

NOTE ON SOME FORMULAS FOR WEIGHTED SUMS OF ZONAL POLYNOMIALS<sup>1</sup>

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1. Introduction. The first purpose of this paper is to prove a stronger result than the previous lemma for the sum of zonal polynomials given in Sugiura and Fujikoshi [5], which played an important role in deriving the asymptotic expansions of the non-null distributions of the likelihood ratio criteria in multivariate analysis by these authors, Sugiura [6] and of the generalized variance by Fujikoshi [1].

THEOREM 1. Let  $C_{\kappa}(Z)$  be the zonal polynomial of degree  $k$  corresponding to a partition  $\kappa = \{k_1, k_2, \dots, k_p\}$  of  $k$  ( $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$ ) for a  $p \times p$  positive definite matrix  $Z$ . Put

$$(1.1) \quad \begin{aligned} a_1(\kappa) &= \sum_{\alpha=1}^p k_{\alpha} (k_{\alpha} - \alpha) \\ a_2(\kappa) &= \sum_{\alpha=1}^p k_{\alpha} (4k_{\alpha} - 6\alpha k_{\alpha} + 3\alpha^2). \end{aligned}$$

Then the following equalities hold:

$$(1.2) \quad \sum_{(\kappa)} a_1(\kappa) C_{\kappa}(Z) = k(k-1) \operatorname{tr} Z^2 (\operatorname{tr} Z)^{k-2}$$

$$(1.3) \quad \begin{aligned} \sum_{(\kappa)} a_1(\kappa)^2 C_{\kappa}(Z) &= k(k-1) [\{\operatorname{tr} Z^2 + (\operatorname{tr} Z)^2\} (\operatorname{tr} Z)^{k-2} \\ &\quad + 4(k-2) \operatorname{tr} Z^3 (\operatorname{tr} Z)^{k-3} + (k-2)(k-3) (\operatorname{tr} Z^2)^2 (\operatorname{tr} Z)^{k-4}] \end{aligned}$$

$$(1.4) \quad \begin{aligned} \sum_{(\kappa)} a_2(\kappa) C_{\kappa}(Z) &= k[(\operatorname{tr} Z)^k + 3(k-1) \{\operatorname{tr} Z^2 + (\operatorname{tr} Z)^2\} (\operatorname{tr} Z)^{k-2} \\ &\quad + 4(k-1)(k-2)(k-3) \operatorname{tr} Z^3 (\operatorname{tr} Z)^{k-3}], \end{aligned}$$

where the symbol  $\sum_{(\kappa)}$  means the sum of all possible partition  $\kappa$  of  $k$ .

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Dividing by  $k!$  on both sides of each of the equations (1.2), (1.3) and (1.4) and summing with respect to  $k$  from zero to infinity, we obtain the lemma given by Sugiura and Fujikoshi [5].

The second purpose of this paper is to give an alternative proof of the following theorem due to Fujikoshi [2], by using a differential equation for zonal polynomials obtained recently by James [4].

THEOREM 2. (Fujikoshi). With the same notation as in Theorem 1, put

$$(1.5) \quad (b)_\kappa = \prod_{\alpha=1}^p (b - \frac{\alpha-1}{2})(b + 1 - \frac{\alpha-1}{2}) \dots (b + k_\alpha - 1 - \frac{\alpha-1}{2}).$$

Then the following equalities hold:

$$(1.6) \quad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa a_1(\kappa) C_\kappa(Z)}{k!} = \frac{b}{2} |I - Z|^{-b} \{(2b + 1) \text{tr}W^2 + (\text{tr}W)^2\}$$

$$(1.7) \quad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa a_1(\kappa)^2 C_\kappa(Z)}{k!} = \frac{b}{4} |I - Z|^{-b} \{(2b + 1)(2b^2 + b + 2)(\text{tr}W^2)^2 \\ + 2(2b^2 + b + 2)\text{tr}W^2(\text{tr}W)^2 + 2(8b^2 + 10b + 5)\text{tr}W^4 + 8(2b + 1)\text{tr}W^3 \text{tr}W \\ + b(\text{tr}W)^4 + 8(2b^2 + 3b + 2) \text{tr}W^3 + 12(2b + 1)\text{tr}W^2 \text{tr}W + 4(\text{tr}W)^3 \\ + 2(2b + 1)(\text{tr}W)^2 + 2(2b + 3) \text{tr}W^2\},$$

where the positive definite matrix  $Z$  is assumed to have characteristic roots less than one, and  $W = Z(I - Z)^{-1}$ .

