Numerical Method in Determining Buckling Loads Due to Fluid Pressure

A.R. van Heteren, M. Former

NUCON Research and Development, P.O. Box 622, NL-3000 AP Rotterdam, The Netherlands

ABSTRACT

Standard commercial finite element codes usually offer two ways for performing stability analyses. As iterative procedures suffer from prohibitively high costs, the eigenvalue formulation is the preferred approach. However, follower-force-effects are normally neglected, resulting in errors of up to 25% in case of idealized externally pressurized cylinders. Adding a pressure-stiffness-matrix overcomes this defect. This modified approach is used for determining the stability performance of externally loaded structures e.g. submarines by which the effect on realistic structures is appreciated.

1. INTRODUCTION

Fluid pressure always acts in a direction normal to the surface of a structure. This means, that, due to the deformation process of the structure, this direction is continuously changing, hence giving rise to the term "follower-force".

To account for this type of loading in finite element codes, two ways are open. The first may be an incremental-iterative procedure, in which for each series of steps the pressure loading is incremented and within each series is iterated till convergence has been achieved; the change of direction of the loading may be automatically adjusted in each step. The computational costs involved, however, are usually as such as to make most engineers look around for another approach, which may be found in the well-known eigenvalue formulation of the problem.

Unfortunately, in most finite element codes commercially available today, the implementation of this approach does not include follower-force-effects, hence producing collapse pressures being in error by a percentage of up to 25 in case of circular externally pressurized cylinders, see fig. 1. This effect merely representing one of the contributions to the total potential energy of the system, it can be added to the contributions already accounted for in the program.

This has been realized for the NASTRAN-code by running part of the job in the usual way, stopping and calculating the follower-force contributions by an independent program, adding this result to the NASTRAN-database and restarting the so modified job. This approach has been employed for determining the overall stability performance of submarines, which project offered the opportunity to appreciate the influence of the effect on realistic structures.

2. THEORETICAL CONSIDERATIONS

With regard to the main characteristics of a loading system, one of the first questions coming to mind concerns its conservativeness. The case of fluid pressure is known to be a conservative system if it is continuously distributed and acts on a fully enclosed volume. This implies:

- the absence of sudden changes in pressure magnitude;
- the free edges of the structure to be unloaded.

Starting from such a situation, the potential of the external loading can be derived, consisting of a part disregarding the change in volume and a contribution accounting for it. The first part is commonly represented in all finite element-method-programs, so need not be discussed here any further. The second part, however, embodies the potential of the load due to the change in volume, thus reflecting the so-called follower-force- or live-load-effects.

For the general shell as depicted in figure 2, Koiter has derived this contribution to the potential energy to be /1,2/:

$$W_p = -\tfrac{1}{2} \int_S p \, \{w(e_1+e_2)-u_1\phi_1-u_2\phi_2\} \, dS, \qquad \text{eq. (1)}$$

in which: S = surface;
p = pressure;
u_1, u_2 = in-surface displacements;
w = normal displacement;
e_1, e_2 = membrane strain measures;
ϕ_1, ϕ_2 = rotational displacements.

The strain measures may be written as /2/:

$$e_1 = \frac{1}{A} \cdot \frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{AB} \cdot \frac{\partial A}{\partial \alpha_2} - \frac{w}{R_1} + \tfrac{1}{2}\phi_1{}^2 + \tfrac{1}{2}\Omega^2 \qquad \text{eq. (2a)}$$

$$e_2 = \frac{1}{B} \cdot \frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{AB} \cdot \frac{\partial B}{\partial \alpha_1} - \frac{w}{R_2} + \tfrac{1}{2}\phi_2{}^2 + \tfrac{1}{2}\Omega^2 \qquad \text{eq. (2b)}$$

$$\phi_1 = \frac{1}{A} \cdot \frac{\partial w}{\partial \alpha_1} + \frac{u_1}{R_1} \qquad \text{eq. (2c)}$$

$$\phi_2 = \frac{1}{B} \cdot \frac{\partial w}{\partial \alpha_2} + \frac{u_2}{R_1} \qquad \text{eq. (2d)}$$

in which: A,B = Lamé parameters;
α_1, α_2 = curvilinear principal coordinates;

$$\Omega = \frac{1}{AB} \cdot \frac{\partial B}{\partial \alpha_1} \cdot u_2 + \frac{1}{A} \cdot \frac{\partial u_2}{\partial \alpha_1} - \frac{1}{AB} \cdot \frac{\partial A}{\partial \alpha_2} \cdot u_1 - \frac{1}{B} \cdot \frac{\partial u_1}{\partial \alpha_2}$$

