UNIVERSITY OF NORTH CAROLINA
Department of Statistics
Chapel Hill, N. C.

AUTOREGRESSIVE MODELS AND TESTING OF HYPOTHESES ASSOCIATED WITH THESE MODELS

by

John William Adams
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Some tests of the hypothesis of independence are given. The alternative hypotheses are autoregressive schemes. In some cases the non-null distributions of the test statistics are given in closed form; in other cases approximations are given which are as accurate as one pleases.

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INTRODUCTION

The general outline of this thesis is the following: Chapter I contains some results in matrices and one or two miscellaneous results; Chapter II contains some tests of independence, the non-null distributions of some of the test statistics (in one case an approximation), and the proof of the consistency of some of the tests; Chapter III contains a comparison of the power functions of two possible tests on data from a two-way classification (no interaction) when a circular model is assumed; Chapter IV contains an approximate analysis of variance for a multi-way classification with interaction when a circular model is assumed. These problems arise out of the following context.

Consider a set of r jointly normally distributed random variables, say $Y_1, Y_2, \ldots, Y_r$. Let the expectations of these random variables be known linear functions of some unknown constants; and let their covariance matrix be $V\sigma^2$, where σ^2 is an unknown scalar multiplier.

When V is known to be the identity matrix I, the above assumptions constitute what is sometimes called the univariate linear hypothesis. The problems usually considered in this situation are those of hypothesis testing and confidence intervals. These problems and their solutions are well known and need not be elaborated here.

If, however, V is not known to be I, then the problems of testing and confidence intervals are made more difficult, and some altogether new problems come up. A few of the many possibilities are the following: (i) the matrix V is known except for the scalar multiplier σ^2; (ii) the matrix V is diagonal with unknown elements; (iii) the matrix V is a patterned matrix with unknown elements.

Case (i) is the case of quasi-independence, and leads to no new problems because a linear transformation can be found which transforms the random variables $Y_1, Y_2, \ldots, Y_r$ into the random variables $X_1, X_2, \ldots, X_r$ whose covariance matrix if $I\sigma^2$, whose expectations are linear functions of the unknown constants, and whose distribution is multivariate normal. Thus Case (i) can always be reduced to the totally independent case.

Case (ii) arises when the variance components model is assumed. The variance components model will not be considered in what follows.

Case (iii) arises when the random variables or their distributions satisfy certain restrictions. For example, assume that $Y_1, Y_2, \ldots, Y_r$ are a sequence of random variables, with zero means, which satisfy the stochastic difference equation,

$$Y_i = \rho Y_{i-1} + \epsilon_i \quad , \qquad i = 1, 2, \ldots, r \quad ,$$

where $|\rho| < 1$, and the ϵ_i are independently and normally distributed with zero means and variance σ^2. It is assumed that Y_1 is normally distributed with mean zero and variance σ^2 and independent of the ϵ_i, then $V = (v_{ij})$, where

$$v_{ij} = \rho^{|i-j|} (1 - \rho^2)^{-1} \quad ,$$

which is a patterned matrix depending on only one unknown parameter. A more general pattern results when the random variables are assumed to be a segment of a stationary time series. Then V is a positive definite matrix with elements constant on all diagonals.

Some of the problems which arise out of Case (iii) are the following: (iv) under a model such that V is the patterned matrix which results when the random variables satisfy the first-order stochastic difference equation, shown above, V depends on one unknown parameter, i.e., on ρ, and the problem is to devise a test for the hypothesis that $\rho = 0$ against the alternative hypothesis that $\rho \neq 0$; (v) test the hypothesis that V is a particular patterned matrix (not the identity matrix) against the alternative hypothesis that it is some other patterned matrix; (vi) analysis of variance, including tests of hypotheses and confidence intervals, under a model in which the random variables associated with the experiment have as covariance matrix a particular patterned matrix.

Problem (iv) has received considerable attention in the literature. A few of these papers are mentioned below. Koopmans [12][1] proposed the first-order serial correlation coefficient as a test statistic. He found the null distribution of this statistic, but its form was not suitable for numerical computation, and he was forced to make use of approximations. The circular serial correlation coefficient of first order was suggested as a possible test statistic, and R. L. Anderson [2] found its null distribution. Dixon [9] and Siddiqui [15, 16] investigated approximations to the distribution of the serial correlation coefficient. T. W. Anderson [5] proved that there is no uniformly most powerful test of the hypothesis $\rho = 0$,

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[1] Numbers in square brackets refer to bibliography.

even against one-sided alternatives, unless the model is modified to a circular type, in which case uniformly most powerful tests do exist for the one-sided alternatives. The appropriate test statistics for these cases are the circular serial correlation coefficients of the first order. For the two-sided alternative, he found a test which is uniformly most powerful within a certain sub-class of tests, and the test statistics are again the circular serial correlation coefficients of first order. Hannan [11] described an exact test first proposed by Ogawa [13].

Chapter II contains results which it is hoped bear on this problem. Section 2.2 describes an intersection test of a particular kind. This test may be useful against more general alternatives than specified in (iv). The non-null distribution will, however, be difficult to find, and it is hard to say against just what alternatives the test will perform best. Section 2.3 describes a second test, and in section 2.4 it is shown how an approximation to the non-null distribution can be found. This test may also be useful against more general alternatives. In section 2.5 a circular model is considered, and a test is proposed whose exact non-null distribution is derived in section 2.6, and whose consistency is proved in section 2.7.

Problem (v) is the most difficult of the three. It includes such problems as testing the hypothesis that the observations derive from an autoregressive scheme of order q against the alternative hypothesis that they derive from a scheme of order q'. Nothing in this thesis sheds any light on this problem. However, a result proved in section 1.2 may be useful in deriving a test for a hypothesis slightly different from the one specified in (v). Suppose that it is desired to test the hypothesis that V is doubly[2] symmetric against the alternative hypothesis that it is not. The result proved in section 1.2 is the following: if V is doubly symmetric and of even dimension r, then there exists an orthogonal matrix H, not depending on V, such that

$$HVH' = \left|\begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array}\right| ,$$

where V_1 and V_2 are both (r/2, r/2) matrices; and if V is not doubly symmetric, then H does not transform V into the form shown above. So if there is available more than one realization of the random variables, the methods of multivariate analysis could be used to test the hypothesis that the transform of V is of the form shown above against the alternative hypothesis that it is not. If this hypothesis were rejected, then one would be in effect rejecting the hypothesis that the observations were deriving from a stationary process. It would seem that such a test could be useful.

A solution to problem (vi) was the original goal of this work. It was desired to find a way of analyzing data which arise from multivariate normal populations subject to a multi-way classification. For example in the balanced two-way classification there are b blocks, t treatments, and r observations in each treatment-block combination

[2] Doubly symmetric means symmetric about both diagonals.

which occurs in the design, and it is assumed that the observations are generated by independent random variables except with in cells (a cell is a particular treatment-block combination) where the r random variables satisfy a stochastic difference equation of the form previously mentioned. It was hoped that procedures could be found which depend on statistics whose central distributions are known and tabulated.

Such classifications arise in the following types of situations. Suppose there are bt identical machines which produce objects in a sequence and that these objects each have a common measurable characteristic. If b lots of homogeneous (within lots) raw material are available, and if t different adjustments of each machine are possible, then it may be required to study the effect of the machine adjustment while eliminating the disturbing effect of heterogeneity in the raw material (between lots) and the possible interaction between the effects of raw material and machine adjustment. It would not be unusual if the observations generated by such a process did not behave like the relaizations of a sequence of independent random variables. In such cases one is obliged to assume a more complex model, and the one chosen here is the stochastic difference equation of first order (in actuality the model assumed is circular, but it is hoped that the effects of this will be negligible when the number of observations in a cell is moderate). Such a model is relatively simple, and it seems reasonable in light of the empirical fact that objects produced one after the other are usually more nearly alike than those objects widely separated in the seuqence of production.

An approximate analysis of a particular case is described in section 4.2. If the observations within a cell derive from a model which T. W. Anderson [5] has called modified circular, then it is possible, when r is odd, to find two linear combinations of the observations within each cell (there will be as many such pairs of linear combinations as there are cells) such that the corresponding pairs of linear combinations of random variables are independent, have non-zero expectations, and have variances which are equal except for a quantity which vanishes as r increases. And if r is even (see section 4.3), it is possible to find two linear combinations of observations within each cell such that the linear combinations of the corresponding random variables have non-zero expectations, have variances which are equal, and have covariance which is zero except for a quantity which vanishes as r increases. Thus, supposing the small quantities mentioned above to be zero, it is possible to analyze the transformed observations as if they were generated by independent, homoscedastic random variables with two random variables associated with each cell. The effects of blocks and interaction can be eliminated by means of the usual analysis.

CHAPTER I

SOME RESULTS IN MATRICES

1.1 Introduction. This chapter contains the proofs of some of the results which will be used in later chapters. In addition, some more or less well-known identities, which are quite frequently used throughout the thesis, are stated without proof.

1.2. A Result in Matrices. If V is an $(r x r)$, doubly symmetric matrix, then, when r is even, there is an $(r x r)$ orthogonal matrix H such that

$$(1.2.1) \quad HVH' = \left| \begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array} \right| ,$$

where V_1 and V_2 are $(r/2 \text{ x } r/2)$ matrices. The matrix H can always be written in the form

$$(1.2.2) \quad H = \frac{1}{\sqrt{2}} \left| \begin{array}{c|c} A & AK \\ \hline B & -BK \end{array} \right| ,$$

where A and B are arbitrary, $(r/2 \text{ x } r/2)$, orthogonal matrices, and K is the $(r/2 \text{ x } r/2)$ matrix with elements

$$(1.2.3) \quad k_{ij} = \begin{cases} 1 & \text{if } i + j = \frac{r}{2} + 1 \\ 0 & \text{otherwise} \end{cases} ;$$

and if V is not doubly symmetric, then there is no matrix H, of the form shown in (1.2.2), which reduces V to the form shown in (1.2.1).

That this result is true can be shown in the following way: first the matrix H is orthogonal. This can be verified by a simple multiplication. The product HH' is

$$(1.2.4) \quad \frac{1}{\sqrt{2}} \left|\begin{array}{c|c} A & AK \\ \hline B & -BK \end{array}\right| \frac{1}{\sqrt{2}} \left|\begin{array}{c|c} A & AK \\ \hline B & -BK \end{array}\right| = \frac{1}{2} \left|\begin{array}{c|c} A(I + KK')A' & A(I-KK')B' \\ \hline B(I - KK')A' & B(I+KK')B' \end{array}\right|$$

but $KK' = I$, and A and B are orthogonal by hypothesis.

Second, suppose that V is an arbitrary symmetric matrix. Then V can be partitioned,

$$(1.2.5) \qquad V = \left|\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}' & V_{22} \end{array}\right| \quad ,$$

where

$$(1.2.6) \quad V_{11} = \left|\begin{array}{cccccc} v_{11} & v_{12} & v_{13} & \cdot & \cdot & \cdot & v_{1,r/2} \\ v_{12} & v_{22} & v_{23} & \cdot & \cdot & \cdot & v_{2,r/2} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ v_{1,r/2} & v_{2,r/2} & v_{3,r/2} & \cdot & \cdot & \cdot & v_{r/2,r/2} \end{array}\right| \quad ,$$

$$(1.2.7) \quad V_{22} = \left|\begin{array}{cccc} v_{(r+2)/2,(r+2)/2} & v_{(r+2)/2,(r+4)/2} & \cdots & v_{(r+2)/2,r} \\ v_{(r+2)/2,(r+4)/2} & v_{(r+4)/2,(r+4)/2} & \cdots & v_{(r+4)/2,r} \\ \vdots & \vdots & & \vdots \\ v_{(r+2)/2,r} & v_{(r+4)/2,r} & \cdots & v_{rr} \end{array}\right|$$

and

$$(1.2.8)\quad V_{12} = \begin{vmatrix} v_{1,(r+2)/2} & v_{1,(r+4)/2} & \cdots & v_{1r} \\ v_{2,(r+2)/2} & v_{2,(r+4)/2} & \cdots & v_{2r} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ v_{r/2,(r+2)/2} & v_{r/2,(r+4)/2} & \cdots & v_{r/2,r} \end{vmatrix}$$

The transform of V is

$$(1.2.9)\quad HVH' = \frac{1}{2}\left|\begin{array}{c|c} A(V_{11}+KV_{12}'+V_{12}K'+KV_{22}K')A' & A(V_{11}+KV_{12}'-V_{12}K'-KV_{22}K')B' \\ \hline B(V_{11}-KV_{12}'+V_{12}K'-KV_{22}K')A' & B(V_{11}-KV_{12}'-V_{12}K'+KV_{22}K')B' \end{array}\right|$$

Since A and B are both of full rank, necessary and sufficient conditions that HVH' be of the form shown in (1.2.1) are that

$$(1.2.10)\quad V_{11} - KV_{12}' + V_{12}K' - KV_{22}K' = 0 ,$$

and that

$$(1.2.11)\quad V_{11} + KV_{12}' - V_{12}K' - KV_{22}K' = 0 .$$

These equations are satisfied if and only if

$$(1.2.12)\quad V_{11} = KV_{22}K' ,$$

and

$$(1.2.13)\quad KV_{12}' = V_{12}K' .$$

The (i, j) element of $KV_{22}K'$ is $v_{r-j+1,r-i+1}$, for $i = 1, 2, \ldots, r/2$, $j \geq i$, and is $v_{r-i+1,r-j+1}$, for $j = 1, 2, \ldots, r/2$, $i \geq j$; the (i, j) element of KV'_{12} is $v_{j,r-i+1}$, for $i, j = 1, 2, \ldots, r/2$; and the (i, j) element of $V_{12}K'$ is $v_{i,r-j+1}$, for $i,j = 1, 2, \ldots, r/2$.

If (1.2.10) and (1.2.11) both hold, then (1.2.12) and (1.2.13) both hold, and it follows that

$$(1.2.14) \quad v_{ij} = \begin{cases} v_{r-j+1,r-i+1} & \text{when } i = 1, 2, \ldots, r/2,\ j \geq i \\ v_{r-i+1,r-j+1} & \text{when } j = 1, 2, \ldots, r/2,\ i \geq j \end{cases},$$

and

$$(1.2.15) \quad v_{i,r-j+1} = v_{j,r-i+1} \qquad i, j = 1, 2, \ldots, r/2 \ .$$

Equation (1.2.15) implies that V_{12} is symmetric about its secondary diagonal and similarly for V'_{12}. Equation (1.2.14) implies that the matrix V^* ,

$$(1.2.16) \qquad V^* = \left|\begin{array}{c|c} V_{11} & 0 \\ \hline 0 & V_{22} \end{array}\right| ,$$

is symmetric about its secondary diagonal. Therefore, (1.2.10) and (1.2.11), taken together, imply that V is symmetric about its secondary diagonal; and, since V is symmetric by hypothesis, this amounts to double symmetry.

If V is doubly symmetric, then

$$(1.2.17)\qquad \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}' & V_{22} \end{array}\right] = \left[\begin{array}{c|c} 0 & K \\ \hline K & 0 \end{array}\right] \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}' & V_{22} \end{array}\right] \left[\begin{array}{c|c} 0 & K \\ \hline K & 0 \end{array}\right]$$

or

$$(1.2.18)\qquad \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}' & V_{22} \end{array}\right] = \left[\begin{array}{c|c} KV_{22}K' & KV_{12}'K' \\ \hline KV_{12}K' & KV_{11}K' \end{array}\right] ,$$

from which it follows that

$$(1.2.19)\qquad V_{11} = KV_{22}K' ,$$

and that

$$(1.2.20)\qquad KV_{12}' = V_{12}K' .$$

This proves the result.

1.2 <u>A Property of the Tshebysheff Orthogonal Polynomials.</u> The Tshebysheff orthogonal polynomials are a set of r polynomials $P_i(x)$,

$$(1.2.1)\qquad P_i(x) = \sum_{j=0}^{i} c_{ij}\, x^j \ , \quad i = 0, 1, 2, \ldots, r-1 ,$$

where the c_{ij} are so determined that

$$(1.2.2)\qquad \sum_{k=1}^{r} \Big(\sum_{j=0}^{i} c_{ij}\, u_k^j\Big)\Big(\sum_{j=0}^{i'} c_{i'j}\, u_k^j\Big) = 0 ,$$

when $i \neq i'$ and $u_k = (2k-r-1)/2$, $k = 1, 2, \ldots, r$.

The numbers P_{ik},

$$(1.2.3)\qquad P_{ik} = \sum_{j=0}^{i} c_{ij}\, u_k^j \ , \quad k = 1, 2, \ldots, r,\ i = 1, 2, \ldots, r-1 ,$$

are tabulated, (see R. L. Anderson and E. E. Houseman [3], and

Fisher and Yates [10]), for various values of r.