2. Proof of Theorem 1. Since the zonal polynomial  $C_\kappa(Z)$  is a homogeneous symmetric polynomial of degree  $k$  with respect to the  $p$  characteristic roots of  $Z$ , it is sufficient to prove the equalities in Theorem 1 and Theorem 2, when  $Z$  is a diagonal matrix  $Y = \text{diag}(y_1, y_2, \dots, y_p)$ . Fujikoshi [2] has shown, in the proof of Theorem 2, that the following differential relations hold:

$$(2.1) \quad C_\kappa(Y) a_1(\kappa) = \text{tr}(Y\partial)^2 C_\kappa(\Sigma) \Big|_{\Sigma=Y}$$

$$(2.2) \quad C_{\kappa}(Y) \{3a_1(\kappa)^2 - a_2(\kappa) + k\} = [3\{\text{tr}(Y\partial)\}^2 + 8\text{tr}(Y\partial)^3]C_{\kappa}(\Sigma) \Big|_{\Sigma=Y},$$

where the symbol  $\partial$  means a matrix of differential operators given by

$$\partial = (\frac{1}{2}(1 + \delta_{ij})\partial/\partial\sigma_{ij}), \text{ operating on a positive definite matrix } \Sigma = (\sigma_{ij}),$$

( $\delta_{ij}$  is a Kronecker delta.) The proof of this formula is based on the asymptotic expression of the equality (1.22) in Sugiura and Fujikoshi [5]. By the

formula  $\Sigma_{(\kappa)} C_{\kappa}(\Sigma) = (\text{tr}\Sigma)^k$  in James [3], we have from (2.1)

$$\Sigma_{(\kappa)} C_{\kappa}(\Sigma) = (\text{tr}\Sigma)^k \text{ in James [3], we have from (2.1)}$$

$$(2.3) \quad \Sigma_{(\kappa)} a_1(\kappa)C_{\kappa}(Y) = \sum_{\alpha=1}^p y_{\alpha}^2 (\partial^2/\partial\sigma_{\alpha\alpha}^2) (\text{tr}\Sigma)^k \Big|_{\Sigma=Y} = k(k-1)\text{tr}Y^2(\text{tr}Y)^{k-2}$$

$$(2.4) \quad \begin{aligned} \Sigma_{(\kappa)} a_1(\kappa)^2 C_{\kappa}(Y) &= \sum_{(\kappa)} \text{tr}(Y\partial)^2 a_1(\kappa)C_{\kappa}(\Sigma) \Big|_{\Sigma=Y} = \text{tr}(Y\partial)^2 k(k-1)(\text{tr}\Sigma^2)(\text{tr}\Sigma)^{k-2} \Big|_{\Sigma=Y} \\ &= k(k-1) \left\{ \sum_{\alpha=1}^p y_{\alpha}^2 (\partial^2/\partial\sigma_{\alpha\alpha}^2) + \frac{1}{2} \sum_{\alpha < \beta} y_{\alpha} y_{\beta} (\partial^2/\partial\sigma_{\alpha\beta}^2) \right\} \text{tr}\Sigma^2 (\text{tr}\Sigma)^{k-2} \Big|_{\Sigma=Y} \\ &= k(k-1) \left\{ 2\text{tr}Y^2(\text{tr}\Sigma)^{k-2} + 4(k-2)\text{tr}Y^3(\text{tr}\Sigma)^{k-3} + (k-2)(k-3)(\text{tr}Y^2)^2 (\text{tr}\Sigma)^{k-4} \right. \\ &\quad \left. + 2 \sum_{\alpha < \beta} y_{\alpha} y_{\beta} (\text{tr}\Sigma)^{k-2} \right\} \Big|_{\Sigma=Y}, \end{aligned}$$

which imply equalities (1.2) and (1.3) respectively. From (2.2) we have

$$(2.5) \quad \begin{aligned} \Sigma_{(\kappa)} a_2(\kappa)C_{\kappa}(Y) &= 3 \sum_{(\kappa)} a_1(\kappa)^2 C_{\kappa}(Y) + k(\text{tr}Y)^k - [3 \left\{ \sum_{\alpha=1}^p y_{\alpha}^4 (\partial^4/\partial\sigma_{\alpha\alpha}^4) \right. \\ &\quad \left. + 2 \sum_{\alpha < \beta} y_{\alpha}^2 y_{\beta}^2 (\partial^4/\partial\sigma_{\alpha\alpha}^2 \partial\sigma_{\beta\beta}^2) \right\} + 8 \sum_{\alpha=1}^p y_{\alpha}^3 (\partial^3/\partial\sigma_{\alpha\alpha}^3) ] (\text{tr}\Sigma)^k \Big|_{\Sigma=Y}, \end{aligned}$$

which yields (1.4) of Theorem 1.

