A simplified procedure for elastic buckling analysis

S.Raff, B.Dolensky & R.Krieg
Kernforschungszentrum Karlsruhe GmbH, Institut für Reaktorentwicklung, Karlsruhe, FR Germany

ABSTRACT

The computational effort to analyse the buckling behaviour of structures with complex geometry using conventional methods is relatively high. Therefore , a simplified procedure is proposed which is based on linear finite element codes that also complex problems can be analysed with a reasonable effort. In addition, the simplified procedure allows to consider the effect of geometrical imperfections on the buckling load.

1. BASIC IDEA OF THE PROCEDURE

Two types of buckling behaviour are distinguished in buckling theory: bifurcation buckling and limit point buckling. Structures optimized against buckling often belong to the bifurcation type. But the technical realization of these structures due to inevitable deviations from the planned or ideal form leads to imperfect geometries. As a consequence, the type of the buckling behaviour changes: the original bifurcation type becomes a limit point type.

The proposed procedure considers the imperfect geometry. However, it is assumed that the ideal structure shows a bifurcation type buckling with linear prebuckling deformations. The analysis of an imperfect structure is obtained usually by the solution of a nonlinear boundary value problem and the buckling load is determined by the load at the limit point. The proposed procedure replaces this analysis by a set of linear boundary value problems which easily can be solved by common FEM codes. Of course, if the deviations from the ideal form approach zero, the solution of the proposed procedure approaches the bifurcation buckling load of the ideal structure.

1.1 Power series solution approach

For purpose of illustration a system with a single degree of freedom is considered. For the dependence between the load parameter P, the imperfection v and the relative displacement w (difference of the displacements of the imperfect and ideal structure) the power series approximation

$$w = P\sum_{i=1}^{I} (v + w)^{i}\beta_{i} \tag{1}$$

is made. The constants β_i serve to adjust the approximation (1) to the actual problem.

Equation (1) can be solved for P

$$P = w/\left[\sum_{i=1}^{I}(v+w)^{i}\beta_i\right] \qquad (2)$$

which gives characteristic buckling trajectories. Fig. 1 shows such trajectories where one or two of the first three coefficients $\beta_1, \beta_2, \beta_3$ describe approximately the most important types of the initial post buckling behaviour. Thus the special power series approach (1) proves to be favourable for the solution of the problem.

The determination of the parameters β_i is carried out using the initial gradients for different arbitrarily chosen imperfections v_k

$$c(v_k) = 1/[dP/dw]_{w=0} = \sum_{i=1}^{I} v_k^{i}\beta_i \qquad k = 1,2,\ldots K \qquad (3)$$

If $c(v_k)$ and v_k $k = 1,2,\ldots K$ are known, where $K = I$, then (3) represents a system of K linear equations for the I parameters β_i. This method of parameter estimation has been chosen because the K quantities

$$c(v_k) = w_k/P \qquad k = 1,2,\ldots K \qquad (3)$$

may be found by solving K linear problems yielding the linear displacements w_k for the arbitrary imperfections v_k under the arbitrary load P. This usually is performed using linear FEM-codes.

Whereas the imperfections v_k, used to determine the coefficients β_i, have been arbitrarily chosen, the specific imperfection of the actual imperfect structure is used now to determine the buckling load. It is the maximum load P_{cr} at the limit point in Fig. 1 and it is calculated using the condition

$$dP/dw = 0 \qquad (5)$$

1.2 Order of approximation and postbuckling behaviour

Using only the first coefficient β_1 ($I = 1$ in (1)), the procedure is called 1. order approximation (O_1-approx.). It is sufficient for the stability analysis of problems showing neutral postbuckling behaviour including linearized buckling problems. The single parameter β_1 is calculated by

$$\beta_1 = c/v \qquad \text{where} \qquad c(v) = w/P. \qquad (6)$$

Here w is the solution of the linear problem with the imperfection v and the load P. In this case, using (5) and (2), P_{cr} is obtained for w and it is given by

$$P_{cr} = 1/\beta_1 \qquad (7)$$

For neutral postbuckling problems β_1 does not depend on v.