Substitution yields, with neglect of the linear terms, in Ω:

$$W_p = -\tfrac{1}{2} \int_S p \cdot \{\frac{w}{A} \cdot \frac{\partial u_1}{\partial \alpha_1} + \frac{wu_2}{AB} \cdot \frac{\partial A}{\partial \alpha_2} - \frac{w^2}{R_1} + \frac{w}{B} \cdot \frac{\partial u_2}{\partial \alpha_2} + \frac{wu_1}{AB} \cdot \frac{\partial B}{\partial \alpha_1} +$$

$$- \frac{w^2}{R_2} - \frac{u_1}{A} \cdot \frac{\partial w}{\partial \alpha_1} - \frac{u_1^2}{R_1} - \frac{u_2}{B} \cdot \frac{\partial w}{\partial \alpha_2} - \frac{u_2^2}{R_2} \} \, dS +$$

$$-\tfrac{1}{4} \int_S pw(\phi_1^2 + \phi_2^2) \, dS. \qquad \text{eq. (3)}$$

The last integral contains cubic terms due to the volume change and can be neglected in case of small finite deflections /1/.
Once having derived this contribution, the total loading potential W can be obtained, which, upon differentiating, yields the generalized force in Lagrangian sense. The internal forces coming from differentiating the elastic energy V in the usual manner, the equations of motion can now be written and hence the eigenvalue approach formulated as follows:

- elastic energy of the structure is $V = \tfrac{1}{2}K_e q^2$,
  in which: - K_e = elastic stiffness;
  - q = generalized coordinate;
- internal force: $\frac{\partial V}{\partial q} = Kq$;
- differentiating and linearizing potential energy W of loading system yields generalized force:

  $\frac{\partial W}{\partial q} = K_G q = \lambda K_g q$;

  in which: - K_G = load stiffness;
  - K_g = do., at some reference load;
  - λ = multiplier;
- an eigenvalue approach may be formulated, reading:

$$(K_e + \lambda K_g)q = 0 \qquad \text{eq. (4a)}$$

or

$$K_e + \lambda K_g = 0 \qquad \text{eq. (4b)}$$

Note, that K_g is here built up from a part K_d already dealt with by the finite element code, and an additional part K_f representing follower force effects as derived by differentiating eq. (3), so eq. (4b) may also read:

$$K_e + \lambda(K_d + K_f) = 0 \qquad \text{eq. (5)}$$

3. IMPLEMENTATION

The implementation of eq. (3) will be demonstrated by the example of the common situation of an uniformly pressurized isoparametric quadrilateral bending element, see fig. 3. In cartesian coordinates, eq. (3) reads:

$$W_p = -\tfrac{1}{2} \int_x \int_y p\left(w.\frac{\partial u}{\partial x} + w.\frac{\partial v}{\partial y} - u.\frac{\partial w}{\partial x} - v.\frac{\partial w}{\partial y}\right) dxdy \qquad \text{eq. (6)}$$

The isoparametric notation for both coordinates and displacements is:

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4; \text{ analogous } y \text{ and } z; \qquad \text{eq. (7a)}$$

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4; \text{ analogous } v \text{ and } w.$$

The interpolation functions N may be:

- linear for x and y (and u and v): eq. (7b)
  $N_1 = \tfrac{1}{4}(1+\xi)(1+\eta)$, N_2, N_3 and N_4 likewise;
- cubic for z (and w):
  $N_1 = \{(2-\xi)(1+\xi)+(2-\eta)(1+\eta)-2\}.(1+\xi)(1+\eta)/8$ etc.

Because of the functions N_i being given in local coordinates ξ and η, the derivatives also have to be expressed in these as follows:

$$\frac{\partial .}{\partial \xi} = \frac{\partial x}{\partial \xi} \cdot \frac{\partial .}{\partial x} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial .}{\partial y} \qquad \text{eq. (8a)}$$

$$\frac{\partial .}{\partial \eta} = \frac{\partial y}{\partial \eta} \cdot \frac{\partial .}{\partial x} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial .}{\partial y}$$

This leads to:

$$\begin{vmatrix} \frac{\partial .}{\partial \xi} \\ \frac{\partial .}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial .}{\partial x} \\ \frac{\partial .}{\partial y} \end{vmatrix} = J \cdot \begin{vmatrix} \frac{\partial .}{\partial x} \\ \frac{\partial .}{\partial y} \end{vmatrix} \qquad \text{eq. (8b)}$$

$$\text{so: } \begin{vmatrix} \frac{\partial .}{\partial x} \\ \frac{\partial .}{\partial y} \end{vmatrix} = J^{-1} \cdot \begin{vmatrix} \frac{\partial .}{\partial \xi} \\ \frac{\partial .}{\partial \eta} \end{vmatrix} = \frac{1}{\det J} \cdot \begin{vmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial .}{\partial \xi} \\ \frac{\partial .}{\partial \eta} \end{vmatrix} \qquad \text{eq. (8c)}$$

Using vector and matrix notation upon substitution of eq. (7) and (8) in eq. (4) yields the pressure-stiffness-matrix, which is a-symmetric on element level. After assembling all elements and applying boundary conditions ensuring conservativeness however, a symmetric system matrix results.
Now taking the NASTRAN-code as an example, the stability analysis procedure is as follows:

- run the static part of the original problem to determine the fundamental state;

- retrieve from the database coordinates and pressure loading of the individual elements;
- determine eq. (3) on the basis of interpolation functions like eq. (7);
- add the resulting matrix K_f to the differential stiffness-matrix K_d already calculated by the original code, see eq. (5);
- restart the problem with K_d replaced by $K_d + K_f$, see eq. (5).