Suppose r is even. The (rxr) matrix P,

$$
(1.2.4)\quad P = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
P_{21} & P_{22} & \cdots & P_{2,r} \\
P_{41} & P_{42} & \cdots & P_{4,r} \\
\vdots & \vdots & & \vdots \\
P_{r-2,1} & P_{r-2,2} & \cdots & P_{r-2,r} \\
P_{11} & P_{12} & \cdots & P_{1,r} \\
P_{31} & P_{32} & \cdots & P_{3,r} \\
\vdots & \vdots & & \vdots \\
P_{r-1,1} & P_{r-1,2} & \cdots & P_{r-1,r}
\end{vmatrix}
$$

is of the form

$$
(1.2.5)\qquad P = \left|\begin{array}{c|c} A & AK \\ \hline B & -BK \end{array}\right| ,
$$

where A and B are (r/2 x r/2) matrices, with the rows in each matrix mutually orthogonal, and K is the (r/2 x r/2) matrix defined in (1.2.3) .

That this statement is true may be seen in the following way: The values of P_{ik}, according to R. L. Anderson [4], are

$$(1.2.6)\quad P_{ik} = \begin{cases} \sum_{j=1}^{\frac{i+1}{2}} c_{i,\,2j-1}\, u_k^{2j-1} & \text{when } i \text{ is odd} \\ \sum_{j=0}^{\frac{i}{2}} c_{i,2j}\, u_k^{2j} & \text{when } i \text{ is even} \end{cases} .$$

When r is even, $u_k = -u_{r-k+1}$, $k = 1, 2, \ldots, r/2$. Thus when i is odd $P_{ik} = -P_{i,r-k+1}$, $k = 1, 2, \ldots, r/2$; and when i is even $P_{ik} = P_{i,r-k+1}$, $k = 1, 2, \ldots, r/2$; and from this fact it follows that P is of the form shown in (1.2.5). That the rows of A are mutually orthogonal follows the properties of the orthogonal polynomials and similarly for B.

The matrix A is

$$(1.2.7)\quad A = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ P_{21} & P_{22} & \cdots & P_{2,r/2} \\ P_{41} & P_{42} & \cdots & P_{4,r/2} \\ \vdots & \vdots & & \vdots \\ P_{r-2,1} & P_{r-2,2} & \cdots & P_{r-2,r/2} \end{vmatrix} ,$$

and the matrix B is

$$(1.2.8)\quad B = \begin{vmatrix} P_{11} & P_{12} & \cdots & P_{1,r/2} \\ P_{31} & P_{32} & \cdots & P_{3,r/2} \\ \vdots & \vdots & & \\ P_{r-1,1} & P_{r-1,2} & \cdots & P_{r-1,r/2} \end{vmatrix} .$$

1.3 The Characteristic Roots of a Matrix. It is required to find the characteristic roots of the (rxr) matrix V,

$$(1.3.1)\quad V = \begin{vmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{r-1} \\ \rho & 1 & \rho & \cdots & \rho^{r-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \rho^{r-1} & \rho^{r-2} & \rho^{r-3} & & 1 \end{vmatrix} \quad \frac{1}{1-\rho^2} ,$$

where ρ is a real number $|\rho| < 1$.

This matrix is positive definite, and its inverse is

$$(1.3.2)\quad V^{-1} = (1+\rho^2)I - \rho^2 B - \rho C ,$$

where I is the (rxr) identity matrix, B is the (rxr) matrix with all elements zero except for $b_{11} = b_{rr} = 1$, and C is the (rxr) matrix with all elements zero except for $c_{ij} = 1$ if $|i-j|=1$.

If λ_k, $k = 1, 2, \ldots, r$, are the roots of V^{-1}, then λ_k^{-1} are the roots of V. The problem is therefore equivalent to that of finding values of λ so that

$$(1.3.3)\quad |(1 + \rho^2 - \lambda)I - \rho^2 B - \rho C| = 0 .$$

Let $\Delta_r(\lambda,\rho)$ be the left-hand side of (1.3.3). Expanding $\Delta_r(\lambda,\rho)$ by minors yields

$$(1.3.4)\quad \Delta_r(\lambda,\rho) = (1-\lambda)^2 D_{r-2}(\lambda,\rho) - 2\rho^2(1-\lambda)D_{r-3}(\lambda,\rho) + \rho^4 D_{r-4}(\lambda,\rho),$$

where $D_r(\lambda,\rho)$ is the determinant of the (rxr) matrix V^*,

$$(1.3.5)\quad V^* = (1+\rho^2-\lambda)I - \rho C \ .$$

Expanding $D_r(\lambda,\rho)$ by minors leads to the difference equation,

$$(1.3.6)\quad D_r(\lambda,\rho)-(1+\rho^2-\lambda)\ D_{r-1}(\lambda,\rho) + \rho^2\ D_{r-2}(\lambda,\rho) = 0 \ ;$$

and it is known, (see Siddiqui [15], Hildebrand [11]), that this difference equation has the solution,

$$(1.3.7)\quad D_r(\lambda,\rho) = \rho^r\ \frac{\sin\ (r+1)\Theta}{\sin\ \Theta} \quad ,$$

where $1+\rho^2-\lambda = 2\rho\cos\Theta$.

Substituting (1.3.7) in (1.3.4) and equating $\Delta_r(\lambda,\rho)$ to zero results in the equation,

$$(1.3.8)\quad (1-\lambda)^2\ \rho^{r-2}\ \frac{\sin(r-1)\Theta}{\sin\ \Theta} - 2(1-\lambda)\rho^{r-1}\ \frac{\sin(r-2)\Theta}{\sin\ \Theta} + \rho^r\ \frac{\sin(r-3)\Theta}{\sin\ \Theta} = 0 \ .$$

If equation (1.3.8) is multiplied by $\sin\ \Theta$, ($\sin\Theta = 0$ is not a solution), and if the complex representation of the sine function is used, equation (1.3.8), after a few manipulations, becomes

$$(1.3.9)\quad e^{i(r-3)\Theta}\ ((1-\lambda)e^{i\Theta} - \rho)^2 = e^{-i(r-3)\Theta}((1-\lambda)e^{-i\Theta} - \rho)^2 \ .$$

Thus

$$(1.3.10)\quad e^{\frac{i(r-3)\Theta}{2}}\ ((1-\lambda)e^{i\Theta}-\rho) = \pm\ e^{\frac{-i(r-3)\Theta}{2}}\ ((1-\lambda)e^{-i\Theta} - \rho) \ ;$$

and this can be expressed in the form

$$(1.3.11)\quad (1-\lambda)\sin\frac{(r-1)}{2}\theta = \rho\sin\frac{(r-3)}{2}\theta ,$$

and

$$(1.3.12)\quad (1-\lambda)\cos\frac{(r-1)}{2}\theta = \rho\cos\frac{(r-3)}{2}\theta ;$$

or in the form

$$(1.3.13)\quad (2\cos\theta-\rho)\sin\frac{(r-1)}{2}\theta = \sin\frac{(r-3)}{2}\theta ,$$

and

$$(1.3.14)\quad (2\cos\theta-\rho)\cos\frac{(r-1)}{2}\theta = \cos\frac{(r-3)}{2}\theta ,$$

since $1+\rho^2-\lambda = 2\cos\theta$. Finally, since

$$(1.3.15)\quad 2\cos\theta\sin\frac{(r-1)}{2}\theta = \sin\frac{(r+1)}{2}\theta + \sin\frac{(r-3)}{2}\theta ,$$

and

$$(1.3.16)\quad 2\cos\theta\cos\frac{(r-1)}{2}\theta = \cos\frac{(r+1)}{2}\theta + \cos\frac{(r-3)}{2}\theta ,$$

are well-known identities, (1.3.13) and (1.3.14) can be reduced to

$$(1.3.17)\quad \sin\frac{(r+1)}{2}\theta = \rho\sin\frac{(r-1)}{2}\theta ,$$

and

$$(1.3.18)\quad \cos\frac{(r+1)}{2}\theta = \rho\cos\frac{(r-1)}{2}\theta .$$

The problem of the roots is solved when values of θ, say θ_k, are found which are solutions to (1.3.17) and (1.3.18). The roots of V^{-1} will be $\lambda_k = 1 - 2\rho\cos\theta_k + \rho^2$.

Apparently there is no closed form solution for (1.3.17) or for (1.3.18) which holds in general. It is possible, however, to find approximate solutions which are as accurate as one cares to make then. An iterative solution is proposed here, and this solution is shown to converge.

The numerical solution of (1.3.17) proceeds in the following way: if $\rho=1$, the solutions are

$$(1.3.19)\quad \theta_k = \frac{(2k-1)\pi}{r}, \qquad k = \begin{array}{l} 1,2,\ldots, r/2, \quad r \text{ even} \\ 1,2,\ldots,(r-1)/2, \; r \text{ odd} \end{array} ;$$

if $\rho = 0$, the solutions are

$$(1.3.20)\quad \theta_k = \frac{2k\pi}{r+1}, \qquad k = \begin{array}{l} 1,2,\ldots, r/2, \quad r \text{ even} \\ 1,2,\ldots,(r-1)/2, \; r \text{ odd} \end{array} ;$$

and if $\rho = -1$, the solutions are

$$(1.3.21)\quad \theta_k = \frac{2k\pi}{r}, \qquad k = \begin{array}{l} 1,2,\ldots,r/2, \quad r \text{ even} \\ 1,2,\ldots,(r-1)/2, \; r \text{ odd} \end{array} .$$

Let R_k be an open interval on the θ-axis,

$$(1.3.22)\quad R_k: \quad \left(\frac{(2k-1)\pi}{r}, \; \frac{2k\pi}{r+1}\right),$$

where $k = 1, 2, \ldots, r/2$ if r is even, and $k = 1, 2, \ldots, (r-1)/2$ if r is odd. The function $f(\theta)$,

$$(1.3.23)\quad f(\theta) = \frac{\sin \frac{(r+1)}{2}\theta}{\sin \frac{(r-1)}{2}\theta},$$

is continuous for $\theta \in R_k$, and

$$(1.3.24) \qquad f(\theta) = \begin{cases} 1 & \theta = \dfrac{(2k-1)\pi}{r} \\ 0 & \theta = \dfrac{2k\pi}{r+1} \end{cases} ;$$

i.e., $f(\theta)$ is unity at the left end point of each R_k, and $f(\theta)$ vanishes at the right end point of each R_k.

If it can be shown that $f'(\theta)$ is strictly negative for $\theta \in R_k$, then it will follow that $f(\theta)$ takes on each value between zero and one exactly once in each interval R_k.

It is to be proved that $f'(\theta)$,

$$(1.3.25) \qquad f'(\theta) = \frac{\sin r\theta - r \sin \theta}{2 \sin^2((r-1)/2)\theta} ,$$

is strictly negative for $\theta \in R_k$. From (1.3.22) it is clear that $\sin \theta > 0$ and $\sin r\theta < 0$ for $\theta \in R_k$. This proves the result, and it can be asserted that (1.3.17) has exactly one solution in each interval R_k, provided that $0 < \rho < 1$.

To find the solution for a given value of ρ, r and k, the following iterative procedure is proposed: Put (1.3.17) in the form,

$$(1.3.26) \qquad \theta = \frac{2}{r+1} \sin^{-1} \left(\rho \sin \frac{(r-1)}{2} \theta\right) .$$

Then choose a point, say θ_o, from the interval R_k, and substitute this point θ_o for θ in the right hand side of (1.3.26). The value of ρ and r are known so (1.3.26) now has a numerical value which can be calculated. This number, say θ_1, is a first approximation of the true solution. A second approximation, say θ_2, is obtained by substituting θ_1 back in (1.3.26). In this way a sequence of approximations is produced.

The conditions under which this sequence converges to a true root are stated in the following theorem from Brand [6]: If $\Theta_0 \in R$, where R is an interval containing a root of the equation $\Theta = g(\Theta)$, and if $g(\Theta)$ is differentiable throughout R, and if $|g'(\Theta)| < M < 1$ for $\Theta \in R$, then the sequence of iterative solution of the equation $\Theta = g(\Theta)$, starting with the point Θ_0, converges to a root of the equation.

That the conditions of this theorem are satisfied in the case of (1.3.26) is easily seen. First $\Theta_0 \in R_k$, and R_k contains exactly one solution. Second, the derivative of the right-hand side of (1.3.26) is

$$(1.3.27)\quad \frac{d}{d\,\Theta}\left(\frac{2}{r+1}\sin^{-1}\left(\rho\sin\left(\frac{r-1}{2}\right)\Theta\right)\right) = \frac{r-1}{r+1}\;\frac{\rho\cos\left(\frac{r-1}{2}\right)\Theta}{\sqrt{1-\rho^2\sin^2\left(\frac{r-1}{2}\right)\Theta}}\;,$$

which exists for all Θ when $|\rho| < 1$; and

$$(1.3.2\ )\qquad \frac{r-1}{r+1}\left|\frac{\rho\cos\left(\frac{r-1}{2}\right)\Theta}{\sqrt{1-\rho^2\sin^2\left(\frac{r-1}{2}\right)\Theta}}\right| < \frac{r-1}{r+1} < 1\;,$$

when $|\rho| < 1$.

If $-1 < \rho < 0$, then the R_k become

$$(1.3.29)\quad R_k:\ \left(\frac{2k\pi}{r+1}\,,\ \frac{2k\pi}{r}\right)\,,\qquad k = \begin{cases} 1,2,\ldots,\ r/2 & \text{if } r \text{ is even} \\ 1,2,\ldots,\ (r-1)/2 & \text{is } r \text{ is odd} \end{cases}\,.$$

In this case,

$$(1.3.30) \quad f(\Theta) = \begin{cases} 0 & \text{if} \quad \Theta = \dfrac{2k\pi}{r+1} \\[2ex] -1 & \text{if} \quad \Theta = \dfrac{2k\pi}{r} \end{cases}$$

To prove that $f'(\Theta) < 0$ for all $\Theta \in R_k$, it is necessary to show that

$$(1.3.31) \quad \sin r\,\Theta - r \sin \Theta < 0 \,,$$

for $\Theta \in R_k$. This is true for the same reason as before, viz. that $\sin \Theta > 0$ and $\sin r\,\Theta < 0$ when $\Theta \in R_k$. So, exactly as before, it turns out that there is exactly one solution in each interval R_k. The proposed iterative procedure still converges since it was proved to converge for $|\rho| < 1$. This completes the analysis of equation (1.3.17).

Consider equation (1.3.1[illegible]). If $\rho = 1$, the solutions are

$$(1.3.32) \quad \Theta_k = \frac{2k\pi}{r} \,, \qquad k = \begin{cases} 0,1,2,\ldots,\ (r-2)/2 \text{ if } r \text{ is even} \\ 0,1,2,\ldots,\ (r-1)/2 \text{ if } r \text{ is odd} \end{cases}$$

if $\rho = 0$, the solutions are

$$(1.3.33) \quad \Theta_k = \frac{(2k-1)\pi}{r+1} \,, \qquad k = \begin{cases} 1,2,\ldots,\ r/2 \text{ if } r \text{ is even} \\ 1,2,\ldots,\ (r+1)/2 \text{ if } r \text{ is odd} \end{cases}$$

and if $\rho = -1$, the solutions are

$$(1.3.34) \quad \Theta_k = \frac{(2k-1)\pi}{r} \,, \qquad k = \begin{cases} 1,2,\ldots,r/2 \text{ if } r \text{ is even} \\ 1,2,\ldots,\ (r+1)/2 \text{ if } r \text{ is odd} \end{cases}$$

For $0 < \rho < 1$, the intervals R_k are

$$(1.3.35)\quad R_k:\ \left(\frac{(2k-2)\pi}{r},\ \frac{(2k-1)\pi}{r+1}\right),\quad k = \begin{cases} 1,2,\ldots,r/2 \text{ if } r \text{ is even} \\ 1,2,\ldots,(r+1)/2 \text{ if } r \text{ is odd} \end{cases}$$

and for $-1 < \rho < 0$, the intervals R_k are

$$(1.3.36)\quad R_k:\ \left(\frac{(2k-1)\pi}{r+1},\ \frac{(2k-1)\pi}{r}\right),\quad k = \begin{cases} 1,2,\ldots,r/2 \text{ if } r \text{ is even} \\ 1,2,\ldots,(r-1)/2 \text{ is } r \text{ is odd} \end{cases}$$

To prove that

$$(1.3.37)\quad \frac{d}{d\,\Theta}\ \frac{\cos\frac{(r+1)}{2}\Theta}{\cos\frac{r-1}{2}\Theta} = \frac{-r\sin\Theta - \sin r\,\Theta}{2\cos^2\frac{(r-1)}{2}\Theta}$$

is always negative for $\Theta \in R_k$, it is only necessary to observe that $\sin\ \Theta > 0$ and $\sin r\,\Theta > 0$ for $\Theta \in R_k$. This proves that there is exactly one solution in each R_k. The proposed iterative solution will converge if

$$(1.3.38)\quad \frac{d}{d\,\Theta}\left(\frac{2}{r+1}\ \cos^{-1}\left(\rho\cos\frac{r-1}{2}\Theta\right)\right) = \frac{r-1}{r+1}\ \frac{\rho\sin\left(\frac{r-1}{2}\right)\Theta}{\sqrt{1-\rho^2\cos^2\left(\frac{r-1}{2}\right)\Theta}}$$

exists for $\Theta \in R_k$ and is less in modulus than some number which is less than one. When $|\rho| < 1$, the derivative exists for all Θ, and

$$(1.3.39)\quad \frac{r-1}{r+1}\left|\frac{\rho\sin\frac{(r-1)}{2}\Theta}{\sqrt{1-\rho^2\cos^2\frac{(r-1)}{2}\Theta}}\right| < \frac{r-1}{r+1}$$

This completes the analysis of equation (1.3.18).

