Displacement v versus dimensionless time τ for the system of two equal dies.

4. CONTACT EQUATION FOR THE LONG STRIP FOUNDATION.

Let the mid-line (Γ) of the slender contact area (Ω) be closed or not and be sufficiently smooth, the maximal width ($2h$) of the area Ω be much smaller than the large linear size ($2l$) and the radius of curvature of the mid-line. It is assumed that the displacement (w) and the contact pressure (σ) vary slowly over the stretch of time comparable to the period $2h/c_2$ of shear wave propagation across the contact area. It is taken into account that the contact pressure σ vary slightly along the countor Γ (over an interval of length h). The smoothness can be broken near the ends.

We generalize in dynamics the asymptotic solutions to the corresponding static problem of the slender die (Kalker [3], Sivashinsky [4], Nayak&Johnson [5]). The idea of the extension of the asymptotic approach to dynamics has been put forward by Lavrov&Slepyan [6].

Let a given normal pressure $\sigma(\underline{x})$ is applied to the thin curvilinear region Ω of the boundary $z = 0$ of the half-space $z \leq 0$. First we represent the load σ in the form

$$\begin{aligned} &\sigma(\underline{x}) = Z(\underline{y})\delta(\underline{x}-\underline{y}) + \left[\sigma(\underline{x}) - Z(\underline{y})\delta(\underline{x}-\underline{y})\right], \\ &Z(\underline{y}) = \int_{-h}^{h} \sigma(\underline{x})\, d\zeta, \quad \underline{x} = \underline{y} + \underline{n}\zeta, \quad \underline{x} \in \Omega, \quad \underline{y} \in \Gamma, \end{aligned} \tag{9}$$

where Z is the net force over the transversal segment $\mid \zeta \mid \leq h(\underline{y})$, $\underline{n}(\underline{y})$ is a normal to the contour Γ, and $\delta(\underline{x})$ is Dirac's delta-function. The vertical displacement w at the points of the area Ω can be evaluated

$$\begin{aligned} w(\underline{x}) &= G^w(\underline{x}) *\!\circ \sigma(\underline{x}) = \\ &= G^w(\underline{x}) * Z(\underline{y}) + G^w(\underline{x}) *\!\circ \left[\sigma(\underline{x}) - Z(\underline{y})\delta(\underline{x}-\underline{y})\right] \sim \\ &\sim G^w(\underline{x}) * Z(\underline{y}) + G^3(\underline{x}) *\!\circ \left[\sigma(\underline{x}) - Z(\underline{y})\delta(\underline{x}-\underline{y})\right], \ \varepsilon \to 0 \end{aligned} \tag{10}$$

Here the asterisk and the circle mean the convolution over the contour Γ and over the transversal (local) segment. The subsequent asymptotic analysis (the Lebesgue theorem on the passage to the limit under the intergal sign and the boundedness of the difference $G^w(\underline{x}) - G^3(\underline{x})$ are used) results in the following estimate of the displacement at the points far-away from the ends $\underline{y} = \underline{y}_{\pm}$

$$\begin{aligned} w(\underline{x}) &\sim \left[G^w(\underline{y}) - G^3(\underline{y})\right] * Z(\underline{y}) + G^3(\underline{x}) * Z(\underline{y}) + \\ &+ G^2(\zeta) \circ \left[\sigma(\underline{x}) - Z(\underline{y})\delta(\underline{x}-\underline{y})\right], \quad \varepsilon \to 0, \\ &\quad h \ll \mid \underline{y} - \underline{y}_{\pm} \mid, \end{aligned} \tag{11}$$

where $G^2(\zeta) = -(1-\nu)/(\pi\mu) \ln|\zeta|$ is Flamant's solution [2].

Then we consider a contact problem. The slow variation of the profiles of interacting bodies allows us to find the transversal distribution of the contact pressure $\sigma = \sigma_0(\underline{y}, \zeta)$ through solving 2D static contact problem with the parameter $\underline{y}$. Thus we obtain the basic equation of contact containing only single integrals

$$\begin{aligned} w(\underline{x}) &\sim \left[G^w(\underline{y}) - G^3(\underline{y})\right] * Z(\underline{y}) + G^3(\underline{x}) * Z(\underline{y}) + \\ &+ Z(\underline{y}) Z_0{}^{-1}(\underline{y}) G^2(\zeta) \circ \sigma_0(\underline{y}, \zeta) - G^2(\zeta) Z(\underline{y}), \ \varepsilon \to 0, \end{aligned} \tag{12}$$

$$Z_0(\underline{y}) = \int_{-h}^{h} \sigma_0(\underline{y}, \zeta) d\zeta .$$

5. VERTICAL VIBRATIONS OF SLENDER DIE.

Motion of a slender rigid die with flat footing $\Omega\{|x| \leq l, |y| \leq h, z = 0\}$ at given vertical displacement $\Delta e^{-i\omega t}$ is governed by the equation (12) of contact which can be rewritten in the form (the factor $e^{-i\omega t}$ is omitted below)

$$\begin{aligned} \Delta &= \left[G^{\omega}(x) - G^{3}(x)\right] * Z(x) + \qquad (13) \\ &+ \frac{1-\nu}{2\pi\mu} \int_{-l}^{l} \frac{Z(\xi) - Z(x)}{\mid x - \xi \mid} d\xi + \frac{1-\nu}{2\pi\mu} Z(x) \left[\ln 4(l^2 - x^2) - A\right] . \end{aligned}$$