It is interesting to note that the first two equations (1.2) and (1.3) in Theorem 1 can also be obtained from the following linear partial differential equation of second degree derived by James [4]:

$$(2.6) \quad \sum_{\alpha=1}^p y_{\alpha}^2 (\partial^2/\partial y_{\alpha}^2) C_{\kappa}(Y) + \sum_{\alpha \neq \beta} y_{\alpha}^2 (y_{\alpha} - y_{\beta})^{-1} (\partial/\partial y_{\alpha}) C_{\kappa}(Y) = \{a_1(\kappa) + (p-1)k\} C_{\kappa}(Y).$$

Summing both sides of the above formula with respect to  $\kappa$  for fixed  $k$ , yields equation (1.2). Operating  $\Sigma_{(\kappa)} a_1(\kappa)$  on both sides of (2.6) yields equation (1.3).

3. Alternative proof of Theorem 2. Noting that

$$(3.1) \quad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_{\kappa} C_{\kappa}(Z)}{k!} = |I - Z|^{-b},$$

when all characteristic roots of positive definite matrix  $Z$  are less than one (James [3]), we can get from (2.6)

$$(3.2) \quad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_{\kappa} a_1(\kappa) C_{\kappa}(Y)}{k!} = \left\{ \sum_{\alpha=1}^p y_{\alpha}^2 (\partial^2 / \partial y_{\alpha}^2) + \sum_{\alpha \neq \beta} y_{\alpha}^2 (y_{\alpha} - y_{\beta})^{-1} (\partial / \partial y_{\alpha}) \right\} |I - Y|^{-b} \\ - (p-1) (d/dt) |I - tY|^{-b} \Big|_{t=1} \\ = b |I - Y|^{-b} \left\{ (b+1) \text{tr} W^2 + \sum_{\alpha \neq \beta} y_{\alpha}^2 (y_{\alpha} - y_{\beta})^{-1} (1 - y_{\alpha})^{-1} - (p-1) \text{tr} W \right\}$$

where  $W = Y(I-Y)^{-1}$ . The second term in the above equation can be simplified by noting  $(I-Y)^{-1} = I + W$  as

$$(3.3) \quad \sum_{\alpha \neq \beta} y_{\alpha}^2 (y_{\alpha} - y_{\beta})^{-1} (1 - y_{\alpha})^{-1} = \sum_{\alpha < \beta} (y_{\alpha} + y_{\beta} - y_{\alpha} y_{\beta}) (1 - y_{\alpha})^{-1} (1 - y_{\beta})^{-1} \\ = \frac{1}{2} \{ 2 \text{tr} W \text{tr}(I + W) - (\text{tr} W)^2 - 2 \text{tr} W(I + W) + \text{tr} W^2 \} \\ = \frac{1}{2} \{ 2(p-1) \text{tr} W + (\text{tr} W)^2 - \text{tr} W^2 \},$$

which implies the first equation (1.6) in Theorem 2. From the differential equation (2.6), we have

$$(3.4) \quad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_{\kappa} a_1(\kappa)^2 C_{\kappa}(Y)}{k!} = \left\{ \sum_{\alpha=1}^p y_{\alpha}^2 \frac{\partial^2}{\partial y_{\alpha}^2} + \sum_{\alpha \neq \beta} \frac{y_{\alpha}^2}{y_{\alpha} - y_{\beta}} \frac{\partial}{\partial y_{\alpha}} \right\} f(Y) |I - Y|^{-b} \\ - (p-1) (d/dt) |I - tY|^{-b} f(tY) \Big|_{t=1},$$

where  $f(Y) = (b/2) \{ (2b + 1) \text{tr} W^2 + (\text{tr} W)^2 \}$  with  $W = Y(I-Y)^{-1}$ . The first term in the right hand side of (3.4) can be verified after some computation as

(3.5)

$$\frac{b}{2} |\mathbf{I} - \mathbf{Y}|^{-b} \{b(b+1)(2b+1)(\text{tr}\mathbf{W}^2)^2 + b(b+1)\text{tr}\mathbf{W}^2(\text{tr}\mathbf{W})^2 + 8(b+1)^2\text{tr}\mathbf{W}^4 \\ + 4(b+1)\text{tr}\mathbf{W}\text{tr}\mathbf{W}^3 + 4(2b^2+5b+3)\text{tr}\mathbf{W}^3 + 4(b+1)\text{tr}\mathbf{W}\text{tr}\mathbf{W}^2 + 4(b+1)\text{tr}\mathbf{W}^2\}.$$