4. RESULTS

The outlined procedure has been applied to the stability analysis of naval submarines, of which large models have been made. Results being of confidential character, relative figures are given rather than absolute values. In all cases, also incremental/iterative analyses have been performed, which include all types of geometrical non-linearities and hence yield the most reliable (and lowest) critical load. Also, eigenvalue analyses, in- as well as excluding follower-force-effects have been performed.
From table 1 and figure 4 follows, that both the total effect of geometrical non-linearities and the follower-force-effect heavily depend on the amount and way of stiffening.
For large problems the costs of this eigenvalue approach are about 40 - 60% less than those of the iterative method, and are much influenced by the number of eigenvalues extracted, the starting point of the extraction process and the distance between the eigenvalues.

5. CONCLUSION

By means of the pressure-stiffness matrix the eigenvalue formulation becomes a cost-saving and reliable approach for performing stability analyses under live loading. Being suited to be used in daily engineering practice, critical pressures may be obtained within at most 6% of error, depending on the complexity of the structure considered.

REFERENCES

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/2/ LOGANATHAN, K., et al., "Finite element representation and pressure stiffness in shell stability analysis", Int. J. for Num. Meth. in Engineering, Vol. 14, 1979, pp. 1413-1429.

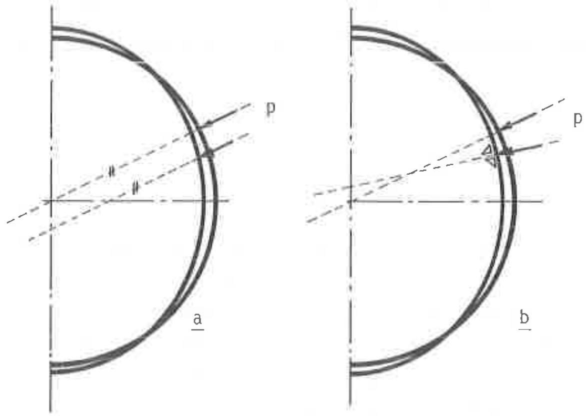


Figure 1: Follower-force-effect

a: dead-load: direction of pressure remains unchanged; $p_k = 4$

b: live-load: direction of pressure changes; $p_k = 3$

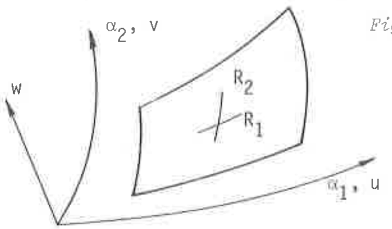


Figure 2: General shell

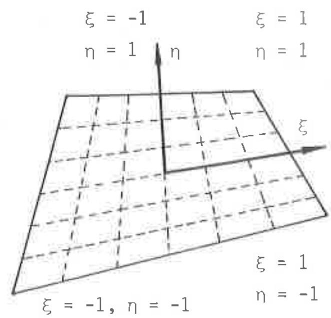
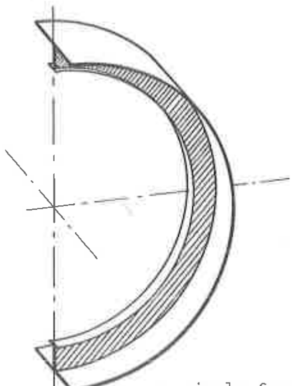


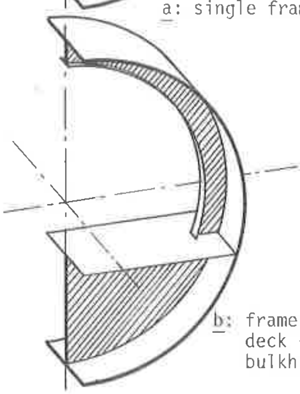
Figure 3: Isoparametric quadrilateral element

Table I: Critical pressures			
calculation method	single frame	frame + deck + blkh	midship section
eigenvalue, dead-load	100%	100%	100%
do., live-load	75%	84%	91%
iter./incr. approach	75%	79%	89%

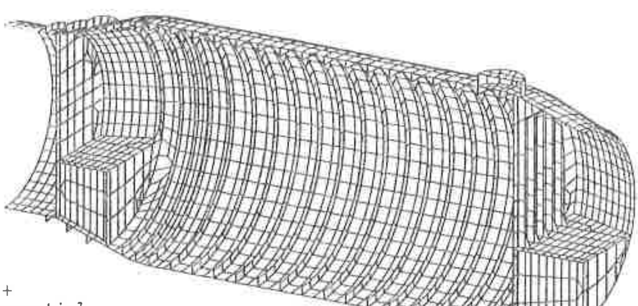
Figure 4: Some submarine geometries



a: single frame



b: frame + deck + partial bulkhead



c: midship section