If the O_1-approx. is applied to non-neutral postbuckling problems using different imperfection values the parameter β_1 turns out to depend on v instead to be constant. Therefore, in general a higher order approximation (O_I-approx. with $I > 1$) has to be performed. But nevertheless, it can be concluded from (2), that for $v \rightarrow 0$ the bifurcation load can be obtained by the O_1-approx. Then the imperfections used in the procedure are introduced only for calculational reason.

1.3 Generalization to multiple degree of freedom systems

To generalize the basic relation (1) for multiple degree of freedom systems an expansion for w(x) and v(x) is introduced using a complete set of orthogonal functions $f_m(x)$

$$w(x) = \sum_{m=1}^{\infty} w_m f_m \qquad \text{and} \qquad v(x) = \sum_{m=1}^{\infty} v_m f_m \tag{8}$$

where w_m are the parameters of the system. For the O_1-approx. the generalized form of (1) is taken to be

$$w(x) = P \sum_{m=1}^{\infty} (v_m + w_m)\ f_m \beta_{m1} \tag{9}$$

If $f_m(x)$ are the eigenmode functions of the buckling problem, then w_m are the modal displacements and v_m the modal imperfections of the system. The coefficients β_{m1} for each buckling mode m are calculated analogously to (6):

$$\beta_{m1} = c_m/v_m \qquad m = 1,\ 2,\ 3 .... \tag{10}$$

The buckling load is calculated using the condition

$$P_{cr} = \underset{m}{\mathrm{Min}}\ 1/\beta_{m1} \tag{11}$$

The imperfection (8) used for the determination of the coefficients must include the critical mode. The larger this mode component is, the more accurate the results are. In principle, to determine the m initial gradients c_m using (10), m linear FEM-calculations have to be performed. But due to their linearity this can be done by one calculation followed by a decomposition of the result using the orthogonal functions of (8). Of course, those modes which are expected to be not critical can be excluded.

1.4 Comments on the error of the approximation

For some simple cases where analytical solutions are available (e.g. an elastic founded rod, a simply supported rectangular plate, a circular ring) it could be shown that the O_1-approx. gives the exact bifurcation values for $v_m \to 0$, if f_m represents the system of eigenfunctions. But an investigation for a more general class of problems led to the conclusion that this is not true in general /2/.

If the assumed mode functions deviate from the real buckling functions, the results of the method are necessarily of approximate nature. Similar to the Rayleigh-Ritz method, the buckling value is affected then by higher buckling modes. But unlike to that method the sign of the error cannot be predicted.

The accuracy of the O_I-approx. ($I > 1$)has not been investigated, so far.

2. APPLICATION OF THE METHOD

At first the application of the O_1-approx. is demonstrated for simply supported perfect rectangular plates with a buckling load in one coordinate direction (see Fig. 2). The fictitious imperfection v orthogonal to the surface, used to calculate the coefficients β_{m1}, is arbitrarily composed of harmonic functions,

$$v = \sum_{m=1}^{10} \sum_{n=1}^{10} v_{mn} \sin\frac{m\pi x}{a} \sin\frac{n'\pi y}{b} \qquad (12)$$

which are the exact buckling modes for this example (a = length; b = width). According to section 1.3 equ. (12) is the basis to calculate the coefficients β_{m1}. The resulting buckling loads for $v \to 0$ and different ratios a/b are shown in Fig. 2. For comparison also reference solutions using classical buckling theory are included. The method gives the correct buckling mode and relatively accurate buckling loads. The errors are mainly due to the discretization of the FEM-calculations.

However, if one edge parallel to the loading direction is free, the eigenfunctions are not harmonic. Nevertheless, harmonic functions have been used as an approximation. Due to the changed kinematic constraints along the free edge $n' = (2n-1)/2$ is introduced into (12). The result of the O_1-approx. is, that the critical buckling mode should have approximately the harmonic form m = 1 and n' = 1/2, independent of the ratio a/b. This is confirmed by the theoretical buckling form. But the buckling loads deviate from the correct value by about 30 %. However, repeating the analysis using a single mode imperfection according to the result of the first calculation, the error is reduced to 3 %. These results are also shown in Fig. 2.