1.4 A Problem of Identification. For r even, consider the $(r\mathrm{x}r)$ matrix,

$$(1.4.1)\quad \frac{1}{\sqrt{2}} \left|\begin{array}{c|c} I & K \\ \hline I & -K \end{array}\right| V^{-1} \left|\begin{array}{c|c} I & K \\ \hline I & -K \end{array}\right| \frac{1}{\sqrt{2}} = \left|\begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array}\right| ,$$

where V^{-1} is the $(r\mathrm{x}r)$ matrix shown in (1.3.2), and K is the $(r/2 \times r/2)$ matrix defined in (1.2.3). It is required to find the characteristic roots of V_1 and V_2.

The roots of V^{-1} can be calculated by the method of the previous section; and, since the roots of V_1 and V_2 are also the roots of V^{-1}, the problem consists of identifying which roots of V^{-1} are also the roots of either V_1 or V_2.

Let $\Delta_{r/2}(\lambda,\rho)$ be the determinant of the matrix $V_1 - \lambda I$,

$$(1.4.2)\quad V_1 = (1+\rho^2)I - \rho C - \rho^2 E ,$$

where C is the $(r/2 \times r/2)$ matrix with all elements zero except for $c_{ij} = 1$ if $|i - j| = 1$, and E is the $(r/2 \times r/2)$ matrix with all elements zero except $e_{11} = 1$. Expanding by minors yields

$$(1.4.3)\quad \Delta_{r/2}(\lambda,\rho) = (1-\lambda)(1-\rho+\rho^2-\lambda)\, D_{r-4/2}(\lambda,\rho) \\ -\rho^2(2-\rho+\rho^2-2\lambda)D_{r-6/2}(\lambda,\rho)+\rho^4 D_{r-8/2}(\lambda,\rho)$$

where $D_r(\lambda,\rho)$ satisfies the difference equation (1.3.6). The solution of (1.3.6) is given by (1.3.7). Substituting (1.3.7) in (1.4.3), recalling that $1 + \rho^2 - \lambda = 2\rho\cos\theta$, and setting $\Delta_{r/2}(\lambda,\rho)$ equal to zero, leads, after a little algebra, to

$$(1.4.4)\quad (2\cos\Theta-\rho)(2\cos\Theta-1)\sin\frac{(r-2)}{2}\Theta$$

$$-(4\cos\Theta-\rho-1)\sin\frac{(r-4)}{2}\Theta+\sin\frac{(r-6)}{2}\Theta=0\ .$$

Repeated application of the trignometric identity,

$$(1.4.5)\quad 2\sin\Theta_1\cos\Theta_2 = \sin(\Theta_1+\Theta_2)+\sin(\Theta_1-\Theta_2)\ ,$$

yields the equation,

$$(1.4.6)\quad 2\sin\frac{\Theta}{2}\left(\cos\frac{(r+1)}{2}\Theta-\rho\cos\frac{(r-1)}{2}\Theta\right) = 0\ .$$

Thus it is seen that the roots of V_1 are those roots of V^{-1} which are derived from equation (1.3.18). The roots of V_2 are, of course, the roots of V^{-1} which are derived from equation (1.3.17).

1.5 The Inverse of a Matrix. It is required to find the inverse of the (rxr) matrix V,

$$(1.5.1)\quad V=(1+\rho^2)I-\rho C\ ,$$

where C is the same matrix that appears in (1.3.2).

Professor Hotelling, in his lectures on least squares, gives a method for finding the inverse of a sub-matrix of a matrix if the inverse of the matrix is known. In this case V is a sub-matrix of the $((r+2)x(r+2))$ matrix V_1 ,

$$(1.5.2)\quad V_1=(1+\rho^2)\,I-\rho^2B-\rho C\ ,$$

where C is an $((r+2)\ x\ (r+2))$ matrix of the same form as appears in (1.5.1), and B is the $((r+2) \times (r+2))$ matrix with all elements zero except for $b_{11} = b_{r+2,\, r+2} = 1$. The inverse of V_1 is

$$(1.5.3)\quad V_1^{-1} = \begin{vmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{r+1} \\ \rho & 1 & \rho & \cdots & \rho^{r} \\ \rho^2 & \rho & 1 & \cdots & \rho^{r-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \\ \rho^{r+1} & \rho^{r} & \rho^{r-1} & \cdots & 1 \end{vmatrix} \quad \frac{1}{1-\rho^2}$$

By applying Professor Hotelling's method, the inverse of the $((r+1) \times (r+1))$ matrix V_2, which is V_1 with the $(r+2)$-nd row and the $(r+2)$-nd column deleted, can be found. By applying the method a second time the inverse of the $(r \times r)$ matrix, which is V_2 with the first row and the first column deleted, can be found. This last inverse is the inverse of V.

The method is this: If v_{ij} are the elements of V_1^{-1} and v^*_{ij} are the elements of V_2^{-1}, then

$$(1.5.4)\qquad v^*_{ij} = v_{ij} - \frac{v_{i,r+2}\, v_{r+2,j}}{v_{r+2,r+2}}$$

In this case

$$(1.5.5)\qquad v_{ij} = \frac{\rho^{|i-j|}}{1-\rho^2}, \quad i = 1, 2, \ldots, r+2,\ j = 1,2,\ldots,r+2\ ;$$

so

$$(1.5.6)\qquad v^*_{ij} = \frac{1}{1-\rho^2}\left(\rho^{|i-j|} - \rho^{2r+4-i-j}\right),$$

$i = 1, 2, \ldots, r+1, \quad j = 1, 2, \ldots, r+1$.

If v_{ij}^{**} are the elements of V^{-1}, then

$$(1.5.7) \quad v_{ij}^{**} = v_{ij}^{*} - \frac{v_{i1}^{*} v_{1j}^{*}}{v_{11}^{*}} \quad ,$$

or

$$(1.5.8) \quad v_{ij}^{**} = \frac{1}{1-\rho^2} (\rho^{|i-j|} - \rho^{2r-i-j+4} - \frac{(\rho^{i-1} - \rho^{2r-i+3})(\rho^{j-1} - \rho^{2r-j+3})}{1-\rho^{2r+2}}$$

$i = 2, 3, \ldots r+1, \; j = 2, 3, \ldots, r+1$,

or

$$(1.5.9) \quad v_{ij}^{**} = \frac{1}{1-\rho^2} (\rho^{|i-j|} - \rho^{2r-i-j+2} - \frac{(\rho^{i} - \rho^{2r-i+2})(\rho^{i} - \rho^{2r-j+2})}{1-\rho^{2r+2}} \quad ,$$

$i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r$.

It is required to find the value of the quantity,

$$(1.5.10) \quad \underline{1}' \, V^{-1} \, \underline{1} = \sum_{ij}^{r} v_{ij}^{**} \quad ,$$

where $\underline{1}$ is the $(r \times 1)$ matrix with all elements one. From (1.5.9), it is seen that

$$(1.5.11) \quad \sum_{ij}^{r} v_{ij}^{**} = \frac{1}{1-\rho^2} \sum_{ij}^{r} (\rho^{|i-j|} - \rho^{2r-i-j+2} - \frac{(\rho^{i} - \rho^{2r-i+2})(\rho^{j} - \rho^{2r-j+2})}{1-\rho^{2r+2}} \quad ,$$

or that

$$(1.5.12)\quad \sum_{ij}^{r} v_{ij}^{**} = \frac{1}{1-\rho^2} \sum_{ij}^{r} \rho^{|i-j|} - \sum_{ij}^{r} \rho^{2r-i-j+2}$$

$$- \frac{1}{1-\rho^2} \sum_{ij}^{r} \frac{(\rho^{i+j} - \rho^{2r-i+j+2} - \rho^{2r+i-j+2} + \rho^{4r-i-j+4})}{1-\rho^{2r+2}}$$

Since

$$(1.5.13)\quad \sum_{ij}^{r} \rho^{2r-i-j+2} = \rho^2 \sum_{i=1}^{r} \rho^{r-i} \sum_{j=1}^{r} \rho^{r-j} = \frac{\rho^2(1-\rho^r)^2}{(1-\rho)^2} \quad ,$$

$$(1.5.14)\quad \sum_{ij}^{r} \rho^{i+j} = \rho^2 \sum_{i=1}^{r} \rho^{i-1} \sum_{j=1}^{r} \rho^{j-1} = \frac{\rho^2(1-\rho^r)^2}{(1-\rho)^2} \quad ,$$

$$(1.5.15)\quad \sum_{ij} \rho^{2r-i+j+2} = \rho^{r+3} \sum_{i=1}^{r} \rho^{r-i} \sum_{j=1}^{r} \rho^{j-1} = \rho^{r+3} \frac{(1-\rho^r)^2}{(1-\rho)^2} \quad ,$$

$$(1.5.16)\quad \sum_{ij} \rho^{2r+i-j+2} = \rho^{r+3} \frac{(1-\rho^r)^2}{(1-\rho)^2} \quad ,$$

$$(1.5.17)\quad \sum_{ij} \rho^{4r-i-j+4} = \rho^{2r+4} \sum_{i=1}^{r} \rho^{r-i} \sum_{j=1}^{r} \rho^{r-j} = \rho^{2r+4} \frac{(1-\rho^r)^2}{(1-\rho)^2} \quad ,$$

it turns out that

$$(1.5.18)\quad \sum_{ij} v_{ij}^{**} = \frac{1}{1-\rho^2} \sum_{ij}^{r} \rho^{|i-j|} - 2\,\frac{\rho^2}{1-\rho^2} \frac{(1-\rho^r)^2}{(1-\rho)^2} \frac{1}{1+\rho^{r+1}} \quad .$$

Finally

$$(1.5.19)\quad \sum_{ij}^{r} \rho^{|i-j|} = r + 2 \sum_{i=1}^{r-1} (r-i)\, \rho^{i} \quad ,$$

but

$$(1.5.20)\quad \sum_{i=1}^{r-1} (r-i)\,\rho^{i} = \sum_{j=1}^{r-1} \left(\sum_{i=1}^{r-j} \rho^{i} \right) = \frac{\rho}{1-\rho}\left((r-1) - \rho\frac{(1-\rho^{r-1})}{1-\rho}\right) .$$

So

$$(1.5.21)\quad \sum_{ij}^{r} \rho^{|i-j|} = \frac{r - 2\rho - r\rho^{2} + 2\rho^{r+1}}{(1-\rho)^{2}} .$$

The final result is

$$(1.5.22)\quad \sum_{ij} v_{ij}^{**} = \frac{r(1-\rho)(1+\rho^{r+1}) - 2\rho(1-\rho^{r})}{(1-\rho)^{3}\ (1+\rho^{r+1})}$$

It is interesting to note that this result can be arrived at in another way. It is true that

$$(1.5.23)\quad \underline{1}'\ V^{-1}\ \underline{1} = \underline{1}'\ H'H\ V^{-1}\ H'H\ \underline{1} ,$$

where H is the $(r\mathrm{x}r)$ orthogonal matrix which reduces V^{-1} to diagonal form. Thus

$$(1.5.24)\quad \underline{1}'\ V^{-1}\ \underline{1} = \underline{1}'\ H'\ D(\lambda_k)\ H\ \underline{1} ,$$

where λ_k are the characteristic roots of V^{-1}. It is known that the characteristic roots of V^{-1} are

$$(1.5.25)\quad \lambda_k = 1 - 2\rho \cos \frac{k\pi}{r+1} + \rho^{2} ,$$

and that the k-th column of H' is $\underline{h}_k$, where

$$(1.5.26)\quad \underline{h}_k' = \frac{2}{r+1}\left(\sin \frac{k\pi}{r+1} , \sin \frac{2k\pi}{r+1} , \ldots, \sin \frac{rk\pi}{r+1}\right) ,$$

$k = 1, 2, \ldots, r.$

Thus

$$(1.5.27) \qquad \underline{1}'\, V^{-1}\, \underline{1} = \frac{2}{r+1} \sum_{k=1}^{r} \frac{\left(\sum_{j=1}^{r} \sin \frac{jk\pi}{r+1}\right)^2}{1-2\rho \cos \frac{k\pi}{r+1} + \rho^2} ,$$

which proves that

$$(1.5.28) \qquad \frac{2}{r+1} \sum_{k=1}^{r} \frac{\left(\sum_{j=1}^{r} \sin \frac{jk\pi}{r+1}\right)^2}{1-2\rho \cos \frac{k\pi}{r+1} + \rho^2} = \frac{r(1-\rho)(1+\rho^{r+1})-2\rho(1-\rho^r)}{(1-\rho)^3\,(1+\rho^{r+1})}$$

when $\rho = 0$ (1.5.28) reduces to the identity,

$$(1.5.29) \qquad \sum_{k=1}^{r} \left(\sum_{j=1}^{r} \sin \frac{jk\pi}{r+1}\right)^2 = \frac{r(r+1)}{2} .$$

1.6 Some Identities. First, (see Titchmarsh [17], and Hildebrand [11]),

$$(1.6.1) \qquad \frac{1-\rho^2}{1-2\rho \cos \Theta + \rho^2} = 1 + 2 \sum_{n=1}^{\infty} \rho^n \cos n\Theta ,$$

and

$$(1.6.2) \qquad \frac{\rho \sin \Theta}{1-2\rho \cos \Theta + \rho^2} = \sum_{n=1}^{\infty} \rho^r \sin n\Theta ,$$

for all Θ and $|\rho| < 1$.

Second, (see Brand [6]),

$$(1.6.3) \qquad \sum_{k=1}^{n} \cos kx = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \cos \frac{(n+1)}{2} x ,$$

$$(1.6.4) \qquad \sum_{k=1}^{n} \sin kx = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \sin \frac{(n+1)}{2} x ,$$

$$(1.6.5) \qquad \sum_{k=1}^{n} \cos (2k-1)x = \frac{\sin 2\,n\,x}{2 \sin x} \quad ,$$

$$(1.6.6) \qquad \sum_{k=1}^{n} \sin (2k-1)x = \frac{1 - \cos 2\,n\,x}{2 \sin x} \quad ,$$

for all x such that the right-hand side of each formula exists.

CHAPTER II

SOME TESTS OF INDEPENDENCE

2.1 Introduction. This chapter contains some tests of independence. The first test is an intersection type test suggested by Professor S. N. Roy. It is based on a set of statistics which are independently distributed as beta random variables under the null hypothesis. The others are based on statistics which have the F distribution under the null hypothesis; and, for these, it is possible, in some cases, to determine the non-central distributions. In other cases the non-central distributions can be approximated with any desired degree of accuracy.

2.2 An Intersection Test. Consider a set of normally distributed random variables, say $Y_1, Y_2, \ldots, Y_r$, with

$$E(Y_k) = \mu_k \quad , \qquad k = 1, 2, \ldots, r \quad , \tag{2.2.1}$$

and

$$\mathrm{Var}(\underline{Y}) = \begin{vmatrix} 1 & v_1 & v_2 & \cdots & v_{r-1} \\ v_1 & 1 & v_1 & \cdots & v_{r-2} \\ v_2 & v_1 & 1 & \cdots & v_{r-3} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{r-1} & v_{r-2} & v_{r-3} & \cdots & 1 \end{vmatrix} \sigma^2 \quad . \tag{2.2.2}$$

A set of observations, say $y_1, y_2, \ldots, y_r$ is given and it is required to test the hypothesis that all the v_i's are zero.

First suppose that $\mu_k = 0$, $k = 1, 2, \ldots, r$. Then, under the null hypothesis, the statistics B_h,

$$B_h = \frac{\sum_{k=h+1}^{r} Y_k^2}{\sum_{k=h}^{r} Y_k^2} \quad , \quad h = 1, 2, \ldots, r-1 \quad , \tag{2.2.3}$$

have beta distributions, with parameters $(1/2, (r-h)/2)$ and the B_h are mutually independent in the statistical sense.