The second term in (3.4) can be written as

(3.6)

$$\frac{b}{2} |\mathbf{I} - \mathbf{Y}|^{-b} [2(2b+1) \sum_{\alpha \neq \beta} y_\alpha^3 (y_\alpha - y_\beta)^{-1} (1 - y_\alpha)^{-3} + 2\text{tr}\mathbf{W} \sum_{\alpha \neq \beta} y_\alpha^2 (y_\alpha - y_\beta)^{-1} (1 - y_\alpha)^{-2} \\ + b\{(2b+1)\text{tr}\mathbf{W}^2 + (\text{tr}\mathbf{W})^2\} \sum_{\alpha \neq \beta} y_\alpha^2 (y_\alpha - y_\beta)^{-1} (1 - y_\alpha)^{-1}].$$

Noting that

(3.7)

$$\sum_{\alpha \neq \beta} \frac{y_\alpha^3}{(y_\alpha - y_\beta)(1 - y_\alpha)^3} = \sum_{\alpha < \beta} \left\{ \frac{y_\alpha^2}{(1 - y_\alpha)^3(1 - y_\beta)} + \frac{y_\alpha y_\beta}{(1 - y_\alpha)^2(1 - y_\beta)^2} + \frac{y_\beta^2}{(1 - y_\alpha)(1 - y_\beta)^3} \right\}$$

$$= \frac{1}{2}\{-3\text{tr}\mathbf{W}^4 + 2\text{tr}\mathbf{W}^3\text{tr}\mathbf{W} + (\text{tr}\mathbf{W}^2)^2 + 2(p-3)\text{tr}\mathbf{W}^3 + 4\text{tr}\mathbf{W}^2\text{tr}\mathbf{W} + (2p-3)\text{tr}\mathbf{W}^2 + (\text{tr}\mathbf{W})^2\}$$

(3.8)

$$\sum_{\alpha \neq \beta} y_\alpha^2 (y_\alpha - y_\beta)^{-1} (1 - y_\alpha)^{-2} = \sum_{\alpha < \beta} \{y_\alpha (1 - y_\alpha)^{-2} (1 - y_\beta)^{-1} + y_\beta (1 - y_\alpha)^{-1} (1 - y_\beta)^{-2}\}$$

$$= -\text{tr}\mathbf{W}^3 + \text{tr}\mathbf{W}^2\text{tr}\mathbf{W} + (\text{tr}\mathbf{W})^2 + (p-2)\text{tr}\mathbf{W}^2 + (p-1)\text{tr}\mathbf{W},$$

we can simplify the second term in (3.4) as

(3.9)

$$\frac{b}{2} |\mathbf{I} - \mathbf{Y}|^{-b} [(b^2+2)\text{tr}\mathbf{W}^2(\text{tr}\mathbf{W})^2 + \frac{1}{2}b(\text{tr}\mathbf{W})^4 + (1-\frac{1}{2}b)(2b+1)(\text{tr}\mathbf{W}^2)^2 \\ - 3(2b+1)\text{tr}\mathbf{W}^4 + 4b\text{tr}\mathbf{W}^3\text{tr}\mathbf{W} + \{8b + 2 + (2b^2 + b + 2)(p-1)\}\text{tr}\mathbf{W}^2\text{tr}\mathbf{W} \\ + \{b(p-1) + 2\}(\text{tr}\mathbf{W})^3 + 2(p-3)(2b+1)\text{tr}\mathbf{W}^3 + (2b+1)(2p-3)\text{tr}\mathbf{W}^2 + \{2b + 1 + \\ 2(p-1)\}(\text{tr}\mathbf{W})^2].$$

The third term in (3.4) can be written as

(3.10)

$$\begin{aligned} (d/dt) |I - tY|^{-b} f(tY) \Big|_{t=1} &= \frac{1}{2}b |I - Y|^{-b} \{2(2b+1)\text{tr}W^3 + (2b^2+b+2)\text{tr}W^2 \text{tr}W \\ &+ b(\text{tr}W)^3 + 2(\text{tr}W)^2 + 2(2b+1)\text{tr}W^2\}. \end{aligned}$$

Substituting (3.5), (3.9), and (3.10) for the right hand side of (3.4), we can derive the second equality (1.7) in Theorem 2.

Fujikoshi [2] obtained a further formula concerning

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} a_2(\kappa) C_{\kappa}(Z) / k! ,$$

based on the equality (2.2). It seems difficult, however, to give an alternative proof from the formula (2.6).

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