It is concluded that, as mentioned in section 1.4, an approximative form of the buckling eigenfunction is sufficient to give satisfactory results for the buckling load. But the calculational effort is twice: a first application gives the approximate buckling form, the second then gives the buckling load.

How the imperfection sensitivity is taken into account is shown for circular cylindrical shells under uniform axial pressure. The results are compared with experimental data /3/ for perfect and imperfect cylinders (radius R = 75 mm, length L = 150 mm, wall thickness s = 0,2 mm).

For non-neutral postbuckling the O_1-approx. gives buckling loads which generally depend on the choice of imperfection amplitude. This is shown in Fig. 3, where the three different types of postbuckling behaviour correspond to different trends in the imperfection sensitivity. Thus the O_1-approx. yields not only the bifurcation load for $v \to 0$, but also the type of the initial postbuckling behaviour of the structure. Especially decreasing buckling loads with increasing imperfection amplitude indicate an unstable postbuckling behaviour which is the reason for imperfection sensitivity of the limit point load. In this case, for an imperfect geometry a higher order approximation is necessary to obtain satisfactory results.

Based on these facts a strategy for the application of increasing orders of approximation can be established. The calculation is done stepwise. At first the properties of a perfect cylinder are considered. The O_1-approx. is performed using the imperfection

$$v = \sum_{m=1}^{20} \sum_{n=1}^{20} v_{mn} \sin\frac{m\pi x}{L} \sin n\varphi \qquad (13)$$

related to the surface of the circular cylinder with the arbitrary coefficients v_{mn} (x and φ are the axial and angular coordinates, respectively). The first 20 coefficients are used because it is expected that they include the critical mode. The result of the first calculation is that the buckling load of the "critical " mode is just somewhat smaller than the buckling load of other modes. Therefore, according to 1.4, the O_1-approx. has been repeated for the critical mode and also for each of the two neighbouring modes by using only the corresponding mode function in

(13). The resulting buckling loads for various imperfection amplitudes, related to the theoretical Donnell buckling load $P_{cr,D}$, are shown in Fig. 4. The mode (m, n) = (7,16) proved to be the critical one for small imperfections.

Because the buckling loads from the O_1-approx. depend on v/s, O_2- and O_3-approx. for the critical mode have been performed. For this no further FEM-calculations are required. The previous calculations can be used for the higher order evaluations, too. The resulting values are included in Fig. 4. The imperfection sensitivity turns out to be rather strong.

To assess the numerical results, Fig. 4 also contains the measured values /3/ for nearly perfect and imperfect cylinders, where the imperfection has the shape of the buckling mode m = 7, n = 16.

The deviation between the Donnell-value and the numerical bifurcation value obtained here, indicated by a ratio $P_{cr}/P_{cr,D} \neq 1.0$, is partly due to simply supported boundary conditions used for the Donnell solution instead of clamped conditions used here. The relatively large deviation between the calculation and the measurement for the nearly perfect cylinder is due to both plastic deformations which occur for $P/P_{cr,D} > 0.6$ and some very small imperfections present even in carefully manufactured samples. As Fig. 4 shows, a very small imperfection of v/s = 0.05 (0.01 mm) would be able to explane the whole discrepancy.

The strong reduction of the buckling values for the modal imperfections v/s = 0.5 and 1 are predicted satisfactorily by the O_2-approx. For imperfection values $1 < v/s < 2$ the O_1-approx. indicates a changing trend towards stable postbuckling behaviour (see Fig. 4, mode (7, 16)). This can be considered by an O_3-approx. It yields the result that no further limit points can be found in the P/w curves. But they are showing a quasi-neutral postbuckling behaviour with quasi constant failure loads (Fig. 4, dashed part of O_3-approx. curve) which seem to correspond to the measured values quite well.

3. CONCLUSIONS

The considered examples show that the power series solution approach is able to predict the buckling load and its imperfection sensitivity also for some nontrivial problems. Knowledge about the buckling behaviour, especially about the shape of the buckling modes, is very useful to shorten the solution procedure and to improve the results. Since only linear problems, using existing high effective FEM-codes, have to be solved within the frame of the proposed strategy, problems of numerical instability are circumvented and the calculational effort may possibly be smaller compared to existing methods.