That the B_h have beta distributions is obvious; that they are independently distributed is proved in the following way: Make the transformation

$$\begin{aligned} Y_1 &= X \cos \Theta_1 \\ Y_k &= X\left(\prod_{i=1}^{k-1} \sin \Theta_i\right) \cos \Theta_k \quad , \quad k = 2, 3, \ldots, r-1 \quad , \\ Y_r &= X \prod_{i=1}^{r-1} \sin \Theta_i \end{aligned} \tag{2.2.4}$$

which has the Jacobian J_r,

$$J_r = X^{r-1} \prod_{i=1}^{r-2} (\sin \Theta_i)^{r-i-1} \quad . \tag{2.2.5}$$

That (2.2.5) is the Jacobian is verified by induction. When $r = 2$,

$$J_r = \begin{vmatrix} \cos \Theta_1 & -X \sin \Theta_1 \\ \sin \Theta_1 & X \cos \Theta_1 \end{vmatrix} = X \quad , \tag{2.2.6}$$

and this is the same as (2.2.5) when $r = 2$. In general

$$(2.2.7)\quad J_r = \begin{vmatrix} \frac{\partial Y_1}{\partial X} & \frac{\partial Y_1}{\partial \Theta_1} & \cdots & \frac{\partial Y_1}{\partial \Theta_{r-1}} \\ \frac{\partial Y_2}{\partial X} & \frac{\partial Y_2}{\partial \Theta_1} & \cdots & \frac{\partial Y_2}{\partial \Theta_{r-1}} \\ \vdots & \vdots & & \\ \frac{\partial Y_r}{\partial X} & \frac{\partial Y_r}{\partial \Theta_1} & \cdots & \frac{\partial Y_r}{\partial \Theta_{r-1}} \end{vmatrix}$$

Look at (2.2.4). It is clear that X is a factor of the last (r-1) columns of J_r, that $\sin \Theta_1$ is a factor of the last (r-2) columns, that $\sin \Theta_2$ is a factor of the last (r-3) columns, etc., and finally that $\sin \Theta_{r-2}$ is a factor of the next to last column. Thus

$$(2.2.\ )\qquad J_r = X^{r-1} \prod_{i=1}^{r-2} (\sin \Theta_i)^{r-i-1} \Delta_r ,$$

where Δ_r is the determinant whose r-th row is

$$(2.2.9)\quad \left(\prod_{i=1}^{r-1} \sin \Theta_i,\ \cos \Theta_1 \prod_{i=2}^{r-1} \sin \Theta_i,\ \cos \Theta_2 \prod_{i=3}^{r-1} \sin \Theta_i ,\right.$$

$$\left. \ldots\ \cos \Theta_{r-2} \sin \Theta_{r-1},\ \cos \Theta_{r-1}\right) .$$

The same factorization of J_{r+1} yields

$$(2.2.10)\quad J_{r+1} = X^r \prod_{i=1}^{r-1} (\sin \Theta_i)^{r-i} \Delta_{r+1} ,$$

where Δ_{r+1} is the determinant whose r-th row is

$$(2.2.11)\quad (\cos\Theta_r \prod_{i=1}^{r-1} \sin\Theta_i,\ \cos\Theta_1 \cos\Theta_r \prod_{i=2}^{r-1} \sin\Theta_i\ ,$$

$$\cos\Theta_2 \cos\Theta_r \prod_{i=3}^{r-1} \sin\Theta_i, \ldots, \cos\Theta_{r-1} \cos\Theta_r, -\sin\Theta_r)\ ,$$

whose (r+1)-st row is

$$(2.2.12)\quad (\prod_{i=1}^{r} \sin\Theta_i,\ \cos\Theta_1 \prod_{i=2}^{r} \sin\Theta_i,\ \cos\Theta_2 \prod_{i=3}^{r} \sin\Theta_i\ ,$$

$$\ldots\ \cos\Theta_{r-1} \sin\Theta_r,\ \cos\Theta_r)\ ,$$

and whose (r+1)-st column is

$$(2.2.13)\quad (0,\ 0,\ \ldots,\ 0,\ -\sin\Theta_r,\ \cos\Theta_r)\ .$$

Compare (2.2.11) with (2.2.9). The first r elements are the same in each except for the constant factor $\cos\Theta_r$ which appears in (2.2.11). Compare (2.2.12) with (2.2.9). The first r elements are the same in each except for the constant factor $\sin\Theta_r$ which appears in (2.2.12). Finally, the first r elements of the first (r-1) rows of Δ_r are the same as the first r elements of the first (r-1) rows of Δ_{r+1}. This can be seen by looking at (2.2.7) and (2.2.4).

So

$$(2.2.14)\quad \Delta_{r+1} = \sin\Theta_r \cos\Theta_r \left| \begin{array}{c|c} \Delta_r & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{-\sin\Theta_r}{\cos\Theta_r} \end{array} \\ \hline \underline{x} & \dfrac{\cos\Theta_r}{\sin\Theta_r} \end{array} \right| ,$$

where $\underline{x}$ is the last row of Δ_r; and, therefore

$$(2.2.15)\quad \Delta_{r+1} = \Delta_r \sin\Theta_r \cos\Theta_r \left(\frac{\cos\Theta_r}{\sin\Theta_r} + \frac{\sin\Theta_r}{\cos\Theta_r}\right) = \Delta_r \,.$$

Since $\Delta_2 = 1$, the result is proved.

Since

$$(2.2.16)\quad X^2 = \sum_{k=1}^{r} Y_k^2 \,,$$

the joint probability density function can be written,

$$(2.2.17)\quad f(X, \Theta_1, \ldots, \Theta_{r-1}) = \text{const.}\ \exp -\frac{1}{2}\,\frac{X^2}{\sigma^2}\; X^{r-1} \prod_{i=1}^{r-2} (\sin\Theta_i)^{r-i-1}$$

This proves that the random variables, X, Θ_1, $\Theta_2, \ldots, \Theta_{r-1}$, are distributed independently. It is easy to verify that

$$(2.2.18)\quad \sin^2\Theta_k = \frac{\sum\limits_{k=h+1}^{r} Y_k^2}{\sum\limits_{k=h}^{r} Y_k^2} \,, \quad h = 1, 2, \ldots, r-1 \,,$$

and this proves that the B_h are distributed independently.

The proposed intersection test is the following: Reject the hypothesis when $\phi(\underline{y}) = 1$ and accept it otherwise, where

$$(2.2.19)\quad \phi(\underline{y}) = \begin{cases} 0 & \text{if } \nu_{1h} < b_h < \nu_{2h},\ h = 1, 2, \ldots, r-1 \\ 1 & \text{otherwise} \end{cases}$$

The b_h are

$$(2.2.20)\quad b_h = \frac{\sum_{k=h+1}^{r} y_k^2}{\sum_{k=h}^{r} y_k^2}, \quad k = 1, 2, \ldots, r-1 ,$$

and the numbers ν_{1h} and ν_{2h} are determined from the relations

$$(2.2.21)\quad P_r(\nu_{1h} < B_h < \nu_{2h}) = (1-\alpha)^{\frac{1}{r-1}}$$

Because of the independence of the B_h

$$(2.2.22)\quad Pr(\phi(\underline{Y}) = 0) = \prod_{h=1}^{r-1} Pr(\nu_{1h} < B_h < \nu_{2h}) = 1 - \alpha ;$$

i.e., the size of the test is α.

If $\mu_k = \mu \neq 0$, $k = 1, 2, \ldots, r$, then the B_h are no longer beta distributed under the null hypothesis, and the test (2.2.19) is not applicable. For this case consider the following test which can be performed when r is even.

Let H be the $(r x r)$ matrix,

$$(2.2.23) \qquad H = \frac{1}{\sqrt{2}} \left|\begin{array}{c|c} A & AK \\ \hline B & -BK \end{array}\right| \quad ,$$

where A and B are the matrices (1.2.7) and (1.2.8), except that both A and B have been made orthogonal by normalizing their rows, and K is the matrix defined by (1.2.3). Then the random vector $\underline{X} = H\,\underline{Y}$, has the multivariate normal distribution; and, from the properties of the orthogonal polynomials, it follows that

$$(2.2.24) \quad E(\underline{X}) = (\sqrt{r}\ \ \mu,\ 0,\ 0,\ \ldots,\ 0);$$

and, from what was proved in section (1.2), it follows that

$$(2.2.25) \quad \mathrm{Var}(\underline{X}) = H\ V\ H'\ \sigma^2 = \left|\begin{array}{c|c} A(V_{11}+V_{12}K')A' & 0 \\ \hline 0 & B(V_{11}-V_{12}K')B' \end{array}\right| \sigma^2$$

where V is the covariance matrix of $\underline{Y}$,

$$(2.2.26) \qquad V = \left|\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}' & V_{11} \end{array}\right| \quad .$$

Therefore, the random vector $\underline{X}^{*} = (x_2,\ x_3,\ \ldots,\ x_r)$ has

$$(2.2.27) \quad E(\underline{X}^{*}) = (0,\ 0,\ \ldots,\ 0) \quad ,$$

and

$$(2.2.28) \quad \mathrm{Var}(\underline{X}^{*}) = \left|\begin{array}{c|c} (A(V_{11}+V_{12}K')A')^{*} & 0 \\ \hline 0 & B(V_{11}-V_{12}K')B' \end{array}\right| \sigma^2 \quad ,$$

where $(A(V_{11} + V_{12}K')A')^*$ is the matrix $A(V_{11} + V_{12} K')A'$ with the first row and first column deleted.

The statistics B_h,

$$(2.2.29) \quad B_h = \frac{\sum_{k=h+1}^{r} X_k^2}{\sum_{k=h}^{r} X_k^2}, \quad h = 2, 3, \ldots, r-1,$$

are mutually independently distributed as beta random variables, with parameters $(1/2, (r-h)/2)$, under the null hypothesis; and test (2.2.19) can be performed on the transformed observations.

The hypothesis tested is that $\mathrm{Var}(\underline{X}^*) = I\,\sigma^2$. Clearly, $B(V_{11} - V_{12}K')B' = I$ if and only if $V_{11} - V_{12}K' = I$, and

$$(2.2.30) \quad V_{11}-V_{12}K' = \begin{vmatrix} 1 & v_1 & \cdots & v_{r-2/2} \\ v_1 & 1 & \cdots & v_{r-4/2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ v_{r-2/2} & v_{r-4/2} & \cdots & 1 \end{vmatrix}$$

$$- \begin{vmatrix} v_{r-1} & v_{r-2} & \cdots & v_{r/2} \\ v_{r-2} & v_{r-3} & \cdots & v_{r-2/2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ v_{r-2/2} & v_{r-4/2} & \cdots & v_1 \end{vmatrix} = I$$

if and only if all the v_i' s are zero. Thus the same hypothesis as before is tested.

2.3 A Second Test of Independence. Consider the same model as in (2.2); i.e. $Y_1, Y_2, \ldots, Y_r$ are multivariate normal with

$$(2.3.1) \quad E(Y_k) = \mu_k, \qquad k = 1, 2, \ldots, r,$$

and

$$(2.3.2) \quad \mathrm{Var}(\underline{V}) = \begin{vmatrix} 1 & v_1 & \cdots & v_{r-1} \\ v_1 & 1 & \cdots & v_{r-2} \\ \vdots & \vdots & & \vdots \\ v_{r-1} & v_{r-2} & & 1 \end{vmatrix} \sigma^2 .$$

For r even, let H be the $(r\mathrm{x}r)$ matrix,

$$(2.3.3) \quad H = \frac{1}{\sqrt{2}} \left| \begin{array}{c|c} I & K \\ \hline I & -K \end{array} \right| ,$$

where I and K are both $(r/2 \times r/2)$. If $\mu_k = 0$, $k = 1, 2, \ldots, r$, then the random vector $\underline{X} = H\,\underline{Y}$ has the multivariate normal distribution with,

$$(2.3.4) \quad E(\underline{X}') = (0, 0, \ldots, 0) ,$$

and

$$(2.3.5) \quad \mathrm{Var}(\underline{X}) = \left| \begin{array}{c|c} V_{11} + V_{12}K' & 0 \\ \hline 0 & V_{11} - V_{12}K' \end{array} \right| \sigma^2$$

where V_{11} and V_{12} are the $(r/2 \times r/2)$ partitions of (2.3.2).

Under the hypothesis of independence, the statistic G,

$$(2.3.6) \qquad G = \frac{\sum_{k=1}^{r/2} X_k^2}{\sum_{k=r+2/2}^{r} X_k^2} \quad ,$$

has the F distribution with (r/2, r/2) degrees of freedom.

The test proposed is the following: Reject the hypothesis if $\phi(\underline{x}) = 1$, where

$$(2.3.7) \qquad \phi(x) = \begin{cases} 0 & \text{if } \nu_2 < g < \nu_1 \\ 1 & \text{otherwise} \end{cases}$$

The numbers ν_2 and ν_1 are the appropriate percentage points of the F distribution and

$$(2.3.8) \qquad g = \frac{\sum_{k=1}^{r/2} x_k^2}{\sum_{k=r+2/2}^{r} x_k^2}$$

where the little x's are the transformed observations.

If $\mu_k = \mu$, k = 1, 2, ..., r, and r is even, let H be the (rxr) matrix,

$$(2.3.9) \qquad H = \frac{1}{\sqrt{2}} \left| \begin{array}{c|c} A & AK \\ \hline B & -BK \end{array} \right| \quad ,$$

where A and B are the same as in (2.2.23). Then the random vector $\underline{X} = H\,\underline{Y}$ has the multivariate normal distribution with

$$(2.3.10) \quad E(\underline{X}') = (\sqrt{r}\ \mu, 0, 0, \ldots, 0)\ ,$$

and

$$(2.3.11) \quad \mathrm{Var}\ (\underline{X}) = \left|\begin{array}{c|c} A(V_{11} + V_{12}K')A' & 0 \\ \hline 0 & B(V_{11} - V_{12}K')B' \end{array}\right| \sigma^2$$

For the same reason as in the previous section the statistic G,

$$(2.3.12) \qquad G = \frac{\sum_{k=2}^{r/2} X_k^2}{\sum_{k=r+2/2}^{r} X_k^2}\ ,$$

has the F distribution with $(r-2/2, r/2)$ degrees of freedom if and only if the hypothesis of independence is true. The test proposed in (2.3.7) is therefore appropriate.

Against specified alternatives, the non-central distribution of G can be approximated with any prescribed degree of accuracy. An example will be given in the next section.

2.4 Non-central Distribution of G(when $\mu = 0$). The approximation may be calculated by a method due to Pachares or a method due to Robbins [14]. The method due to Robbins is used here since Pachares' work is unpublished.

Consider the following alternative to the hypothesis of independence:

$$(2.4.1) \qquad \mathrm{Var}(\underline{Y}) = \begin{vmatrix} 1 & \rho & \cdots & \rho^{r-1} \\ \rho & 1 & & \rho^{r-2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \rho^{r-1} & \rho^{r-2} & \cdots & 1 \end{vmatrix} \frac{\sigma^2}{1-\rho^2} \quad ,$$

where $|\rho| < 1$. This model arises in the case where the Y's satisfy the stochastic difference equation,

$$(2.4.2) \qquad Y_k = \rho \, Y_{k-1} + \epsilon_k \quad , \quad k = 2, 3, \ldots, r \; ,$$

and the ϵ_k are independently and identically distributed as $N(0, \sigma^2)$.

The random vector $\underline{X} = H \, \underline{Y}$, where H is the matrix (2.3.3), has the multivariate normal distribution with

$$(2.4.3) \qquad E(\underline{X}') = (0, 0, \ldots, 0) \; ,$$

and

$$(2.4.4) \qquad \mathrm{Var}(\underline{X}) = \left| \begin{array}{c|c} V_{11}+V_{12}K' & 0 \\ \hline 0 & V_{11}-V_{12}K' \end{array} \right| \quad \sigma^2 \quad ,$$

where V_{11} and V_{12} are the $(r/2 \times r/2)$ partitions of the matrix (2.4.1)

If ρ is known, there is an $(r\mathrm{x}r)$, orthogonal matrix L,

$$(2.4.5) \qquad L = \left| \begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array} \right| \quad ,$$

such that $L_1(V_{11} + V_{12}K')L_1' = D(\lambda_k)$ and $L_2(V_{11} - V_{12}K')L_2' = D(\eta_k)$; and from what was proved in section (1.4), the values of λ_k and η_k can be approximated, with any desired degree of accuracy, when ρ is known.

The random vector $\underline{Z} = H\,\underline{X}$ has the multivariate normal distribution with

$$E(\underline{Z}') = (0, 0, \ldots, 0) \quad , \tag{2.4.6}$$

and

$$\mathrm{Var}(\underline{Z}) = \left|\begin{array}{c|c} D(\lambda_k) & 0 \\ \hline 0 & D(\eta_k) \end{array}\right| \sigma^2 . \tag{2.4.7}$$

Thus the statistic G has the same distribution as the statistic S,

$$S = \frac{\sum_{k=1}^{r/2} \lambda_k \, T_k^2}{\sum_{k=(r+2)/2}^{r} \eta_{r-k+1} \, T_k^2} \quad , \tag{2.4.8}$$

where $T_1, T_2, \ldots, T_r$ are independently and identically distributed as $N(0, 1)$; and

$$\Pr(S < c) \tag{2.4.9}$$

can be approximated with any desired degree of accuracy by a method due to Robbins [14].