It is an equation for unknown $Z(x)$. From 2D problem for a flat footing $\mid y \mid \leq h$ we have

$$\sigma_0(\zeta) = 1/\left(\pi\sqrt{h^2 - \zeta^2}\right), A = 2\int_{-h}^{h} \sigma_0(\zeta) \ln \mid \zeta \mid \, d\zeta / Z_0 = \ln(h^2/4).$$

We next consider a die with a flat footing interacting with a bottom $z = -d$ of a groove $P\{\mid x \mid \leq l, \mid y \mid \leq h, -d \leq z \leq 0\}$, $h/d = O(1)$. We look for the asymptotics of the displacement as a sum $w = w_1 + w_2$ of two terms. The first term $w_1 = G^{\omega} * Z(x)$ is a displacement caused by a normal load $Z(x)\delta(y)$ applied to the boundary $z = 0$ of the half-space (without groove). Then we consider a half-plane $z \leq 0$ (without the groove) under the normal load $-Z(x)\delta(y)\delta(z)$ with a parameter x. The load creates a stress field σ_1 at the mentally separated surface of the groove. After that we consider 2D static problem for the half-plane with the groove. Its boundary is under the self-equilibrated load $\sigma_1 + Z(x)\sigma_0/Z_0$ (the contact pressure σ_0 is calculated numerically). This 2D statement is correct and allows us to get the second unknown displacement w_2. Finally we obtain that the eqn (13) is valid, the parameter A is given by ($Y \equiv y^2 + 4d^2$)

$$A = \frac{1}{Z_0} \int_{-h}^{h} \sigma_0(y) \left\{ \ln Y + \frac{1}{16(1-\nu)} \left[(3 - 4\nu) \ln \frac{y^2}{Y} + \frac{12d^2}{Y} - \frac{32d^4}{Y^2} \right] \right\} dy.$$

Equation (13) is solved numerically for a range of values of the dimensionless frequency $p = \omega l / c_2$ for Poisson's ratio $\nu = 1/4$. The aspect ratio is taken equal to $h/l = 1/100$. The figure 2 illustrates the dependencies of real and imaginary parts of the dimensionless dynamic compliance $\psi = 2\pi\mu l \Delta / [(1-\nu)(Z * 1)]$ on the dimensionless frequency p. The compliance ψ was calculated for shallow ($d = 0$) and deep ($d = h$) footing. As it is seen from the figure 2 the curves for the shallow footing coincide qualitatively with the corresponding curves [7]–[11].

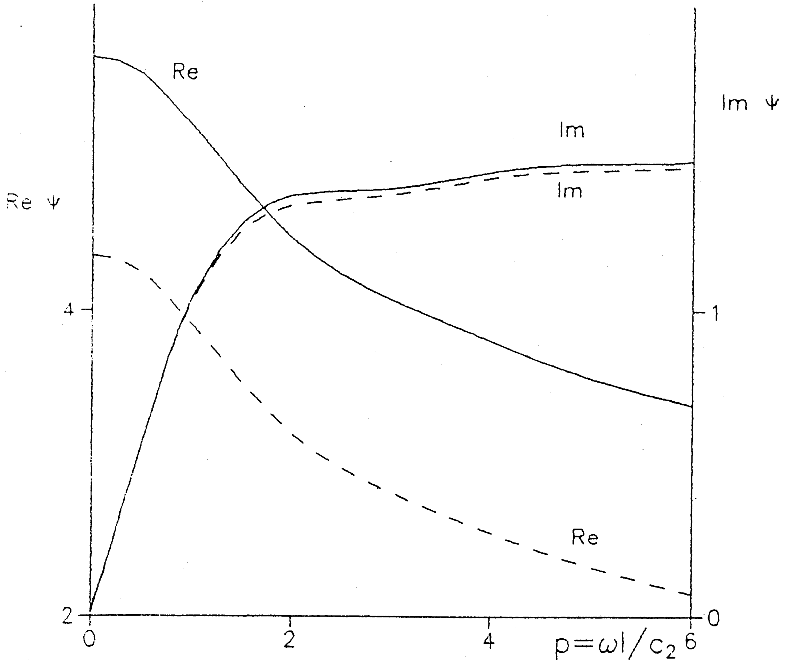

Fig.2 Real and imaginary parts of complex dynamic compliance versys the frequency for $d = 0$ (solid) and $d = h$ (dashed).

6. CONCLUSIONS.

Asymptotic approach proposed and the derived equations lay the groundwork for the asymptotic theory of soil-structure dynamic interaction. On the one hand the derived asymptotic equations (and the proposed method of the derivation) can be used for the development of powerful numerical solutions of new problems. On the other hand they allow one to test available software.

The asymptotic equations derived have the reduced dimension (in comparison with the original equations of elastodynamics) and are suitable for numerical solutions, their kernels do not contain improper integrals of the oscillating functions.

It should be noted that the approach proposed can be applied to dynamic interaction between the soil and the deep point-supports (when the depth is much smaller than the distance). The extention of our analysis to die-wave interaction, to initial-value problem, to an arbitrary number of interacting bodies, to a motion involving 3 components of the displacement and the to general contact conditions creates no problems. The approach above can be successfully applied to the contact between deformable bodies of arbitrary profiles and to the thin inclusion embedded near the boundary.

The detailed analysis of the stress concentration requires first to solve the asymptotic equations (of elastodynamics!) and then to solve the corresponding 3D problem of elastostatics (3D static problems are easy to calculate numerically).

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