REFERENCES

/1/ D. Bushnell, Buckling of Shells - Pitfall for Designers. AIAA Journal, 19 (1981), p. 1183-1226

/2/ T. Malmberg, Theoretical Assessment of a Proposal for the Simplified Determination of Critical Loads of Elastic Shells. KfK 4113 (1986)

/3/ N. Waeckel, J.F. Jullien, P. Ledermann, Experimental Studies on the Instability of Cylindrical Shells with Initial Geometric Imperfections, in 'Recent Advances in Nuclear Component Testing and Theoretical Studies on Buckling', PVP-Vol 89 (1984)

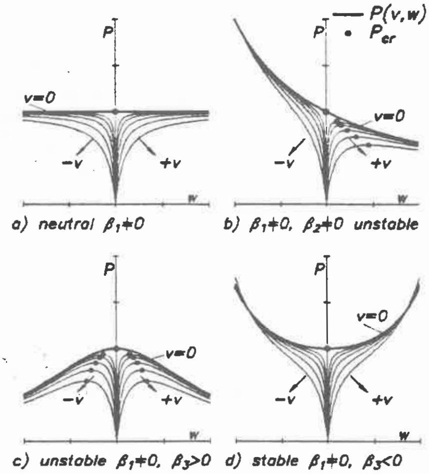

Fig. 1: Characteristic types of buckling behaviour for perfect (v=0) and imperfect (v≠0) geometries

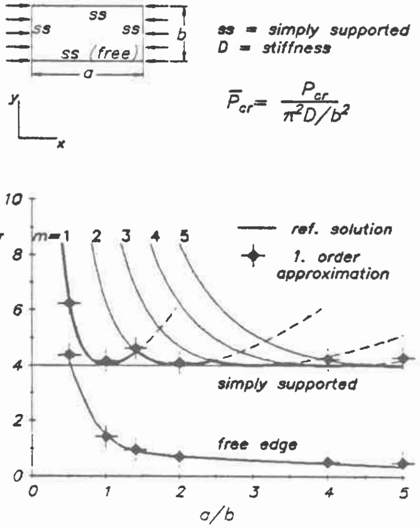

Fig. 2: Bifurcation buckling of rectangular plates

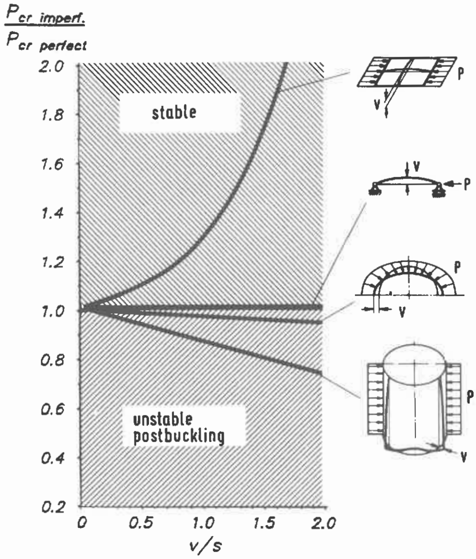

Fig. 3: 1. order approximation and initial postbuckling behaviour

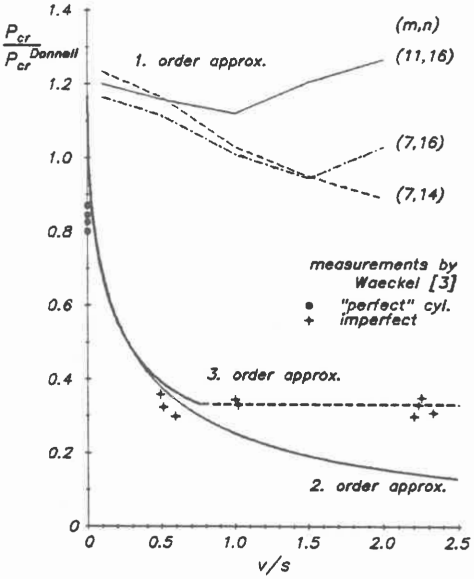

Fig. 4: Imperfection sensitivity of circular cylindrical shells under uniform axial pressure