Robbins' Theorem expresses the probability,

$$(2.4.10) \quad \Pr \quad \frac{a(\chi_o^2 + a_1\chi_1^2 + \ldots + a_r\ \chi_r^2)}{\chi_{r+1}^2 + a_{r+2}\ \chi_{r+2}^2 + \ldots\ \ a_{r+s}\ \chi_{r+s}^2} < c \quad ,$$

where $a, a_1, \ldots, a_r, a_{r+2}, \ldots, a_{r+s}$ are positive constants and $a_k \geq 1$, $k = 1, 2, \ldots, r, r+2, \ldots, r+s$, and the χ^2's are independent chi-square variates, each with any number of degrees of freedom whatever, as an infinite series; and he gives a formula for calculating the maximum error committed by truncating the series after any given number of terms. The maximum error tends to zero as more and more terms are included, so any degree of accuracy can be attained if a large enough number of terms is included.

It is clear that $\Pr(S < c)$ can be approximated by the application of Robbins' Theorem, since

$$(2.4.11) \quad S = \frac{\min(\lambda_k)}{\min(\eta_k)} \quad \frac{\sum_{k=1}^{r/2} \frac{\lambda_k\ T_k^2}{\min(\lambda_k)}}{\sum_{k=(r+2)/2}^{r} \frac{T_k^2\ \eta_{r-k+1}}{\min(\eta_k)}}$$

is the same form as the quantity which appears in (2.4.10). The λ's and the η's are, of course, positive since they are roots of positive definite matrices, and the quantity $(\min(\lambda_k))/(\min(\eta_k))$ plays the role of a.

2.5 A Test of Independence With a Circular Model. Suppose that $Y_1, Y_2, \ldots, Y_r$ are a set of random variables with the multivariate normal distribution and

$$(2.5.1)\quad E(\underline{Y}') = (0,\ 0,\ \ldots,\ 0),$$

and

$$(2.5.2)\quad \mathrm{Var}(\underline{Y}) = V\,\sigma^2 ,$$

where

$$(2.5.3)\quad V^{-1} = ((1+\rho^2)\,I - \rho C)\,\frac{1}{\sigma^2} ,$$

with

$$(2.5.4)\quad C = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{vmatrix}$$

This is the so-called circular model (see T. W. Anderson [5]).

The matrix V is (rxr), and ther is an (rxr), orthogonal matrix H such that

$$(2.5.5)\quad H\,V\,H' = D(\lambda_k),$$

where, when r is even,

$$(2.5.6)\quad \lambda_k = \begin{cases} 1-2\,\rho\cos\dfrac{2(k-1)\pi}{r} + \rho^2 , & k = 1,2,\ldots,(r+2)/2 \\[2ex] 1-2\,\rho\cos\dfrac{(2k-r-2)\pi}{r} + \rho^2, & k = (r+4)/2,(r+6)/2,\ldots,r \end{cases}$$

and, when r is odd

$$(2.5.7)\quad \lambda_k = \begin{cases} 1-2\,\rho\cos\dfrac{2(k-1)\pi}{r} + \rho^2, & k = 1,2,\ldots,(r+1)/2 \\[2ex] 1-2\,\rho\cos\dfrac{(2k-r-1)\pi}{r} + \rho^2, & k = (r+3)/2,(r+5)/2,\ldots,r \end{cases}$$

If r is even, the k-th row of H is

$$(2.5.8)\quad \left(\cos(k-1)\frac{2\pi}{r},\ \cos(k-1)\frac{4\pi}{r},\ \ldots,\ \cos(k-1)\frac{2r\pi}{r}\right)\sqrt{\frac{2}{r}}\ ,$$

$k = 2, 3, \ldots, (r+2)/2$, and

$$(2.5.9)\quad \left(\sin(2k-r-2)\frac{\pi}{r},\ \sin(2k-r-2)\frac{2\pi}{r},\ \ldots,\ \sin(2k-r-2)\frac{r\pi}{r}\right)\sqrt{\frac{2}{r}}\ ,$$

$k = (r+4)/2, (r+6)/2, \ldots, r$. The first row is a row of constants each $1/\sqrt{r}$. If r is odd, the k-th row of H is

$$(2.5.10)\quad \left(\cos(k-1)\frac{2\pi}{r},\ \cos(k-1)\frac{4\pi}{r},\ \ldots,\ \cos(k-1)\frac{2r\pi}{r}\right)\ \frac{2}{r}\ ,$$

$k = 2, 3, \ldots, (r+1)/2$, and

$$(2.5.11)\quad \left(\sin(2k-r-1)\frac{\pi}{r},\ \sin(2k-r-1)\frac{2\pi}{r},\ \ldots,\ \sin(2k-r-1)\frac{r\pi}{r}\right)\sqrt{\frac{2}{r}}$$

$k = (r+3)/2, (r+5)/2, \ldots, r$. The first row is a row of constants each $1/\sqrt{r}$. These results are given by T. W. Anderson [5].

The random vector $\underline{X} = H\,\underline{Y}$ has the multivariate normal distribution; and, since all rows of H sum to zero except the first,

$$(2.5.12)\quad E(\underline{X}') = (\sqrt{r},\ 0,\ 0,\ \ldots,\ 0)\ ;$$

and

$$(2.5.13)\quad \mathrm{Var}(\underline{X}) = D^{-1}(\lambda_k)\,\sigma^2 .$$

If r is even and $r/2$ is even, the statistic G_1,

$$(2.5.14)\quad G_1 = \frac{r-2}{r}\;\frac{\sum\limits_{k=2}^{\frac{r+4}{2}} X_k^2 + \sum\limits_{k=r+4/2}^{\frac{3r+4}{2}} X_k^2}{\sum\limits_{k=r+8/4}^{\frac{r+2}{2}} X_k^2 + \sum\limits_{3r+8/4}^{r} X_k^2},$$

has the F distribution with $(r/2, (r-2)/2)$ degrees of freedom if and only if $\rho = 0$; and if $r/2$ is odd, the statistic G_2,

$$(2.5.15)\quad G_2 = \frac{r}{r-2}\;\frac{\sum\limits_{k=2}^{(r+2)/4} X_k^2 + \sum\limits_{k=(r+4)/2}^{(3r+2)/4} X_k^2}{\sum\limits_{k=(r+6)/4}^{(r+2)/2} X_k^2 + \sum\limits_{k=(3r+6)/4}^{r} X_k^2}$$

has the F distribution with $((r-2)/2, r/2)$ degrees of freedom if and only if $\rho = 0$.

If r is odd and $(r+1)/2$ is even, the statistic G_3,

$$(2.5.16)\quad G_3 = \frac{r-3}{r+1}\;\frac{\sum\limits_{k=2}^{(r+5)/4} X_k^2 + \sum\limits_{k=(r+3)/2}^{(3r+3)/4} X_k^2}{\sum\limits_{k=(r+9)/4}^{(r+1)/2} X_k^2 + \sum\limits_{k=(2r+7)/4}^{r} X_k^2},$$

has the F distribution with $((r+1)/2\ (r-3)/2)$ degrees of freedom if and only if $\rho = 0$; and if $(r+1)/2$ is odd, the statistic G_4,

$$(2.5.17) \quad G_4 = \frac{\sum_{k=2}^{(r+3)/4} X_k^2 + \sum_{k=(r+3)/2}^{(3r+1)/4} X_k^2}{\sum_{k=(r+7)/4}^{(r+1)/2} X_k^2 + \sum_{k=(3r+5)/4}^{r} X_k^2} ,$$

has the F distribution with $((r-1)/2, (r-1)/2)$ degrees of freedom if and only if $\rho = 0$.

Consequently, these statistics can be used to test the hypothesis of independence, i.e., that $\rho = 0$.

There is a uniformly most powerful test of the hypothesis $\rho = 0$ for this model (see T. W. Anderson [5]), but it depends on the serial correlation coefficient, and the percentage points of the distribution of the serial correlation coefficient are not generally available. Moreover, the non-central distribution of G_3 and G_4 can be found, and the power function of these tests provides lower bounds for the power function of the uniformly most powerful test. The power function of the uniformly most powerful test is, of course, not known.

The motivation for choosing the statistics G_1, G_2, G_3, G_4 as they were chosen should be made clear. In each case, the choice was made in such a way that the variances of the X_k's appearing in the numerator of the G's were all larger (or smaller) than the variances of the X_k's appearing in the denominator, the larger (or smaller) depending on $\rho > 0$ (or $\rho < 0$). It seemed that this choice of all possible choices would have maximum power. Also, for this choice, it is possible to determine the exact non-central distributions of G_3 and G_4 in closed form.

2.6 The Non-central Distribution of G_3. Let

$$(2.6.1) \quad Q_1 = \frac{2}{r+1}\left(\sum_{k=2}^{(r+5)/4} X_k^2 + \sum_{k=(r+3)/2}^{(3r+4)/4} X_k^2 \right),$$

and

$$(2.6.2) \quad Q_2 = \frac{2}{r-3}\left(\sum_{k=(r+9)/4}^{(r+1)/2} X_k^2 + \sum_{k=(3r+7)/4}^{r} X_k^2 \right).$$

By checking (2.5.6) and (2.5.7) it is easy to see that

$$(2.6.3) \quad \mathrm{Var}(X_k^2) = \mathrm{Var}(X^2_{k+(r-1)/2}),\ k = 2, 3, \ldots, (r+5)/4, \text{ and}$$

that

$$(2.6.4) \quad \mathrm{Var}(X_k^2) = \mathrm{Var}(X^2_{k+(r-1)/2}),\ k = (r+9)/4, (r+13)/4, \ldots (r+1)/2 .$$

Thus $(r+1)Q_1/2$ has the same distribution as a weighted sum of independent chi-squares, each with two degrees of freedom, where the weights are

$$(2.6.5) \quad c_k = \left(1 - 2\rho\cos(k-1)\frac{2\pi}{r} + \rho^2\right)^{-1}\frac{1}{\sigma^2},$$

$k = 2, 3, \ldots, (r+5)/4$, and $(r-3)Q_2/2$ has a like distribution with weights

$$(2.6.6)\quad C_k = (1 - 2\rho \cos(k-1) \ \frac{2\pi}{r} + \rho^2)^{-1} \frac{1}{\sigma^2} \ ,$$

$k = (r+9)/4, (r+13)/4, \ldots, (r+1)/2$.

The distributions of the quantities $(r+1)Q_1/2$ and $(r-3)Q_2/2$ are quite easily found. Their characteristic functions are

$$(2.6.7)\quad \phi_1(u) = \prod_{k=2}^{(r+5)/4} (1 - 2\,C_k\,u\,i)^{-1} \quad ,$$

and

$$(2.6.8)\quad \phi_2(u) = \prod_{k=(r+9)/4}^{(r+1)/2} (1 - 2\,C_k\,u\,i)^{-1} \ .$$

From a well known theorem of complex variables (see Ahlfors [1]) it follows that there exist b_k such that

$$(2.6.9)\quad \phi_1(u) = \sum_{k=2}^{(r+5)/4} b_k(1 - 2\,C_k\,u\,i)^{-1}$$

and

$$(2.6.10)\quad \phi_2(u) = \sum_{k=(r+9)/4}^{(r+1)/2} b_k\,(1 - 2\,C_k\,u\,i)^{-1}$$

are identities in u for all complex numbers such that both sides of the equations (2.6.9) and (2.6.10) exist.

To find the b_k an artifice described by Courant [7] is employed. Multiply both sides of (2.6.9) and (2.6.10) by the quantity $(1 - 2\,C_{k'}\,u\,i)$. They become

$$(2.6.11) \quad \prod_{\substack{k=2\\k\neq k'}}^{(r+5)/4} (1 - 2\,C_k\,u\,i)^{-1} \equiv b_{k'} + (1 - 2\,C_{k'}\,u\,i) \sum_{\substack{k=2\\k\neq k'}}^{(r+5)/4} b_k \,(1 - 2\,C_k\,u\,i)^{-1} ,$$

and

$$(2.6.12) \quad \prod_{\substack{k=(r+9)/4\\k\neq k'}}^{(r+1)/2} (1 - 2\,C_k\,u\,i)^{-1} \equiv b_{k'} + (1-2\,C_{k'}u\,i) \sum_{\substack{k=(r+9)/4\\k\neq k'}}^{(r+1)/2} b_k \,(1 - 2\,C_k\,u\,i)^{-1} .$$

Equations (2.6.11) and (2.6.12) are identities in u; hence choose u so that $(1 - 2\,C_k\,u\,i) = 0$, and the result is

$$(2.6.13) \quad b_{k'} = \prod_{\substack{k=2\\k\neq k'}}^{(r+5)/4} \left(\frac{C_{k'}}{C_{k'}-C_k}\right), \; k = 2, 3, \ldots, (r+5)/4 ,$$

and

$$(2.6.14) \quad b_{k'} = \prod_{\substack{k=(r+9)/4\\k\neq k'}}^{(r+1)/2} \left(\frac{C_{k'}}{C_{k'}-C_k}\right), \; k = (r+9)/4, (r+13)/4, \ldots (r+1)/2 .$$

The inversion of $\phi_1(u)$ and $\phi_2(u)$ in the form shown in (2.6.9) and (2.6.10) is immediate,

$$(2.6.15) \quad \Pr((r+1)Q_1/2 < c) = 1 - \sum_{k=2}^{(r+5)/4} b_k \, e^{-c/(2c_k)} ,$$

$$(2.6.16) \quad \Pr((r-3)Q_2/2 < c) = 1 - \sum_{k=(r+9)/4}^{(r+1)/2} b_k \; e^{-c/(2c_k)} ,$$

where $c \geq 0$.

Finally,

$$(2.6.17) \quad \Pr(G_3 < c) = \Pr(\frac{r+1}{2} \; Q_1 < \frac{r+1}{r-3} \, c \; \frac{r-3}{2} \; Q_2) ,$$

or

$$(2.6.18) \quad \Pr(G_3 < c) = \int_0^{\infty} (1 - \sum_{k=2}^{(r+5)/4} b_k e^{-\frac{r+1}{r-3} \frac{c}{2c_k} x}) \sum_{k=(r+9)/4}^{(r+1)/2} \frac{b_k}{2c_k} \; e^{-\frac{x}{2c_k}} \; d\,x ,$$

or

$$(2.6.19) \quad \Pr(G_3<c) = 1 - \frac{1}{2} \int_0^{\infty} \sum_{\ell=2}^{(r+5)/4} \; \sum_{k=(r+9)/4}^{(r+1)/2} \frac{b_\ell \, b_k}{c_k} \; e^{-\frac{x}{2} (\frac{r+1}{r-3} \frac{c}{c_\ell} + \frac{1}{c_k})} \; d\,x ,$$

or

$$(2.6.20) \quad \Pr(G_3<c) = 1 - \sum_{\ell=2}^{(r+5)/4} \; \sum_{k=(r+9)/4}^{(r+1)/2} \frac{(r-3) b_\ell \, b_k \, c_\ell}{(r+1) c \, C_k + (r-3) C_\ell}$$

which, of course, has meaning only when $c \geq 0$.

The distribution of G_4 may be found in excatly the same way. It turns out to be

$$(2.6.21)\quad \Pr(G_4<c) = 1 - \sum_{\ell=2}^{(r+3)/4} \sum_{k=(r+7)/4}^{(r+1)/2} \frac{b_\ell\, b_k\, c_\ell}{C\, C_k + C_\ell} ,$$

$$c \geq 0 ,$$

where

$$(2.6.22)\quad C_k = (1 - 2\rho \cos (k-1)\frac{2\pi}{r} + \rho^2)^{-1} \frac{1}{\sigma^2} ,$$

$k = (r+9)/4,\ (r+13)/4,\ \ldots\ (r+1)/2,$ and

$$(2.6.23)\quad C_\ell = (1 - 2\rho \cos (\ell-1)\frac{2\pi}{r} + \rho^2)^{-1} \frac{1}{\sigma^2} ,$$

$\ell = 2, 3, \ldots, (r+3)/4$. The b's are

$$(2.6.24)\quad b_k = \prod_{\substack{k'=(r+7)/4 \\ k'\neq k}}^{(r+1)/2} \left(\frac{C_k}{C_k - C_{k'}}\right),\ k = (r+7)/4,\ (r+11)/4, \ldots,$$

$$(r+1)/2 ,$$

and

$$(2.6.25)\quad b_\ell = \prod_{\substack{\ell'=2 \\ \ell'\neq\ell}}^{(r+3)/4} \left(\frac{C_\ell}{C_\ell - C_{\ell'}}\right) ,\ \ell = 2, 3, \ldots (r+3)/4 .$$

The non-central distributions of G_1 and G_2 are not so easy to find. However, the modified statistics G_1' and G_2' ,

$$(2.6.26) \quad G_1' = \frac{r-4}{r} \frac{\sum_{k=2}^{(r+4)/4} X_k^2 + \sum_{k=(r+4)/2}^{(3r+4)/4} X_k^2}{\sum_{k=(r+8)/4}^{r/2} X_k^2 + \sum_{k=(3r+8)/4}^{r} X_k^2} ,$$

and

$$(2.6.27) \quad G_2' = \frac{\sum_{k=2}^{(r+2)/2} X_k^2 + \sum_{k=(r+4)/2}^{(3r+2)/4} X_k^2}{\sum_{k=(r+6)/4}^{r/2} X_k^2 + \sum_{k=(3r+6)/4}^{r} X_k^2} ,$$

have distributions which can be found by the method used to find the distributions of G_3 and G_4.

2.7 <u>The Consistency of These Tests</u>. Consider G_1 and let

$$(2.7.1) \quad Q_1 = \frac{2}{r} \left(\sum_{k=2}^{(r+4)/4} X_k^2 + \sum_{k=(r+4)/2}^{(3r+4)/4} X_k^2 \right) ,$$

and

$$(2.7.2) \quad Q_2 = \frac{2}{r-2} \left(\sum_{k=(r+8)/4}^{(r+2)/2} X_k^2 + \sum_{k=(3r+8)/4}^{r} X_k^2 \right) .$$

By checking (2.5.6) and (2.5.7) it is easy to see that

$$(2.7.3) \quad E(Q_1) = \frac{4\sigma^2}{r} \sum_{k=2}^{(r+4)/4} \left(1 - 2\rho \cos(k-1) \frac{2\pi}{r} + \rho^2\right)^{-1} ,$$

and that

$$(2.7.4) \quad E(Q_2) = \frac{2\sigma^2}{r-2} \left\{ \frac{1}{1+\rho^2} + 2 \sum_{k=(r+8)/4}^{r/2} \left(1-2\rho\cos(k-1) \frac{2\pi}{r} + \rho^2\right)^{-1} \right\}$$

Recall (1.6.1). The limit of $E(Q_1)$ as r becomes large can be written

$$(2.7.5)\quad \lim_{r\to\infty} E(Q_1) = \lim_{r\to\infty} \frac{4\sigma^2}{r} \sum_{k=1}^{r/4} \frac{1}{1-\rho^2}\left(1 + 2\sum_{n=1} \rho^n \cos\frac{2nk\pi}{r}\right),$$

or

$$(2.7.6)\quad \lim_{r\to\infty} E(Q_1) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} + 2\lim_{r\to\infty}\frac{1}{r}\sum_{k=1}^{r/4}\sum_{n=1} \rho^n \cos\frac{2nk\pi}{r}\right).$$

Since the series on the right-hand side of (2.7.6) is absolutely convergent for $\rho < 1$, (2.7.6) can be rearranged to the form

$$(2.7.7)\quad \lim_{r\to\infty} E(Q_1) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} + 2\sum_{n=1} \rho^n \lim_{r\to\infty}\sum_{k=1}^{r/4} \frac{1}{r}\cos\frac{2nk\pi}{r}\right).$$

Making use of formula (1.6.3), (2.7.7) becomes

$$(2.7.8)\quad \lim_{r\to\infty} E(Q_1) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} + 2\sum_{n=1} \rho^n \lim_{r\to\infty}\frac{1}{r}\,\frac{\sin\frac{n\pi}{4}}{\sin\frac{n\pi}{r}}\right.$$

$$\left.\cos\left(\frac{r+4}{r}\right)\frac{n\pi}{4}\right).$$

But

$$(2.7.9)\quad \lim_{r\to\infty}\frac{1}{r}\,\frac{\sin\frac{n\pi}{4}}{\sin\frac{n\pi}{r}}\cos\left(\frac{r+4}{4}\right)\frac{n\pi}{4} = \frac{\sin\frac{n\pi}{2}}{2n\pi},$$

so that (2.7.8) becomes

$$(2.7.10)\quad \lim_{r\to\infty} E(Q_1) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} + \frac{1}{\pi}\sum_{n=1} \frac{\rho^n}{n}\sin\frac{n\pi}{2}\right).$$

The sum S,

$$(2.7.11)\quad S = \sum_{n=1}^{\infty} \frac{\rho^n}{n} \sin\frac{n\pi}{2},$$

can be evaluated in the following way. Observe that

$$(2.7.12)\quad \int_0^{\rho} \left(\sum_{n=1}^{\infty} x^{n-1} \sin\frac{n\pi}{2}\right) dx = \sum_{n=1}^{\infty} \frac{\rho^n}{n} \sin\frac{n\pi}{2},$$

This is surely true since a power series converges uniformly within its interval of convergence. By formula (1.6.2) it turns out that

$$(2.7.13)\quad \sum_{n=1}^{\infty} x^{n-1} \sin\frac{n\pi}{2} = \frac{1}{1+x^2}, \qquad x < 1 .$$

Thus

$$(2.7.14)\quad \sum_{n=1}^{\infty} \frac{\rho^n}{n} \sin\frac{n\pi}{2} = \int_0^{\rho} \frac{d\,x}{1+x^2} = \tan^{-1}\rho .$$

Therefore,

$$(2.7.15)\quad \lim_{r \to \infty} E(Q_1) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} + \frac{1}{\pi}\tan^{-1}\rho\right) .$$

Next, by making use of (1.6.1) again,

$$(2.7.16)\quad \lim_{r \to \infty} E(Q_2) = \lim_{r \to \infty} \frac{4\sigma^2}{r-2} \sum_{k=1}^{(r-4)/4} \frac{1}{1-\rho^2}\left(1 + 2\sum_{n=1}^{\infty} \rho^n \cos\,(r+4k)\frac{n\pi}{2r}\right) .$$

Inverting the order of summation and making use of (1.6.3) and (1.6.4) leads to

$$(2.7.17)\quad \lim_{r\to\infty} E(Q_2) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} + 2\sum_{n=1}^{\infty} \rho^n \lim_{r\to\infty} \frac{1}{r-2}\, \frac{\sin(\frac{r-4}{4})\frac{n\pi}{r}}{\sin\frac{n\pi}{r}} \cos\frac{3n\pi}{4}\right).$$

But

$$(2.7.18)\quad \lim_{r\to\infty} \frac{1}{r-2}\, \frac{\sin(\frac{r-4}{4})\frac{n\pi}{r}}{\sin\frac{n\pi}{r}} \cos\frac{3n\pi}{4} = -\frac{\sin\frac{n\pi}{2}}{2n\pi}\ .$$

Thus

$$(2.7.19)\quad \lim_{r\to\infty} E(Q_2) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\rho^n}{n} \sin\frac{n\pi}{2}\right),$$

or, recalling (2.7.14),

$$(2.7.20)\quad \lim_{r\to\infty} E(Q_2) = \frac{4\sigma^2}{1-\rho^2}\left(\frac{1}{4} - \frac{1}{\pi}\tan^{-1}\rho\right).$$

If

$$(2.7.21)\quad \lim_{r\to\infty} \frac{1}{r}\left(\sum_{k=1}^{r/4} \left(1-2\rho\cos\frac{2k\pi}{r} + \rho^2\right)^{-2}\right)^{\frac{1}{2}} = 0,$$

and

$$(2.7.22)\quad \lim_{r\to\infty} \frac{1}{r}\left(\sum_{k=1}^{(r-4)/4} \left(1-2\rho\cos\frac{(r+4k)\pi}{2r} + \rho^2\right)^{-2}\right)^{\frac{1}{2}} = 0,$$

then $Var(Q_1)$ and $Var(Q_2)$ tend to zero and Tchebycheff's inequality applies to Q_1 and Q_2.

Making use of (1.6.1), (2.7.21) becomes

$$(2.7.23)\quad \lim_{r\to\infty} \frac{1}{r^2}\sum_{k=1}^{r/4} \left(1 + 4\sum_{n=1}^{\infty} \rho^n \cos\frac{2nk\pi}{r} + 4\left(\sum_{n=1}^{\infty} \rho^n\cos\frac{2nk\pi}{r}\right)^2\right) \frac{1}{1-\rho^2}\ ;$$

but this limit is zero since

$$(2.7.24) \quad \left| \sum_{n=1}^{\infty} \rho^n \cos \frac{2nk\pi}{r} \right| \leq \frac{|\rho|}{1-|\rho|} \quad \text{for} \quad \rho < 1 \quad \text{and}$$

$r > 0$. The same reasoning on the case of (2.7.22) leads to the same result.

This means that

$$(2.7.25) \quad \lim_{r \to \infty} \Pr \left\{ \left| Q_1 - \frac{\sigma^2}{1-\rho^2} \left(1 \pm \frac{4}{\pi} \tan^{-1} \rho\right) \right| < \epsilon \right\} = 0 ,$$

and that

$$(2.7.26) \quad \lim_{r \to \infty} \Pr \left\{ \left| Q_2 - \frac{\sigma^2}{1-\rho^2} \left(1 - \frac{4}{\pi} \tan^{-1} \rho\right) \right| > \epsilon \right\} = 0,$$

for every $\epsilon > 0$.

There is a theorem in Cramer [8] which says that if two random variables, say X_n and Y_n, are converging in probability to points, say C_1 and C_2, then the ratio X_n/Y_n converges in probability to the point C_1/C_2 if $C_2 \neq 0$.

Therefore,

$$(2.7.27) \quad \frac{Q_1}{Q_2} \xrightarrow[p]{} \frac{\pi + 4 \tan^{-1} \rho}{\pi - 4 \tan^{-1} \rho} ,$$

But

$$(2.7.28) \quad \frac{\pi + 4 \tan^{-1} \rho}{\pi - 4 \tan^{-1} \rho} = 1$$

if and only if $\rho = 0$.

This proves the consistency of the test based on the statistic G_1. The consistency of the other tests (those based on G_2, G_3, G_4) will follow through similar arguments.

CHAPTER III

ANALYSIS OF VARIANCE WHEN THERE IS NO INTERACTION

3.1 Introduction. This chapter contains the results of some investigations into the problem of analyzing the data from a two-way classification when the random errors are subject to certain patterns of dependence. For a circular model the power functions of two possible analyses are compared.

3.2 Testing for Treatment Effects Without Interaction. Consider the following situation: The random variables $\underline{Y}_{ij}$, $\underline{Y}_{ij}$ is $(r\mathrm{x}1)$, $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, b$, are independently and identically distributed as multivariate normal with

$$(3.2.1) \quad E(\underline{Y}_{ij}) = (\mu + \tau_i + \beta_j)\,\underline{1}\,, \quad \text{all } (i, j)\,,$$

and

$$(3.2.2) \quad \mathrm{Var}(\underline{Y}_{ij}) = \begin{vmatrix} 1 & \rho & \cdots & \rho^{r-1} \\ \rho & 1 & \cdots & \rho^{r-2} \\ \vdots & \vdots & & \vdots \\ \rho^{r-1} & \rho^{r-2} & \cdots & 1 \end{vmatrix} \qquad \frac{\sigma^2}{1-\rho^2}$$

The parameters μ, τ_i's, β_j's, σ^2, and ρ ($\rho < 1$) are unknown. A set of observations, say $\underline{y}_{ij}$, all (i, j), is given, and it is required to test the hypothesis of no treatment effects.

One thing that can be done is as follows: The random variables

X_{ij},

$$(3.2.3)\quad X_{ij} = \frac{1}{r}\ \underline{1}'\ \underline{Y}_{ij}\ ,$$

are independently distributed as normal with

$$(3.2.4)\quad E(X_{ij}) = \mu + \tau_i + \beta_j\ ,$$

and

$$(3.2.5)\quad \mathrm{Var}\ (X_{ij}) = \frac{r-2\rho-r\rho^2 + 2\rho^{r+1}}{(1-\rho^2)\ (1-\rho)^2}\quad \frac{\sigma^2}{r^2}\ .$$

The latter follows from (1.5.21) .

The X_{ij}'s form a two-way classification with one observation per cell, each observation with the same variance; and the usual analysis of variance can be performed, along with the usual tests. In fact, this analysis will be valid even if the structure of the dependence is not the same as shown in (3.2.2) .

Another possibility arises if the model is changed slightly. Suppose that

$$(3.2.6)\quad \mathrm{Var}(\underline{Y}_{ij}) = ((1 + \rho^2)I - \rho C)^{-1}\ \sigma^2\ ,$$

where C is the (rxr) matrix with all elements zero except for $c_{ij} = 1$ if $i-j = 1$. This is a modification of the circular model described by T. W. Anderson [5].

There exists an (rxr) orthogonal matrix H such that

$$(3.2.7)\quad H\ ((1 + \rho^2)\ I - \rho C)^{-1}\ H' = D(\lambda_k)\ .$$

The k-th row of H, say $\underline{h}_k'$, is (see T. W. Anderson [5]):

$$(3.2.8) \quad \left(\sin \frac{k\pi}{r+1}, \sin \frac{2k\pi}{r+1}, \ldots, \sin \frac{rk\pi}{r+1}\right)\sqrt{\frac{2}{r+1}} \quad ,$$

$k = 1, 2, \ldots, r,$ and

$$(3.2.9) \quad \lambda_k = (1 - 2\rho \cos \frac{k\pi}{r+1} + \rho^2)^{-1} \quad ,$$

$k = 1, 2, \ldots, r$.

Let $\underline{X}_{ij} = H\,\underline{Y}_{ij}$, all (i, j). Then,

$$(3.2.10) \quad E(\underline{X}'_{ij}) = \frac{(\mu + \tau_i + \beta_j)}{\sqrt{\frac{r+1}{2}}} \left(\sum_{\ell=1}^{r} \sin \frac{\ell\pi}{r+1}, \sum_{\ell=1}^{r} \sin \frac{2\ell\pi}{r+1}, \ldots, \sum_{\ell=1}^{r} \sin \frac{r\ell\pi}{r+1} \right) ,$$

and

$$(3.2.11) \quad Var(\underline{X}_{ij}) = D(\lambda_k)\,\sigma^2 \ .$$

Referring to (1.6.4), it is seen that

$$(3.2.12) \quad E(X_{ijk}) = 0 \qquad \text{for } k \text{ even.}$$

Consider the random variables $\underline{Z}_{ij}$,

$$(3.2.13) \quad \underline{Z}_{ij} = \begin{cases} (X_{ij1}, X_{ij3}, \ldots, X_{ijr-1}) , & r \text{ even ;} \\ \\ (X_{ij1}, X_{ij3}, \ldots, X_{ijr}) , & r \text{ odd .} \end{cases}$$

The random vector $\underline{Z}_{ij}$ is $(r/2x1)$ or $((r+1)/2 \times 1)$ depending on whether r is even or odd, and

$$(3.2.14)\quad E(\underline{Z}'_{ij}) = \frac{(\mu + \tau_i + \beta_j)}{\sqrt{\frac{r+1}{2}}} \left(\sum_{\ell=1}^{r} \sin \frac{\ell\pi}{r+1} \,,\; \sum_{\ell=1}^{r} \sin \frac{3\ell\pi}{r+1} \,, \ldots, \sum_{\ell=1}^{r} \sin \frac{p\ell\pi}{r+1} \right) ,$$

where $p = r - 1$ or r depending on whether r is even or odd. Also

$$(3.2.15)\quad \mathrm{Var}(\underline{Z}_{ij}) = D(\lambda_k)\,\sigma^2 ,$$

where

$$(3.2.16)\quad \lambda_k = \left(1 - 2\rho \cos \frac{(2h-1)\pi}{r+1} + \rho^2\right)^{-1} ,$$

$h = 1, 2, \ldots, r/2$ if r is even, $h = 1, 2, \ldots, (r+1)/2$ if r is odd.

The random variables Z_{ijh} are a two-way classification for each h, with variance per observation $\lambda_h\sigma^2$. Thus the usual analysis of variance can be performed for each value of h. For each analysis there is a function of the observations, say $\phi(\underline{z}_h)$, which provides the criterion for rejecting or not rejecting the hypothesis; i.e., the hypothesis is rejected when $\phi(\underline{z}_h) = 1$ and not otherwise.

The following intersection test is proposed. Let

$$(3.2.17)\quad \phi(z_1, z_2, \ldots, z_q) = \begin{cases} 0 & \text{if all } \phi(\underline{z}_h) = 0 , \\ 1 & \text{otherwise} , \end{cases}$$

where $q = r/2$ or $(r+1)/2$ depending on whether r is even or odd. The hypothesis is rejected when $\phi(\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_q) = 1$.

The test can be made of size α in the following way:

(3.2.18) $\Pr(H_o \text{ rejected by } h\text{-th test } H_o) = \nu$;

then, since the tests are independent,

(3.2.19) $\Pr(H_o \text{ not rejected by any of } q \text{ tests } | H_o) = (1-\nu)^q$,

and

(3.2.20) $\Pr(H_o \text{ rejected by at least one of } q \text{ tests } | H_o) = 1-(1-\nu)^q$.

Thus

(3.2.31) $\alpha = 1 - (1 - \nu)^q$,

or

(3.2.22) $\nu = 1 - (1 - \alpha)^{1/q}$.

Therefore, if each individual test is of size ν, where ν is given by 3.2.22, then the intersection test is of size α. This method of fixing the size of the test is, of course, arbitrary.

The question now arises as to how the power function of the proposed intersection test compares with the power function of the test on the means (by test on the means is meant the test on the two-way classification formed by averaging the observations in each cell).

For the modified circular model the test on the means would be by the ordinary F test with variance per observation

(3.2.23) $\frac{\sigma^2}{r^2} \; \underline{1}' \; V \; \underline{1} = \left(\frac{r(1-\rho)(1+\rho^{r+1})-2\rho(1-\rho^r)}{(1-\rho)^3 \; (1+\rho^{r+1})}\right) \frac{\sigma^2}{r^2}$.

The formula (3.2.23) follows from (1.5.22).

3.3 Comparison of the Power Functions. It will be shown that the α-size test on the means is locally more powerful than the α-size intersection test at least for most values of α. It has, however, been suggested by Professor Roy that the intersection test is likely to be more powerful at other values of the non-centrality parameter.

For small Δ^2 and σ^2 taken as one (σ^2 can be taken as one without loss of generality since it cancels out in the comparison), the power function of the test on the means is

$$(3.3.1)\quad \beta = ((1-\alpha) + a_1 K \Delta^2 + \delta) \; e^{-K\Delta^2} \quad ,$$

where

$$(3.3.2)\quad K = \frac{r^2(1-\rho)^3(1+\rho^{r+1})}{r(1-\rho)(1+\rho^{r+1})-2\rho(1-\rho^r)} \quad ,$$

$$(3.3.3)\quad a_1 = \frac{\binom{\frac{b(t-1)}{2}}{1}}{\binom{\frac{t-1}{2}}{1}} \cdot \frac{1}{b-1} \; \frac{1}{B(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2})} \int_0^{F_{1-\alpha}} \frac{(\frac{F}{b-1})^{\frac{t-1}{2}}}{(1+\frac{F}{b-1})^{\frac{b(t-1)+2}{2}}} \, dF \, ,$$

and δ is a small number vanishing at the same rate as Δ^4.

Similarly, for small Δ^2, the power function of the intersection test is

$$(3.3.4)\quad \beta^* = 1 - \prod_{h=1}^{q} \beta_h \quad ,$$

where

$$(3.3.5)\quad \beta_h \doteq \left((1-\alpha)^{1/q} + \frac{b_1}{\lambda_h} + \delta_h\right) e^{-\frac{\Delta^2}{\lambda_k}} ,$$

$h = 1, 2, \ldots, q$, and b_1 is the same as a_1 except that $F_{1-\alpha}$ is replaced by $F_{(1-\alpha)^{1/q}}$, and the δ_h are all small numbers vanishing at the same rate as Δ^4. Therefore,

$$(3.3.6)\quad \prod_{h=1}^{q} \beta_h = \left((1-\alpha) + b_1(1-\alpha)^{\frac{q-1}{q}} E\,\Delta^2 + \delta^*\right) e^{-E\Delta^2} ,$$

where

$$(3.3.7)\quad E = \sum_{h=1}^{q} \frac{1}{\lambda_h} ,$$

and δ^* is a small number vanishing at the same rate as Δ^4 .

It can be proved that

$$(3.3.8)\quad (1-\alpha)^{\frac{q-1}{q}} b_1 \geq a_1 ,$$

where α is any number between zero and one and q is an integer. The quantities a_1 and b_1 have been defined, and

$$(3.3.9)\quad 1 - \alpha = \frac{\frac{1}{b-1}}{B\left(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2}\right)} \int_0^{F_{1-\alpha}} \frac{\left(\frac{F}{b-1}\right)^{\frac{t-3}{2}}}{\left(1 + \frac{F}{b-1}\right)^{\frac{b(t-1)}{2}}} \, dF ,$$

$$(3.3.10) \quad (1-\alpha)^{1/q} = \frac{\frac{1}{b-1}}{B(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2})} \int_0^{F_{(1-\alpha)^{1/q}}} \frac{(\frac{F}{b-1})^{\frac{t-3}{2}}}{(1+\frac{F}{b-1})^{\frac{b(t-1)}{2}}} \, d\,F \; .$$

Make the transformation

$$(3.3.11) \quad x = \frac{\frac{F}{b-1}}{1+\frac{F}{b-1}} \; .$$

Then (3.3.8) becomes

$$(3.3.12) \quad (1-\alpha)^{\frac{q-1}{q}} \int_0^{X_{(1-\alpha)^{1/q}}} X^{\frac{t-1}{2}} (1-X)^{\frac{(b-1)(t-1)-2}{2}} \, d\,X \geq$$

$$\int_0^{X_{1-\alpha}} X^{\frac{t-1}{2}} (1-X)^{\frac{(b-1)(t-1)-2}{2}} \, d\,X \; ;$$

and (3.3.9) and (3.3.10) become

$$(3.3.13) \quad (1-\alpha) = \frac{1}{B(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2})} \int_0^{X_{1-\alpha}} X^{\frac{t-3}{2}} (1-X)^{\frac{(b-1)(t-1)-2}{2}} \, d\,x$$

and

$$(3.3.14) \quad (1-\alpha)^{1/q} = \frac{1}{B(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2})} \int_0^{X_{(1-\alpha)^{1/q}}} X^{\frac{t-3}{2}} (1-X)^{\frac{(b-1)(t-1)-2}{2}} \, d\,x \; .$$

The integrals on either side of (3.3.12) can be evaluated by parts. The result is that

$$(3.3.15)\quad \int_0^{X_{(1-\alpha)}} X^{\frac{t-1}{2}} (1-X)^{\frac{(b-1)(t-1)-2}{2}} \, dX$$

$$= (1-\alpha)\, X_{1-\alpha}\, B\left(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2}\right) - \int_0^{X_{1-\alpha}} u(x)\, dx ,$$

and that

$$(3.3.16)\quad \int_0^{X_{(1-\alpha)^{1/q}}} X^{\frac{t-1}{2}} (1-X)^{\frac{(b-1)(t-1)-2}{2}} \, dx$$

$$= (1-\alpha)^{1/q}\, X_{(1-\alpha)^{1/q}}\, B\left(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2}\right) - \int_0^{X_{(1-\alpha)^{1/q}}} v(x) dx,$$

where

$$(3.3.17)\quad v(x) = \int_0^{X} y^{\frac{t-3}{2}} (1-y)^{\frac{(b-1)(t-1)-2}{2}} \, dy .$$

Thus, since $X_{(1-\alpha)^{1/q}} \geq X_{(1-\alpha)}$, it follows that (3.3.12) reduces to

$$(3.3.18)\quad (1-\alpha)\, B\left(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2}\right) \left(X_{(1-\alpha)^{1/q}} - X_{(1-\alpha)}\right)$$

$$+ \left(1-(1-\alpha)^{\frac{q-1}{q}}\right) \int_0^{X_{1-\alpha}} v(x)\, dx - (1-\alpha)^{\frac{q-1}{q}} \int_{X_{1-\alpha}}^{X_{(1-\alpha)^{1/q}}} v(x)\, dx \geq 0 .$$

The function $v(x)$ is positive and monotone increasing. Therefore, by the law of the mean,

$$(3.3.19) \qquad \int_{X_{(1-\alpha)}}^{X_{(1-\alpha)^{1/q}}} v(x)\,dx \le (X_{(1-\alpha)^{1/q}} - X_{(1-\alpha)})(1-\alpha)^{1/q} \, B\left(\frac{t-1}{2}, \frac{(b-1)(t-1)}{2}\right),$$

and it follows that (3.3.18) is true whereby (3.3.12) is true.

Now assume that

$$(3.3.20) \qquad ((1-\alpha) + b_1(1-\alpha)^{\frac{q-1}{q}} E\,\Delta^2 + \delta^*)\, e^{-E\Delta^2} \ge ((1-\alpha) + a_1 K \Delta^2 + \delta)\, e^{-K\Delta^2} .$$

If $K \ge E$, then (3.3.20) is true when Δ^2 is small because of (3.3.12) and the fact that δ^* and δ are vanishing at the same rate as Δ^4. If $K < E$, then (3.3.20) can be put in the form

$$(3.3.21) \qquad (b_1(1-\alpha)^{\frac{q-1}{q}} E + (1-\alpha)(K-E) - a_1K)\,\Delta^2 + \delta^{**} \ge 0 ,$$

where δ^{**} is vanishing at the same rate as Δ^4, and (3.3.21) will be true for small Δ^2 if and only if

$$(3.3.22) \qquad b_1(1-\alpha)^{\frac{q-1}{q}} E + (1-\alpha)(K-E) - a_1 K \ge 0 .$$

The constant E is positive. Then from (3.3.8), it follows that (3.3.22) will hold when

$$(3.3.23) \qquad a_1 \ge 1 - \alpha .$$

From the Cauchy-Schwarz inequality it follows that

$$(3.3.24) \qquad a_1 \geq \frac{\binom{\frac{b(t-1)}{2}}{1}}{\binom{\frac{t-1}{2}}{1}} (1-\alpha)^2 \quad .$$

Thus (3.3.23) will hold when

$$(3.3.25) \qquad \binom{\frac{b(t-1)}{2}}{1} (1-\alpha) \geq \binom{\frac{t-1}{2}}{1} \quad ,$$

which is true except for α near one.

CHAPTER IV

ANALYSIS OF VARIANCE IN THE PRESENCE OF INTERACTION

4.1 Introduction. This chapter contains the development of a method for analyzing a two-way classification with interaction when the within-cell errors have a modified circular dependence.

4.2 The Analysis. Suppose that the random variables $\underline{Y}_{ij}$, $\underline{Y}_{ij}$ is $(r\times1)$, $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, b$, are independently and identically distributed as multivariate normal with

$$(4.2.1)\quad E(\underline{Y}_{ij}) = (\mu + \tau_i + \beta_j + \gamma_{ij})\ \underline{1}\ , \quad \text{all } (i, j),$$

and

$$(4.2.2)\quad \operatorname{Var}(\underline{Y}_{ij}) = (I - \rho C)^{-1}\sigma^2\ ,$$

where C is the $(r\times r)$ matrix with all elements zero except for $c_{ij} = 1$ if $|i-j| = 1$.

Let $\underline{X}_{ij} = H\,\underline{Y}_{ij}$, where H is the orthogonal matrix defined by (3.2.8). Then

$$(4.2.3)\quad E(\underline{X}'_{ij}) = \frac{(\mu + \tau_i + \beta_j + \gamma_{ij})}{\sqrt{\frac{r+1}{2}}}\left(\sum_{\ell=1}^{r} \sin\frac{\ell\pi}{r+1}\ ,\ \sum_{\ell=1}^{r} \sin\frac{2\ell\pi}{r+1}\ ,\ \ldots\ ,\ \sum_{\ell=1}^{r} \sin\frac{r\ell\pi}{r+1}\ \right),$$

and

(4.2.4) $\quad \mathrm{Var}(\underline{X}_{ij}) = D(\lambda_k)\,\sigma^2$,

where

(4.2.5) $$\lambda_k = \frac{1}{1-2\rho\cos\frac{k\pi}{r+1}+\rho^2}\ , \quad k = 1, 2, \ldots, r\ .$$

Now consider the random variables $\underline{Z}_{ij}$,

(4.2.6) $\quad \underline{Z}'_{ij} = (X_{ij1}, X_{ij3}, \ldots, X_{ijp})$,

where $p=r$ if r is odd and $p = r-1$ if r is even. The expectation of $\underline{Z}'_{ij}$ is

(4.2.7) $$E(\underline{Z}'_{ij}) = \frac{(\mu + \tau_i + \beta_j + \gamma_{ij})}{\sqrt{\frac{r+1}{2}}}\left(\mathrm{Cot}\,\frac{1}{2}\frac{\pi}{r+1}\ ,\ \mathrm{Cot}\,\frac{3}{2}\frac{\pi}{r+1}\ ,\ \ldots,\ \mathrm{Cot}\,\frac{p}{2}\frac{\pi}{r+1}\right),$$

and

(4.2.8) $\quad \mathrm{Var}(\underline{Z}_{ij}) = D(\lambda^*_\ell)\,\sigma^2$,

where

(4.2.9) $$\lambda^*_\ell = \frac{1}{1-2\rho\cos\frac{(2\ell-1)\pi}{r+1}+\rho^2}\ ,$$

$\ell = 1, 2, \ldots, (r+1)/2$ if r is odd and $\ell = 1, 2, \ldots, r/2$ is r is even.

It is required to find (qx1) vectors $\underline{\ell}_1$ and $\underline{\ell}_2$ such that

(4.2.10) $\quad \mathrm{Var}\,(\underline{\ell}'_1\,\underline{Z}_{ij}) = \mathrm{Var}\,(\underline{\ell}'_2\,\underline{Z}_{ij})$,

and

$$(4.2.11) \qquad \operatorname{Cov}\left(\underline{\ell}_1' \, \underline{Z}_{ij}, \; \underline{\ell}_2' \, \underline{Z}_{ij}\right) = 0 \, .$$

The integer q is $r/2$ if r is even and $(r+1)/2$ if r is odd.

From (1.6.1) it follows that

$$(4.2.12) \qquad \lambda_\ell^* = \frac{\sigma^2}{1-\rho^2} \left\{ 1 + 2 \sum_{n=1} \rho^n \cos \frac{n(2\ell-1)\pi}{r+1} \right\} \, .$$

If $\underline{\omega}$ is the $(m\text{x}1)$ vector whose j-th element is ρ^{j-1}, and W is the $(m\text{x}q)$ matrix whose (n,j) element is the coefficient of ρ^{i-1} in the expansion (4.2.12) of λ_{2j-1}^*, then the j-th element in the $(1\text{x}q)$ vector $\underline{\omega}'W$ is the m-th order approximation of λ_{2j-1}^*.

Suppose that r is odd. Then the $(r-1)/2$ -th row of W is

$$(4.2.13) \qquad \underline{w}'_{\frac{r-1}{2}} = \left(\cos \frac{r-1}{2} \frac{\pi}{r+1}, \; \cos \frac{r-1}{2} \frac{3\pi}{r+1}, \; \ldots, \; \cos \frac{r-1}{2} \frac{r\pi}{r+1}\right) \cdot \frac{2\sigma^2}{1-\rho^2} \, ,$$

and

$$(4.2.14) \qquad \underline{w}_n' \, \underline{w}_{\frac{r-1}{2}} = \sum_{\ell=1}^{(r+1)/2} \cos\,(2\ell-1)\frac{n\pi}{r+1} \cos\,(2\ell-1)\frac{r-1}{2}\frac{\pi}{r+1}$$

Making use of a well known trignometric identity, (4.2.14) becomes

$$(4.2.15) \qquad \underline{w}_n' \, \underline{w}_{\frac{r-1}{2}} = \frac{1}{2} \sum_{\ell=1}^{(r+1)/2} \left(\cos\,(2\ell-1)\left(n + \frac{r-1}{2}\right)\frac{\pi}{r+1} + \cos\,(2\ell-1)\left(n - \frac{r-1}{2}\right)\frac{\pi}{r+1}\right) ,$$

which, upon application of (1.6.5), becomes

$$(4.2.16)\quad \underline{w}'_n\,\underline{w}_{\frac{r-1}{2}} = \frac{\sin\,(n+\frac{r-1}{2})\,\pi}{4\,\sin\,(n+\frac{r-1}{2})\,\frac{\pi}{r+1}} + \frac{\sin\,(n-\frac{r-1}{2})\,\pi}{4\,\sin\,(n-\frac{r-1}{2})\,\frac{\pi}{r+1}}$$

It is clear that the right-hand side of (4.2.16) is zero with the exception of the cases where

$$(4.2.17)\quad n = \begin{cases} s(r+1) - \frac{r-1}{2}, & s = 1, 2, \ldots, \\ s(r+1) + \frac{r-1}{2}, & s = 1, 2, \ldots, \end{cases}$$

and for these cases

$$(4.2.18)\quad \underline{w}'_n\ \underline{w}_{\frac{r-1}{2}} = \frac{(r+1)}{4}\,(-1)^s \quad .$$

Suppose $\underline{\ell}_1$ and $\underline{\ell}_2$ are taken as the following: If q is odd,

$$(4.2.19)\quad \begin{aligned} \ell_{1,2j-1} &= 2\sqrt{\frac{\cos\,(\frac{r-1}{r+1})\,(4j-3)\,\frac{\pi}{2}}{(r+1)^2}} \\ \ell_{1,2j} &= 0 \end{aligned} \quad ,\quad j = 1, 2, \ldots, (r+3)/4,$$

and

$$\ell_{2,2j-1} = 0$$

(4.2.20)

$$\ell_{2,2j} = 2\sqrt{\frac{-\cos\left(\frac{r-1}{r+1}\right)(4j-1)\frac{\pi}{2}}{(r+1)^2}}\quad , j = 1,2,\ldots,(r-1)/4;$$

and, if q is even,

$$\ell_{1,2j-1} = 2\sqrt{\frac{\cos\left(\frac{r-1}{r+1}\right)(4j-3)\frac{\pi}{2}}{(r+1)^2}}\quad , j = 1, 2, \ldots, (r+1)/4,$$

(4.2.21)

$$\ell_{1,2j} = 0$$

and

$$\ell_{2,2j-1} = 0$$

(4.2.22)

$$\ell_{2,2j} = 2\sqrt{\frac{-\cos\left(\frac{r-1}{r+1}\right)(4j-1)\frac{\pi}{2}}{(r+1)^2}}\quad , j = 1,2,\ldots,(r+1)/4 .$$

That no ambiguity can arise in this choice of ℓ_1 and ℓ_2 is seen as follows.

(4.2.23) $$\cos\frac{r-1}{r+1}(4j-3)\frac{\pi}{2} = \sin(4j-3)\frac{\pi}{2}\sin(4j-3)\frac{\pi}{r+1} ,$$

but $\sin(4j-3)\frac{\pi}{2} = 1$ and $\sin(4j-3)\frac{\pi}{r+1} > 0$ for $j = 1, 2, \ldots, (r+3)4$.
Similarly,

(4.2.24) $$\cos\frac{(r-1)}{r+1}(4j-1)\frac{\pi}{2} = \sin(4j-1)\frac{\pi}{2}\sin(4j-1)\frac{\pi}{r+1}$$

but $\sin(4j-1)\frac{\pi}{2} = -1$ and $\sin(4j-1)\frac{\pi}{r+1} > 0$ for $j = 1, 2, \ldots, (r+1)/4$.

If g_1 and g_2 are the (qx1) vectors whose j-th elements are ℓ_{1j}^2 and ℓ_{2j}^2, then

$$(4.2.25)\quad \mathrm{Var}\,(\underline{\ell}_1'\,\underline{Z}_{ij}) - \mathrm{Var}(\,\underline{\ell}_2'\;\underline{Z}_{ij}) = \lim_{m \to} \underline{\omega}'\,W\,(\underline{g}_1 - \underline{g}_2)\,.$$

However,

$$(4.2.26)\quad \underline{g}_1 - \underline{g}_2 = \frac{4}{(r+1)^2}\;\underline{w}_{\frac{r-1}{2}}\quad ,$$

and, therefore, it follows from (4.2.16) and (4.2.18) that

$$(4.2.27)\ \lim_{m\to\infty} \underline{\omega}'\,W\,(\underline{g}_1 - \underline{g}_2) = \frac{\sigma^2}{1-\rho^2}\,\Big(\sum_{\Delta=1}(-1)^s\;\frac{\rho}{r+1}^{s(r+1)-\frac{(r-1)}{2}}$$

$$+\sum_{s=0}(-1)^s\;\frac{\rho}{r+1}^{s(r+1)+\frac{r-1}{2}}\qquad ,$$

or that

$$(4.2.28)\ \lim_{m\to\infty} \underline{\omega}'W(\underline{g}_1 - \underline{g}_2) = \frac{\sigma^2}{1-\rho^2}\quad \frac{\rho^{\frac{r-1}{2}}}{r+1}\quad \frac{1+\rho^2}{1-\rho^{r+1}}\quad .$$

Therefore,

$$(4.2.29)\ \lim_{m\to\infty}\ (\mathrm{Var}(\underline{\ell}_1'\,\underline{Z}_{ij}) - \mathrm{Var}(\underline{\ell}_2'\,\underline{Z}_{ij})) = 0\quad .$$

Since, for q odd,

$$(4.2.30)\quad \mathrm{Var}(\underline{\ell}_1'\,\underline{Z}_{ij}) = \frac{4}{(r+1)^2}\sum_{j=1}^{(r+3)/4}\frac{\sin\,(4j-3)\,\frac{\pi}{r+1}}{1-2\,\rho\,\cos(4j-3)\,\frac{\pi}{r+1}+\rho^2}\quad \sigma^2$$

and

$$(4.2.31)\quad \mathrm{Var}(\underline{\ell}_2' \, \underline{Z}_{ij}) = \frac{4}{(r+1)^2} \sum_{j=1}^{(r-1)/4} \frac{\sin(4j-1)\frac{\pi}{r+1}}{1-2\rho\cos(4j-1)\frac{\pi}{r+1} + \rho^2} \sigma^2 ,$$

and, for q even ,

$$(4.2.32)\quad \mathrm{Var}(\underline{\ell}_1' \, \underline{Z}_{ij}) = \frac{4}{(r+1)^2} \sum_{j=1}^{(r+1)/4} \frac{\sin(4j-3)\frac{\pi}{r+1}}{1-2\rho\cos(4j-1)\frac{\pi}{r+1} + \rho^2} \sigma^2 ,$$

it follows that $\mathrm{Var}(\underline{\ell}_1' \, \underline{Z}_{ij})$ and $\mathrm{Var}(\underline{\ell}_2' \, \underline{Z}_{ij})$ both tend to zero as r increases. However, it is clear from (4.2.28) that the difference between the variances tends to zero at a faster rate; i.e.,

$$(4.2.34)\quad \lim_{r \to \infty} \frac{\mathrm{Var}(\underline{\ell}_1' \, \underline{Z}_{ij}) - \mathrm{Var}(\underline{\ell}_2' \, \underline{Z}_{ij})}{\mathrm{Var}(\underline{\ell}_k' \, \underline{Z}_{ij})} = 0$$

for $k = 1$ and $k = 2$. Finally, it follows from (4.2.19), (4.2.20), (4.2.21) and (4.2.22) that

$$(4.2.35)\quad \mathrm{cov}\,(\underline{\ell}_1' \, \underline{Z}_{ij}, \underline{\ell}_2' \, \underline{Z}_{ij}) = 0 .$$

Suppose that r is even. Let $\underline{Z}_{ij}^{*} = D(\ \sin \frac{k\pi}{r+1})\underline{X}_{ij}^{*}$. Then

$$(4.2.36)\quad E(\underline{Z}_{ij}^{*\,\prime}) = \frac{(\mu + \tau_i + \beta_j + \gamma_{ij})}{\frac{r+1}{2}} \left(\sin \frac{\pi}{r+1} \cot \frac{1}{2}\frac{\pi}{r+1} , 0,\right.$$

$$\left. \sin \frac{3\pi}{r+1} \ \mathrm{Cot}\, \frac{3}{2}\frac{\pi}{r+1} , 0, \ldots, \ \sin \frac{(r-1)\pi}{r+1} \ \mathrm{Cot}\, \frac{(r-1)\pi}{2(r+1)} , 0\right),$$

and

$$(4.2.37) \quad \operatorname{Var}(\underline{Z}^{*}_{ij}) = D(\lambda_k \sin \frac{k\pi}{r+1}) \ \sigma^2 .$$

From (1.6.2) it follows that

$$(4.2.38) \quad \lambda_k \sin \frac{k\pi}{r+1} = \sum_{n=1} \rho^{n-1} \sin \frac{nk\pi}{r+1} .$$

Let $\underline{\omega}$ be the (mxl) vector whose j-th element is ρ^{j-1}, and W be the (mxr) matrix whose (n,j) element is the coefficient of ρ^{i-1} in (4.2.38). Then $\underline{\omega}' W \sigma^2$ is the (lxr) vector whose k-th element is the m-th order approximation of the k-th element in the diagonal matrix on the right-hand side of (4.2.37). W is

$$(4.2.39) \quad \underline{w}'_r \underline{w}_n = \sum_{k=1}^{r} \sin \frac{nk\pi}{r+1} \sin \frac{rk\pi}{r+1} ,$$

which is the same as

$$(4.2.40) \quad \underline{w}'_r \underline{w}_n = \frac{1}{2} \sum_{k=1}^{r} (\cos (n-r) \frac{k\pi}{r+1} - \cos (n+r) \frac{k\pi}{r+1}) ;$$

and this, by (1.6.3), is

$$(4.2.41) \quad \underline{w}'_r \underline{w}_n = \frac{1}{2} \left(\frac{\sin \frac{r}{2} \frac{n-r}{r+1} \pi}{\sin \frac{1}{2} \frac{n-r}{r+1} \pi} \cos \frac{(n-r)}{2} \pi \right.$$

$$\left. - \frac{\sin \frac{r}{2} \frac{n+r}{r+1} \pi}{\sin \frac{1}{2} \frac{n+r}{r+1} \pi} \cos \frac{(n+r)}{2} \pi \right) .$$

Finally, it turns out that $\underline{w}'_r \, \underline{w}_n = 0$ with the exception of the cases

$$(4.2.42) \qquad n = \begin{cases} 2s\,(r+1) + r & , \quad s = 0,\ 1,\ 2,\ \ldots, \\ 2s\,(r+1) - r & , \quad s = 1,\ 2,\ 3,\ \ldots, \end{cases}$$

and, in these cases,

$$(4.2.43) \qquad \underline{w}'_r \, \underline{w}_n = \begin{cases} \frac{r}{2} & n = r,\ 3r+2,\ 5r+4,\ \ldots, \\ -\frac{r}{2} & n = r+2,\ 3r+4,\ 5r+6,\ \ldots\ . \end{cases}$$

Let

$$(4.2.44) \qquad \begin{aligned} \ell^*_{1,2j-1} &= \sqrt[2]{\frac{\sin\,(2j-1)\,\frac{r\pi}{r+1}}{r^2}} \\ \ell^*_{1,2j} &= 0 \end{aligned} \qquad , \ j = 1,\ 2,\ \ldots,\ \frac{r}{2}\ ,$$

and

$$(4.2.45) \qquad \begin{aligned} \ell^*_{2,2j-1} &= 0 \\ \ell^*_{2,2j} &= \sqrt[2]{\frac{-\sin\frac{2jr}{r+1}\pi}{r^2}} \end{aligned} \qquad , \ j = 1,\ 2,\ \ldots,\ \frac{r}{2}\ .$$

There can be no ambiguity in this choice of $\underline{\ell}^*_1$ and $\underline{\ell}^*_2$ since

$$(4.2.46) \qquad \sin(2j-1)\,\frac{r\pi}{r+1} = -\sin(2j-1)\,\frac{\pi}{r+1}\,\cos\,(2j-1)\pi \quad ,$$

which is positive since $\cos(2j-1)\pi = -1$ and $\sin\,(2j-1)\,\frac{\pi}{r+1} > 0$ for $j = 1, 2, \ldots, r/2$. Similarly,

$$(4.2.47)\quad \sin\frac{2jr\pi}{r+1} = -\ \sin\frac{2j\pi}{r+1}\ \cos 2j\pi \ ,$$

and the right-hand side is negative since $\cos 2j\pi = 1$ and $\sin\frac{2j\pi}{r+1} > 0$ for $j = 1, 2, \ldots, r/2$.

If $\underline{g}_1$ and $\underline{g}_2$ are the (rx1) vectors whose j-th elements are ℓ^{*2}_{1j} and ℓ^{*2}_{2j}, then

$$(4.2.48)\quad \mathrm{Var}(\underline{\ell}^{*\prime}_1\,\underline{Z}^{*}_{ij}) - \mathrm{Var}(\underline{\ell}^{*\prime}_2\,\underline{Z}^{*}_{ij}) - \mathrm{Var}(\underline{\ell}^{*\prime}_2\,\underline{Z}^{*}_{ij}) = \lim_{m \to}\ \underline{\omega}'W(\underline{g}_1 - \underline{g}_2)$$

Making use of (4.2.41) and (4.2.43), it follows that

$$(4.2.49)\quad \lim_{m\to\infty}\ \underline{\omega}'W\,(\underline{g}_1 - \underline{g}_2) = \frac{\sigma^2}{1-\rho^2}\quad \frac{2}{r}\,\Big(\sum_{s=0}\ \rho^{2s(r+1)+r}$$

$$-\sum_{s=1}\ \rho^{2s(r+1)-r}\ \Big)\ ,$$

or that

$$(4.2.50)\quad \lim_{m\to\infty}\ \underline{\omega}'\,W(\underline{g}_1 - \underline{g}_2) = \frac{2}{r}\ \frac{\sigma^2\,\rho^{2r}}{1+\rho^{2r+2}}\quad .$$

Since

$$(4.2.51)\quad \mathrm{Var}(\underline{\ell}^{*\prime}_1\ \underline{Z}^{*}_{ij}) = \frac{4}{r^2}\sum_{k=1}^{r/2}\ \frac{\sin\,(2k-1)\,\frac{r\pi}{r+1}}{1-2\rho\,\cos(2k-1)\,\frac{\pi}{r+1} + \rho^2}\ \ \sigma^2\ ,$$

and

$$(4.2.52)\quad \operatorname{Var}(\underline{\ell}_2^{*\prime}\underline{Z}_{ij}^*) = \frac{4}{r^2}\sum_{k=1}^{r/2}\frac{\sin\frac{2kr\pi}{r+1}}{1-2\rho\cos(2k-1)\frac{\pi}{r+1}+\rho^2}\,\sigma^2 ,$$

it follows that $\operatorname{Var}(\underline{\ell}_1^{*\prime}\underline{Z}_{ij}^*)$ and $\operatorname{Var}(\underline{\ell}_2^{*\prime}\underline{Z}_{ij}^*)$ and both tend to zero as r increases. However, it is clear from (4.2.50) that the difference between the variances tends to zero at a faster rate; i.e.,

$$(4.2.53)\quad \lim_{r\to\infty}\frac{\operatorname{Var}(\underline{\ell}_1^{*\prime}\underline{Z}_{ij}) - \operatorname{Var}(\underline{\ell}_2^{*\prime}\underline{Z}_{ij}^*)}{\operatorname{Var}(\underline{\ell}_k^{*\prime}\underline{Z}_{ij}^*)} = 0$$

for $k = 1$ and $k = 2$.

Let

$$(4.2.54)\quad \begin{aligned} U_{1ij} &= \tfrac{1}{2}\,((\underline{\ell}_1^{*\prime}\underline{Z}_{ij}^*) + (\underline{\ell}_2^{*\prime}\underline{Z}_{ij}^*)) , \\ U_{2ij} &= \tfrac{1}{2}\,((\underline{\ell}_1^{*\prime}\underline{Z}_{ij}^*) - (\underline{\ell}_2^{*\prime}\underline{Z}_{ij}^*)) . \end{aligned}$$

Then, since

$$(4.2.55)\quad E(\underline{\ell}_1^{*\prime}\underline{Z}_{ij}^*) = C(\mu + \tau_i + \beta_j + \gamma_{ij}) ,$$

where $\sqrt{\frac{r+1}{2}}\,C$ is the product of the vector $\underline{\ell}_1^*$ and the vector on the right-hand side of (4.2.36), and

$$(4.2.56)\qquad E(\underline{\ell}_2^{*\prime}\underline{Z}_{ij}^*) = 0 ,$$

it follows that

$$(4.2.57)\quad E(U_{1ij}) = E(U_{2ij}) = \frac{C}{2}(\mu + \tau_i + \beta_j + \gamma_{ij})$$

and that

$$(4.2.58)\quad \mathrm{Var}(U_{1ij}) = \mathrm{Var}(U_{2ij}) = \frac{1}{4}(\mathrm{Var}(\underline{\ell}_1^{*'}\, \underline{Z}_{ij}) + \mathrm{Var}(\underline{\ell}_2^{*'}\, \underline{Z}_{ij}))$$

Finally,

$$(4.2.59)\quad \mathrm{Cov}(U_{1ij}, U_{2ij}) = \frac{1}{4}(\mathrm{Var}(\underline{\ell}_1^{*'}\underline{Z}_{ij}) - \mathrm{Var}(\underline{\ell}_2^{*}\, \underline{Z}_{ij}))\ .$$

The U_{1ij} and U_{2ij} are a set of random variables subject to a two-way classification with two random variables per cell and constant error variance. If $\mathrm{Cov}(U_{1ij}, U_{2ij})$ is negligible, then the regular analysis of variance, which takes account of interaction between blocks and treatments, may be performed.

For the case when r is odd,

$$(4.2.60)\quad E(\underline{\ell}_1'\, \underline{Z}_{ij}) = C_1(\mu + \tau_i + \beta_j + \gamma_{ij})\ ,$$

and

$$(4.2.61)\quad E(\underline{\ell}_2'\, \underline{Z}_{ij}) = C_2(\mu + \tau_i + \beta_j + \gamma_{ij})\ ,$$

where C_1 and C_2 are constants determined by forming the scalar products of the vectors $\underline{\ell}_1$ and $\underline{\ell}_2$ with the vector on the right-hand side of (4.2.7). Thus $\underline{\ell}_1\, \underline{Z}_{ij}$ and $\underline{\ell}_2\, \underline{Z}_{ij}$ are a set of independent random variables subject to a two-way classification with two random variables per cell. If the difference between $\mathrm{Var}(\underline{\ell}_1' \underline{Z}_{ij})$ and $\mathrm{Var}(\underline{\ell}_2' \underline{Z}_{ij})$ is negligible, then a regular analysis of variance, which takes account of the interaction between blocks and treatments, may be performed.

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